Can one control systematic errors of hadron parameters obtained with QCD sum rules?

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We study the possibility to control the systematic errors of the bound-state parameters obtained with SVZ sum rules, making use of the harmonic-oscillator potential model. In this case, one knows the exact solution for the polarization operator $\Pi(\mu)$, which allows one to obtain (i) OPE to any order (ii) the spectrum of states.

We apply the standard sum-rule machinery for extracting the parameters of the ground state, and check the accuracy of the extracted values by comparing with the exact known values. In this way we probe the accuracy of the method.

A QCD sum-rule calculation of hadron parameters involves two steps:

(i) one calculates the operator product expansion (OPE) series for a relevant correlator, and obtains the sum rule which relates this OPE to the sum over hadronic states.

(ii) one extracts the parameters of the ground state by a numerical procedure.

The first step lies fully within QCD and allows (at least in principle) a rigorous treatment of the uncertainties.

The second step lies beyond QCD: even if several terms of the OPE for the correlator are known precisely, the hadronic parameters might be extracted by a sum rule only within some error, which may be treated as a systematic error of the method. For many applications of sum rules, especially in flavor physics, one needs rigorous error estimates of the theoretical results for comparing theoretical predictions with the experimental data.

To study the systematic errors of the ground-state parameters extracted with SVZ sum rules, one needs a model where the exact values are known. The harmonic-oscillator potential model is an ideal example: in this case, one knows the exact solution for the polarization operator $\Pi(\mu)$, which allows one to obtain (i) OPE to any order and (ii) the spectrum of states.

MODEL:

$$H = H_0 + V(r), \quad H_0 = \vec{p}^2/2m, \quad V(r) = \frac{m\omega^2\vec{r}^2}{2}, \quad G(E) = (H - E)^{-1}.$$

Polarization operator $\Pi(E)$ defined through the full Green function $G(E)$:

$$\Pi(E) = (2\pi/m)^{3/2} \langle \vec{r}_f = 0|G(E)|\vec{r}_i = 0\rangle,$$

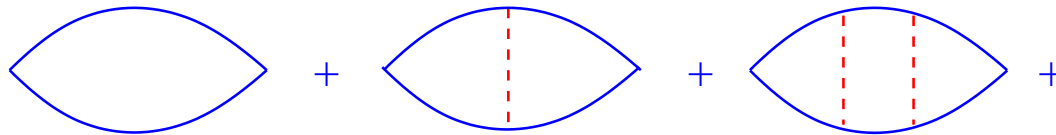**The Borel transformed $\Pi(\mu)$ is the evolution operator in imaginary time $1/\mu$:**

$$\Pi(\mu) = (2\pi/m)^{3/2} \langle \vec{r}_f = 0| \exp(-H/\mu)|\vec{r}_i = 0\rangle = \left(\frac{\omega}{\sinh(\omega/\mu)}\right)^{3/2}.$$

OPE:

Expanding in inverse powers of μ gives the OPE series for $\Pi(\mu)$ to any order:

$$\Pi_{\text{OPE}}(\mu) \equiv \Pi_0(\mu) + \Pi_1(\mu) + \Pi_2(\mu) + \cdots = \mu^{3/2}\left[1 - \frac{\omega^2}{4\mu^2} + \frac{19}{480}\frac{\omega^4}{\mu^4} - \frac{631}{120960}\frac{\omega^6}{\mu^6} + \cdots\right].$$

Each term may be also calculated from the perturbative expansion of $G(E)$:

