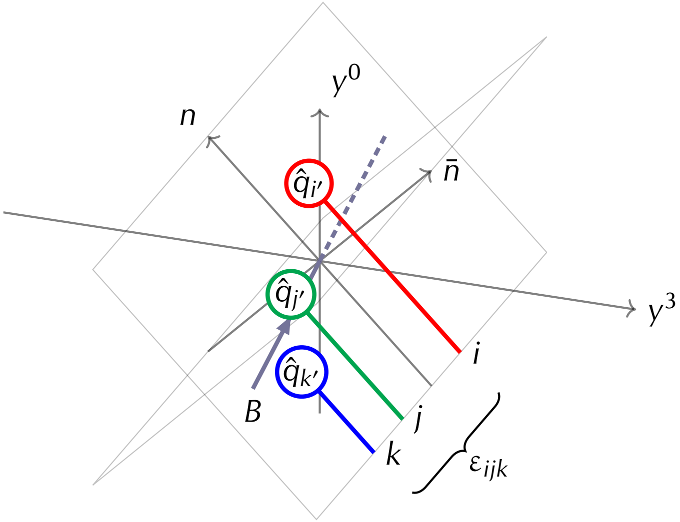
Connecting Baryon Light-Front Wave Functions to Quasi-TMD Correlators in Lattice QCD

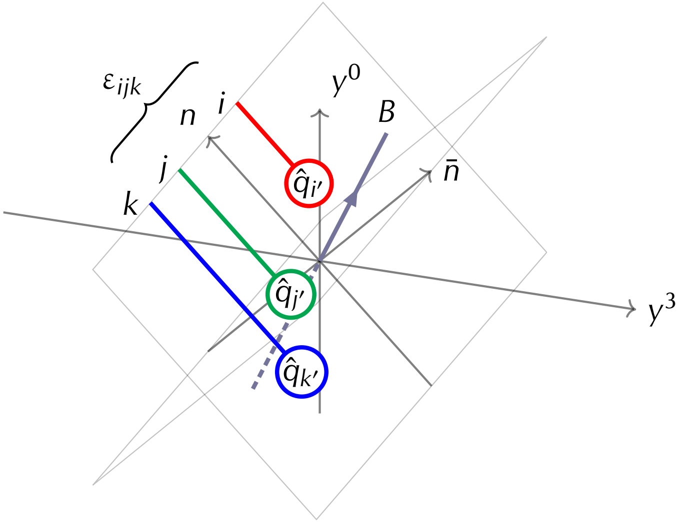
Andrea Schiavi
Simone Rodini, Barbara Pasquini

$$\widetilde{\Phi}_{3q;B}\left(\left\{y_q\right\}\right) = \langle 0|\varepsilon_{ijk}\left[\pm\infty^- + \infty_\perp, \pm\infty^- + b_1\right]_{ii''}\left[\pm\infty^- + b_1, y_1^- n + b_1\right]_{i''i'}$$
$$\times\left[\pm\infty^- + \infty_\perp, \pm\infty^- + b_2\right]_{jj''}\left[\pm\infty^- + b_2, y_2^- n + b_2\right]_{j''j'}$$
$$\times\left[\pm\infty^- + \infty_\perp, \pm\infty^- + b_3\right]_{kk''}\left[\pm\infty^- + b_3, y_3^- n + b_3\right]_{k''k'}$$
$$\times q_{i'}\left(y_1^- n + b_1\right) q_{j'}\left(y_2^- n + b_2\right) q_{k'}\left(y_3^- n + b_3\right)|B(P,S)\rangle,$$

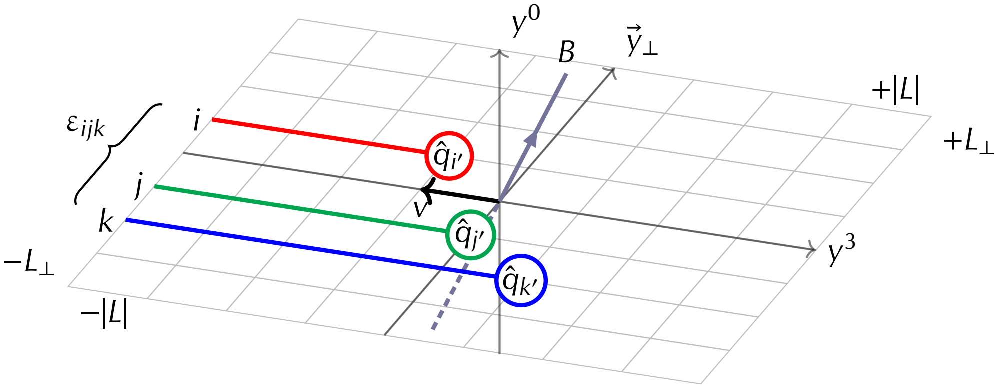
$$n^2 = 0 = \bar{n}^2, \quad \bar{n}n = 1, \qquad \bar{n}y = y^-, \quad ny = y^+, \quad \bar{n}b = 0 = nb,$$

