

Lattice determination of the hadronic contribution to $(g - 2)_\mu$

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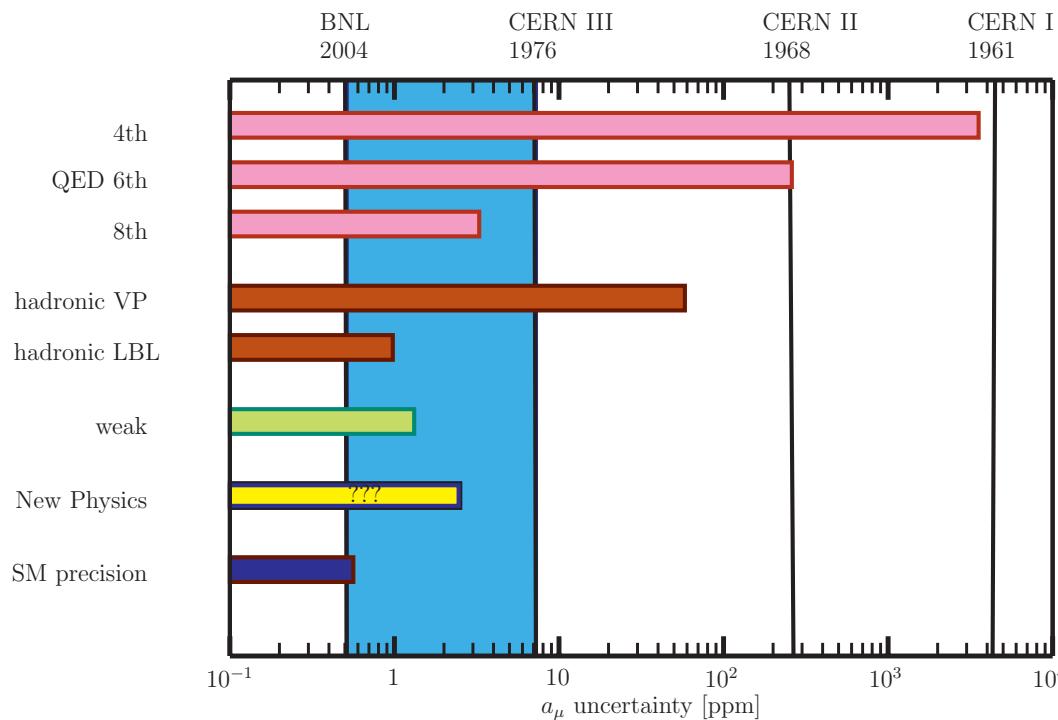


Hadronic contributions to $(g - 2)_\mu$

- Muon anomalous magnetic moment: $a_\mu = \frac{1}{2}(g - 2)_\mu$

$$a_\mu = \begin{cases} 116\,592\,080(63) \cdot 10^{-11} & \text{Experiment} \\ 116\,591\,790(65) \cdot 10^{-11} & \text{SM prediction*} \end{cases} \quad (3.2\sigma \text{ tension})$$

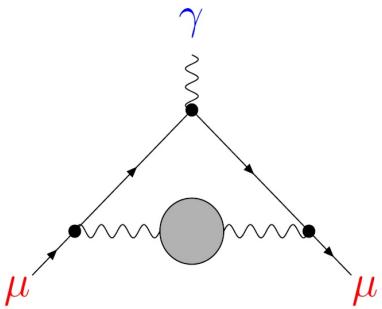
- Experimental sensitivity versus individual contributions:



[*Jegerlehner & Nyffeler, Phys Rept 477 (2009) 1]

Hadronic vacuum polarisation

- Leading contribution:



- Phenomenological approach:

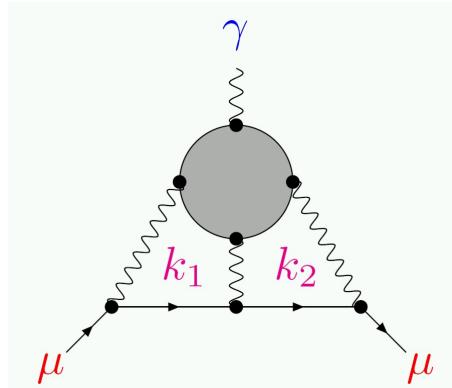
$$a_\mu^{\text{VP;had}} = \left(\frac{\alpha m_\mu}{3\pi} \right)^2 \left\{ \int_{m_\pi^2}^{E_{\text{cut}}^2} ds \frac{R_{\text{had}}^{\text{data}}(s) \hat{K}(s)}{s^2} + \int_{E_{\text{cut}}^2}^\infty ds \frac{R_{\text{had}}^{\text{pQCD}}(s) \hat{K}(s)}{s^2} \right\}$$

$$\Rightarrow a_\mu^{\text{VP;had}} = \begin{cases} (690.75 \pm 4.72) \cdot 10^{-10} & (\text{combined } e^+e^- \text{-data}) \\ (690.96 \pm 4.65) \cdot 10^{-10} & (e^+e^- \text{ and } \tau \text{-data}) \end{cases}$$

- After accounting for $\rho - \gamma$ mixing the 3σ -tension persists

[Jegerlehner & Szafron, Eur Phys J C71 (2011) 1632]

Hadronic light-by-light scattering



- Evaluate:
- Effective field theory description: ChPT + extensions
- Model-dependence introduced
- Current estimate:

[Jegerlehner & Nyffeler 2009]

$$a_\mu^{\text{LbL;had}} = (11.6 \pm 3.9) \cdot 10^{-10}$$

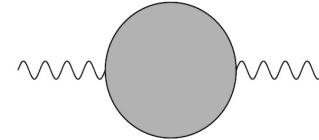
Rôle of lattice calculations

- Lattice QCD: “*ab initio*” method; eliminate model-dependence
- **Hadronic VP:**
lattice calculations must aim for a total accuracy of $\lesssim 0.5\%$
- **Hadronic LbL:**
lattice calculations with overall accuracy of $25 - 30\%$ will have major impact
- Major technical difficulties must be overcome
→ Long-term research programme

Hadronic Vacuum Polarisation

Lattice approach to hadronic vacuum polarisation

- Euclidean vacuum polarisation tensor:



$$\Pi_{\mu\nu}(q) = i \int d^4x e^{iq \cdot (x-y)} \langle J_\mu(x) J_\nu(y) \rangle \equiv (q_\mu q_\nu - g_{\mu\nu} q^2) \Pi(q^2)$$

$$J_\mu(x) = \sum_{q=u,d,s,\dots} Q_q \bar{q}(x) \gamma_\mu q(x)$$

- Determine α_μ^{had} from convolution integral:

$$a_\mu^{\text{had}} = 4\pi^2 \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dq^2 f(q^2) \{ \Pi(q^2) - \Pi(0) \}$$

$$f(q^2) = \frac{m_\mu^2 q^2 Z^3 (1 - q^2 Z)}{1 + m_\mu^2 q^2 Z^2}, \quad Z = -\frac{q^2 - \sqrt{q^4 + 4m_\mu^2 q^2}}{2m_\mu^2 q^2}$$

