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Experimental status of the S0 wave at low energy: poles and scattering length

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1. Introduction.

This talk is based on the following papers:

PY05: Peláez, J. R., and Ynduráin, F. J., *Phys. Rev.* **D71**, 074016 (2005)
KPY06: Kamiński, R., Peláez, J. R., and Ynduráin, F. J., *Phys. Rev.* **D74**, 014001 (2006) and (E), **D74**, 079903 (2006)
GMPY05: García-Martín, R., Peláez, J. R., and Ynduráin, F. J., hep-ph/0701025 (to be published in Phys. Rev. D.)
KPY07: Kamiński, R., Peláez, J. R., and Ynduráin, F. J., *IV Internat. Conference on Quarks and Nuclear Physics*, Madrid, June 2006 (hep-ph/0610315), and to appear very soon.

Low energy: $4M_\pi^2 \leq s^{1/2} \lesssim 4m_K^2$ (actually, $s^{1/2} \leq 932$ MeV);
Intermediate energy: $4m_K^2 \lesssim s^{1/2} \leq 1420$ MeV.

We will not concern ourselves with higher energies here.

Experimental situation circa 1976:

Low and intermediate energy:

[Berkeley]: Protopopescu, S. D., et al., *Phys Rev.* **D7**, 1279, (1973) several Solutions.

[Cern-Munich]: Hyams, B., et al., *Nucl. Phys.* **B64**, 134, (1973), 1 Sol.; Estabrooks, P., and Martin, A. D., *Nucl. Physics*, **B79**, 301, (1974), 2 Sol.; Grayer, G., et al., *Nucl. Phys.* **B75**, 189, (1974), 4 Sol. (2, A and B).

Only $I = 2$: Losty, M. J., et al. *Nucl. Phys.*, **B69**, 185 (1974); Hoogland, W., et al. *Nucl. Phys.*, **B126**, 109 (1977); Durusoy, N. B., et al., *Phys. Lett.* **B45**, 517 (1973); the isospin 2 p.w.a. was reasonably well determined.

Intermediate energy:
Hyams, B., et al., *Nucl. Phys.* **B100**, 205, (1975): 4 Sol.

Enormous uncertainties:

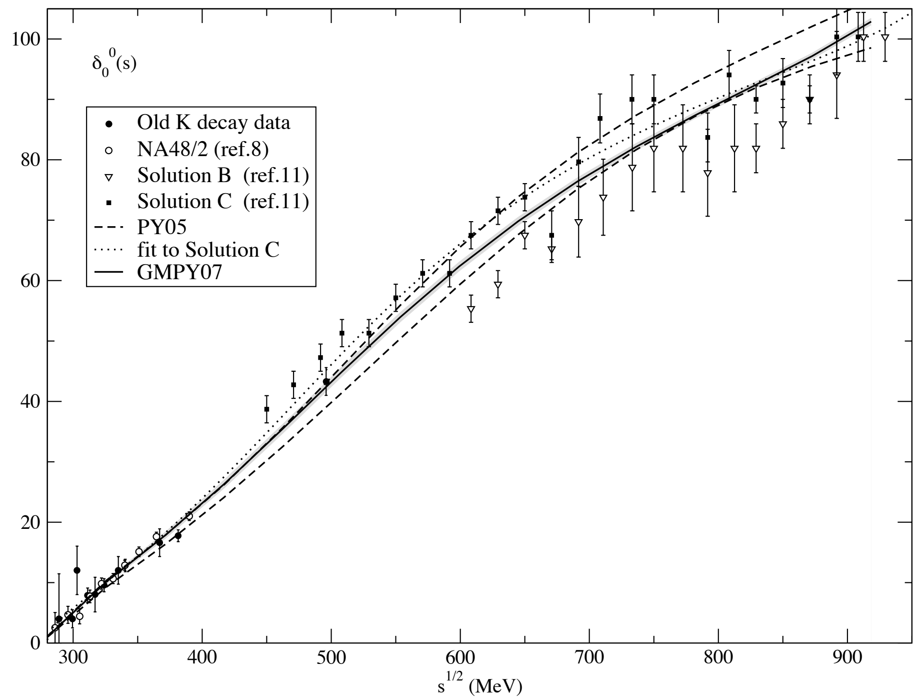
$$\text{Protopopescu}: \quad a_0^{(0)} = \begin{cases} 0.14 \\ 0.6. \end{cases}$$

$$\text{Protopopescu}: \delta_0^0(m_K) - \delta_0^2(m_K) = \begin{cases} 40 \pm 4 \\ 49 \pm 4 \end{cases}$$

Basdevant, Froggatt and Petersen (Roy Eqs.), PL **41B** 178 (1972):

$$\delta_0^0(m_K) - \delta_0^2(m_K) = \begin{cases} 36 \pm 5 \\ 38 \pm 5 \\ 47 \pm 5 \end{cases}$$

Similarly for the Cern-Munich analysis:



Data for Solutions B and C of Grayer et al. (with the same P, D, F waves) shown.

A **minimum** error of 8 - 13 degrees, dominated by *systematic* errors, had to be admitted.

