Off−light−cone effects in heavy−to−light form factors

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We report the first systematic analysis of the off-LC effects in sum rules for heavy-to-light form factors. These effects are studied in a model with scalar constituents: it allows a technically simple analysis which has the essential features of the QCD calculation.

The correlator relevant for the extraction of the heavy-to-light form factor is calculated in two different ways:

(i) via the full Bethe–Salpeter amplitude of the light meson;

(ii) by performing the expansion of the BS amplitude near the light cone $x^2 = 0$.

We demonstrate:

(a) the contributions to the cut correlator from the LC term $x^2 = 0$ and the off-LC terms $x^2 \neq 0$ have the same order in the $1/m_Q$ expansion.

(b) The cut LC correlator, corresponding to $x^2 = 0$, overestimate the full correlator, the difference being $\sim \Lambda_{\rm QCD}/\delta$, with $\delta \sim 1$ GeV - the effective continuum threshold. Numerically, this difference is $10 \div 20\%$.

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The Bethe–Salpeter amplitude and its expansion near the light cone:

$$\Psi_{\rm BS}(x,p') = \langle 0|T\varphi(x)\varphi(0)|M(p')\rangle = \Psi(x^2, xp', p'^2 = M^2).$$

As a function of the variable xp', the amplitude may be represented by the Fourier integral

$$\Psi_{\rm BS}(x,p') = \int_0^1 d\xi \exp(-i\xi p' x) K(x^2,\xi),$$

ξ-integration runs from 0 to 1 as follows from the analytic properties of Feynman diagrams. Nakanishi proposed to parametrize the kernel $K(x^2,\xi)$ as

$$K(x^2,\xi) = \frac{1}{(2\pi)^4 i}\int_0^\infty dz\, G(z,\xi)\int d^4k' \frac{\exp(-ik'x)}{[z+m^2-\xi(1-\xi)M^2-k'^2-i0]^3}.$$

Near the LC $x^2=0$:

$$K(x^2,\xi) = g_0(\xi) \;+\; x^2\Big[g_1(\xi)+\log(-x^2m^2)h_1(\xi)\Big] + x^4\Big[g_2(\xi)+\log(-x^2m^2)h_2(\xi)\Big] + \cdots.$$

The functions g_n and h_n may be expressed in terms of $G(z,\xi)$, e.g.

$$g_0(\xi) \;=\; \frac{1}{32\pi^2}\int_0^\infty dz\, G(z,\xi)\frac{1}{z+m^2-\xi(1-\xi)M^2},$$

Terms $\log(-x^2)$ **are related to divergence of higher** $k_\perp^2$**-moments of** Ψ_{LC} **(only** 0**-th moment is finite). Introduce cut-off in** $k_\perp$ **integrals, or introduce a regulator** λ **in coordinate space:**

$\log(-x^2m^2) \to \log(\lambda - x^2m^2)$.

For $\lambda \neq 0$**, represent** $\log(\lambda - x^2m^2)$ **as power series in** x^2**:**

$$K(x^2, \xi, \lambda) = \phi_0(\xi, \lambda) + x^2\phi_1(\xi, \lambda) + x^4\phi_2(\xi, \lambda) + \cdots,$$

$$\begin{aligned}
\phi_0(\xi) &= g_0(\xi), \\
\phi_1(\xi, \lambda) &= g_1(\xi) + \log(\lambda)h_1(\xi), \\
\phi_2(\xi, \lambda) &= g_2(\xi) + \log(\lambda)h_2(\xi) - \frac{1}{\lambda}m^2h_1(\xi).
\end{aligned}$$

BS wave function: power series with $\log(-x^2)$ **with** ***finite*** **coefficients** $\Rightarrow$ **pure power series with singular** λ**-dependent coefficients for** $\lambda \to 0$**.**

$$\Psi_{BS}(x, p') = \sum_{n=0}^{\infty}(x^2)^n \int_0^1 d\xi \exp(-ip'x\xi)\phi_n(\xi, \lambda).$$

