Phase structure, critical points, and susceptibilities in Nambu-Jona-Lasinio type models

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- Susceptibilities and critical exponents
- Partial and effective restoration of chiral symmetry
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Motivation

- QCD has a rich phase structure with challenging questions:
  - order of phase transition
  - evidence of critical end point (CEP) or tricritical point (TCP)
  - deconfinement, chiral symmetry restoration and existence of quark gluon plasma

- Status of lattice QCD\(^1\):
  - progress at \( \mu = 0 \)
  - difficulties at \( \mu \neq 0 \)
  - still no consensus on location of the CEP

- Phenomenological models can give valuable information

QCD thermodynamics is becoming more transparent due to combination research in several areas: perturbative QCD, lattice calculations and effective models

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**Model and formalism**

- **Lagrangian** of the SU(3) NJL model:

\[
\mathcal{L} = \bar{q} (i\partial \cdot \gamma - \hat{m}) q + \frac{g_S}{2} \sum_{a=0}^{8} [(\bar{q} \lambda^a q)^2 + (\bar{q} i\gamma_5 \lambda^a q)^2] + g_D [\det[\bar{q} (1 + \gamma_5) q] + \det[\bar{q} (1 - \gamma_5) q]]
\]

**Methodology:**

- **Bosonization** of the quark effective action using Feynman path integrals

  Minimization of the effective action

  \[\downarrow\]

  **Gap equations:**

  \[M_i = m_i - 2 g_S \langle \langle \bar{q}_i q_i \rangle \rangle - 2 g_D \langle \langle \bar{q}_j q_j \rangle \rangle \langle \langle \bar{q}_k q_k \rangle \rangle\]

- Effective action for the scalar and pseudoscalar mesons

  \[\downarrow\]

  **Meson propagators**, \(g_{M\bar{q}q}, f_M,\ldots\)

**Generalization of the NJL model to finite T and \(\mu\): Matsubara formalism**

Equations of state of quark matter

The thermodynamic potential of the grand canonical ensemble, $\Omega(T, \mu_i)$, comes directly from the effective action

- Our model of strong interacting matter can simulate a hot and dense fireball created in a heavy-ion collision
  ⇒ Chemical equilibrium condition: $\mu_u = \mu_d = \mu_s = \mu_B$

- The relevant equations of state for the entropy $S$, the pressure $p$, the particle number $N_i$, and the internal energy $E$, follow from expressions like the Gibbs-Duhem relation: $\Omega(T, \mu_i) = E - TS - \sum_{i=u,d,s} \mu_i N_i$

Pressure as a function of the density at different temperatures. The points $A$ and $B$ illustrate the Gibbs criteria.
• $m_u = m_d = 0$: chiral restoration happens via a second-order phase transition at $\mu_B = 0$, and a first-order one at $T = 0 \Rightarrow$ a TCP is found.

• $m_u = m_d = 5.5$ MeV: the second-order phase transition becomes a smooth crossover $\Rightarrow$ the CEP is located at $T^{CEP} = 79.9$ MeV and $\mu_B^{CEP} = 331.7$ MeV.
• $m_u = m_d = m_s = 0$: the phase transition is first-order for all $\mu_B$ and $T$

• $m_u = m_d = 0$, $m_s \geq m_s^{crit}$: the transition is second-order at $\mu_B = 0$, and first-order at $T = 0 \Rightarrow$ a TCP is found

• $m_u = m_d = 5.5$ MeV, $m_s = 140.7$ MeV: the second-order phase transition becomes a smooth crossover $\Rightarrow$ the CEP is located at $T^{CEP} = 67.7$ MeV and $\mu_B^{CEP} = 318.4$ MeV

SU(2) and SU(3) results at $\mu_B = 0$ are in agreement with what is expected: chiral phase transition in the chiral limit is second-order for $N_f = 2$ and first-order for $N_f \geq 3
Baryon number susceptibility $\chi_B$ in the vicinity of the CEP

Influence of the nature of the phase transition on the baryon number susceptibility $\chi_B = \frac{1}{V} \frac{\partial \rho_B}{\partial \mu} = -\frac{1}{V} \frac{\partial^2 \Omega}{\partial \mu^2}$, for different temperatures around the CEP: $T^\text{CEP} = 67.7$ MeV and $T = T^\text{CEP} \pm 10$ MeV

- $T < T^\text{CEP}$: the transition is first-order and, consequently, $\chi_B$ has a discontinuity
- $T = T^\text{CEP}$: the slope of $\rho_B$ tends to infinity at $\mu_B = \mu_B^\text{CEP}$, and the susceptibility $\chi_B$ diverges
- $T > T^\text{CEP}$: $\rho_B$ changes in a continuous way, and the discontinuity of $\chi_B$ disappears at the transition line (crossover)
Specific heat $C$ in the vicinity of the CEP

Influence of the nature of the phase transition on the specific heat

$$C = \frac{T}{V} \frac{\partial S}{\partial T} = -\frac{T}{V} \frac{\partial^2 \Omega}{\partial T^2},$$

for different chemical potentials around the CEP: $\mu_B^{CEP} = 318.4$ MeV and $\mu_B = \mu_B^{CEP} \pm 10$ MeV

- $\mu_B > \mu_B^{CEP}$: the transition is **first-order** and, consequently, $C$ has a discontinuity
- $\mu_B = \mu_B^{CEP}$: the specific heat **diverges**
- $\mu_B < \mu_B^{CEP}$: the **discontinuity of $\chi_B$ disappears** at the transition line (crossover)
**Critical exponents**

Critical behavior of baryon susceptibility $\chi_B$ in the vicinity of the CEP by using a linear logarithmic fit $\ln \chi_B = -\epsilon \ln |\mu_B - \mu_B^{CEP}| + c_1$