However, in the region 810 MeV $\leq s^{1/2} \leq$ 952 MeV, all the various experiments are compatible, so we have a reliable set of data in this region:

$$\delta_0^{(0)}(0.870^2\ \mathrm{GeV}^2) = 91 \pm 9^\circ\,; \quad \delta_0^{(0)}(0.910^2\ \mathrm{GeV}^2) = \ 99 \pm 6^\circ\,;$$
$$\delta_0^{(0)}(0.935^2\ \mathrm{GeV}^2) = 109 \pm 8^\circ\,; \quad \delta_0^{(0)}(0.912^2\ \mathrm{GeV}^2) = 103 \pm 8^\circ\,;$$
$$\delta_0^{(0)}(0.929^2\ \mathrm{GeV}^2) = 112.5 \pm 13^\circ\,; \quad \delta_0^{(0)}(0.952^2\ \mathrm{GeV}^2) = 126 \pm 16^\circ\,;$$
$$\delta_0^{(0)}(0.810^2\ \mathrm{GeV}^2) = 88 \pm 6^\circ\,; \quad \delta_0^{(0)}(0.830^2\ \mathrm{GeV}^2) = 92 \pm 7^\circ\,;$$
$$\delta_0^{(0)}(0.850^2\ \mathrm{GeV}^2) = 94 \pm 6^\circ\,.$$

Errors here include systematic errors, estimated by comparing different determinations.

A series of improvements: at intermediate energy,

$\pi\pi \to \bar{K}K$ data: Wetzel, W., et al., *Nucl. Phys.* **B115**, 208 (1976); Cohen, D. et al., *Phys. Rev.* **D22**, 2595 (1980); Etkin, E. et al., *Phys. Rev.* **D25**, 1786 (1982), all compatible.

This reduces to 1 the 4 solutions in Hyams 75.

Pion form factor [Novosibirsk + LEP data; see de Trocóniz and FJY, *Phys. Rev.*, **D65**, 093001, (2002) and **D71**, 073008 (2005)].

This gives P wave virtually exactly.

$Ke4$ experiments: Rosselet, L., et al. *Phys. Rev.* **D15**, 574 (1977); Pislak, S., et al. *Phys. Rev. Lett.*, **87**, 221801 (2001);
And, above all, CERN/SPS experiment (NA48/2); Bloch-Devaux, B., presented at QCD06 in Montpellier (France), 3-7 July 2006 and Masetti, L., presented at ICHEP06 in Moscow (Russia), 26 July to 2 August 2006.

Very precise S0 wave below 400 MeV.

$K2\pi$ **experiments relate** $K \to 2\pi$ **decays to** $\delta_0^0(m_K) - \delta_0^2(m_K)$

This provides a very accurate value for S0-S2 at the key energy of 500 MeV:

$$\delta_0^{(0)}(m_K^2) - \delta_0^{(2)}(m_K^2) = \begin{cases} 58.0 \pm 4.3\,(\text{exp}) \pm 1.6\,(\text{rad})\ [\text{P.Pascual \& FJY}]\ 1974 \\ 57.27 \pm 0.82\,(\text{exp}) \pm 3\,(\text{rad}) \pm 1(\text{ch.p.app.})\ [\text{KLOE+C.}] \end{cases}$$

[P.Pascual & FJY: NP **B83**, 362 (1974); V. Cirigliano et al., Eur. Phys. J. **C33** ,369 (2004)+KLOE]

In addition, for the difference $a_0^{(0)} - a_0^{(2)}$ one has results form the following independent experimental determinations: from pionic atoms [Adeva, B., Romero Vidal, A. and Vázquez Doce, O., *Eur. Phys. J.* **31**, 522 (2007)] that give

$$a_0^{(0)} - a_0^{(2)} = 0.280 \pm 0.013\ (\mathrm{St.}) \pm 0.008\ (\mathrm{Syst.})\ M_\pi^{-1}$$

and from $K_{3\pi}$ decays [Cabibbo, N., and Isidori, G., *JHEP* 0503:021 (2005); NA48 Experiment: see e.g. Balev, S., arXiv: 0705.4183 v2 (2007)] that imply

$$a_0^{(0)} - a_0^{(2)} = 0.268 \pm 0.010\ (\mathrm{St.}) \pm 0.013\ (\mathrm{Syst.})\ M_\pi^{-1}.$$

Since these experiments are independent, and compatible, we can combine them (adding statistic and systematic errors in quadrature) to find

$$a_0^{(0)} - a_0^{(2)} = 0.274 \pm 0.011\ \ M_\pi^{-1} \quad [\ \textbf{Pionic atoms} + K_{3\pi}\ \textbf{decays}].$$

