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Combined Partial Wave Analysis

1. The two pseudoscalar particle final states from pion induced reactions reveal very limited number of states. e.g. CERN-Munich:
   \[ \rho(770), \ f_2(1270), \ f_0(980), \ \rho_3(1690), \ f_4(2050) \]
   \[ \rho(1450) \ ?, \ \rho(1700) \ ?, \ f_0(1500), \ f_0(1750) \ ? \]
   \[ f_0(1370), \ f_2(1560), \ f_0(1710), \ \rho(1900), \ f_2(1950), \ f_2(2010), \ f_0(2020) \]

2. A large set of the resonances had been discovered in the analysis of the proton-antiproton annihilation and in \( \pi N \) interaction into multi-body final states.

3. In many cases only combined analysis of the reactions with multi-body and two body final states can produce an unambiguous result.

Problem:
The pion induced reactions are not a two particle collision reactions. They are \( \pi N \rightarrow \pi \pi N \) reactions which is much more complicated process.
### Fitted data: Two body reactions:

<table>
<thead>
<tr>
<th>Reaction</th>
<th>Experiment</th>
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<th>Experiment</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi^+\pi^- \rightarrow \pi^+\pi^-$ (all waves)</td>
<td>CERN-München</td>
<td>$\pi\pi \rightarrow \pi^0\pi^0$ (S-wave)</td>
<td>GAMS</td>
</tr>
<tr>
<td>$\pi\pi \rightarrow \pi^0\pi^0$ (S-wave)</td>
<td>GAMS</td>
<td>$\pi\pi \rightarrow \eta\eta$ (S-wave)</td>
<td>GAMS</td>
</tr>
<tr>
<td>$\pi\pi \rightarrow K\bar{K}$ (S-wave)</td>
<td>BNL</td>
<td>$\bar{K}^+\pi^+ \rightarrow K^-\pi^+$ (S-wave)</td>
<td>LASS</td>
</tr>
</tbody>
</table>

### Fitted data: Three body reactions from Crystal Barrel: (L-liquid, G-gaseous targets):

<table>
<thead>
<tr>
<th>Reaction</th>
<th>Target</th>
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</tr>
</thead>
<tbody>
<tr>
<td>$\bar{p}p \rightarrow \pi^0\pi^0\pi^0$</td>
<td>(L) $H_2$</td>
<td>$\bar{p}p \rightarrow \pi^+\pi^0\pi^-$</td>
<td>(L) $H_2$</td>
<td>$\bar{p}p \rightarrow K_SK_S\pi^0$</td>
<td>(L) $H_2$</td>
</tr>
<tr>
<td>$\bar{p}p \rightarrow \pi^0\eta\eta$</td>
<td>(L) $H_2$</td>
<td>$\bar{p}n \rightarrow \pi^0\pi^0\pi^-$</td>
<td>(L) $D_2$</td>
<td>$\bar{p}p \rightarrow K^+K^-\pi^0$</td>
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<td>$\bar{p}p \rightarrow \pi^0\pi^0\eta$</td>
<td>(L) $H_2$</td>
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<td>(L) $D_2$</td>
<td>$\bar{p}p \rightarrow K_LK^\pm\pi^\mp$</td>
<td>(L) $H_2$</td>
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<td>$\bar{p}p \rightarrow \pi^0\pi^0\pi^0$</td>
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The data are available from: PWA.HISKP.UNI-BONN.DE
Problems in the analysis of the $\pi N \rightarrow XN$ reactions

The $\pi N$ reaction with large energy of initial pion should be described by t-exchanges. However:

1. There is no analysis of the data based on the particle exchanges: there are only models.

2. There is no a solid analysis which preserves unitarity and includes all known states in P and D-waves. This is should be important at small $t$ where the $\pi$ exchange is a dominant one and data are close to the unitarity limit.

3. Most of the models have problems at large $t$ where exchanges of particles with large spin play a significant role.
Cross section for the reactions $\pi N \rightarrow \pi \pi N, KN, \eta \eta N$

\[
d\sigma = \frac{(2\pi)^4 |A|^2}{8\sqrt{s_{\pi N}|p_2|}} d\Phi(p_1 + p_2, k_1, k_2, k_3)
\]

\[
d\Phi(p_1 + p_2, k_1, k_2, k_3) = (2\pi)^3 d\Phi(P, k_1, k_2) d\Phi(p_1 + p_2, P, k_3) ds,
\]