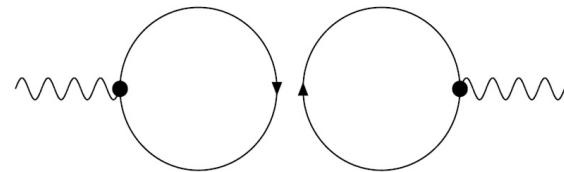
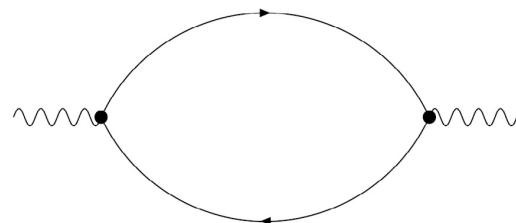
Problems for lattice calculations:

- Convolution integral dominated by momenta near m_μ :

maximum of $f(q^2)$ located at: $(\sqrt{5} - 2)m_\mu^2 \approx 0.003 \text{ GeV}^2$

lowest momentum transfer: $\left(\frac{2\pi}{T}\right)^2 \approx 0.06 \text{ GeV}^2$

- Subtracted amplitude enters convolution integral: $\hat{\Pi}(q^2) \equiv \Pi(q^2) - \Pi(0)$
→ determine $\Pi(0)$ via extrapolation $q^2 \rightarrow 0$
- Contributions from quark disconnected diagrams



Large noise-to-signal ratio

ChPT and the rôle of disconnected diagrams

[Della Morte & Jüttner, JHEP 11 (2010) 154]

Two key results:

- Connected and disconnected contributions have individual continuum and finite-volume limits in *partially quenched* QCD
- Disconnected contribution strongly suppressed relative to the connected one:

Combination $\hat{\Pi}(q^2) \equiv \Pi(q^2) - \Pi(0)$ enters convolution integral:

$$\Rightarrow \frac{\Pi_{\text{disc}}(q^2) - \Pi_{\text{disc}}(0)}{\Pi_{\text{conn}}(q^2) - \Pi_{\text{conn}}(0)} = -\frac{1}{10} \quad \text{in partially quenched ChPT}$$

Systematic effects in lattice calculations

Lattice artefacts:

$$\left\langle \frac{m_N}{f_\pi} \right\rangle^{\text{lat}} = \left\langle \frac{m_N}{f_\pi} \right\rangle^{\text{cont}} + O(a^p), \quad p \geq 1$$

→ requires **extrapolation** to continuum limit, $a \rightarrow 0$

Finite volume effects:

- Mass estimates distorted by finite box size
- Rule of thumb:
 $L \approx 2.5 - 3 \text{ fm}$ and $m_\pi L > 3 - 4$ sufficient for many purposes

Unphysical quark masses:

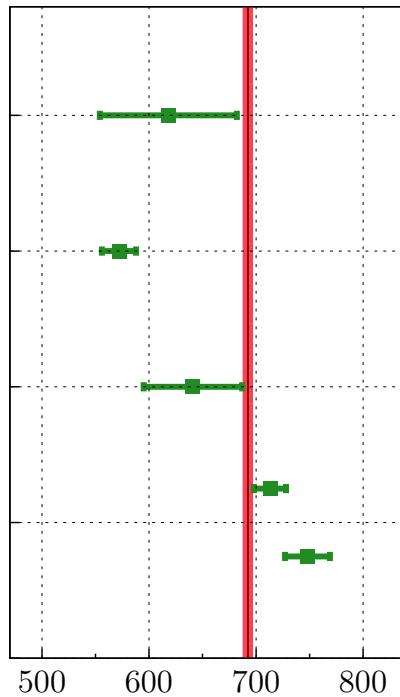
- Perform **chiral extrapolations** to physical values of m_u, m_d
- Use Chiral Perturbation Theory (ChPT) as a guide

Recent lattice calculations

$a_\mu^{\text{VP;had}} / 10^{-10}$	errors	N_f	action	Collab.
713(15)				
748(21)	stat	2+1	stagg.	Aubin & Blum 2006
572(16)	stat	2	tmQCD	ETMC 2011
641(46)	stat, sys	2+1	DWF	UKQCD 2011
618(64)	stat+sys	$2+1_q$	Wilson	Mainz 2011

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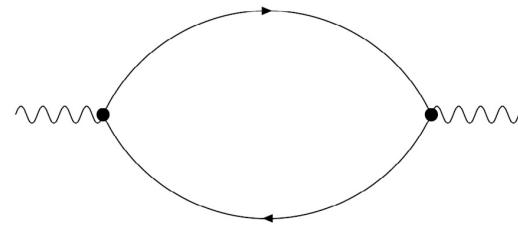


Current accuracy **not competitive**
with phenomenological approach!

Mainz lattice calculation

- Focus on **connected** contribution:

$$\Pi(q^2) = \frac{i \int d^4x e^{iq \cdot x} \langle J_\mu^{rs}(x) J_\nu^{sr}(0) \rangle}{q_\mu q_\nu - g_{\mu\nu} q^2}$$



- $\hat{\Pi}(q^2) \equiv \Pi(q^2) - \Pi(0)$ enters convolution integral

→ requires extrapolation to $q^2 = 0$

- Lattice momenta: $q_\mu = n_\mu \frac{2\pi}{L_\mu}$, $n_\mu = 0, 1, \dots, L_\mu/a - 1$

$$L = 2.5 \text{ fm}, \quad T = 2L \quad \Rightarrow \quad q^2 \gtrsim 0.06 \text{ GeV}^2$$

→ Lack of accurate data points near $q^2 = 0$

→ Extrapolation to $q^2 = 0$ not well controlled

Twisted boundary conditions

[Bedaque 2004; de Divitiis, Petronzio & Tantalo 2004; Flynn, Jüttner & Sachrajda 2005]

- Apply “twisted” spatial boundary conditions;

Impose periodicity up to a phase $\vec{\theta}$:

$$\psi(x + L\hat{e}_k) = e^{i\theta_k} \psi(x) \quad \Rightarrow \quad q_k = n_k \frac{2\pi}{L} + \frac{\theta_k}{L}$$