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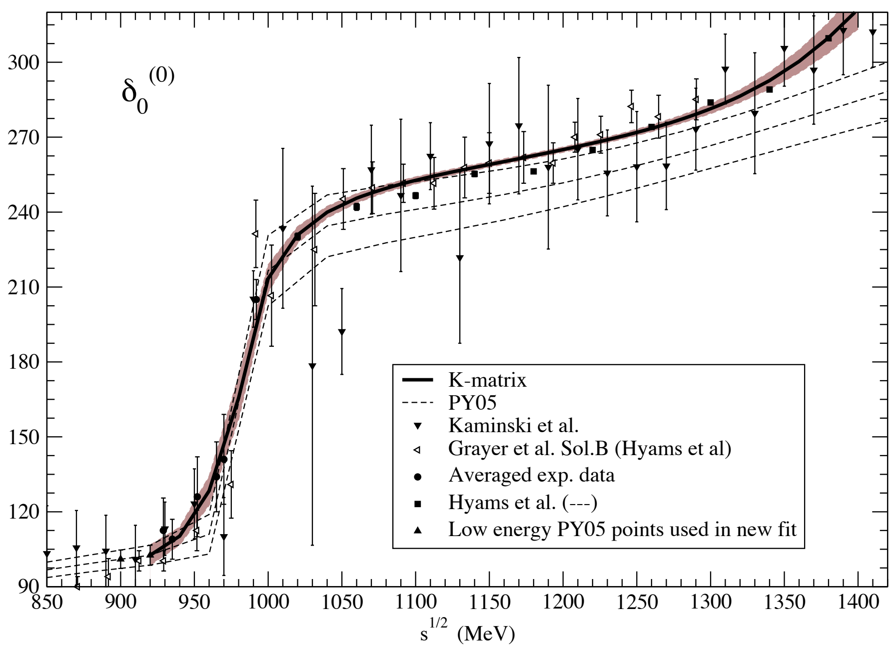
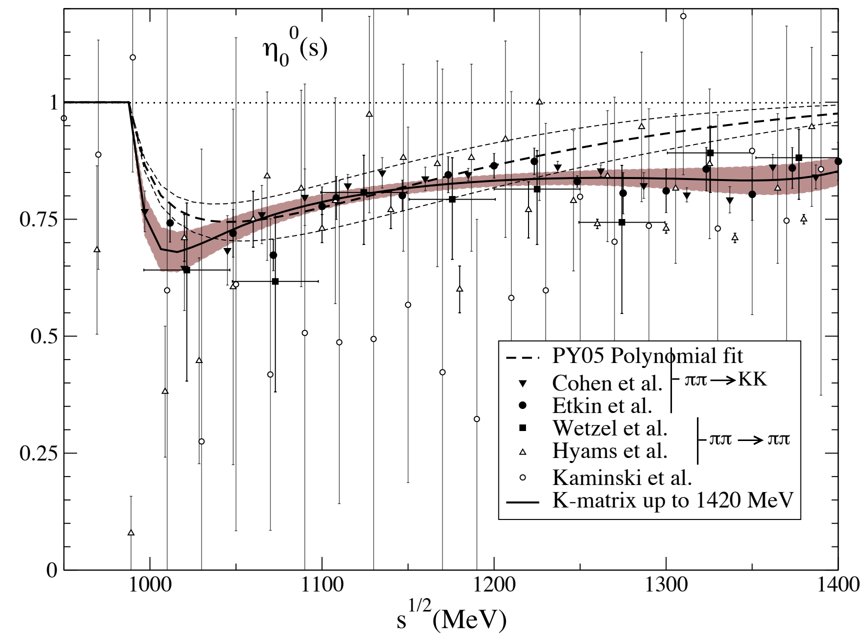
2. The intermediate energy region

Here one uses the well-known K-matrix method. Note that this **cannot** reliably be used below $s = 4(m_K^2 - M_\pi^2)$, as problems with the l.h.cut in the reaction $\pi\pi \to \bar{K}K$ will arise.

The details may be found in KPY06; they are summarized in the following figures:

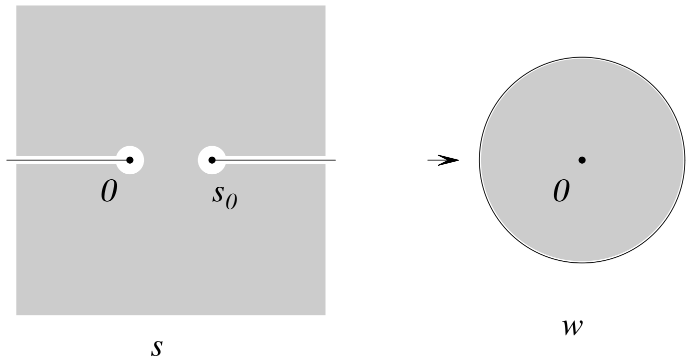


K-matrix fit to $\delta_0^{(0)}$ (solid line and dark area). Dotted lines: the fit in PY05.



K-matrix fit to $\eta_0^{(0)}$ using (2.4), with error given by the shaded area. The dotted lines represent the central values and error limits of the old fit in PY05.

An unfortunate fact is that this representation is not very accurate above $s^{1/2} \simeq 1380$ MeV as it does not take into account the contribution of the 4π states, which become dominant there.



The mapping $s \to w$.

3. Low energy: The conformal mapping method

How can one correlate (and parametrize) these data, without introducing theoretical assumptions, besides causality and unitarity?

The analyticity and unitarity properties of the S0 partial wave amplitude imply that the *effective range function*

$$\psi(s) \equiv (2k/s^{1/2}) \cot \delta_0^{(0)}(s)$$

is analytic in the complex plane cut from $-\infty$ to 0 and from where inelasticity matters s_0, to $+\infty$ (figure): **it has not the elastic cut**. For the S0 wave, one can take $s_0 = 4m_K^2$.

The standard method to deal with this situation is to make a mapping

$$s \to w(s) = \frac{\sqrt{s} - \sqrt{4m_K^2 - s}}{\sqrt{s} + \sqrt{4m_K^2 - s}}:$$

the cut plane is mapped into the unit disk; the analyticity properties of the effective range function in the complex plane in the variable s are *strictly equivalent* to convergence of a Taylor series for ψ in the variable w.