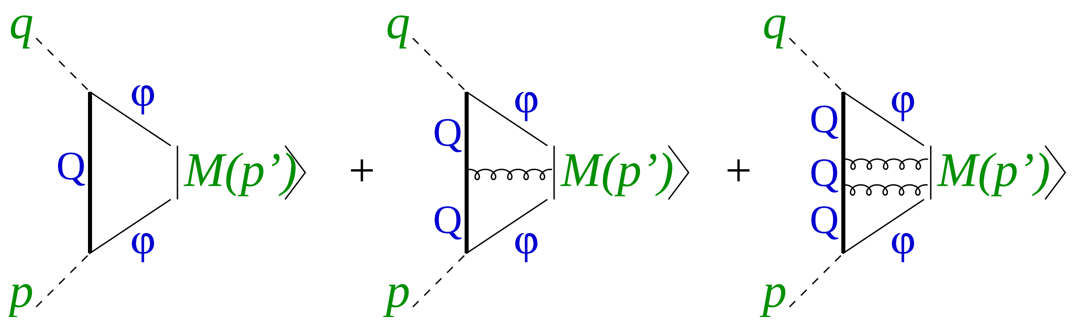
Important for us:

- **The first term, corresponding to** $x^2 = 0$**, does not depend on** λ**.**
- **The full BS amplitude does not depend on** λ**, too.**

BUT: Any truncation of the series leads to $\Psi_{BS}(x, p', \lambda)$ **such that it is singular at** $\lambda \to 0$**.**

Heavy-to-light correlator and the decay form factor from sum rule

$$\Gamma(p^2,q^2) = i\int d^4x \exp(ipx)\langle 0|T\left(J^\dagger_{\rm interp}(x)J_{\rm weak}(0)\right)|M(p')\rangle, \qquad J_{\rm interp} = J_{\rm weak} = \varphi(x)Q(x)$$



(i) Dispersion representation in p^2 (ii) Borel transform in p^2 to extract $F_{M_Q\to M}$

$$\Gamma_{\rm th}(p^2,q^2) = \int \frac{ds}{s-p^2-i0}\Delta_{\rm th}(s,q^2) \qquad \Rightarrow \qquad \hat\Gamma_{\rm th}(\mu_B^2,q^2) = \int ds \exp(-s/2\mu_B^2)\Delta_{\rm th}(s,q^2)$$

The sum rule: $$\boxed{f_{M_Q}F_{M_Q\to M}(q^2) = \exp\left(M_Q^2/2\mu_B^2\right)\hat\Gamma_{\rm th}(\mu_B^2,q^2,s_0(\mu_B^2,q^2))}$$

with the cut correlator $$\hat\Gamma_{\rm th}(\mu_B^2,q^2,s_0) = \int ds\,\theta(s<s_0)\exp\left(-s/2\mu_B^2\right)\Delta_{\rm th}(s,q^2).$$

For $q^2 \ll m_Q^2$, first diagram gives the main contribution; full propagator $\mathcal{D}_Q \to$ free propagator

$$\Gamma_{\rm th}(p^2,q^2) = \frac{1}{(2\pi)^4}\int d^4k d^4x \exp(ipx-ikx)\frac{1}{m_Q^2-k^2-i0}\langle 0|T\varphi(x)\varphi(0)|M(p')\rangle.$$

We can proceed further in two different ways:

A. First, use the full BS amplitude in momentum space:

$$\Gamma_{\rm th}(p^2,q^2) = \frac{1}{(2\pi)^4}\int d^4k \frac{\Psi_{\rm BS}(k,p')}{m_Q^2-(p-k)^2-i0} \qquad \Rightarrow \qquad \Delta_{\rm th}(s,q^2)$$