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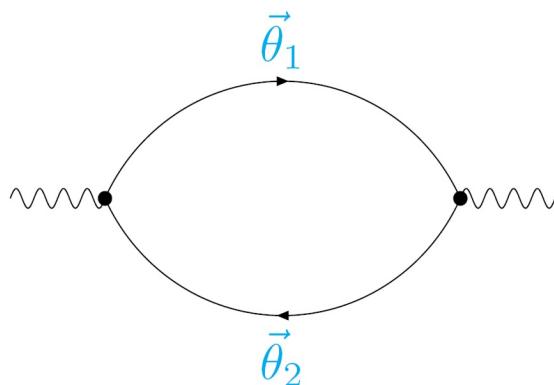
- Imposing twisted boundary conditions in **valence** sector only:

exponentially small finite-volume effects

[Sachrajda & Villadoro, Phys Lett B609 (2005) 73]

- Can tune q^2 to any desired value

→ Compute **connected** contribution to $\Pi(q^2)$



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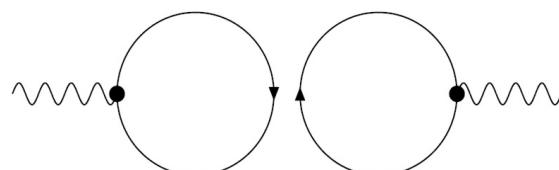
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[Sachrajda & Villadoro, Phys Lett B609 (2005) 73]

- Effect of twist angle cancels in **disconnected** contribution to $\Pi(q^2)$

→ Compute disconnected diagrams for Fourier modes only;

→ Validate their relative suppression



Run Table

- Discretisation: $N_f = 2$ flavours of $\mathcal{O}(a)$ improved Wilson quarks
- 3 lattice spacings: $a = 0.08, 0.065, 0.05 \text{ fm}$
- Pion masses: $m_\pi = 195 - 700 \text{ MeV}$

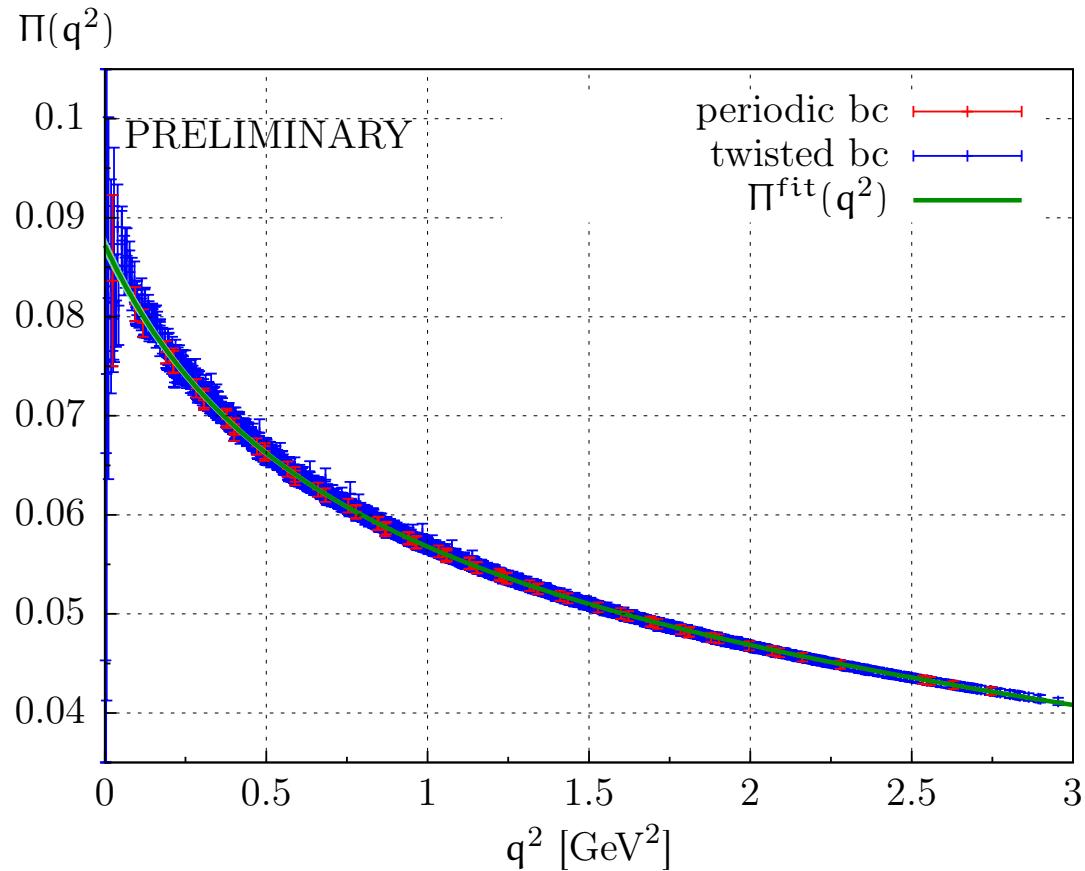
β	$a[\text{fm}]$	lattice	$L[\text{fm}]$	masses	$m_\pi L$	Labels
5.20	0.08	$32^3 \cdot 64$	2.6	4 masses	4.7 – 7.9	A2 – A5
5.20	0.08	$48^3 \cdot 96$	3.8	1 mass	5.4	B6
5.30	0.065	$32^3 \cdot 64$	2.0	3 masses	4.7 – 7.9	E3 – E5
5.30	0.065	$48^3 \cdot 96$	3.2	2 masses	5.0, 4.2	F6, F7
5.30	0.065	$64^3 \cdot 128$	4.2	1 mass	4.0	G8
5.50	0.05	$48^3 \cdot 96$	2.5	4 masses	4.2 – 7.7	N3 – N5, N6
5.50	0.05	$64^3 \cdot 128$	3.4	1 mass	4.2	O7

New ensembles: reduction of uncertainties due to lattice artefacts and chiral extrapolations

Update on $a_\mu^{\text{VP;had}}$

[Mainz Group, to appear]

