NNLO $\pi^{\pm}$ and $K^{\pm}$ Fragmentation Functions

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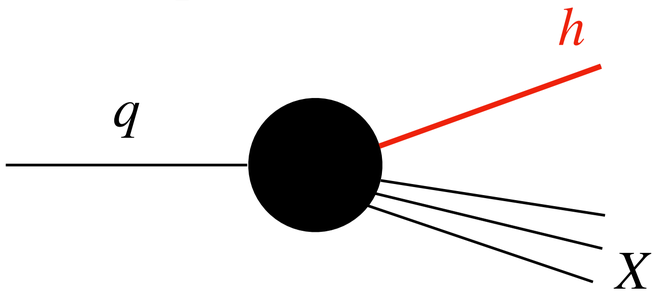
Supervisors:
Alessandro Bacchetta
Marco Radici

MAP MEETING, Pavia, 27/04/2026

Collinear Fragmentation Functions

Non perturbative $q \to hX$



$h = \pi^+, \pi^-, K^+, K^-$

$$z_{1,2} = \frac{2E_{1,2}}{Q}$$

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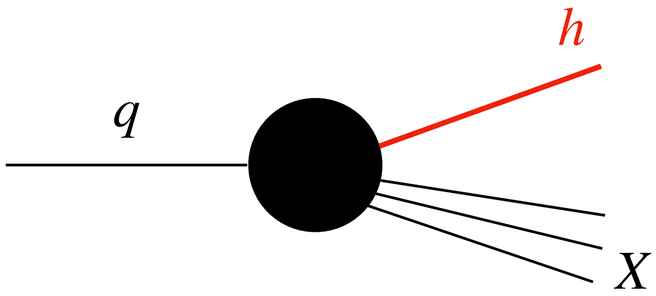

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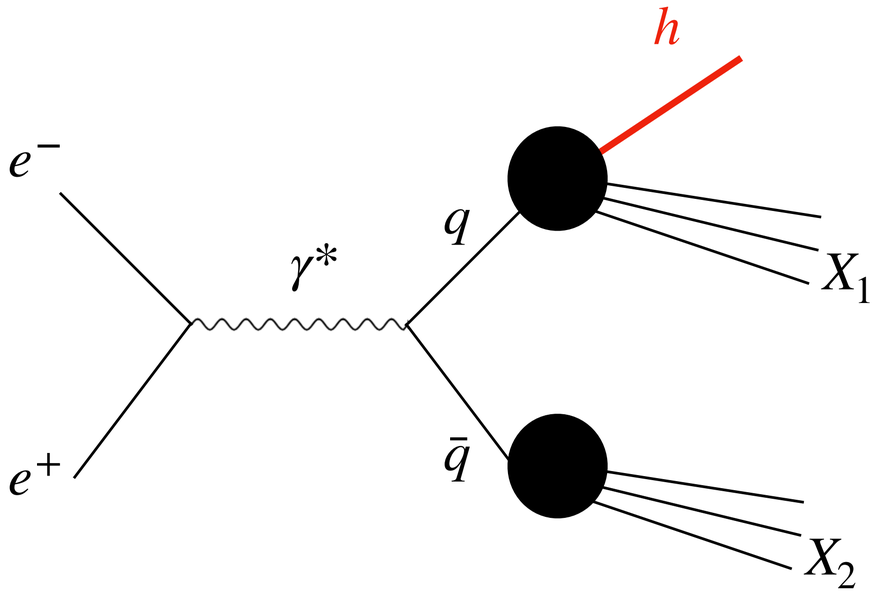
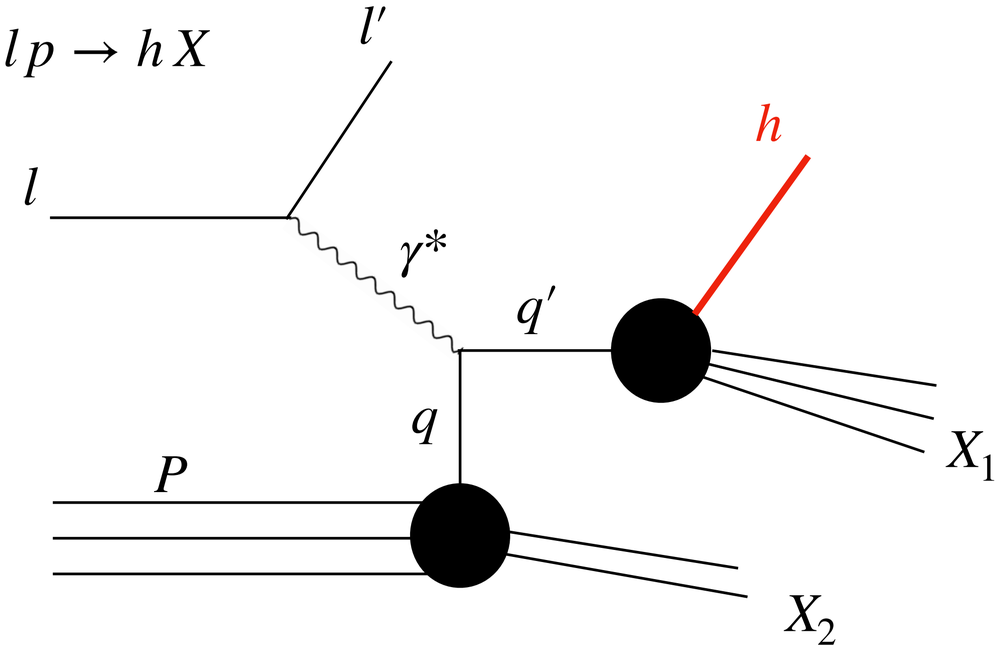
SIA