B. Substitute the LC expansion of $\Psi_{\rm BS}(x,p')$:

$$\begin{aligned}\Gamma_{\rm th}(p^2,q^2) &= \frac{1}{(2\pi)^4}\int d^4k\, d^4x \exp\left(ipx-ikx\right)\frac{1}{m_Q^2-k^2-i0}\sum_{n=0}^{\infty}(x^2)^n\int_0^1 d\xi \exp(-ip'x\xi)\phi_n(\xi,\lambda)\\ &= \int_0^1 \frac{d\xi\ \phi_0(\xi,\lambda)}{m_Q^2-p^2(1-\xi)+M^2\xi(1-\xi)-q^2\xi} - 8m_Q^2\int_0^1 \frac{d\xi\ \phi_1(\xi,\lambda)}{\left[m_Q^2-p^2(1-\xi)+M^2\xi(1-\xi)-q^2\xi\right]^3}+\cdots.\end{aligned}$$

Hereafter set $q^2=0$ and $M=0$.

The Borel parameter $\mu_B^2 \to m_Q\beta$. In the standard SR analysis $\beta \simeq 1$ GeV.

The uncut Borel image:

$$\exp\left(M_Q^2/2\beta m_Q\right)\hat{\Gamma}_{\rm th}(\beta) \simeq \int_0^1 \frac{d\xi}{1-\xi}\left[\phi_0(\xi)-\frac{\phi_1(\xi,\lambda)}{\beta^2(1-\xi)^2}+\cdots\right]\exp\left(-\frac{m_Q}{2\beta}\xi\right).$$

The cut Borel image:

Heavy meson mass: $M_Q = m_Q + \varepsilon_Q$**; Effective threshold** $s_0 = (m_Q + \delta)^2, \quad \delta \simeq 0.5 - 1$ **GeV.**

$$\exp\left(M_Q^2/2\beta m_Q\right)\hat{\Gamma}_{\rm th}(\beta,\delta) \simeq \int\limits_0^{2\delta/m_Q} \frac{d\xi}{1-\xi}\left[\phi_0(\xi) - \frac{\phi_1(\xi,\lambda)}{\beta^2(1-\xi)^2} + \cdots\right]\exp\left(-\frac{m_Q}{2\beta}\xi\right)$$
$$- \; 4\exp\left(\frac{\varepsilon_Q - \delta}{\beta}\right)\left[\frac{\phi_1(\xi_0)}{m_Q^2} + \frac{\phi_1'(\xi_0)}{4m_Q\beta} + \frac{\phi_1'(\xi_0)}{m_Q^2}\right] + \cdots, \qquad \xi_0 = 2\delta/m_Q.$$

For massless-boson exchange at small distances, one finds

$$\phi_0(\xi) \simeq \xi,\ \phi_1(\xi) \simeq \xi, \quad \cdots.$$

In the limit $m_Q \to \infty$**, all contributions** $n = 0, 1, \ldots$ **behave as** $1/m_Q^2$**.**
In the limit $\beta \to \infty$**, due to surface terms, all contributions** $n = 0, 1, \ldots$ **have the same order.**

For the realistic case of the interaction dominated by massless-boson exchange at short distances, the LC contribution does not dominate the cut correlator parametrically.

There are models in which off-LC contributions are power-suppressed compared to $n = 0$**:**

For instance, for $\phi_n(\xi) \sim \xi^{n+1}$ **corresponding to the light-cone wave function**

$\Psi_{\rm LC}(\xi, k_\perp^2) = \exp\left(\frac{-k_\perp^2}{2\beta_M^2\xi(1-\xi)}\right)$, β_M **the size parameter of the light meson.**