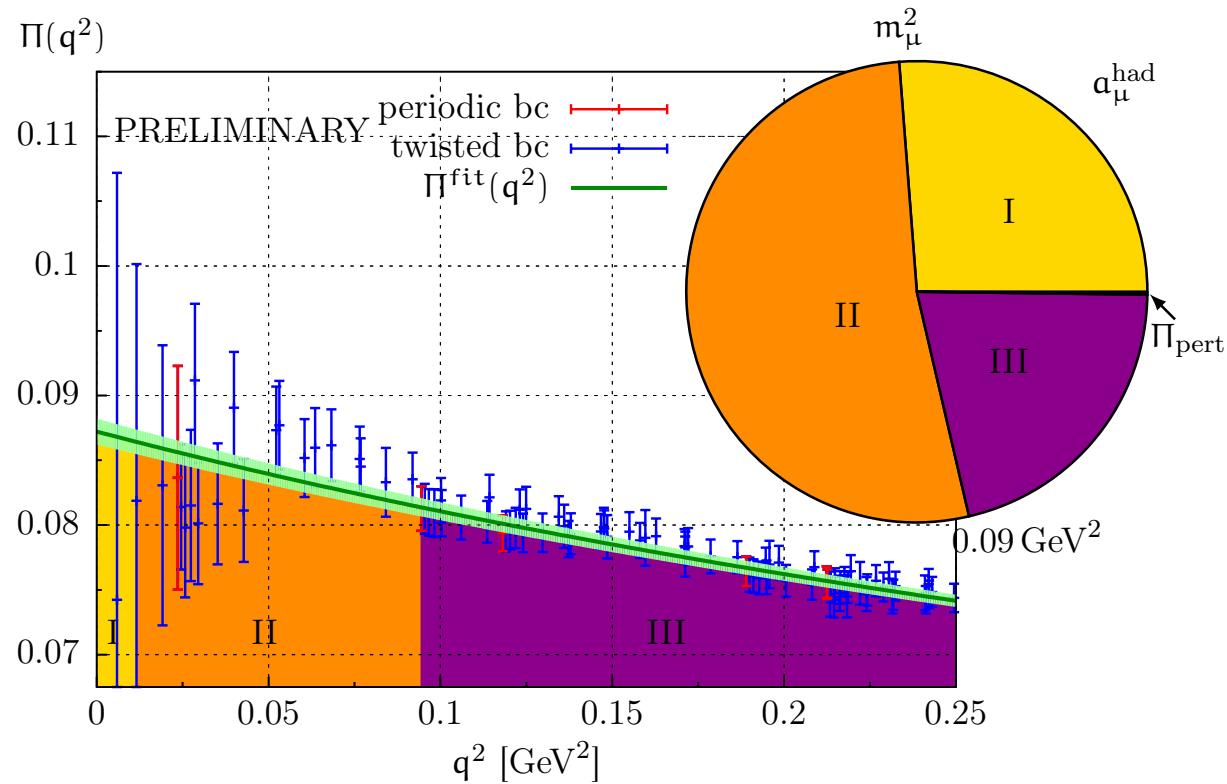
- $\Pi(q^2)$ on ‘G8’ ensemble: $m_\pi \approx 195 \text{ MeV}$, $a = 0.065 \text{ fm}$, $L = 4.0 \text{ fm}$



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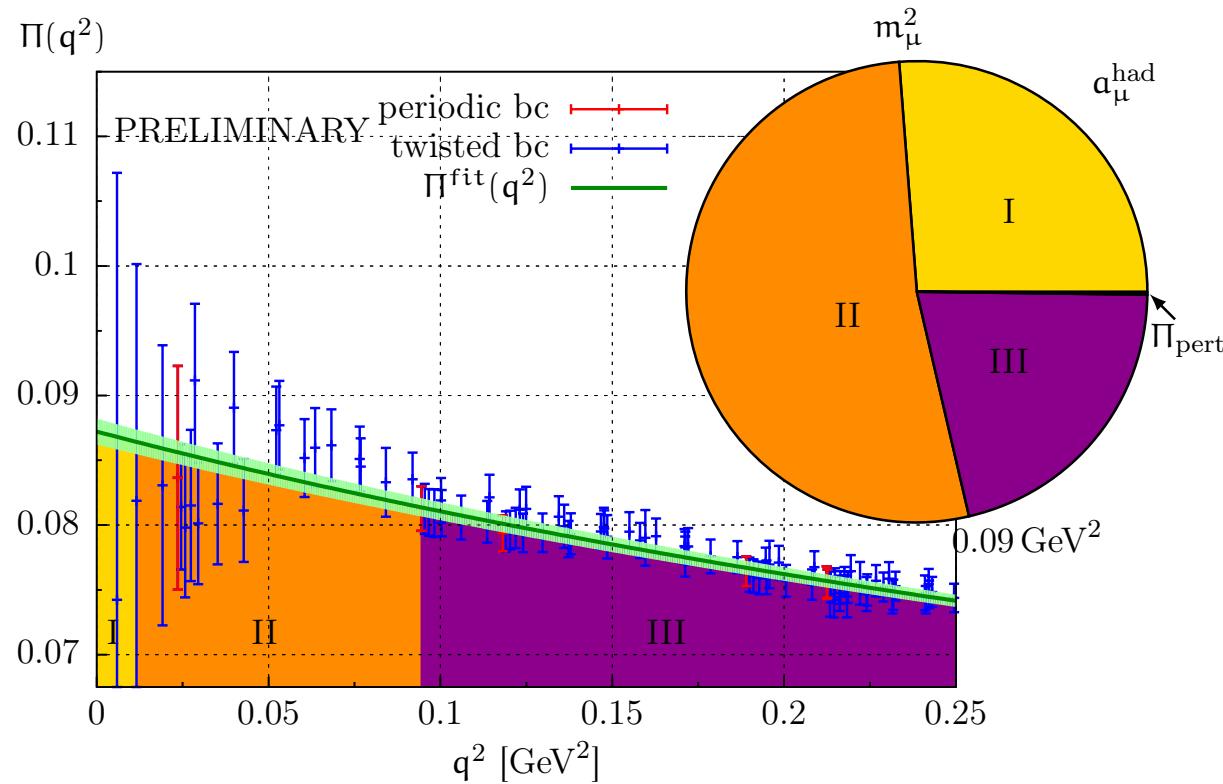
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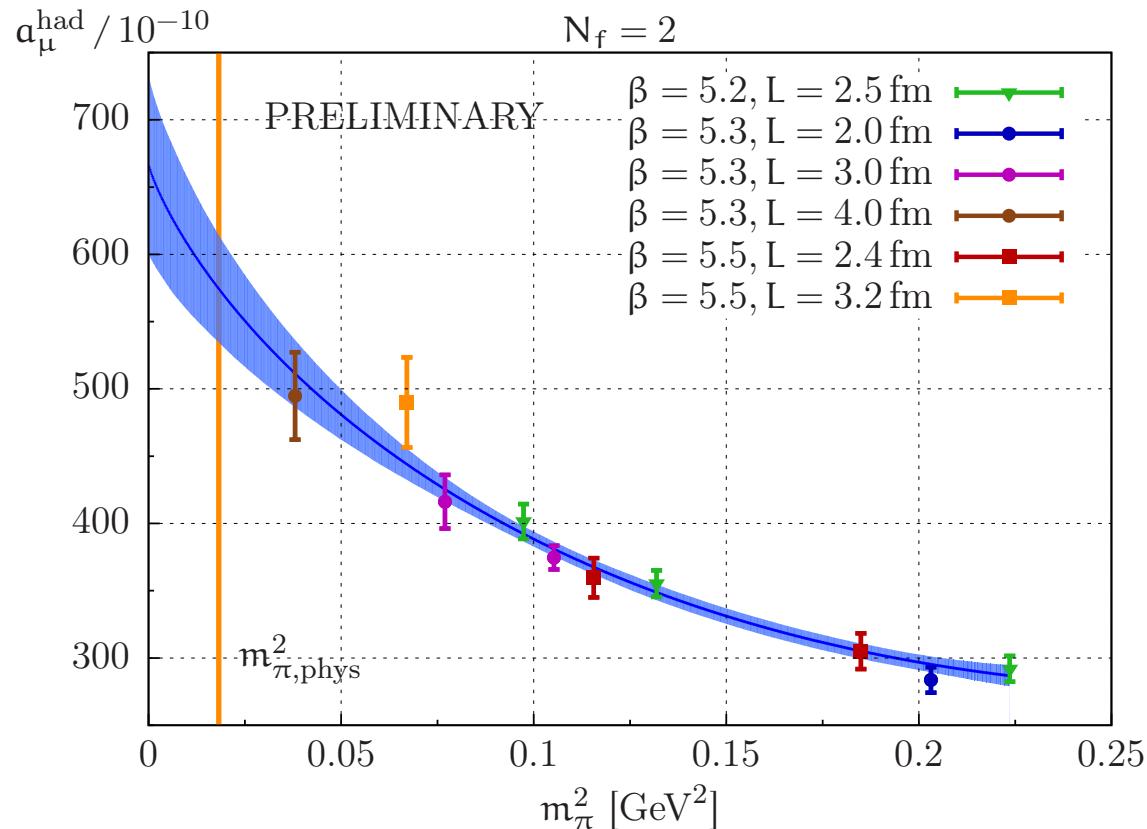
q^2 -dependence of $\Pi(q^2)$ described by [2,3] Padé ansatz