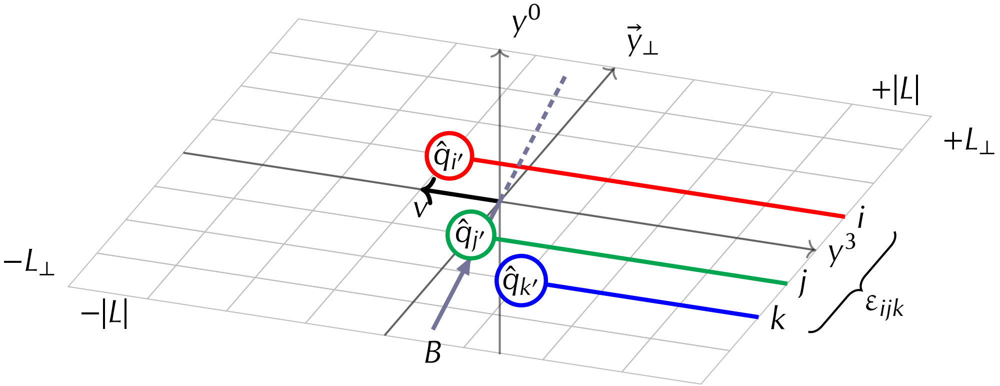
$$[y_{\text{fin}}, y_{\text{in}}] = P\left[\exp\left(ig\int_{y_{\text{in}}}^{y_{\text{fin}}} dy^\mu A_\mu(y)\right)\right].$$


y^0
n
$\bar{n}$
$\hat{q}_{i'}$
$\hat{q}_{j'}$
$\hat{q}_{k'}$
y^3
i
j
k
B
ε_{ijk}



ε_{ijk}
i
n
j
k
y^0
B
$\bar{n}$
y^3
$\hat{q}_{i'}$
$\hat{q}_{j'}$
$\hat{q}_{k'}$


y^0
B
$\vec{y}_\perp$
$+|L|$
$+L_\perp$
ε_{ijk}
i
j
k
$\hat{q}_{i'}$
$\hat{q}_{j'}$
$\hat{q}_{k'}$
v
y^3
$-L_\perp$
$-|L|$



y^0
$\vec{y}_\perp$
$+|L|$
$+L_\perp$
$\hat{q}_{i'}$
v
$\hat{q}_{j'}$
$\hat{q}_{k'}$
i
j
k
y^3
ε_{ijk}
$-L_\perp$
$-|L|$
B

$$\widetilde{\Omega}_{3q;B}\left(\left\{y_q = l_q v + b_q\right\}\right) = \langle 0|\varepsilon_{ijk} J_i(y_1)\, J_j(y_2)\, J_k(y_3)\,|B(P,S)\rangle,$$

$$J(y) = [Lv + L_\perp, Lv + b]\,[Lv + b, lv + b]\,q(lv + b),$$

$$v = \frac{n - \bar{n}}{\sqrt{2}}.$$

$$\widetilde{\Omega}_{3q;B}\left(\left\{y_q = l_q v + b_q\right\}\right) = \langle 0|\varepsilon_{ijk} J_i(y_1)\, J_j(y_2)\, J_k(y_3)\,|B(P,S)\rangle,$$

$$J(y) = [Lv + L_\perp, Lv + b]\,[Lv + b, lv + b]\,q(lv + b),$$

$$v = \frac{n - \bar{n}}{\sqrt{2}}.$$

$$\widetilde{\Omega}_{3q;B}\left(\left\{y_q\right\}\right) = \int [DqDA]\, e^{iS_{\mathrm{QCD}}[q,A]}\varepsilon_{ijk} J_i(y_1)\, J_j(y_2)\, J_k(y_3)\, \phi(P,S).$$

$$q \mapsto \psi + q_{\bar{n}} + q_v, \qquad A^\mu \mapsto B^\mu + A^\mu_{\bar{n}} + A^\mu_v,$$

L. F. Abbott (1982) Introduction to the Background Field Method.

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$$(\partial^+, \partial^-, \partial_\perp)(q_{\bar{n}}, A_{\bar{n}}) \lesssim \left(1, \lambda^2, \lambda\right) P^+ (q_{\bar{n}}, A_{\bar{n}}),$$
$$(\partial^+, \partial^-, \partial_\perp)(q_v, A_v) \lesssim \left(\lambda^2, \lambda^2, \lambda\right) P^+ (q_v, A_v),$$

$$\lambda = (M/P^+) \ll 1.$$

L. F. Abbott (1982) Introduction to the Background Field Method.

$$\widetilde{\Gamma}\left[\widetilde{B}_c^\mu; A_{\bar{n}}^\mu + A_v^\mu\right] = \Gamma\left[\widetilde{B}_c^\mu + A_{\bar{n}}^\mu + A_v^\mu\right] \Rightarrow \widetilde{\Gamma}\left[0; A_{\bar{n}}^\mu + A_v^\mu\right] = \Gamma\left[A_{\bar{n}}^\mu + A_v^\mu\right].$$