$$\Pi_0(E) \qquad\qquad \Pi_1(E) \qquad\qquad \Pi_2(E)$$

with e.g. $\Pi_0(\mu) = \int\limits_0^\infty dz\rho_0(z) \exp(-z/\mu), \quad \rho_0(z) = \frac{2}{\sqrt{\pi}} \sqrt{z}.$
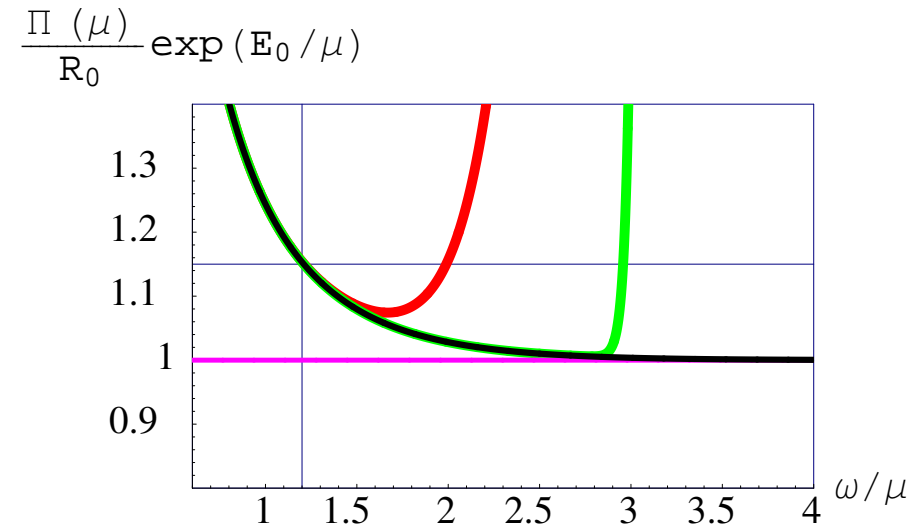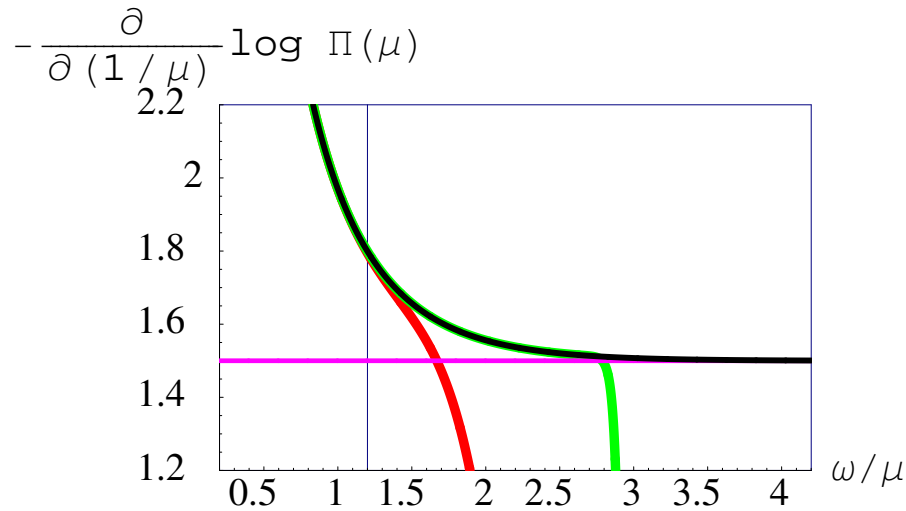
The "phenomenological" representation for $\Pi(\mu)$ is in the basis of hadron eigenstates:

$$\Pi(\mu) = \sum_{n=0}^{\infty} R_n \exp(-E_n/\mu),$$

E_n = energy of the n-th bound state, $R_n = (2\pi/m)^{3/2}|\Psi_n(\vec{r}=0)|^2$.

$$E_0 = \tfrac{3}{2}\omega, \quad R_0 = 2\sqrt{2}\omega^{3/2}, \qquad E_1 = \tfrac{7}{2}\omega, \quad R_1 = 3\sqrt{2}\omega^{3/2}.$$

How to calculate E_0 and R_0 from $\Pi(\mu)$ known numerically?

Black - exact $\Pi(\mu)$; Red - OPE with 4 power corrections, Green - OPE with 100 power corrections.