[Blum, Golterman, Peris, arXiv:1205.3695]

Update on $a_\mu^{\text{VP;had}}$

[Mainz Group, to appear]

- Pion mass dependence: $N_f = 2$, $m_\pi \lesssim 475 \text{ MeV}$

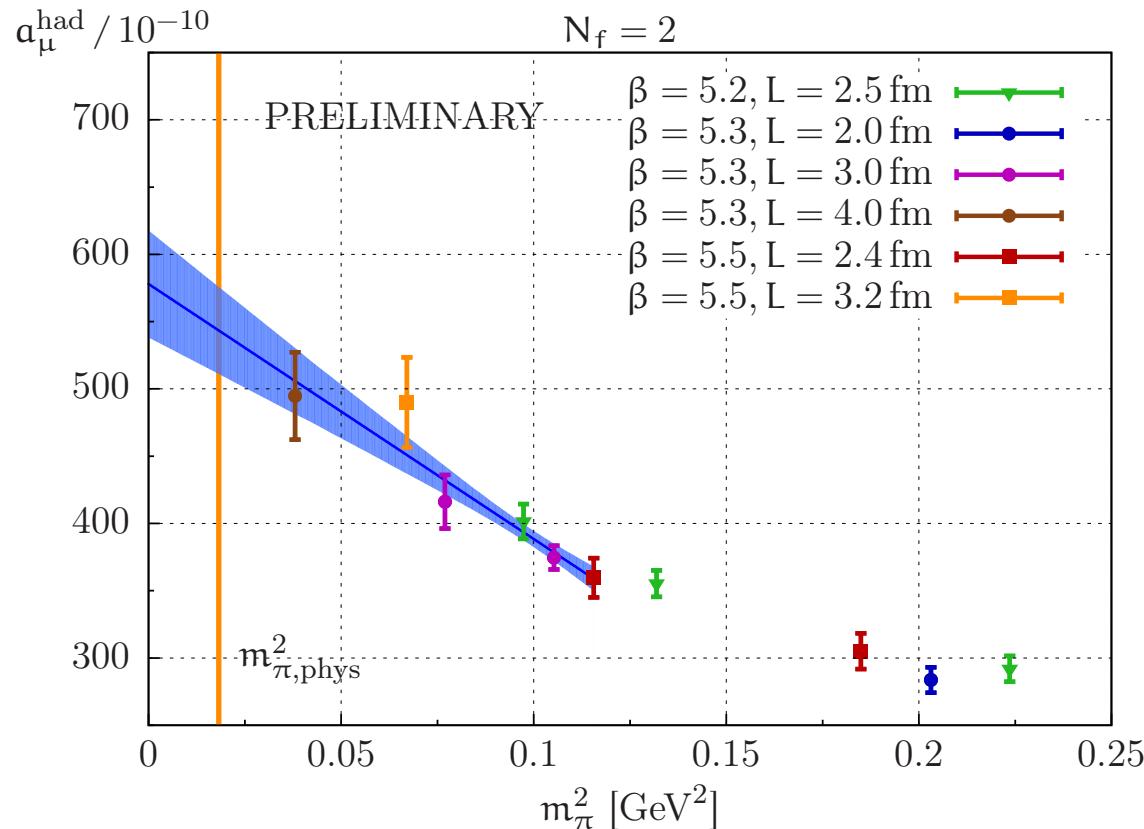


- Chiral fit: $a_\mu^{\text{had}}(m_\pi^2) = A + Bm_\pi^2 + Cm_\pi^2 \ln m_\pi^2$

Update on $a_\mu^{\text{VP;had}}$

[Mainz Group, to appear]