To improve this convergence, it is convenient to separate off those zeros and poles of the effective range function that lie on the real axis. Of these we have one of each: a pole due to the so-called Adler zero of the partial wave amplitude, lying near the left hand cut, at $s = \frac{1}{2}z_0^2$, $z_0 \simeq M_\pi$ (with M_π the charged pion mass); and then there is a zero due to the phase shift crossing $\pi/2$ for an energy μ_0 near 780 MeV. Thus, we can, in all generality, write the following parametrization:

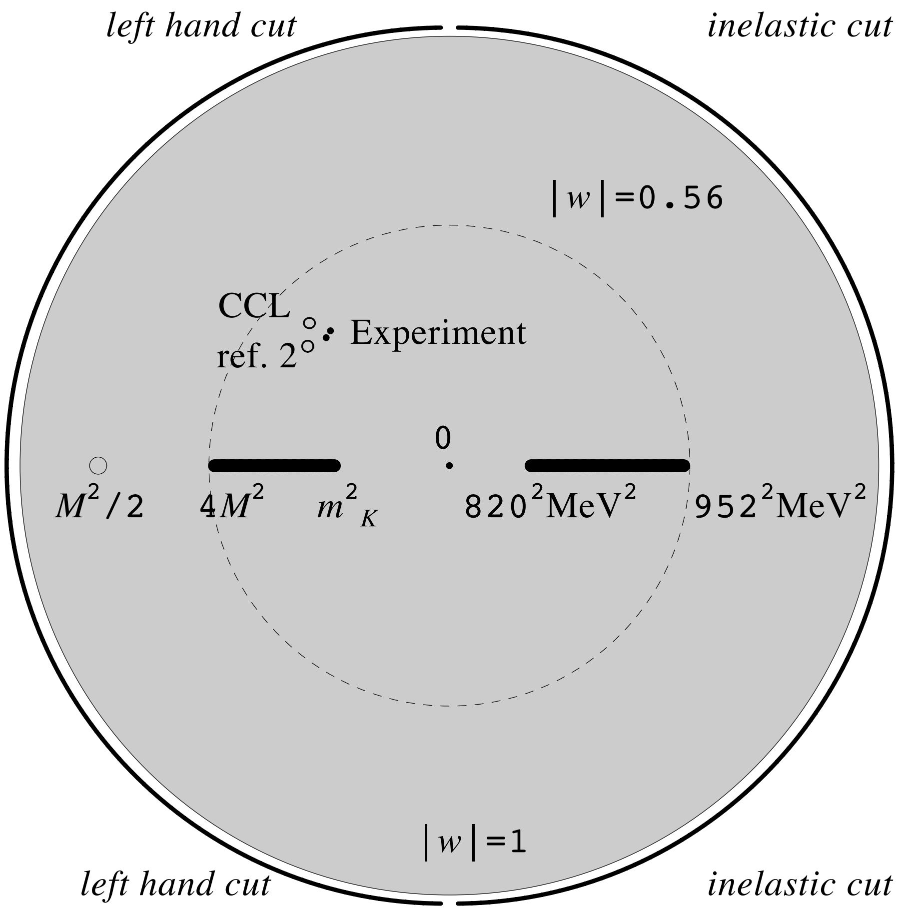
$$\cot\delta(s) = \frac{s^{1/2}}{2k}\,\frac{M_\pi^2}{s-\frac{1}{2}z_0^2}\,\frac{\mu_0^2-s}{\mu_0^2}\big\{B_0+B_1w(s)+\cdots\big\}.$$

or, if not separating the zero (we will need one more parameter),

$$\cot\delta(s) = \frac{s^{1/2}}{2k}\,\frac{M_\pi^2}{s-\frac{1}{2}z_0^2}\,\big\{B_0+B_1w(s)+B_2w(s)^2\cdots\big\}. \tag{1}$$

The key point for us is that this converges in all the cut plane: therefore, (1) can be used as it is to evaluate the scattering length $a_0^{(0)}$ or to find the location of the σ resonance.

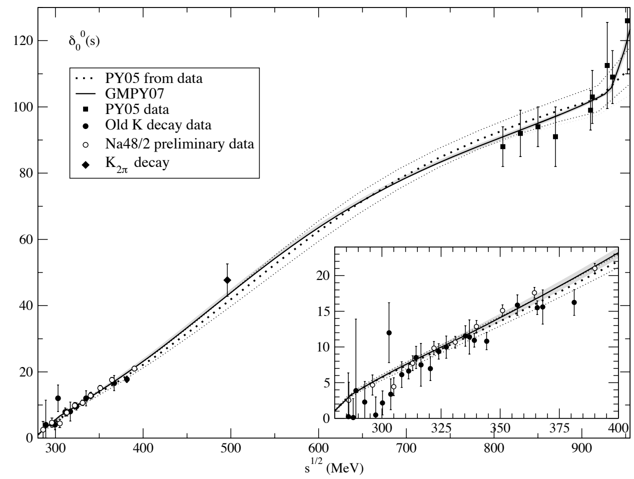
Perhaps it should be emphasized tha this isn neither a model nor theory, but merely an approximation.



The w disk, $|w| < 1$. The dashed line is the line $|w| = 0.56$. The thick lines are the regions where one has reliable experimental data. The sigma pole is also shown.

As a matter of fact, all the experimental points (and also the σ resonance) we have discussed are located, in the w plane, inside the disk $|w| \leq 0.56$, hence we only expect an error of $|B_4|\,|w|^4 \lesssim 3\%$ on the real axis[1] and of $\lesssim 6\%$ inside the $|w| \leq 0.56$.

[1] B_4 can be estimated in terms of the known B_3.



The phase shift, $\delta_0^{(0)}$. The dotted line is the value given in PY05; the continuous line is the fit to it, as given here.