SUM RULE

The equality of the correlator calculated in the hadron basis and in the "quark" basis:

$$R_0 e^{-E_0/\mu} + \int_{z_{\text{cont}}}^{\infty} dz \rho_{\text{phen}}(z) e^{-z/\mu} = \int_0^{\infty} dz \rho_0(z) e^{-z/\mu} + \mu^{3/2} \left[-\frac{\omega^2}{4\mu^2} + \frac{19}{480}\frac{\omega^4}{\mu^4} - \frac{631}{120960}\frac{\omega^6}{\mu^6} + \cdots \right].$$

<u>Effective continuum threshold $z_{\text{eff}}(\mu)$</u> (cannot be a μ-independent constant!)

$$\Pi_{\text{cont}}(\mu) = \int_{z_{\text{cont}}}^{\infty} dz \, \rho_{\text{phen}}(z) \, \exp(-z/\mu) = \int_{z_{\text{eff}}(\mu)}^{\infty} dz \, \rho_0(z) \, \exp(-z/\mu).$$

Rewrite the sum rule in the form:

$$R_0 \exp(-E_0/\mu) = \Pi(\mu, z_{\text{eff}}(\mu)) \equiv \frac{2}{\sqrt{\pi}} \int_0^{z_{\text{eff}}(\mu)} dz \sqrt{z} \exp(-z/\mu) + \mu^{3/2} \left[-\frac{\omega^2}{4\mu^2} + \frac{19}{480}\frac{\omega^4}{\mu^4} - \frac{631}{120960}\frac{\omega^6}{\mu^6} + \cdots \right].$$

The cut correlator $\Pi(\mu, z_{\text{eff}}(\mu))$ satisfies the equation:

$$E(\mu) \equiv -\frac{d}{d(1/\mu)} \log \Pi(\mu, z_{\text{eff}}(\mu)) = E_0.$$

The cut correlator governs the extraction of the ground-state: parameters.

$$R_0 \exp(-E_0/\mu) = \omega^{3/2} \frac{2}{\sqrt{\pi}} \int_0^{z_{\mathrm{eff}}(\mu)/\omega} dz' \sqrt{z'} \exp(-z'\omega/\mu) + \omega^{3/2} \frac{\mu^{3/2}}{\omega^{3/2}} \left[-\frac{\omega^2}{4\mu^2} + \frac{19}{480}\frac{\omega^4}{\mu^4} - \frac{631}{120960}\frac{\omega^6}{\mu^6} + \cdots \right].$$

Constraints on $z_{\mathrm{eff}}(\mu)$. Expand both sides near $\omega/\mu = 0$: the l.h.s. contains only integer powers of ω/μ, power corrections on the r.h.s. contain only odd powers of $\sqrt{\omega/\mu}$. Therefore $z_{\mathrm{eff}}(\mu)$ cannot be constant:

$$z_{\mathrm{eff}}(\mu) = \omega \left[\bar{z}_0 + \bar{z}_1 \sqrt{\frac{\omega}{\mu}} + \bar{z}_2 \frac{\omega}{\mu} + \cdots \right].$$

Inserting this series in the SR above:

$$R_0 = \frac{4}{3\sqrt{\pi}} \bar{z}_0^{3/2} \omega^{3/2}, \quad \bar{z}_1 = \frac{\sqrt{\pi}}{8\sqrt{\bar{z}_0}}, \quad R_0 E_0 = \frac{2}{5\sqrt{\pi}} \bar{z}_0^{5/2} \omega^{5/2} - \frac{\omega^{5/2}}{2\sqrt{\pi}} (\bar{z}_1^2 + 4\sqrt{\bar{z}_0}\bar{z}_2).$$

For any E_0 and R_0 within a broad range of values there exists a function $z_{\mathrm{eff}}(\mu, E_0, R_0)$ which solves the sum rule *exactly*.

Setting $E_0 = \frac{3}{2}\omega$ and $R_0 = 2\sqrt{2}\omega^{3/2}$, the equations above yield the following solution for the exact effective continuum threshold in the HO model: $\bar{z}_0 = 2.418$, $\bar{z}_1 = 0.142$, $\bar{z}_2 = -0.081$, etc.