- Pion mass dependence: $N_f = 2$, $m_\pi \lesssim 360 \text{ MeV}$

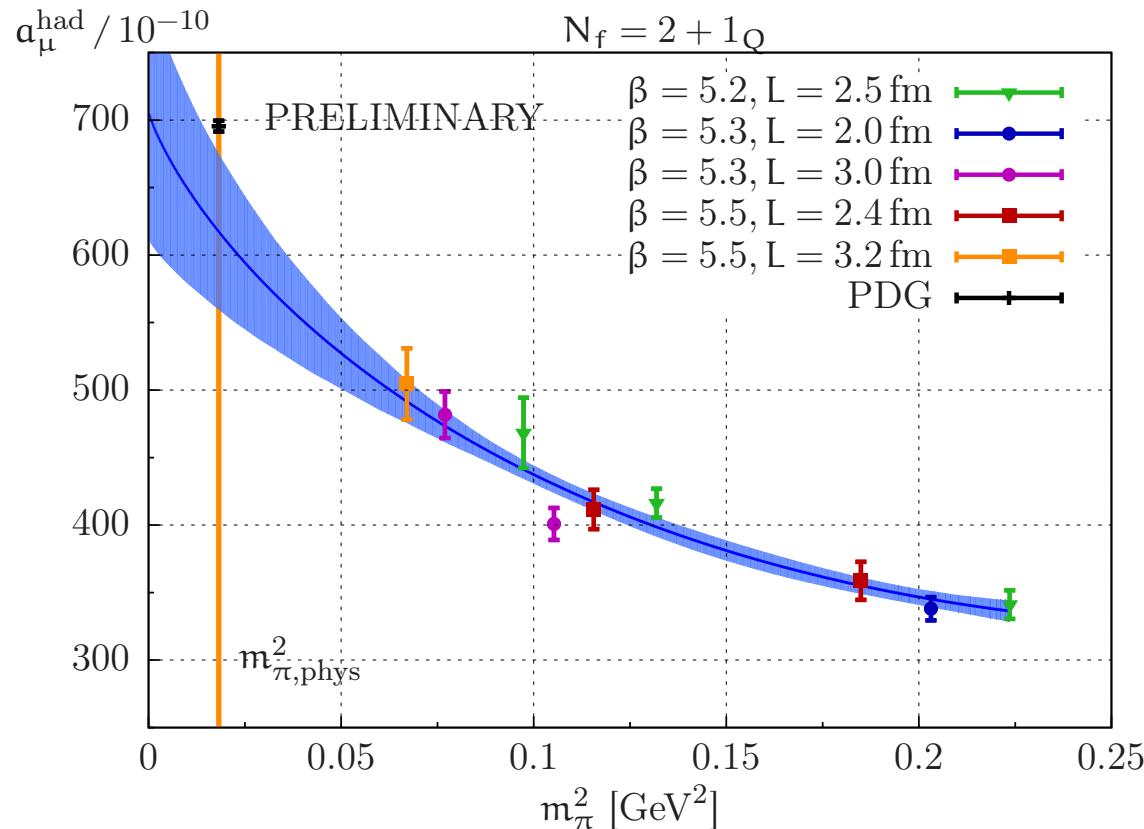


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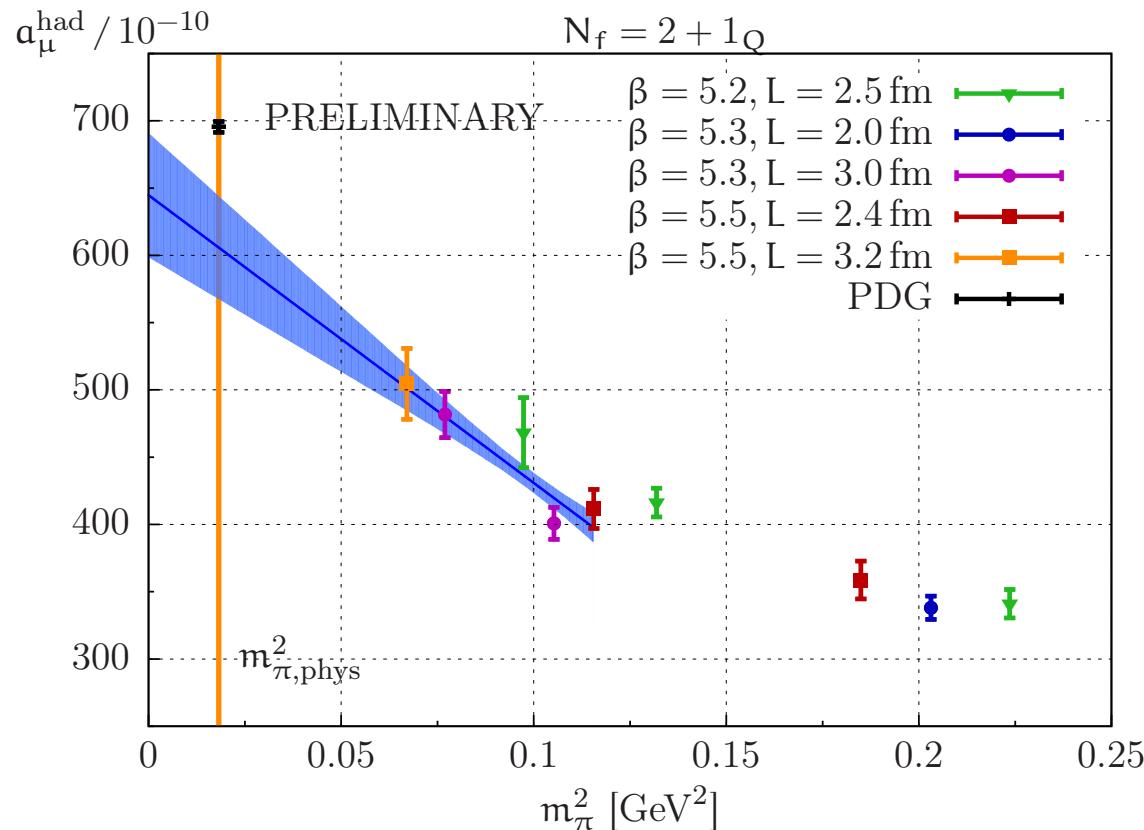


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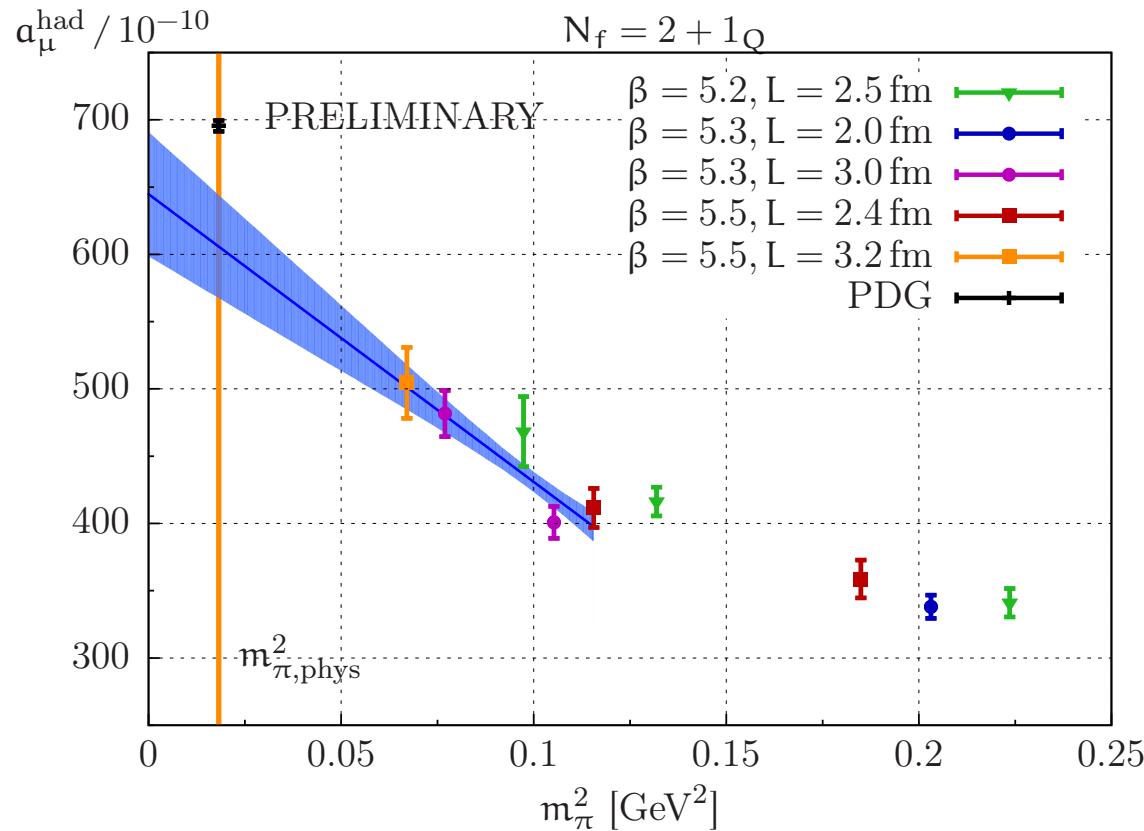