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$$\int [Dq_{\bar{n}} DA_{\bar{n}}][Dq_v DA_v]\, e^{iS_{\mathrm{QCD}}[q_{\bar{n}}, A_{\bar{n}}]} e^{iS_{\mathrm{QCD}}[q_v, A_v]} O(y_1, y_2, y_3)\, \phi(P, S),$$

$$O(y_1, y_2, y_3) = \int [D\psi DB]\, e^{iS_{\mathrm{int}}[\psi, q_{\bar{n}}, q_v, B, A_{\bar{n}}, A_v]} \varepsilon_{ijk} J_i(y_1)\, J_j(y_2)\, J_k(y_3),$$

$$S_{\mathrm{int}} = S_{\mathrm{QCD}}[\psi + q_{\bar{n}} + q_v, B + A_{\bar{n}} + A_v] - S_{\mathrm{QCD}}[q_{\bar{n}}, A_{\bar{n}}] - S_{\mathrm{QCD}}[q_v, A_v].$$

$$(\partial^+, \partial^-, \partial_\perp)(q_{\bar{n}}, A_{\bar{n}}) \lesssim \big(1 > \lambda^2, \lambda^2, \lambda\big) P^+(q_{\bar{n}}, A_{\bar{n}}),$$
$$(\partial^+, \partial^-, \partial_\perp)(q_v, A_v) \lesssim \big(\qquad \lambda^2, \lambda^2, \lambda\big) P^+(q_v, A_v),$$

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$$(\partial^+, \partial^-, \partial_\perp)(q_v, A_v) \lesssim \big(\qquad \lambda^2, \lambda^2, \lambda\big) P^+ (q_v, A_v),$$

$$\widetilde{\Omega}_{3q;B}\left(\left\{y_q\right\}\right) = \frac{\int [Dq_{\bar{n}} DA_{\bar{n}}][Dq_v DA_v]\, e^{iS_{\rm QCD}[q_{\bar{n}},A_{\bar{n}}]} e^{iS_{\rm QCD}[q_v,A_v]} O\left(\left\{y_q\right\}\right)\, \phi(P,S)}{S\left(\left\{y_q\right\}\right)},$$

$$O\left(\left\{y_q\right\}\right) = \int [D\psi DB]\, e^{iS_{\rm int}[\psi, q_{\bar{n}}, q_v, B, A_{\bar{n}}, A_v]} \varepsilon_{ijk} J_i(y_1)\, J_j(y_2)\, J_k(y_3).$$

$$\partial_\mu B^\mu + t^I f_{IJK}\left(A^J_{\bar{n},\nu} + A^J_{v,\rho}\right) B^{\rho,K}.$$

$$A^+_{\bar{n}} = 0, \qquad \lim_{y^- \to \mathrm{sign}(L)\infty} A^\mu_{\bar{n},\perp}(y) = 0,$$
$$A^-_{v} = 0, \qquad \lim_{y^+ \to \mathrm{sign}(L)\infty} A^\mu_{v,\perp}(y) = 0.$$

A. Vladimirov et al. (2022) 2109.09771.

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$$A^-_{v} = 0, \qquad \lim_{y^+\to \mathrm{sign}(L)\infty} A^\mu_{v,\perp}(y) = 0.$$

$$q_{\bar{n}} = \left(\frac{1}{2}\gamma^-\gamma^+ + \frac{1}{2}\gamma^+\gamma^-\right) q_{\bar{n}} := \xi_{\bar{n}} + \eta_{\bar{n}} \overset{\text{EOM}}{\Longrightarrow} \eta_{\bar{n}} \sim \lambda\xi_{\bar{n}}.$$

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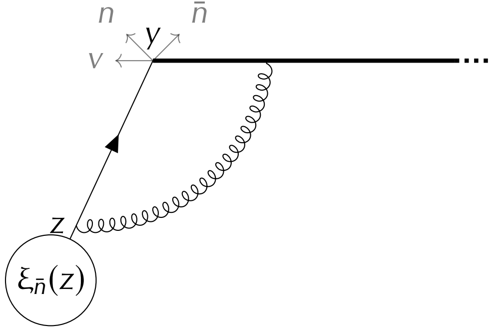
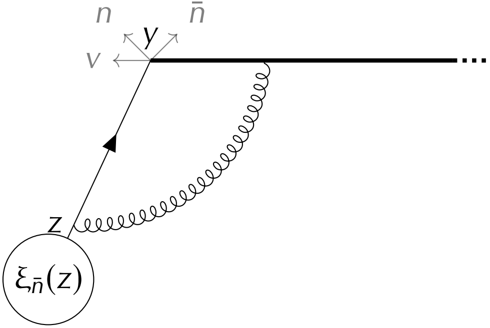
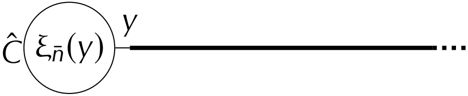
$$ig\int_0^L d\theta v^\mu B_\mu(\theta v + y)\,\psi(y)\, ig\int d^D z \bar{\psi} \not{B} \xi_{\bar{n}}(z).$$



$$ig\int_0^L d\theta v^\mu B_\mu(\theta v+y)\,\psi(y)\,ig\int d^Dz\,\bar{\psi}\,\not{B}\,\xi_{\bar{n}}(z).$$