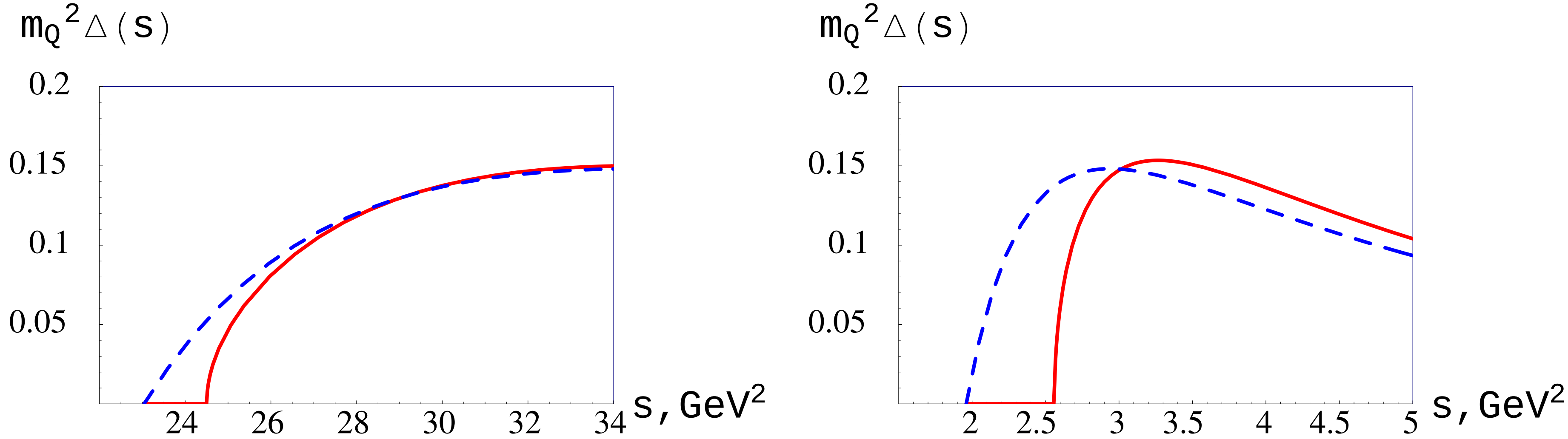
Numerical results

For the BS kernel $G(z,\xi) = m^2\delta(z)\xi(1-\xi)$ explicit expressions for $\Delta_{\rm th}(s)$ and $\Delta_{\rm LC}(s) = \Delta^{n=0}(s)$.

Parameters:

For beauty-meson $m_Q = 4.8$ GeV. For charm-meson $m_Q = 1.4$ GeV.

The light-quark mass m appears in the framework of the BS equation, i.e., the constituent quark mass. $m = 150$ MeV for beauty-meson decay, and $m = 200$ MeV for charm-meson decay.



$m_Q^2\Delta_{\rm th}(s)$ **(solid red line);** $m_Q^2\Delta_{\rm LC}(s)$ **(dashed blue line). Left - beauty, right - charm.**

The thresholds in $\Delta_{\rm th}$ and $\Delta_{\rm LC}$ do not coincide: in the LC spectral density the threshold is m_Q^2 whereas in the full spectral density it is $(m_Q+m)^2$. The region near the threshold provides the main contribution to the cut Borel-transformed correlators, therefore the mismatch of the thresholds is responsible for the nonvanishing of the off-LC effects in sum rules.

How to fix δ in the effective continuum threshold?

Fix δ by the standard procedure: for some value of the Borel parameter β

$$\langle s(\beta,\delta)\rangle \equiv \frac{\int\limits_{s_{\rm low}}^{(m_Q+\delta)^2} ds\,\exp\left(-s/2m_Q\beta\right)\; s\,\Delta(s,q^2)}{\int\limits_{s_{\rm low}}^{(m_Q+\delta)^2} ds\,\exp\left(-s/2m_Q\beta\right)\;\Delta(s,q^2)} = M_Q^2.$$