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- Analysis in progress
- Increase statistics on new ensembles

Lattice versus dispersion relations

[Bernecker & Meyer, EPJA 47 (2011) 148]

- Relation between vacuum polarisation and R -ratio in the Euclidean domain:

$$\hat{\Pi}(Q^2) \equiv \Pi(Q^2) - \Pi(0) = \frac{Q^2}{12\pi^2} \int_0^\infty ds \frac{R(s)}{s(s + Q^2)}$$

- Lattice calculations yield $\Pi(Q^2) - \Pi(0)$

→ Use parameterisation of measured $R(s)$ and evaluate the integral

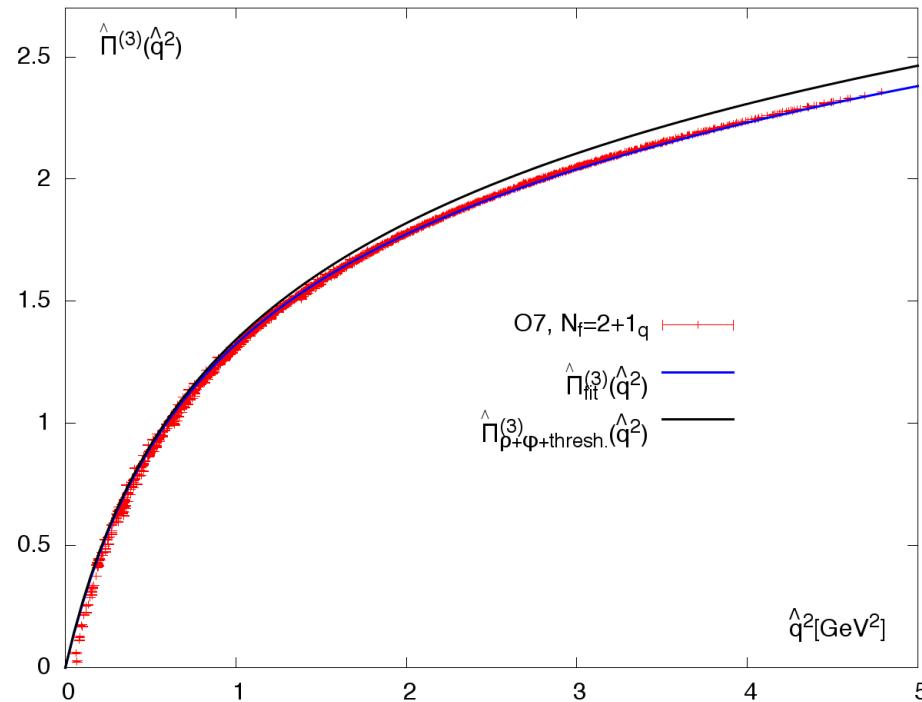
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O7 ensemble: $m_\pi = 270 \text{ MeV}$, $a = 0.050 \text{ fm}$, $N_f = 2 + 1_q$



[Mainz Group, in progress]

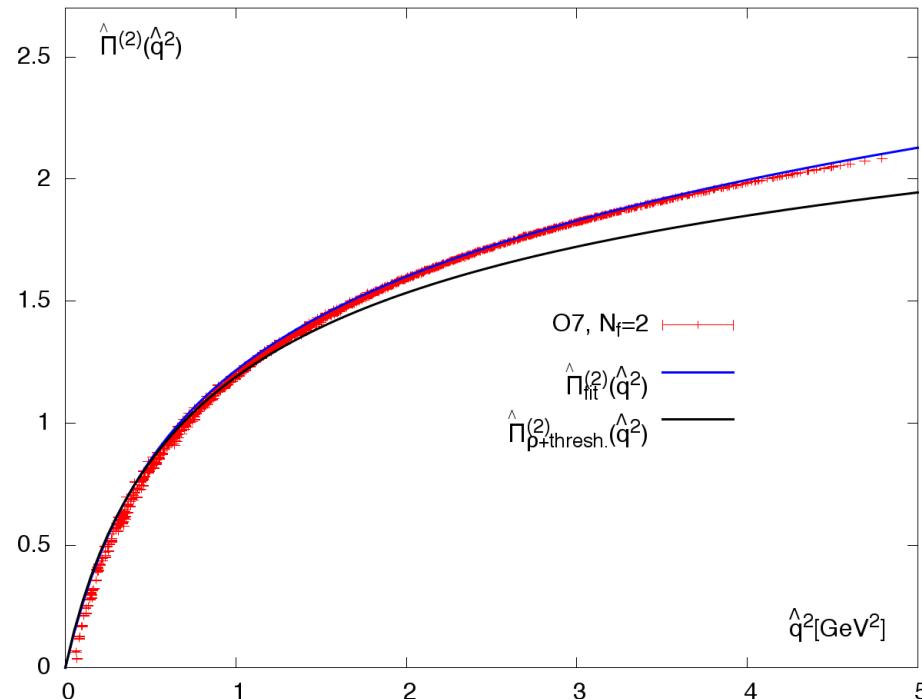
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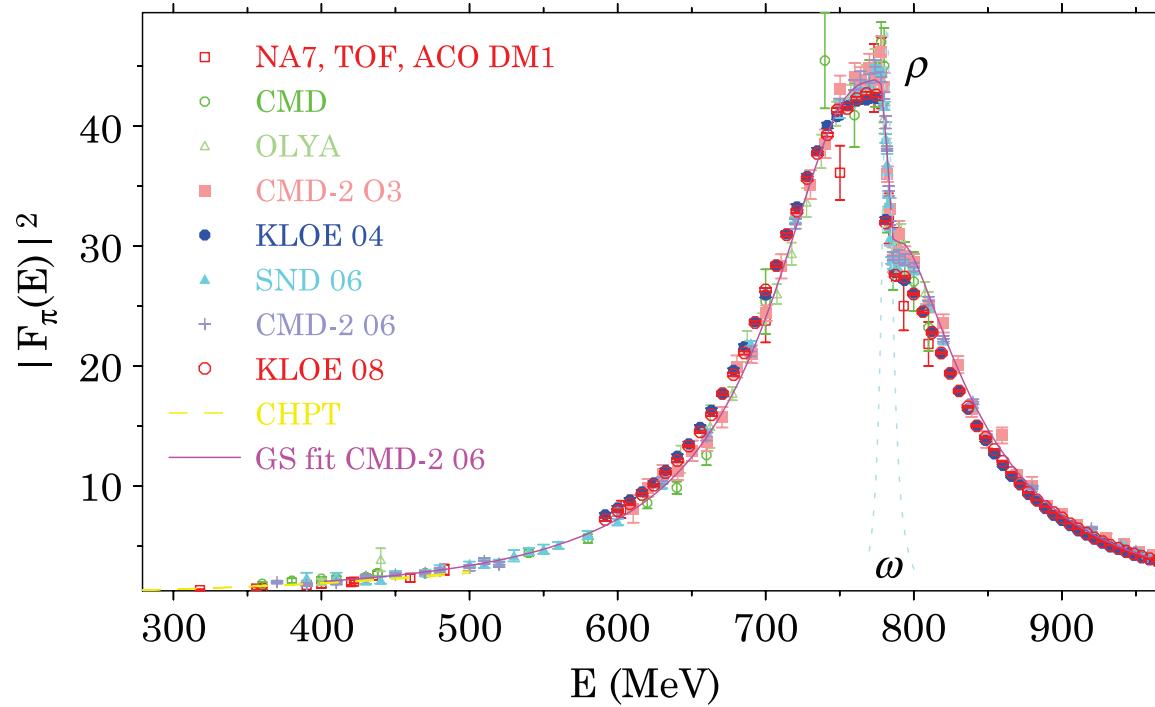
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[Mainz Group, in progress]