SU(2) NJL model

SU(3) NJL model

We use two paths parallel to $\mu_B$-axis: one from lower and other from higher $\mu_B$ towards the CEP

\[ \epsilon = 0.66 \pm 0.01 \]
\[ \epsilon' = 0.68 \pm 0.01 \]

- The exponents $\epsilon$ and $\epsilon'$ are consistent with the mean field theory prediction $\epsilon(\epsilon') = 2/3$
- $\epsilon \approx \epsilon'$ means that the size of the critical region is independent of the way we approach the CEP
Critical behavior of specific heat $C$ in the vicinity of the CEP by using a linear logarithmic fit $\ln C = -\alpha \ln |T - T^{CEP}| + c_2$

**SU(2) NJL model**

![Graph SU(2) NJL model](image)

**SU(3) NJL model**

![Graph SU(3) NJL model](image)

- $T > T^{CEP}$: $\alpha'$ are **consistent** with mean field predictions ($\alpha' \simeq \epsilon' = 2/3$)
- $T < T^{CEP}$: we obtain **nontrivial** values of $\alpha$, and a clear change of the slope of data points around $|T - T^{CEP}| = 0.3$ MeV in the SU(2) case
- The **nontrivial results** of $\alpha$ can be interpreted as a crossover of different universality classes

### Comparison with universality results

<table>
<thead>
<tr>
<th>Quantity</th>
<th>critical exponents/path</th>
<th>SU(2) NJL</th>
<th>SU(3) NJL</th>
<th>Universality/mean field</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\epsilon$</td>
<td>$\rightarrow$·</td>
<td>$0.66 \pm 0.01$</td>
<td>$0.67 \pm 0.01$</td>
<td>$2/3$</td>
</tr>
<tr>
<td>$\chi_B$</td>
<td>$\epsilon'$/$\leftarrow$·</td>
<td>$0.66 \pm 0.01$</td>
<td>$0.68 \pm 0.01$</td>
<td>$2/3$</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>$\uparrow$·</td>
<td>$\alpha = 0.59 \pm 0.01$</td>
<td>$0.61 \pm 0.01$</td>
<td>$2/3$</td>
</tr>
<tr>
<td>$\alpha_1$</td>
<td>$\uparrow$</td>
<td>$0.45 \pm 0.01$</td>
<td>$0.45 \pm 0.02$</td>
<td>$1/2$</td>
</tr>
<tr>
<td>$C$</td>
<td>$\alpha'$/$\downarrow$·</td>
<td>$0.69 \pm 0.01$</td>
<td>$0.67 \pm 0.01$</td>
<td>$2/3$</td>
</tr>
</tbody>
</table>

The arrow $\rightarrow$· ($\uparrow$) indicates the path in the $\mu_B(T)$– direction to the CEP (TCP) for $\mu_B < \mu_B^{CEP}$ ($T < T^{TCP}$).

- Exponents of $\chi_B$: $\epsilon$ and $\epsilon'$ are **consistent** with the mean field predictions.

- The exponents of $C$ are **sensitive** to the way we approach the CEP:
  - $\alpha'$ are **consistent** with the mean field predictions.
  - $\alpha$ have **nontrivial** values, which is a clear evidence that the CEP is affected by the TCP.
  - This effect of **crossover of different universality classes** also induces a change of the **slope of data points**, clearly visible in the SU(2) case.
Partial and effective restoration of chiral symmetry

The mesonic behavior in hot and dense medium can provide indications about possible restoration of symmetries.

- Criterion for partial restoration of chiral symmetry in the first-order region:
  - discontinuity of several observables at some critical $\mu_B$

- Criterion for partial restoration of chiral symmetry in the crossover region:
  - inflection point of the quark masses, $\partial^2 M / \partial T^2 = 0$, or of the quark condensates

- Criterion for effective restoration of chiral symmetry: the $\sigma$ and the pion (chiral partners) become degenerate in mass
Degeneracy of \((\pi^0, \sigma)\) for different temperatures around the CEP

The mesonic behavior around the CEP reflects the nature of the phase transition

\[ T = 40 \text{ MeV} < T^{CEP} \]: a **discontinuity** in the meson masses at \(\mu_B \approx 350\) MeV signalizes a **first-order** phase transition

\[ T = T^{CEP} = 67.7 \text{ MeV} \]: the **sharp decrease** (increase) of the sigma (pion) meson masse at \(\mu_B = \mu_B^{CEP}\) reflects the nature of the **second-order** phase transition

\[ T = 100 \text{ MeV} > T^{CEP} \]: we have a **crossover** and the mesons masses have a **smooth behavior**

For all cases, the **effective restoration of chiral symmetry occurs** after the phase transition takes place
Conclusions

- The model calculation reproduces qualitative features of the QCD phase diagram at $\mu_B = 0$: for $m_i = 0$ the chiral transition is second-order for $N_f = 2$, and first-order for $N_f \geq 3$.

- The critical exponents for $\chi_B$ around the CEP, in both $N_f = 2$ and $N_f = 3$ NJL models, are consistent with the mean field values $\epsilon = \epsilon' = 2/3$.

- For the specific heat, the nontrivial values of $\alpha$ ($1/2 < \alpha < 2/3$) around the CEP can be interpreted as a crossover from a tricritical universality class to a Ising-like universality class.

- Nevertheless the nontrivial values of $\alpha$ at the TCP and at the CEP are consistent within the NJL model, for both SU(2) and SU(3) versions.

- We define the line in the $T - \mu_B$ plane of partial restoration of chiral symmetry as points of inflexion (discontinuity) for quark masses, and define the line of effective restoration as that one where the masses of chiral partners become degenerate.