Assuming that amplitude depends only on $t$ and $s$:

\[
d\Phi(p_1 + p_2, P, k_3) = \frac{1}{(2\pi)^5} \frac{dt}{8|p_2|\sqrt{s_{\pi N}}} \quad t = (k_3 - p_2)^2
\]

and

\[
d\Phi(P, k_1, k_2) = \frac{1}{(2\pi)^5} \rho(s)d\Omega \quad \rho(s) = \frac{1}{16\pi} \frac{2|k_1|}{\sqrt{s}},
\]
Then:

\[
\frac{d\sigma}{dt} = \frac{(2\pi)^4 |A|^2 (2\pi)^3}{8|\vec{p}_2| \sqrt{s_{\pi N}}} \frac{1}{(2\pi)^5} \frac{dt 2M dM d\Phi(P, k_1, k_2)}{8|\vec{p}_2| \sqrt{s_{\pi N}}} = \frac{(M |A|^2 \rho) dt dM d\Omega}{(2\pi)^3 32 |\vec{p}_2|^2 s_{\pi N}}
\]

Unitarity relation:

\[ImA = \rho(s)|A|^2\]

And the cross section can be expressed in the terms of spherical functions:

\[
\frac{d^4\sigma}{dt dM d\Omega} = N \sum_l \left( < Y_l^0 > Y_l^0(\Omega) + \sum_{m=0}^l 2 < Y_l^m > Re Y_l^m(\Omega) \right)
\]
CERN-Munich approach

The CERN-Munich model was developed for the analysis of the data on $\pi^- p \rightarrow \pi^+ \pi^- n$ reaction and based partly on the absorption model but mostly on the phenomenological observations.

$$|A|^2 = | \sum_{J=0} A_J^0 Y_J^0 + \sum_{J=1} A_J^- ReY_J^1 |^2 + | \sum_{J=1} A_J^+ ReY_J^1 |^2$$

Additional assumptions:

1) helicity 1 amplitudes are equal for natural and unnatural exchanges:

$$A_J^(-) = A_J^ (+)$$

2) The ratio of the $A_J^(-)$ and the $A_J^0$ amplitudes is a polynomial over mass of the two pion system which does not depend on $J$ up to total normalization.

$$A_J^(-) = \frac{A_J^0}{C_J \sum_{n=0} b_n M^n}$$
The amplitude squared can be rewritten via density matrices

\[ \rho_{00} = A_0^0 A_m^0, \quad \rho_{01} = A_0^0 A_m^(-), \quad \rho_{11} = 2A_n^- A_m^-, \]

as:

\[ |A|^2 = \sum_{J=0}^{J} Y_J^0 \left( \sum_{n,m} d_{n,m,J}^{0,0,0} \rho_{00}^{nm} + d_{n,m,J}^{1,1,0} \rho_{11}^{nm} \right) + \sum_{J=0}^{J} ReY_J^1 \left( \sum_{n,m} d_{n,m,J}^{1,0,1} \rho_{10}^{nm} + d_{n,m,J}^{0,0,1} \rho_{11}^{nm} \right) \]

where

\[ d_{n,m,J}^{i,k,l} = \frac{\int d\Omega \ ReY_n^i(\Theta, \varphi) ReY_m^k(\Theta, \varphi) ReY_J^l(\Theta, \varphi)}{\int d\Omega \ ReY_J^l(\Theta, \varphi) ReY_J^l(\Theta, \varphi)} \]

Substituting such amplitude into cross section one can directly fit the moments \( < Y_J^m > \).
Description of the CERN-Munich data

\[ \pi^- p \rightarrow \pi^+ \pi^- n \]

\[ A_{1j} = K_{1m}(I - i\hat{\rho}(s)\hat{K})^{-1}_{mj} \]
GAMS, VES and BNL approach

The Cern-Munich approach does not work for large $t$ and does not work for many other final states.

The $\pi N$ data are decomposed as a sum of amplitudes with angular dependence defined by spherical functions:

$$|A^2| = |\sum_{J=0} A^0_J Y^0_J + \sum_{J=1} A^-_J \sqrt{2} Re Y^1_J|^2 + |\sum_{J=1} A^+_J \sqrt{2} Im Y^1_J|^2$$

Here the $A^0_J$ functions are called $S_0, P_0, D_0, F_0$ . . . , the $A^-_J$ functions defined as $P_-, D_-, F_-, \ldots$ and the $A^+_J$ functions as $P_+, D_+, F_+, \ldots$.