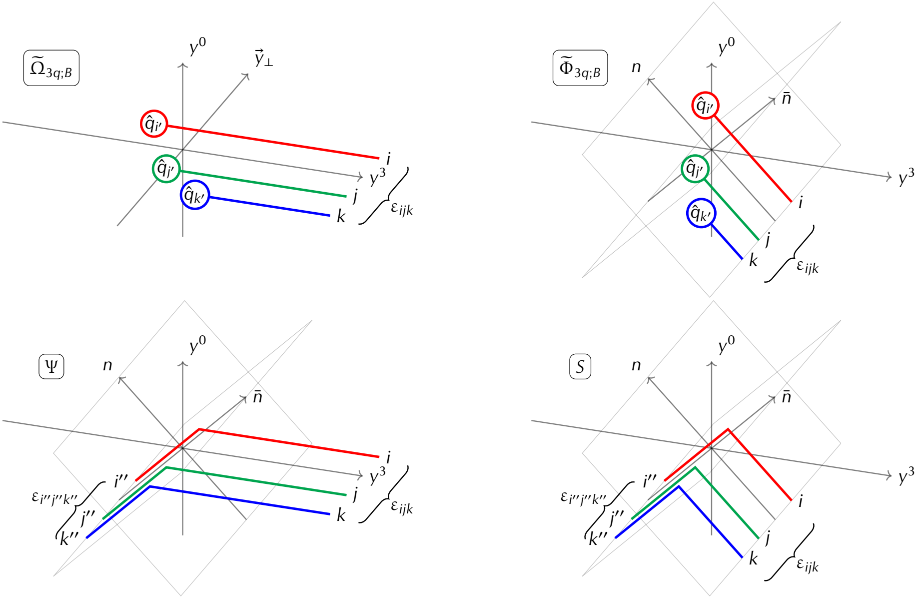


$$\widetilde{\Omega}_{3q;B}\left(\left\{y_q = l_q v + b_q\right\}\right) \to \frac{\Psi(b_1, b_2, b_3)\,\hat{C}\hat{C}\hat{C}\,\widetilde{\boldsymbol{\Phi}}_{\boldsymbol{3q;B}}\left(\left\{y_q^- n + b_q\right\}\right)}{S(b_1, b_2, b_3)}.$$

$$\widetilde{\Omega}_{3q;B}\big(\{y_q = l_q v + b_q\}\big) \to \frac{\Psi(b_1, b_2, b_3)\,\hat{C}\hat{C}\hat{C}\,\widetilde{\boldsymbol{\Phi}}_{\boldsymbol{3q;B}}\Big(\{y_q^- n + b_q\}\Big)}{S(b_1, b_2, b_3)}.$$

$$\Omega_{3q;B}\Big(\{y_q^+, x_q, b_q\}\Big) \to \frac{\Psi(b_1, b_2, b_3)\,C(x_1)\,C(x_2)\,C(x_3)\,\boldsymbol{\Phi}_{\boldsymbol{3q;B}}\big(\{x_q, b_q\}\big)}{S(b_1, b_2, b_3)},$$

$$x_q = \frac{p_q^+}{P^+}, \quad x_1 + x_2 + x_3 - 1.$$


$\widetilde{\Omega}_{3q;B}$
y^0
$\vec{y}_\perp$
$\hat{q}_{i'}$
$\hat{q}_{j'}$
$\hat{q}_{k'}$
y^3
i
j
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ε_{ijk}
$\widetilde{\Phi}_{3q;B}$
n
y^0
$\bar{n}$
$\hat{q}_{i'}$
$\hat{q}_{j'}$
$\hat{q}_{k'}$
y^3
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j
k
ε_{ijk}
Ψ
n
y^0
$\bar{n}$
y^3
i
j
k
ε_{ijk}
i''
j''
k''
$\varepsilon_{i''j''k''}$
S
n
y^0
$\bar{n}$
y^3
i
j
k
ε_{ijk}
i''
j''
k''
$\varepsilon_{i''j''k''}$

$$D = 4 - 2\epsilon \qquad \leftrightarrow \qquad \mu.$$

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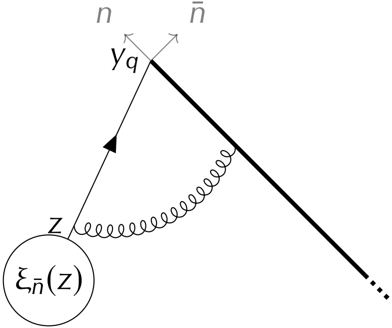
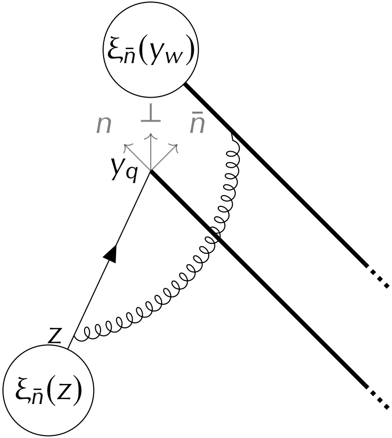
$$[Ln + y, y] \mapsto P\left[\exp\left(ig\int_0^L d\sigma n^\mu A_\mu(\sigma n + y)\, e^{-|\sigma|\delta^+}\right)\right],$$
$$[y, L\bar{n} + y] \mapsto P\left[\exp\left(ig\int_L^0 d\chi \bar{n}^\mu A_\mu(\chi \bar{n} + y)\, e^{-|\chi|\delta^-}\right)\right],$$

$$\delta^\pm > 0, \quad 2\delta^+\delta^- = \delta^2 \qquad \leftrightarrow \qquad \left\{\nu_q^+, \nu_q^-\right\}, \quad 2\nu_q^+\nu_q^- = \nu_q^2.$$