3. A few observables: The scattering length, σ pole, Phase difference on the K

We find the following results: a phase shift, at low energy, as shown in the figure:

If we *also* include medium energy data, as in ref. 5, we find

$$B_0 = 4.3 \pm 0.3, \quad B_1 = -26.7 \pm 0.6, \quad B_2 = -14.1 \pm 1.4$$

and

$$a_0^{(0)} = 0.231 \pm 0.009, \quad b_0^{(0)} = -0.288 \pm 0.009 \quad \textbf{[Experiment]}.$$

This is compatible, but somewhat displaced, with the estimate given by CGL based on chiral perturbation theory and Roy equations,

$$a_0^{(0)} = 0.220 \pm 0.005, \quad b_0^{(0)} = -0.280 \pm 0.004 \quad \text{[Ch. P. Theory]}.$$

For the parameters of the σ pole: we find

$$M_\sigma = 484{\pm}6\ (\text{St.}){\pm}11\ (\text{Sys.}), \quad \Gamma_\sigma/2 = 255{\pm}8\ (\text{St.}){\pm}2\ (\text{Sys.}); \quad \text{Experiment.}$$

to be compared with the theoretical estimates,

$$M_\sigma = 441 \pm 17\ \text{MeV}, \quad \Gamma_\sigma/2 = 230 \pm 15\ \text{MeV}; \quad \text{Theory, Ref. 2}$$

$$M_\sigma = 441^{+16}_{-8}\ \text{MeV}, \quad \Gamma_\sigma/2 = 279^{+9}_{-12.5}\ \text{MeV}; \quad \text{Theory, Ref. 3.}$$

$$M_\sigma = 470 \pm 50\ \text{MeV}, \quad \Gamma_\sigma/2 = 285 \pm 25\ \text{MeV}; \quad \text{Theory, Ref. 4}$$

2.- Unitarized CH.P.T.: Peláez, J. R., *Modern Phys. Lett. A*, **19**, 2879 (2004); Oller, J. A., Oset, E., and Peláez, J. R., *Phys. Rev.* **D59**, 074001 (1999) and Erratum, *ibid.* **D60**, 099906 (1999); A. Dobado and Peláez, J. R., *Phys. Rev.* **D56**, 3057 (1997); Gómez-Nicola, A., and Peláez, J. R., *Phys. Rev.* **D65**, 054009 (2002).

3.- CH.P.T. + Roy Eqs.: Caprini, I., Colangelo, G., and Leutwyler, H. *Phys. Rev. Lett.* **96**, 132001 (2006).

4.- Unitarized CH.P.T.: Zhou, Z. Y., et al. *JHEP* 0502, 043 (2005).

Thus, agreement at the 2 σ level.

For the difference $\delta_0^{(0)}(m_K^2) - \delta_0^{(2)}(m_K^2)$:

$$\delta_0^{(0)}(m_K^2) - \delta_0^{(2)}(m_K^2) = \begin{cases} 50.1 \pm 1.8 & [\textbf{not}\ \text{using}\ K2\pi] \\ 51.7 \pm 1.3 & [\textbf{using}\ K2\pi] \\ 57.3 \pm 4.8 & [\text{exp., KLOE+C.}] \end{cases}$$

One can compare this with

$$\delta_0^{(0)}(m_K^2) - \delta_0^{(2)}(m_K^2) = 47.7 \pm 1.5 \quad [\text{CGL}]$$

4. Compatibility tests: FDR and Roy equations

$$\Delta_{00}(s) \equiv \operatorname{Re} F_{00}(s) - F_{00}(4M_\pi^2) \\ - \frac{s(s-4M_\pi^2)}{\pi} \text{P.P.} \int_{4M_\pi^2}^{\infty} \mathrm{d}s' \, \frac{(2s'-4M_\pi^2) \operatorname{Im} F_{00}(s')}{s'(s'-s)(s'-4M_\pi^2)(s'+s-4M_\pi^2)}$$

and an identical one, with $00 \to 0+$; and

$$\Delta_1(s) \equiv \operatorname{Re} F^{(I_t=1)}(s,0) - \frac{2s-4M_\pi^2}{\pi} \text{P.P.} \int_{4M_\pi^2}^{\infty} \mathrm{d}s' \, \frac{\operatorname{Im} F^{(I_t=1)}(s',0)}{(s'-s)(s'+s-4M_\pi^2)}.$$

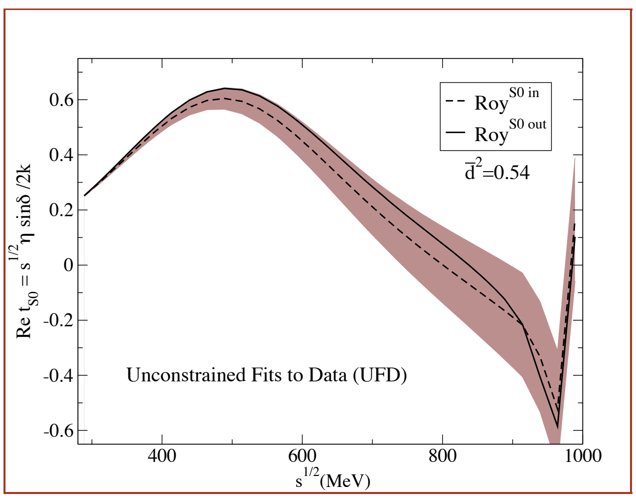
These quantities would vanish, $\Delta_i = 0$, if the dispersion relations were exactly satisfied.