Setting E_0 equal to its exact value does not help much in extracting R_0:
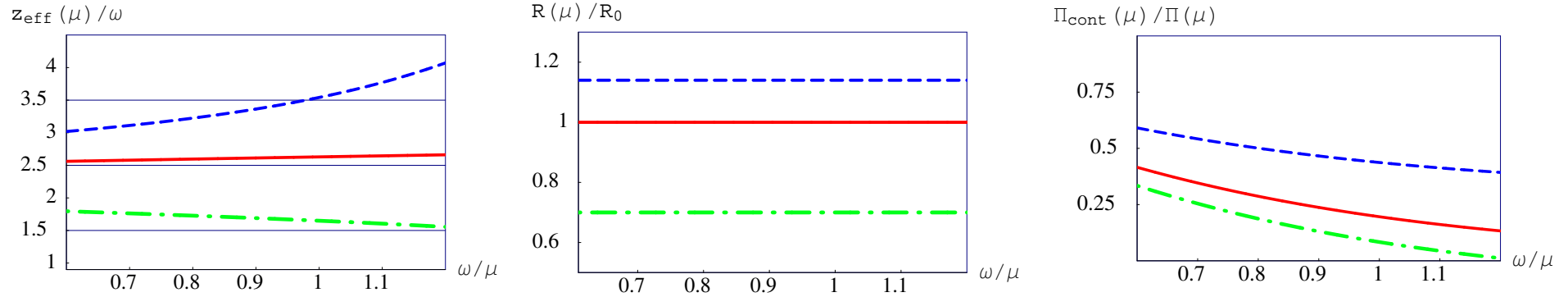
As might be expected, in a limited range of μ OPE *alone* cannot say anything about ground-state parameters. What really matters is the continuum contribution, i.e. $z_{\mathrm{eff}}(\mu)$

> **Without constraints on the effective continuum threshold the results obtained from OPE in a limited range of μ are not restrictive.**

In several important problems, the contribution of hadron continuum is not known:

- Calculation of heavy-hadron observables

- Properties of exotic hadrons

Still, QCD sum rules are being extensively applied to these problems, and give predictions.

How reliable and accurate are the error estimates supplied for these predictions?

A CLOSER LOOK AT THE STANDARD PROCEDURE:

Let us work with 3 power corrections: then in the region $\omega/\mu < 1.1$ one has

$$\frac{\Pi_{\text{OPE}}(\mu) - \Pi(\mu)}{\Pi(\mu)} \leq 0.5\%$$

We know the ground-state parameters, so we fix $0.7 < \omega/\mu$, where the ground state gives more than 60% of the full correlator.

So the "fiducial" range is $0.7 < \omega/\mu < 1.1$.

We shall seek the (approximate) solution to the equation

$$R \exp(-E/\mu) = \Pi(\mu, z_{\text{eff}}(\mu)) \equiv \Pi_{\text{OPE}}(\mu) - \int_{z_{\text{eff}}(\mu)}^{\infty} dz \rho_0(z) \exp(-z/\mu)$$

in the range $0.7 < \omega/\mu < 1.1$.

We set $E = E_0 = \frac{3}{2}\omega$ and denote as R the values extracted from the sum rule. The notation $R_0 = 2\sqrt{2}\omega$ is reserved for the known exact value.

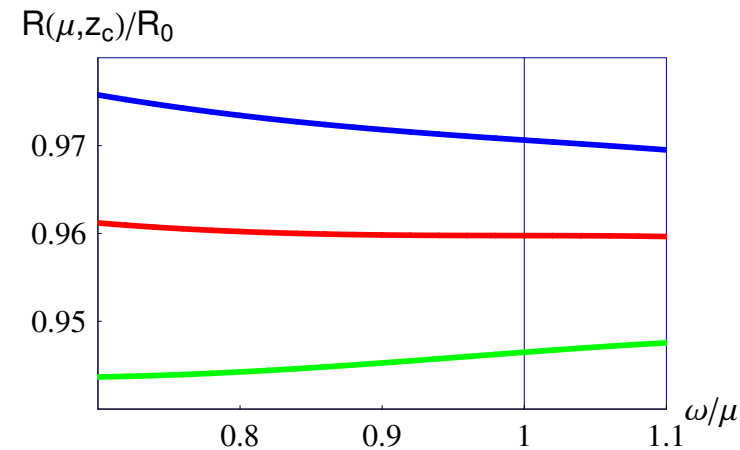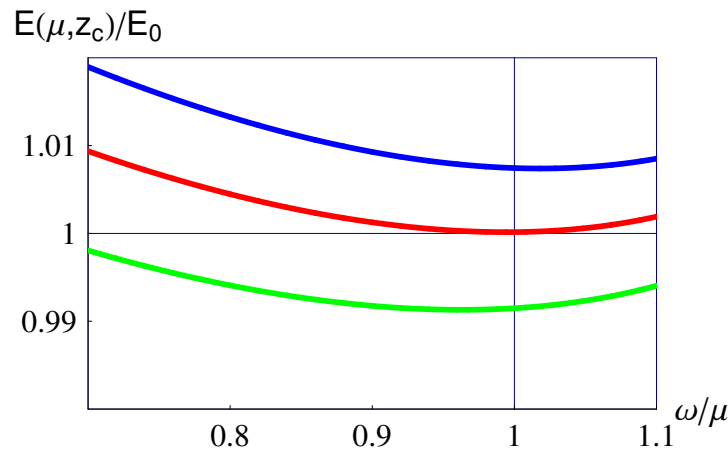**Only by imposing constraints on $z_{\mathrm{eff}}(\mu)$ one can obtain predictions. The standard procedure:**