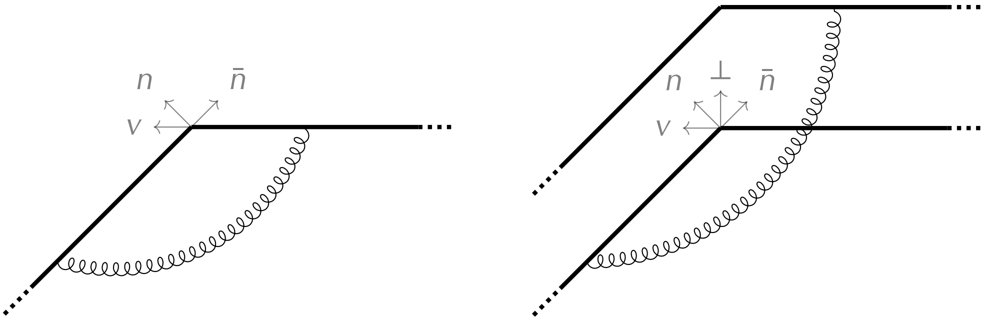
n
n̄
y_q
z
$\xi_{\bar{n}}(z)$



$\xi_{\bar{n}}(y_w)$
n
⊥
n̄
y_q
z
$\xi_{\bar{n}}(z)$

$$Z_{\Phi 1;\overline{\mathrm{MS}}}\left(\frac{\delta^+}{p_q^+}\right) = 1 + a_s C_F \frac{1}{\epsilon}\left(2 + 2\ln\left(\frac{\delta^+}{i\,\mathrm{sign}(L)\,p_q^+}\right)\right) + \mathcal{O}\left(a_s^2\right).$$

$$R_{\overline{\mathrm{MS}}}\left(\frac{\delta^+}{v_q^+}\right) = 1 + 2a_s \frac{C_F}{N-1}\sum_{\substack{w=1\\ w\neq q}}^{N} \ln\left(\frac{-\left(b_w - b_q\right)^2}{4}\mu^2\right)\ln\left(\frac{\delta^+}{v_q^+}\right) + \mathcal{O}\left(a_s^2\right).$$


n
n̄
v
n
⊥
n̄
v

$$Z_{\Psi 1;\overline{\mathrm{MS}}}\left(\frac{\sqrt{-v^2}\delta^-}{\mu v^-}\right) = 1 - a_s C_F \left(\frac{1}{\epsilon^2} - \frac{1}{\epsilon}\ln\left(\frac{\sqrt{-v^2}\delta^-}{\mu v^-}\right)^2\right) + \mathcal{O}\left(a_s^2\right).$$

$$R_{\overline{\mathrm{MS}}}\left(\frac{\delta^-}{v_q^-}\right) = 1 + 2a_s \frac{C_F}{N-1} \sum_{\substack{w=1 \\ w\neq q}}^{N} \ln\left(\frac{-\left(b_q - b_w\right)^2}{4}\mu^2\right) \ln\left(\frac{\delta^-}{v_q^-}\right) + \mathcal{O}\left(a_s^2\right).$$

$$S_{\overline{\mathrm{MS}}}(\delta^2) = 1 - 2a_s \frac{C_F}{N-1} \sum_{\substack{q,w=1 \\ w\neq q}}^{N} \left(\frac{1}{\epsilon^2} - \left(\frac{1}{\epsilon} + \ln\left(\frac{-\left(b_q - b_w\right)^2}{4} \mu^2 \right) \right) \ln\left(\frac{2\delta^+\delta^-}{\mu^2} \right) - \frac{1}{2} \ln^2\left(\frac{-\left(b_q - b_w\right)^2}{4} \mu^2 \right) - \frac{\pi^2}{12} \right) + \mathcal{O}\left(a_s^2\right).$$

$$S_{\overline{\mathrm{MS}}}(\delta^2) = 1 - 2a_s \frac{C_F}{N-1} \sum_{\substack{q,w=1 \\ w\neq q}}^{N} \left(\frac{1}{\epsilon^2} - \left(\frac{1}{\epsilon} + \ln\left(\frac{-\left(b_q - b_w\right)^2}{4} \mu^2 \right) \right) \ln\left(\frac{2\delta^+ \delta^-}{\mu^2} \right) - \frac{1}{2} \ln^2\left(\frac{-\left(b_q - b_w\right)^2}{4} \mu^2 \right) - \frac{\pi^2}{12} \right) + \mathcal{O}\left(a_s^2\right).$$

$$S_{\overline{\mathrm{MS}}}(\delta^2) = R_{\overline{\mathrm{MS}}}\left(\delta^+, \left\{\nu_q^+\right\}\right) Z_{R;\overline{\mathrm{MS}}}\left(\delta^+, \left\{\nu_q^+\right\}\right) R_{\overline{\mathrm{MS}}}\left(\delta^-, \left\{\nu_q^-\right\}\right) Z_{R;\overline{\mathrm{MS}}}\left(\delta^-, \left\{\nu_q^-\right\}\right) \times Z_{S;\overline{\mathrm{MS}}}\left(\left\{\nu_q^2\right\}\right) S_{0;\overline{\mathrm{MS}}}\left(\left\{\nu_q^2\right\}\right).$$

$$R_{\overline{\mathrm{MS}}}\left(\delta^{\pm},\left\{\nu_q^{\pm}\right\}\right)=\prod_{q=1}^{N} R_{\overline{\mathrm{MS}}}\left(\frac{\delta^{\pm}}{\nu_q^{\pm}}\right).$$