No assumptions that helicity 1 amplitudes with natural and unnatural exchanges are equal each to another.
At small $t$: $|t| < 0.1$:
At large $t$: $|t| > 0.4$:
The S-wave has a very prominent structure at large $|t|$. 
Reggezied exchanges:

\[ A_{\pi p \to \pi \pi n}^{(\text{pion trajectories})} = \sum_{\pi_j} A(\pi \pi_j \to \pi \pi) R_{\pi_j} \left( s_{\pi N}, q^2 \right) \varphi_n^+ \left( \vec{\sigma} \vec{p}_\perp \right) \varphi_p \ g_{pn}^{(\pi_j)}. \]

\[ A_{\pi p \to \pi \pi n}^{(a_1-\text{trajectories})} = \sum_{a_1^{(j)}} A(\pi a_1^{(j)} \to \pi \pi) R_{a_1^{(j)}} \left( s_{\pi N}, q^2 \right) \varphi_n^+ \left( \vec{\sigma} \vec{n}_z \right) \varphi_p \ g_{pn}^{(a_1^{(j)})}. \]

\[ R_{\pi_j} \left( s_{\pi N}, q^2 \right) = \exp \left( -i \frac{\pi}{2} \alpha_{\pi}^{(j)}(q^2) \right) \frac{(s_{\pi N}/s_{\pi N0})^{\alpha_{\pi}^{(j)}(q^2)}}{\sin \left( \frac{\pi}{2} \alpha_{\pi}^{(j)}(q^2) \right) \Gamma \left( \frac{1}{2} \alpha_{\pi}^{(j)}(q^2) + 1 \right)} \]

\[ R_{a_1^{(j)}} \left( s_{\pi N}, q^2 \right) = i \exp \left( -i \frac{\pi}{2} \alpha_{a_1}^{(j)}(q^2) \right) \frac{(s_{\pi N}/s_{\pi N0})^{\alpha_{a_1}^{(j)}(q^2)}}{\cos \left( \frac{\pi}{2} \alpha_{a_1}^{(j)}(q^2) \right) \Gamma \left( \frac{1}{2} \alpha_{a_1}^{(j)}(q^2) + \frac{1}{2} \right)} \]
The $\pi$ and $a_1$ exchanges do not interfere if the polarization of nucleons is not detected. Indeed $\pi N$ coupling is described by $\gamma_5$ (singlet) while $a_1$ coupling is proportional to $\gamma_\mu$ (triplet).

$$S_p[(m_N + \hat{k}_3)\gamma_5(m_N + \hat{p}_2)\gamma_\mu] = 0$$

This is true for $\pi$ and $a_2$ exchanges.

It means that

$$|S|^2 = |S_\pi|^2 + |S_{a_1}|^2$$

And $D_p$-wave could only interfere with $S_{a_1}$.

Already $a_1$ and $\pi$ exchanges produced a more complicated structure.
Nice features of reggeized $a_1$ exchange:

$$A(\pi a_1^{(j)} \rightarrow \pi \pi) = \sum_J \epsilon_{\beta}^{(-)} \left[ A^{(J+)}_{\pi a_1^{(j)} \rightarrow \pi \pi} X^{(J+1)}_{\beta} + A^{(J-)}_{\pi a_1^{(j)} \rightarrow \pi \pi} Z_{\mu_1 \ldots \mu_J}^{\beta} \right] X^{(J)}_{\nu_1 \ldots \nu_J},$$

$$A(\pi a_1^{(k)} \rightarrow \pi \pi) = \sum_J \alpha_J |\vec{p}|^{J-1} |\vec{k}|^J \left( W_0^{(J)} Y_0^J (\Theta, \varphi) + W_1^{(J)} Re Y_1^J (\Theta, \varphi) \right)$$

where:

$$W_0^{(J)} = -N_J 0 \left( k_{3z} - \frac{|\vec{p}|}{2} \right) \left( |\vec{p}|^2 A^{(J+)}_{\pi a_1^{(k)} \rightarrow \pi \pi} - A^{(J-)}_{\pi a_1^{(k)} \rightarrow \pi \pi} \right)$$

(1)

$$W_1^{(J)} = -\frac{N J 1}{J (J+1)} k_{3x} \left( |\vec{p}|^2 J A^{(J+)}_{\pi a_1^{(k)} \rightarrow \pi \pi} + (J+1) A^{(J-)}_{\pi a_1^{(k)} \rightarrow \pi \pi} \right)$$

Then $< Y_j^2 >$ moments in the cross section are $(k_{3x} / k_{3z})^2$.