Pion form factor in the time-like domain [H.B. Meyer, PRL 107 (2011) 072002]

- Lattice QCD restricted to space-like momentum transfers
- Experimentally determined form factor not directly accessible



Pion form factor in the time-like domain [H.B. Meyer, PRL 107 (2011) 072002]

- Lattice QCD restricted to space-like momentum transfers
- Connect Euclidean observables with **spectral function** of the vector current:

$$\varrho(s) = \frac{1}{48\pi^2} \left(1 - \frac{4m_\pi^2}{s}\right)^{3/2} |F_\pi(\sqrt{s})|^2,$$

$$|F_\pi(\sqrt{s})|^2 = \left(q\phi'(q) + k \frac{\partial}{\partial k} \delta_1(k)\right) \frac{3\pi E^2}{k^5} |A_\phi|^2$$

E : invariant mass of two pions on a torus

$\delta_1(k)$: scattering phase shift

A_ϕ : vector current matrix element between vacuum and two-pion state

Current issues for hadronic VP on the lattice

- Increase overall accuracy of direct determinations:
 - Noise-reduction techniques, e.g. “all-mode averaging”
[Blum, Izubuchi & Shintani, arXiv:1208.4349]
 - Calculate $\Pi(q^2)$ directly at $q^2 = 0$ *[de Divitiis, Petronzio & Tantalo, arXiv:1208.5914]*
- Improve description of q^2 -dependence
- Evaluate quark-disconnected diagrams
- Explore methods to test dispersive approach using lattice data

Hadronic Light-by-light Scattering

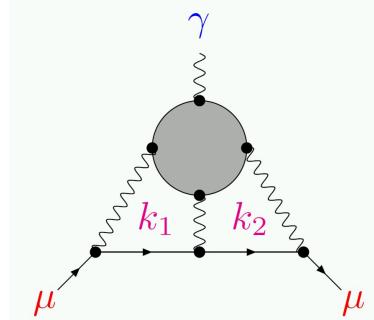
Lattice approach to hadronic light-by-light scattering

- Brute force:

- compute 4-point function

- “integrate” over internal momenta

- extrapolate $q^2 \rightarrow 0$



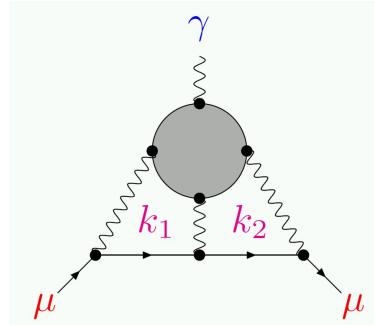
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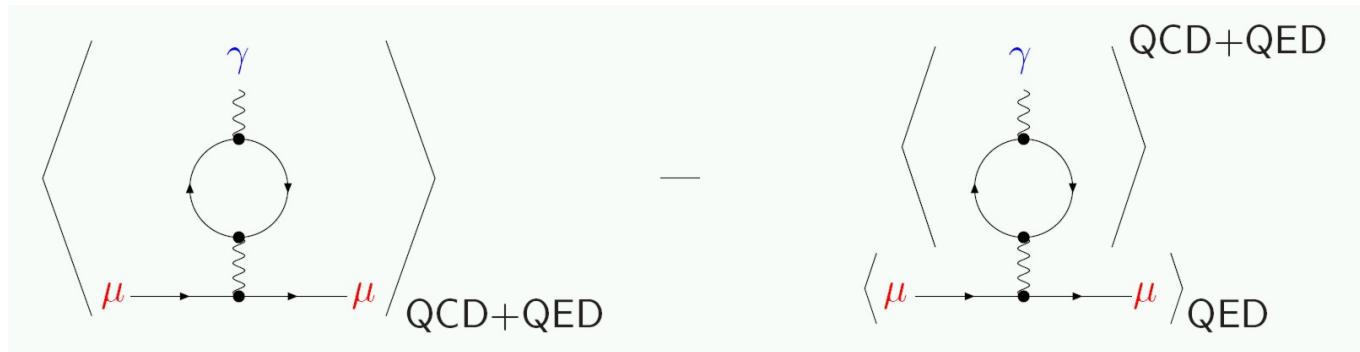
“integrate” over internal momenta

extrapolate $q^2 \rightarrow 0$



- Simultaneous QCD+QED simulations

[Blum, Chowdury et al.]



Difference yields desired contribution up to $O(\alpha^3)$

- Statistical noise? QED quenching effects?

Summary

- Hadronic contributions to $(g - 2)_\mu$: “hot topic” in lattice QCD
- Many challenges ahead:

Hadronic VP:

- push precision of direct determinations **below 1% level**
- test dispersive approach

Hadronic LbL:

- explore unconventional ideas