$$Z_{R;\overline{\mathrm{MS}}}\left(\delta^{\pm},\left\{\nu_q^{\pm}\right\}\right)=\prod_{q=1}^{N} Z_{R;\overline{\mathrm{MS}}}\left(\frac{\delta^{\pm}}{\nu_q^{\pm}}\right)=\prod_{q=1}^{N}\left(1+2a_s C_F\frac{1}{\epsilon}\ln\left(\frac{\delta^{\pm}}{\nu_q^{\pm}}\right)\right)+\mathcal{O}\left(a_s^2\right).$$

$$Z_{S;\overline{\mathrm{MS}}}\left(\left\{\nu_q^{2}\right\}\right)=\prod_{q=1}^{N} Z_{S;\overline{\mathrm{MS}}}\left(\nu_q^{2}\right)=\prod_{q=1}^{N}\left(1-2a_s C_F\left(\frac{1}{\epsilon^2}-\frac{1}{\epsilon}\ln\left(\frac{\nu_q^2}{\mu^2}\right)\right)\right)+\mathcal{O}\left(a_s^2\right).$$

$$\Omega_{\text{bare}} = Z_J^3 \Omega(\mu).$$

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$$\frac{\Psi_{\rm bare} C(x_1)\, C(x_2)\, C(x_3)\, \Phi_{\rm bare}}{S}$$

$$= \frac{R\left(\delta^-, \left\{\nu_q^-\right\}\right) Z_{\Psi 1}^3\left(\frac{\sqrt{-v^2}\,\delta^-}{\mu v^-}\right) R\left(\delta^+, \left\{\nu_q^+\right\}\right) \left(\prod_{q=1}^3 Z_{\Phi 1}\left(\frac{\delta^+}{p_q^+}\right)\right)}{R\left(\delta^+, \left\{\nu_q^+\right\}\right) Z_R\left(\delta^+, \left\{\nu_q^+\right\}\right) R\left(\delta^-, \left\{\nu_q^-\right\}\right) Z_R\left(\delta^-, \left\{\nu_q^-\right\}\right) Z_S\left(\left\{\nu_q^2\right\}\right)}$$

$$\times \frac{\Psi\left(\mu, \left\{\nu_q^-\right\}\right) C(x_1) C(x_2) C(x_3)\, \Phi\left(\mu, \left\{\nu_q^+\right\}\right)}{S_0\left(\left\{\nu_q^2\right\}\right)},$$

$$\Omega_{\text{bare}} = Z_J^3 \Omega(\mu).$$

$$\frac{\Psi_{\text{bare}} C(x_1)\, C(x_2)\, C(x_3)\, \Phi_{\text{bare}}}{S}$$

$$= \frac{R\left(\delta^-, \left\{\nu_q^-\right\}\right) Z_{\Psi 1}^3 \left(\frac{\sqrt{-\nu^2}\,\delta^-}{\mu \nu^-}\right) R\left(\delta^+, \left\{\nu_q^+\right\}\right) \left(\prod_{q=1}^3 Z_{\Phi 1}\left(\frac{\delta^+}{p_q^+}\right)\right)}{R\left(\delta^+, \left\{\nu_q^+\right\}\right) Z_R\left(\delta^+, \left\{\nu_q^+\right\}\right) R\left(\delta^-, \left\{\nu_q^-\right\}\right) Z_R\left(\delta^-, \left\{\nu_q^-\right\}\right) Z_S\left(\left\{\nu_q^2\right\}\right)}$$

$$\times \frac{\Psi\left(\mu, \left\{\nu_q^-\right\}\right) C(x_1) C(x_2) C(x_3)\, \Phi\left(\mu, \left\{\nu_q^+\right\}\right)}{S_0\left(\left\{\nu_q^2\right\}\right)},$$

$$\Omega_{\text{bare}} = Z_J^3 \Omega(\mu).$$

$$\begin{aligned}
&\frac{\Psi_{\text{bare}} C(x_1)\, C(x_2)\, C(x_3)\, \Phi_{\text{bare}}}{S} \\
= \quad &\frac{Z^3_{\Psi 1}\left(\frac{\sqrt{-v^2}\,\delta^-}{\mu v^-}\right)\left(\prod_{q=1}^3 Z_{\Phi 1}\left(\frac{\delta^+}{p_q^+}\right)\right)}{Z_R\left(\delta^+, \left\{\nu_q^+\right\}\right) Z_R\left(\delta^-, \left\{\nu_q^-\right\}\right) Z_S\left(\left\{\nu_q^2\right\}\right)} \\
&\times \frac{\Psi\left(\mu, \left\{\nu_q^-\right\}\right) C(x_1) C(x_2) C(x_3)\, \Phi\left(\mu, \left\{\nu_q^+\right\}\right)}{S_0\left(\left\{\nu_q^2\right\}\right)},
\end{aligned}$$