We find, with $\bar{d}^2 = \text{average}\,(\Delta^2/\delta\Delta^2)$,

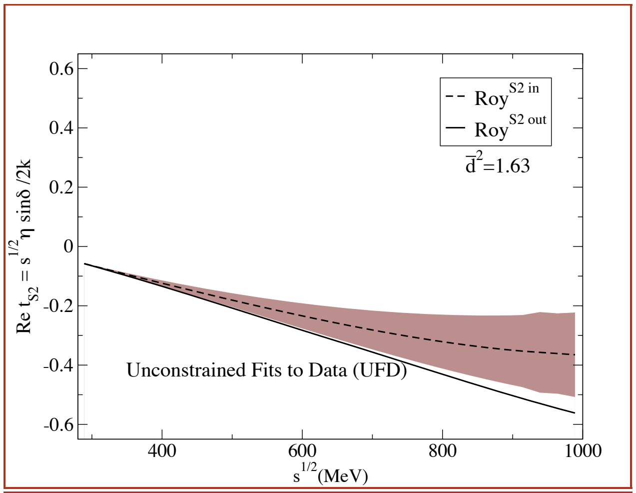
	$s^{1/2} \leq 932$ MeV	$s^{1/2} \leq 1420$ MeV
$\pi^0\pi^0$ FDR	$\bar{d}^2 = 0.12$	$\bar{d}^2 = 0.29$
$\pi^0\pi^+$ FDR	$\bar{d}^2 = 0.89$	$\bar{d}^2 = 0.90$
$I_t = 1$ FDR	$\bar{d}^2 = 0.67$	$\bar{d}^2 = 1.89.$

and, for the Roy equations,

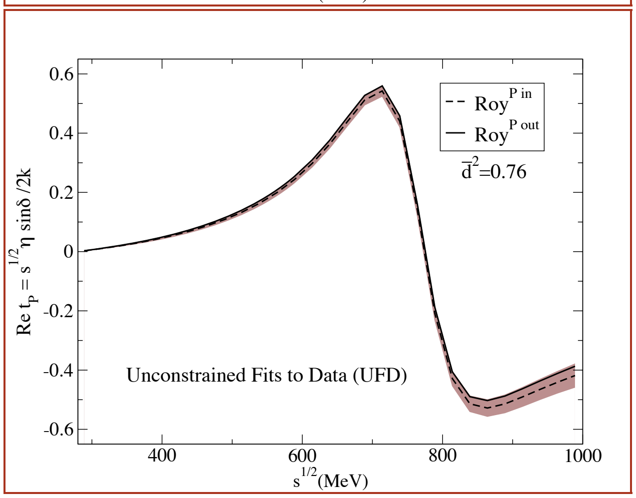
$$\bar{d}^2_{S0} = 0.54, \quad \bar{d}^2_{S2} = 1.63, \quad \bar{d}^2_P = 0.77.$$



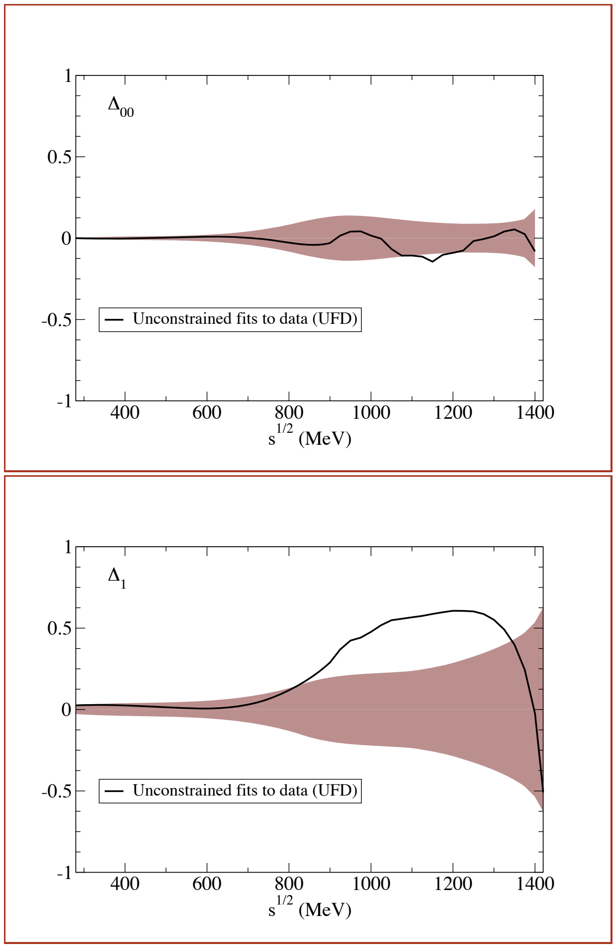
Fulfillment of the Roy equation for the S0 wave.
Continuous line: the result of the dispersive integral.
Dashed line: real part, with the dark band the error band.



Fulfillment of the Roy equation for the S2 wave.
Continuous line: the result of the dispersive integral.
Dashed line: real part, with the dark band the error band.



Fulfillment of the Roy equation for the P wave.
Continuous line: the result of the dispersive integral.
Dashed line: real part, with the dark band the error band.



Fulfillment of dispersion relations. The error bands are also shown.

5. Observables using info on other waves. Constrained fits. The scattering length

If we use data on the other waves (and Regge parameters at high energy) we can repeat the fits, imposing fulfillment of FDR and Roy equations (and two sum rules), within experimental errors; i.e., adding a weight $\sum_i \bar{d}_i^2$ in the fits to data. We call this CFD. The S0 wave is left essentially unaltered; only the D2 wave changes by a bit more than one sigma.