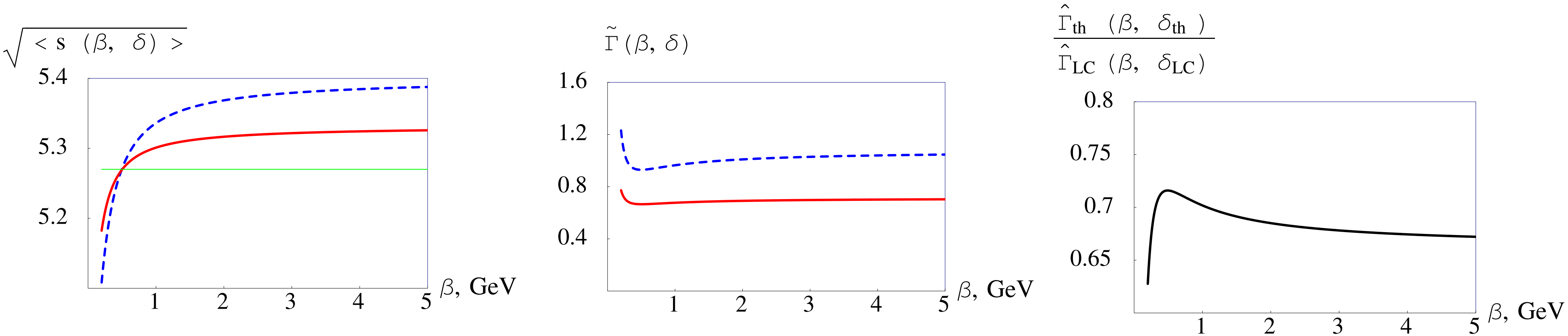
The next transparency gives:

$$\widetilde{\Gamma}(\beta,\delta) = m_Q^2 \exp\left(M_Q^2/2m_Q\beta\right)\hat{\Gamma}(\beta,\delta)$$

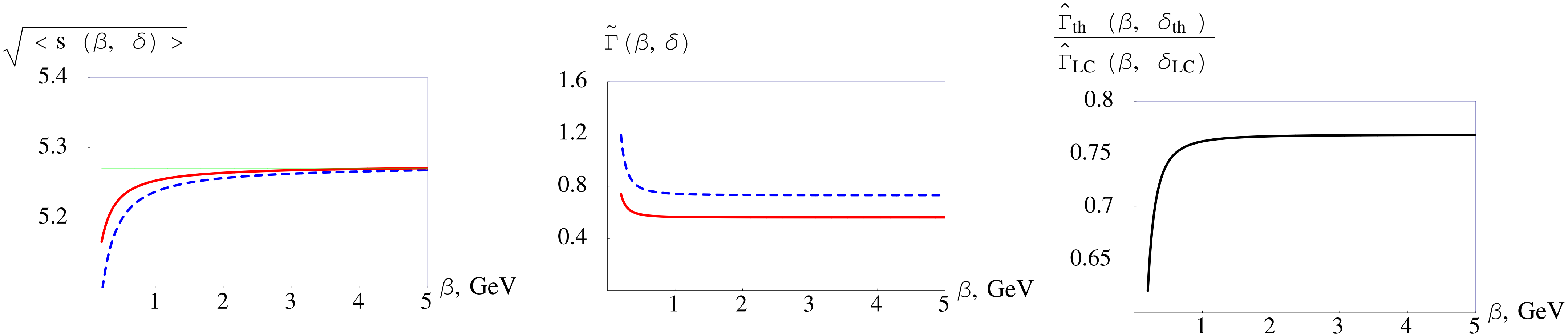
BEAUTY-MESON DECAY:

$m_Q = 4.8$ **GeV,** $m = 150$ **MeV,** δ **fixed from** $\sqrt{\langle s \rangle} = M_Q = 5.27$ **GeV at two different values of** β**:**