$$\Omega_{\text{bare}} = Z_J^3\,\Omega(\mu).$$

$$\begin{aligned}
&\frac{\Psi_{\text{bare}} C(x_1)\, C(x_2)\, C(x_3)\, \Phi_{\text{bare}}}{S} \\
&= \frac{Z_{\Psi 1}^3\left(\frac{\sqrt{-v^2}\,\delta^-}{\mu v^-}\right)\left(\prod_{q=1}^3 Z_{\Phi 1}\left(\frac{\delta^+}{p_q^+}\right)\right)}{Z_R\left(\delta^+, \left\{v_q^+\right\}\right) Z_R\left(\delta^-, \left\{v_q^-\right\}\right) Z_S\left(\left\{v_q^2\right\}\right)} \\
&\quad\times \frac{\Psi\left(\mu, \left\{v_q^-\right\}\right) C(x_1) C(x_2) C(x_3)\, \Phi\left(\mu, \left\{v_q^+\right\}\right)}{S_0\left(\left\{v_q^2\right\}\right)},
\end{aligned}$$

$$\Omega_{\text{bare}} = Z_J^3 \Omega(\mu).$$

$$\frac{\Psi_{\text{bare}} C(x_1)\, C(x_2)\, C(x_3)\, \Phi_{\text{bare}}}{S}$$

$$= \frac{\prod_{q=1}^{3} Z_{\Psi 1}\left(\frac{\sqrt{-v^2}\,\nu_q{}^{-}}{\mu v^{-}}\right) Z_{\Phi 1}\left(\frac{\nu_q^{+}}{p_q^{+}}\right)}{Z_S\left(\left\{\nu_q^2\right\}\right)} \frac{\Psi C C C \Phi}{S_0\left(\left\{\nu_q^2\right\}\right)},$$

$$\Omega_{\text{bare}} = Z_J^3 \Omega(\mu).$$

$$\frac{\Psi_{\text{bare}} C(x_1)\, C(x_2)\, C(x_3)\, \Phi_{\text{bare}}}{S}$$

$$= \left(\prod_{q=1}^{3} \frac{Z_{\Psi 1}\left(\frac{\sqrt{-v^2}\nu_q^-}{\mu v^-} \right)}{Z_S^{\frac{1}{2}}\left(\nu_q^2\right)} \frac{Z_{\Phi 1}\left(\frac{\nu_q^+}{p_q^+} \right)}{Z_S^{\frac{1}{2}}\left(\nu_q^2\right)} \right) \frac{\Psi}{S_0^{\frac{1}{2}}\left(\left\{\nu_q^2\right\}\right)} CCC \frac{\Phi}{S_0^{\frac{1}{2}}\left(\left\{\nu_q^2\right\}\right)},$$

$$\Omega_{\text{bare}} = Z_J^3 \Omega(\mu).$$

$$\begin{aligned}
&\frac{\Psi_{\text{bare}} C(x_1)\, C(x_2)\, C(x_3)\, \Phi_{\text{bare}}}{S} \\
:=\ &\left(\prod_{q=1}^{3} Z_{\Psi 1,\text{sub}}\left(\frac{\bar{\zeta}_q}{\mu^2}\right) Z_{\Phi 1,\text{sub}}\left(\frac{\zeta_q}{\mu^2}\right)\right) \\
&\times \Psi_{\text{sub}}\left(\mu, \left\{\bar{\zeta}_q\right\}\right) C(x_1)\, C(x_2)\, C(x_3)\, \Phi_{\text{sub}}\left(\mu, \left\{\zeta_q\right\}\right),
\end{aligned}$$

$$\Omega_{\text{bare}} = Z_J^3 \Omega(\mu).$$

$$\begin{aligned}
&\frac{\Psi_{\text{bare}} C(x_1)\, C(x_2)\, C(x_3)\, \Phi_{\text{bare}}}{S} \\
:= &\left(\prod_{q=1}^{3} Z_{\Psi 1,\text{sub}}\left(\frac{\bar{\zeta}_q}{\mu^2}\right) Z_{\Phi 1,\text{sub}}\left(\frac{\zeta_q}{\mu^2}\right) \right) \\
&\times \Psi_{\text{sub}}\left(\mu, \left\{\bar{\zeta}_q\right\}\right) C(x_1)\, C(x_2)\, C(x_3)\, \Phi_{\text{sub}}\left(\mu, \left\{\zeta_q\right\}\right),
\end{aligned}$$

$$\zeta_q = 2\left(x_q P^+\right)^2 \frac{\nu_q^-}{\nu_q^+}, \qquad \bar{\zeta}_q = 2 \frac{(\mu v^-)^2}{-v^2} \frac{\nu_q^+}{\nu_q^-}.$$

$$C(x) = 1 + 2a_s C_F \frac{1-\epsilon}{1-2\epsilon}\Gamma(-\epsilon)\,\Gamma(2\epsilon)\left(\frac{-v^2}{(iss_x 2|x|v^- P^+)^2}\right)^{\epsilon} + \mathcal{O}\left(a_s^2\right).$$

$$\mathrm{pole}\left[C_{\mathrm{NLO}}\left(x_q\right)\right]_{\overline{\mathrm{MS}}} + \left(Z_{\Psi 1,\mathrm{sub};\overline{\mathrm{MS}}}\left(\frac{\bar{\zeta}_q}{\mu^2}\right) Z_{\Phi 1,\mathrm{sub};\overline{\mathrm{MS}}}\left(\frac{\zeta_q}{\mu^2}\right) Z^{-1}_{J;\overline{\mathrm{MS}}}\right)_{\mathrm{NLO}} = 0.$$