1. ANSATZ: $z_{\mathrm{eff}}(\mu) \to z_c = \mathrm{const.}$

2. Impose a criterion for fixing z_c: e.g. one calculates

$$E(\mu, z_c) = -\frac{d}{d(1/\mu)} \log \Pi(\mu, z_c).$$

This now depends on μ due to approximating $z_{\mathrm{eff}}(\mu)$ with a constant. Then, one determines μ_0 and z_c as the solution to the system of equations

$$E(\mu_0, z_c) = E_0, \qquad \frac{\partial}{\partial \mu} E(\mu, z_c)|_{\mu=\mu_0} = 0,$$

$$z_c = 2.4\omega \text{ (green)}; \quad z_c = 2.454\omega \text{ (red)}; \quad z_c = 2.5\omega \text{ (blue).}$$

> **The sum-rule estimate for R is obtained as follows:**
> **The "right" values:** $z_c = 2.454\omega$ **(red)**, $\mu = \mu_0 = \omega \quad \Rightarrow \quad R = R(z_c, \mu_0)$**.**

We obtained:

1. A very good description of $\Pi(\mu)$ in the full range $0.7 \leq \mu/\omega \leq 1.1$: better than 1% accuracy.

2. The deviation of the $E(\mu, z_c)$ from E_0: less than 1% accuracy.

3. Extreme stability of $R(\mu, z_c)$ in the fiducial range: much better than 1% accuracy

Nevertheless, a 4% error in the extracted value of R!

How to guess these 4%? As seen from the plot, it would be *incorrect* to estimate the error, e.g., from the range covered by R when varying the Borel parameter μ within the fiducial interval.

In the model under consideration the sum rules give good estimates for the parameter R_0. This might be due to the following specific features of the model:

(i) a large gap between the ground state and the first excitation that contributes to the sum rule;

(ii) an almost constant exact effective continuum threshold in a wide range of μ.

Whether or not the same good accuracy may be achieved in QCD, where the features mentioned above are absent, is not obvious.

Even in this simple model, one cannot control the accuracy of the extracted value of R.

CONCLUSIONS

1. The knowledge of the correlator in a limited range of the Borel parameter μ is not sufficient for an extraction of the ground-state parameters with a controlled accuracy, even if the ground-state mass is known precisely: Rather different models for the correlator in the form ground state plus an effective continuum lead to the same correlator.

2. A sum-rule extraction of the ground-state parameters without knowing the hadron continuum suffers from uncontrolled systematic uncertainties (not to be confused with the uncertainties related to errors in quark masses, α_s, renormalization point, condensates, etc; the latter errors are usually properly taken into account).

An important notice: the independence of the extracted hadron parameters from the Borel mass does not guarantee the extraction of their true values.

3. A typical sum-rule analysis of HEAVY-MESON observables belongs to this class of problems: in this case, the hadron continuum is not known and is modeled by an effective continuum threshold treated as a fit parameter.

In this case one may perhaps obtain quite reasonable central values, but no estimates of systematic errors for hadron parameters obtained with sum rules can be given.

The impossibility to control the systematic uncertainties is an obstacle for using the results from QCD sum rules for precision physics, such as electroweak physics.