I. δ **fixed at** $\beta = 0.5$ **GeV:** $\delta_{\rm LC} = 0.96$ **GeV,** $\delta_{\rm th} = 0.79$ **GeV.**



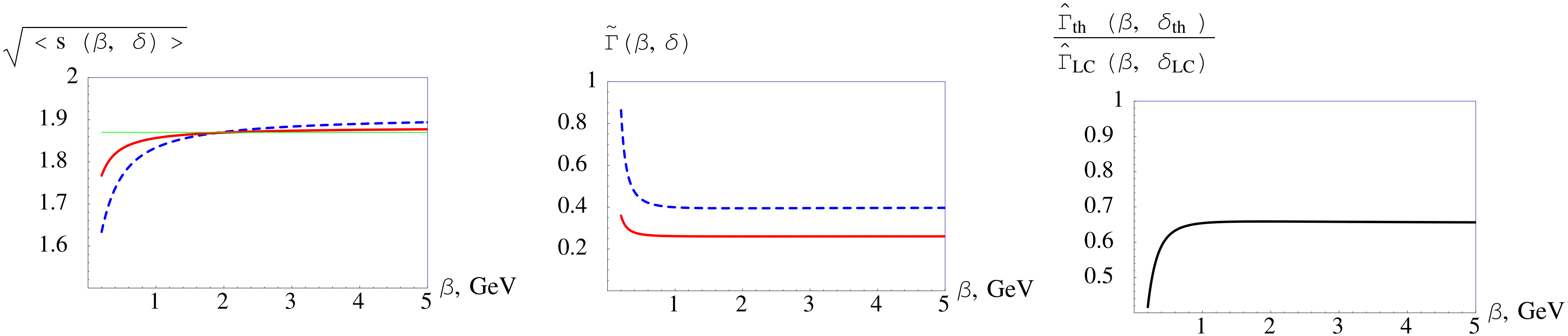
II. δ **fixed at** $\beta = 4$ **GeV:** $\delta_{\rm LC} = 0.755$ **GeV,** $\delta_{\rm th} = 0.69$ **GeV.**



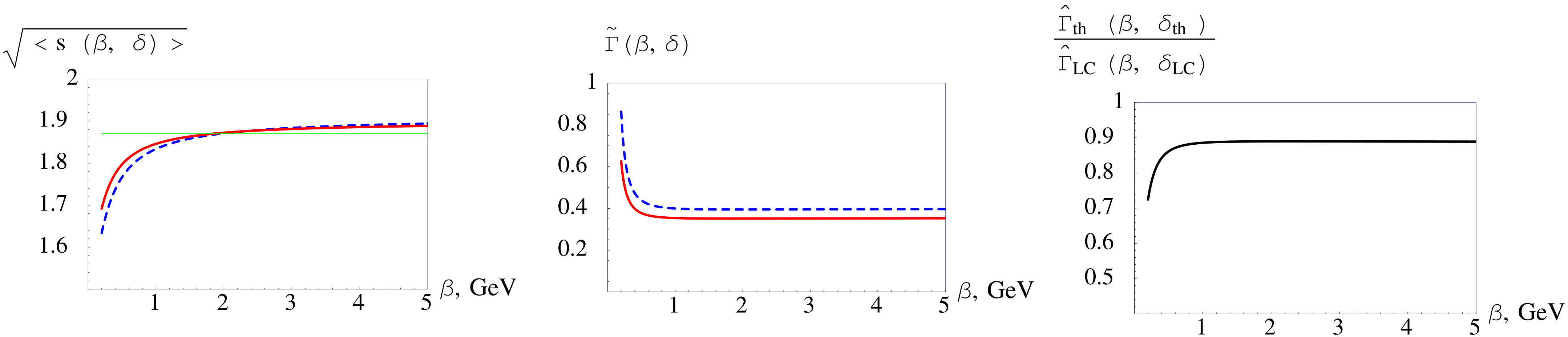
<u>CHARM-MESON DECAY</u>:

$m_Q = 1.4$ **GeV,** δ **fixed from** $\sqrt{\langle s\rangle} = M_Q = 1.87$ **GeV at** $\beta = 2$ **GeV.** $\delta_{\rm LC} = 0.93$ **GeV.**

I. $m = 200$ **MeV,** $\delta_{\rm th} = 0.72$ **GeV.**



II. $m = 100$ **MeV,** $\delta_{\rm th} = 0.85$ **GeV.**



Explicit analytic results may be obtained in the limit $\beta \gg m_Q \to \infty$:

$$
\begin{aligned}
m_Q^2 \hat{\Gamma}_{\rm LC}(\beta \to \infty, \delta_{\rm LC}) &= 2\delta_{\rm LC}^2 + O(\delta_{\rm LC}^3/m_Q), \\
m_Q^2 \hat{\Gamma}_{\rm th}(\beta \to \infty, \delta_{\rm th}) &= 2\delta_{\rm th}^2 - m^2 \left[\log\left(\frac{4\delta_{\rm th}^2}{m^2}\right) + 1 \right] + O(m^4/\delta_{\rm th}^2) + O(\delta_{\rm th}^3/m_Q).
\end{aligned}
$$