However the contribution to $< Y_j^0 >$ could be rather large already at small $t$. 
The exchange which interferes with pion exchange is either $\pi_2$ or double meson exchange.

$$A^{(\pi_2)}_{\pi p \to \pi \pi n} = A_{\alpha \beta}(\pi \pi_2 \to \pi \pi) \varepsilon^{(a)}_{\alpha \beta} R_{\pi_2}(s_{\pi N}, q^2) \varepsilon^{(a)}_{\alpha' \beta'} X^{(2)}_{\alpha' \beta'}(p_2) \left( \varphi_n (\vec{\sigma} \vec{p}_\perp) \varphi_p \right) g^{(\pi_2)}_{pn}.$$ 

However leading contribution is taken into account by $\pi$-trajectory exchange.

Double exchange:

Interferes with $\pi$ exchange and $a_1$ exchange but should be small at large $-t-$. 
K-matrix for S and D-waves.

In the K-matrix form the unitarity condition is satisfied if:

$$A^J_{m \rightarrow n}(s) = \sum_i \hat{K}^J_{m i}(I - i\hat{\rho}^J(s)\hat{K}^J)_{i n}^{-1}$$

where $\hat{\rho}$ is the diagonal matrix with elements:

$$\rho^J_{ii}(s) = \frac{2\sqrt{-k_{i\perp}^2}}{\sqrt{s}}(-k_{i\perp}^2)^J.$$

In the present work we parameterized the elements of the K-matrix as follows:

$$K^J_{mn} = \sum_\alpha \frac{1}{B_J(k_{m\perp}^2, r_\alpha)} \left( \frac{g^\alpha(J) g^\alpha(J)}{M_\alpha^2 - s} \right) \frac{1}{B_J(k_{n\perp}^2, r_\alpha)} + \frac{f^{(J)}_{mn}}{B_J(k_{m\perp}^2, r_0) B_J(k_{n\perp}^2, r_0)}$$

and P-vector as

$$P^J_m = \sum_\alpha \frac{1}{B_J(k_{m\perp}^2, r_\alpha)} \left( \frac{\Lambda^\alpha(J) g^\alpha(J)}{M_\alpha^2 - s} \right) \frac{1}{B_J(k_{n\perp}^2, r_\alpha)} + \frac{F^{(J)}_n}{B_J(k_{m\perp}^2, r_0) B_J(k_{n\perp}^2, r_0)}$$
S-wave K-matrix: 5 poles 5 channels: $\pi\pi$ $K\bar{K}$, $\eta\eta$ $\eta\eta'$ and $4\pi$

D-wave K-matrix 4 poles and 5 channels: $\pi\pi$, $K\bar{K}$, $\eta\eta$, $4\pi$ and $\omega\omega$.

Under 2 particle threshold we used:

1) analytical continuation of the phase volume

2) subtracted dispersion integral:

$$\rho(s < 4m^2) = (s - 4m^2) \int_{4m^2}^{\infty} ds' \frac{\rho(s')}{\pi} \frac{1}{B^2(k^2, r^2)(s' - 4m^2)(s' - s)}$$

Masses of the S-wave K-matrix poles:
650, 1210 ($\pi\pi$, $K\bar{K}$), 1275 ($\pi\pi$, $4\pi$), 1650($\pi\pi$, $K\bar{K}$, $4\pi$), 1800 ($K\bar{K}$, $4\pi$) MeV.

Masses of the D-wave K-matrix poles:
1280 ($\pi\pi$), 1510 ($K\bar{K}$), 1560 ($\omega\omega$, $4\pi$), 1920 ($\pi\pi$, $\omega\omega$, $4\pi$)
The description of $p\bar{p} \rightarrow 3\pi^0$ CB-LEAR data
The description of $p\bar{p} \rightarrow \pi^0\pi^0\eta$ CB-LEAR data

(in liquid hydrogen)

(in gaseous hydrogen)
The description of $p\bar{p} \rightarrow \pi^0\eta\eta$ CB-LEAR data
The description of $\pi N \rightarrow \pi^0 \pi^0 N$ (E852)
The description of $\pi N \rightarrow \pi^0 \pi^0 N$ (E852)
The description of $\pi N \rightarrow \pi^0 \pi^0 N$ (E852)
The S-wave pole position in the complex S-plane
The D-wave pole position in the complex S-plane
S-wave at large $|t|$ due to $\pi$ and $a_1$ exchanges
CONCLUSION

1. The combined K-matrix analysis of proton-antiproton annihilation into three mesons and the $\pi N \rightarrow \pi^0\pi^0 N$ data at different t-intervals shows a good compatibility.

2. The $f_0(1370)$, $f_2(1560)$ and $f_2(1980)$ are needed for the data description.

3. At large $|t|$ the signal from $f_0(1370)$ due to $a_1$-exchange is clearly seen. But systematic check of all possibilities is not done yet.

4. The $\pi N$ interaction provide a very important information about t-channel exchanges. It is very pity that it is not explored systematically.