Now, however, we can test data involving waves other than S0. Using the Olsson sum rule one can get a very precise number for the $a_0^{(2)}$ scattering length, finding

$$a_0^{(2)} = -0.0436 \pm 0.0043 \quad \text{[CFD]}$$

This can be compared with CGL: perfect agreement

$$a_0^{(2)} = -0.0444 \pm 0.0011 \quad \text{[CGL]}$$

Combinig this with the $a_0^{(0)}$ scattering length, we can get a prediction for the difference $a_0^{(0)} - a_0^{(2)}$, **which was not used in our fits**. We get

$$a_0^{(0)} - a_0^{(2)} = 0.271 \pm 0.012 \quad \text{[CFD]},$$

Experimentally,

$$a_0^{(0)} - a_0^{(2)} = 0.274 \pm 0.011 \quad [\text{ Pionic atoms} + K_{3\pi} \text{ decays}],$$

agreeing very well with our result

By comparison, the number from CGL is

$$a_0^{(0)} - a_0^{(2)} = 0.264 \pm 0.007 \quad [\text{ CGL}].$$

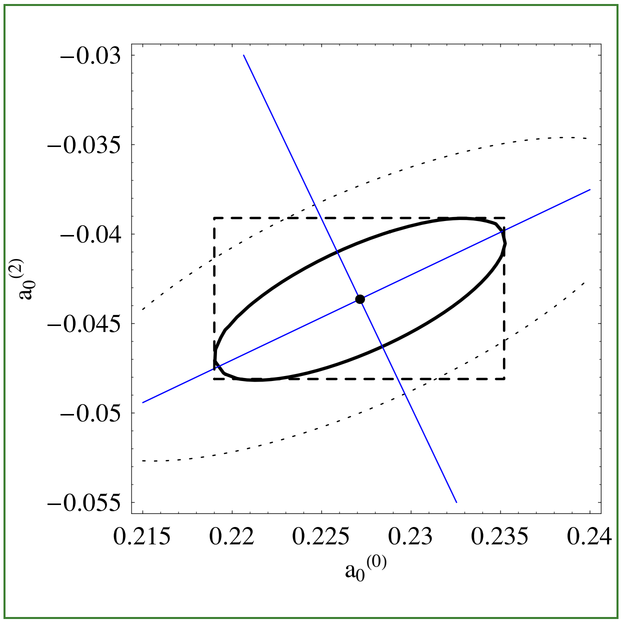
Alternatively, we can use

$$a_0^{(0)} - a_0^{(2)} = 0.274 \pm 0.011 \quad [\text{ Pionic atoms} + K_{3\pi} \text{ decays}],$$

and the Olsson sum rule, that gives

$$2a_0^{(0)} - 5a_0^{(2)} = 0.670 \pm 0.018$$

to determine $a_0^{(0)}$, $a_0^{(2)}$. This gives an **ellipse**,



Ellipse in the $a_0^{(0)}$, $a_0^{(2)}$ plane corresponding to 1-sigma (continuous line).

The corresponding scattering lengths are

$$a_0^{(0)} = 0.227 \pm 0.008, \quad a_0^{(2)} = -0.0436 \pm 0.0043.$$

To compare:

$$a_0^{(0)} = 0.220 \pm 0.005 \quad [\text{CGL}].$$

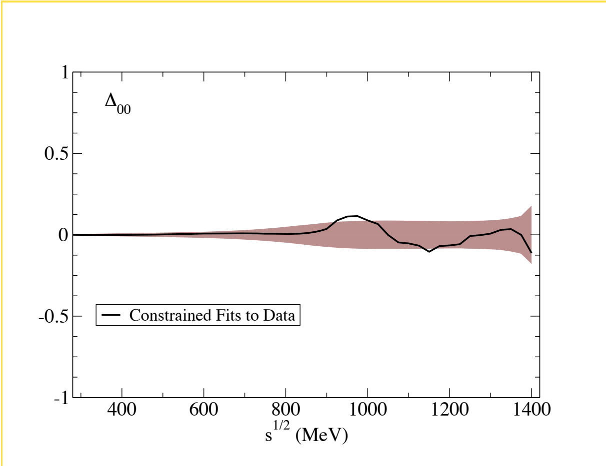
The S0 wave of CGL is slightly distorted (at the 1 to 2-sigma level), probably because of the faulty input in ACGL, CGL.

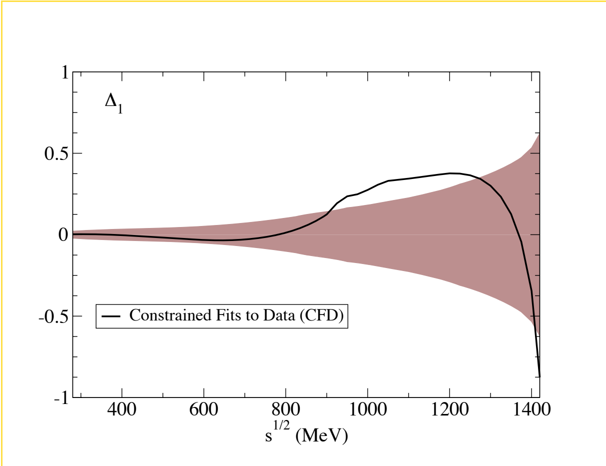
FDR and Roy are now virtually perfect: we have

		$s^{1/2} \le 932$ MeV	$s^{1/2} \le 1420$ MeV
$\pi^0\pi^0$	FDR	$\bar{d}^2 = 0.13$	$\bar{d}^2 = 0.32$
$I_t = 1$	FDR	$\bar{d}^2 = 0.22$	$\bar{d}^2 = 0.73,$

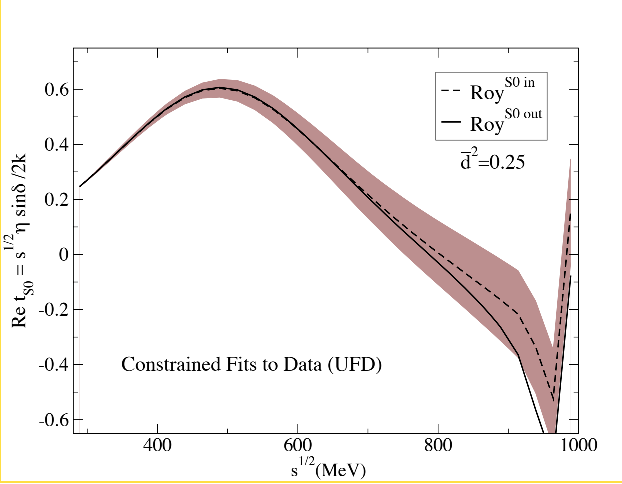
and, for the Roy equations,

$$\bar{d}^2_{S0} = 0.25, \quad \bar{d}^2_{S2} = 0.26, \quad \bar{d}^2_{P} = 0.01.$$