Fixing $\delta_{\rm th}$ and $\delta_{\rm LC}$ by the discussed procedure, we express them via ε_Q ($M_Q = m_Q + \varepsilon_Q$)

$$
\delta_{\rm LC} = \frac{3}{2}\varepsilon_Q, \qquad \delta_{\rm th} = \frac{3}{2}\varepsilon_Q - \frac{2m^2}{3\varepsilon_Q}\left[\log\left(\frac{3\varepsilon_Q}{m}\right) - 1 \right] + \cdots,
$$

We then obtain

$$
\begin{aligned}
m_Q^2 \hat{\Gamma}_{\rm LC}(\beta \to \infty, \delta_{\rm LC}) &= \frac{9}{2}\varepsilon_Q^2, \\
m_Q^2 \hat{\Gamma}_{\rm th}(\beta \to \infty, \delta_{\rm th}) &= \frac{9}{2}\varepsilon_Q^2 - 6m^2 \log\left(\frac{3\varepsilon_Q}{\sqrt{e}m}\right) + \cdots,
\end{aligned}
$$

and

$$
\frac{\hat{\Gamma}_{\rm th}(\beta \to \infty, \delta_{\rm th})}{\hat{\Gamma}_{\rm LC}(\beta \to \infty, \delta_{\rm LC})} = 1 - \frac{4m^2}{3\varepsilon_Q^2} \log\left(\frac{3\varepsilon_Q}{\sqrt{e}m}\right) + \cdots.
$$

Here dots denote terms containing higher powers of m/ε_Q. We have compared the correlators evaluated at different values of the cut parameters $\delta_{\rm LC}$ and $\delta_{\rm th}$. This is relevant if one wants to understand the error due to taking into account only the light-cone ($x^2 = 0$) contribution to the correlator and neglecting terms containing higher powers of x^2.

<u>Lessons and conclusions</u>:

1. The off-LC effects in the cut correlator are *not parametrically suppressed* **compared to the LC contribution.** *In heavy-to-light decays, there exists no sensible limit in which the cut LC correlator coincides with the cut full correlator***.**

Numerically, LC provides the bulk of the full correlator, but the difference between the cut full and the cut LC correlators always remains nonvanishing.

2. The Borel curves for the full and the LC correlators have similar shapes, but LC correlator *overestimates* **the full correlator, at small q^2 by $10 \div 20\%$ in a wide range of the heavy-quark mass relevant for charm and beauty decays.**

The similarity of the Borel curves for the full and the LC correlators implies that the systematic difference between the correlators cannot be diminished by a relevant choice of the criterion for extracting the heavy-to-light form factor.

3. The difference between $\widetilde{\Gamma}_{\rm LC}$ and $\widetilde{\Gamma}_{\rm th}$ increases with increasing mass of the light spectator quark. Therefore, this difference is expected to be greater for the heavy mesons B_s and D_s, containing the strange s-quark, than for B and D. This prompts that the error in the predictions for the form factors related to off-LC effects is greater for strange heavy mesons.

4. In QCD:

$$\frac{\hat{\Gamma}_{\rm th}(\beta,\delta_{\rm th})}{\hat{\Gamma}_{\rm LC}(\beta,\delta_{\rm LC})} = 1 - O\left(\frac{\Lambda_{\rm QCD}}{\delta}\right).$$

We believe our numerical estimates for higher-twist effects are realistic estimates for QCD.

5. The extracted values of the form factor depend on two ingredients:

(i) the field-theoretic calculation of the relevant correlator.

(ii) the technical "extraction procedure" which is external to the underlying field theory.

The second ingredient introduces a systematic error which is very hard to control in any version of QCD sum rules, even if the correlator is known exactly.

In LCSRs, also the first ingredient contains uncertainties related to higher-twist (including off-LC) effects, which are not suppressed by large parameters. Therefore further study of these effects is necessary.