$$C(x) = 1 + 2a_s C_F \frac{1-\epsilon}{1-2\epsilon}\Gamma(-\epsilon)\,\Gamma(2\epsilon)\left(\frac{-v^2}{(iss_x 2|x|v^- P^+)^2}\right)^{\epsilon} + \mathcal{O}\left(a_s^2\right).$$

$$\mathrm{pole}\left[C_{\mathrm{NLO}}\left(x_q\right)\right]_{\overline{\mathrm{MS}}} + \left(Z_{\Psi 1,\mathrm{sub};\overline{\mathrm{MS}}}\left(\frac{\bar{\zeta}_q}{\mu^2}\right) Z_{\Phi 1,\mathrm{sub};\overline{\mathrm{MS}}}\left(\frac{\zeta_q}{\mu^2}\right) Z^{-1}_{J;\overline{\mathrm{MS}}}\right)_{\mathrm{NLO}} = 0.$$

$$\Omega_{3q;B}(\mu) = \Psi_{\mathrm{sub}}\left(\mu, \left\{\bar{\zeta}_q\right\}\right) C_0(x_1)\, C_0(x_2)\, C_0(x_3)\, \boldsymbol{\Phi}_{\mathbf{sub,3}\boldsymbol{q};\boldsymbol{B}}\left(\mu, \left\{\zeta_q\right\}\right),$$

$$C_{0;\overline{\mathrm{MS}}}(x) = 1 + a_s C_F\left(\frac{\pi^2}{12} - 2 - \ln\left(\frac{-v^2\mu^2}{(2xv^- P^+)^2}\right) - \frac{1}{2}\ln^2\left(\frac{-v^2\mu^2}{(2xv^- P^+)^2}\right)\right) + a_s C_F i\pi \mathrm{sign}(L)\,\mathrm{sign}(x) + \mathcal{O}\left(a_s^2\right).$$

$$\mu^2 \frac{\partial}{\partial \mu^2} \Phi_{\text{sub}}\left(\mu, \left\{\zeta_q\right\}\right) = \sum_{q'=1}^{3} \gamma_{\Phi 1}\left(x_{q'}, \frac{\zeta_{q'}}{\mu^2}\right) \Phi_{\text{sub}}\left(\mu, \left\{\zeta_q\right\}\right),$$

$$\gamma_{\Phi 1}\left(x_q, \frac{\zeta_q}{\mu^2}\right) \overset{\overline{\text{MS}}}{=} a_s C_F \left(2 - \frac{1}{2} - \ln \frac{\zeta_q}{\mu^2} - 2 \ln\left(i \operatorname{sign}(L) \operatorname{sign}\left(x_q\right)\right)\right) + \mathcal{O}\left(a_s^2\right).$$

$$\zeta_q \frac{\partial}{\partial \zeta_q} \Phi_{\text{sub}}\left(\mu, \left\{\zeta_q\right\}\right) = \frac{\nu_q^+}{2} \frac{\partial \ln\left(R\left(\frac{\delta^+}{\nu_q^+}\right)\right)}{\partial \nu_q^+}$$

$$= -\frac{1}{4} \sum_{\substack{w=1 \\ w \neq q}}^{3} \mathcal{D}\left(\left(b_q - b_w\right)^2, \mu\right) \Phi_{\text{sub}}\left(\mu, \left\{\zeta_q\right\}\right),$$

$$\mathcal{D}_{\overline{\text{MS}}}(b^2, \mu) = 2 a_s C_F \ln\left(\frac{-b^2}{4} \mu^2\right) + \mathcal{O}\left(a_s^2\right).$$

$$d \ln \Phi_{\rm sub}\left(\mu, \left\{\zeta_q\right\}\right) = \sum_{q=1}^{3} \gamma_{\Phi 1}\left(x_q, \frac{\zeta_q}{\mu^2}\right) \frac{d\mu^2}{\mu^2} - \frac{1}{4} \sum_{\substack{q,w=1 \\ w\neq q}}^{3} \mathcal{D}\left(\left(b_q - b_w\right)^2, \mu\right) \frac{d\zeta_q}{\zeta_q}.$$

$$d \ln \Phi_{\text{sub}}\left(\mu, \left\{\zeta_q\right\}\right) = \sum_{q=1}^{3} \gamma_{\Phi 1}\left(x_q, \frac{\zeta_q}{\mu^2}\right) \frac{d\mu^2}{\mu^2} - \frac{1}{4} \sum_{\substack{q,w=1 \\ w \neq q}}^{3} \mathcal{D}\left(\left(b_q - b_w\right)^2, \mu\right) \frac{d\zeta_q}{\zeta_q}.$$

$$\frac{\Phi_{\text{sub}}\left(\mu, \left\{\zeta_q\right\}\right)}{\Phi_{\text{sub}}\left(\mu_0, \left\{\zeta_{q,0}\right\}\right)} = \prod_{q=1}^{3} \left(\frac{\zeta_q}{\zeta_{q,0}}\right)^{-\frac{1}{4} \sum_{\substack{w=1 \\ w \neq q'}}^{3} \mathcal{D}\left(\left(b_{q'} - b_w\right)^2, \mu\right)} \exp \int \frac{d\mu^2}{\mu^2} \gamma_{\Phi 1}\left(x_q, \frac{\zeta_q}{\mu^2}\right).$$