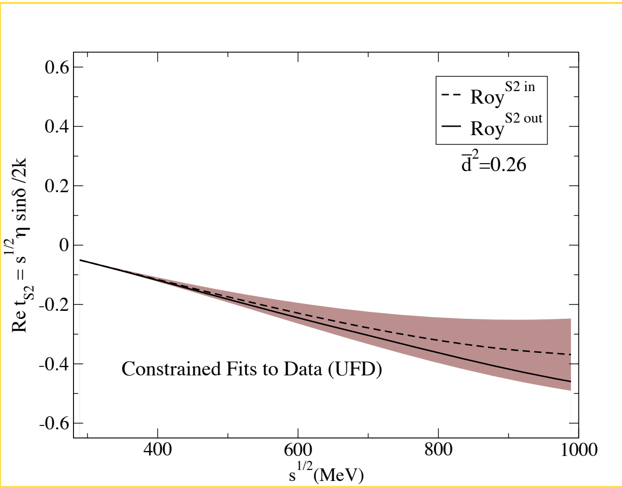




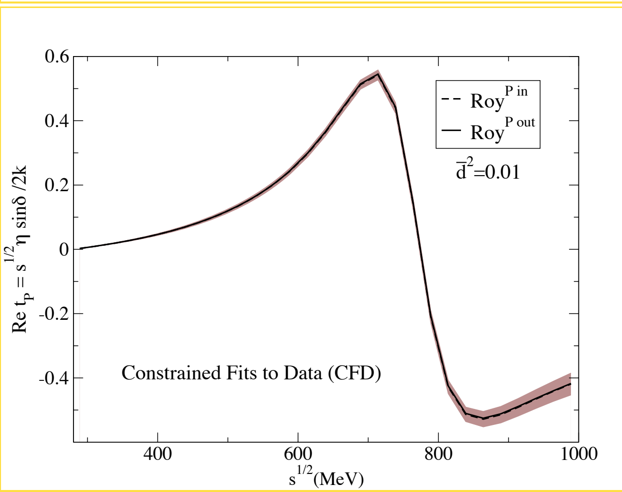
Fulfillment of dispersion relations, with the central parameters with CFD. The error bands are also shown.



Fulfillment of the Roy equation for the S0 wave.
Continuous line: the result of the dispersive integral.
Dashed line: real part, with the dark band the error band.



Fulfillment of the Roy equation for the S2 wave.
Continuous line: the result of the dispersive integral.
Dashed line: real part, with the dark band the error band.



Fulfillment of the Roy equation for the P wave.
Continuous line: the result of the dispersive integral.
Dashed line: real part, with the dark band the error band.

Besides sum rules and dispersion relations, another independent test of our amplitudes is the Adler sum rule that relates the pion decay constant to pion-pion scattering amplitudes with one pion off its mass sell. This has been recently evaluated with our scattering UDF amplitudes, and a very satisfactory fulfillment of the sum rule is found; the discrepancy Δ_π that measures the accuracy with which the sum rule is fulfilled (and which should vanish if it was satisfied exactly) is found to be

$$\Delta_\pi = 0.021 \pm 0.023$$

[CGL would get $\Delta_\pi = 0.033$].

6. Including isospin breaking corrections

Note added in proof

The precision of the fits is such that we have to worry about isospin breaking corrections. We have had notice of a calculation by J. Gasser, "Theoretical progress on cusp effect and Kl4," [talk at KAON07, May 21-25, 2007, Frascati, Italy], as yet unpublished * in which account is taken, for Ke4 decays, of the fact that in the real world isospin is broken. A full analysis is missing; but, if we take the results of the calculation at face value, we can repeat our best fit incorporating these corrections. We then find:

$$\begin{array}{lll} B_0 = 3.80 \pm 0.34, & B_1 = -27.1 \pm 0.8, & B_2 = -8.3 \pm 1.8; \\ a_0^{(0)} = 0.211 \pm 0.010\ M_\pi^{-1}, & b_0^{(0)} = 0.278 \pm 0.010\ M_\pi^{-3}; & \\ M_\sigma = 481 \pm 7\ \text{MeV}; & \Gamma_\sigma/2 = 237 \pm 5\ \text{MeV}. & \end{array}$$

$$M_\sigma = 484 \pm 6\ (\text{St.}) \pm 11\ (\text{Sys.}), \quad \Gamma_\sigma/2 = 255 \pm 8\ (\text{St.}) \pm 2\ (\text{Sys.}); \quad \text{Experiment};$$

$$a_0^{(0)} = 0.231 \pm 0.009, \quad b_0^{(0)} = -0.288 \pm 0.009 \quad [\text{Experiment}].$$

The phase shift itself moves very little, by less than 1^o.

When fitting including requirement of FDR and Roy equations, the S0 wave moves closer to the one given before, and the scattering length becomes quite compatible with what we found neglecting isospin violations: we now get

$$a_0^{(0)} = 0.219 \pm 0.011,$$

and the phase shift moves by less than 0.5^o.

Although, as stated, a full analysis will be necessary, it does not seem that including isospin breaking corrections will much affect our results.

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