$e^+e^- \to hX$



SIDIS

$lp \to hX$



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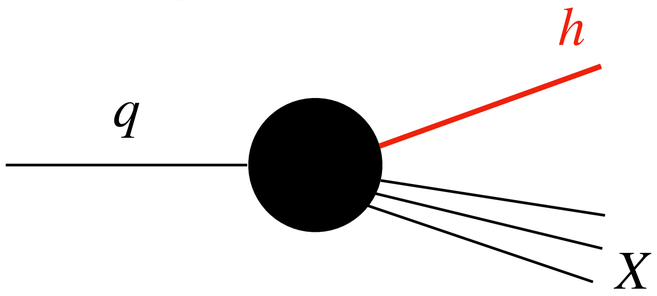

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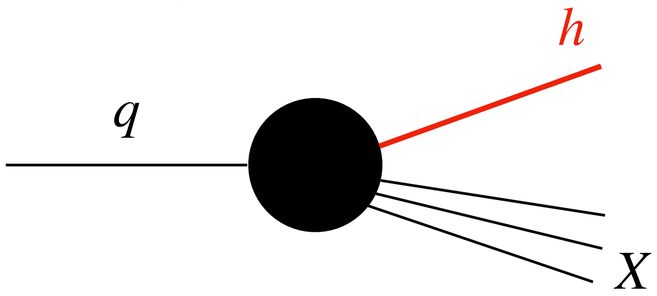

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SIA

$e^+e^- \to h\,X$

Total p.ti = 377

$h^\pm$	**BELLE**	**ALEPH**	$z_{min} < z < 0.9$
$h^\pm$	**BABAR**	**DELPHI**	$z_{min} = 0.075$
$h^\pm$	**TASSO**	**OPAL**	$z_{min} = 0.02$ **if** $\sqrt{S} \sim M_Z$
$h^\pm$	**TPC**	**SLD**	

SIDIS

$l\,p \to h\,X$

Total p.ti = 322

$h^- d$	**HERMESS**	$0.2 < z < 0.8$
$h^+ p$	**HERMESS**	$Q^2 > 2\,\mathrm{GeV}$
h^-	**COMPASS**	
h^+	**COMPASS**	

BELLE, BABAR K $z_{min} = 0.2$

Parameterisation

Neural Network parameterisation → $[1,20,7]$ π → **7 independent output nodes**

$[1,20,7]$ K

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Example for pions: $|\pi^+> = |u\bar{d}>$

$$D_1^{u\to\pi^+} \quad D_1^{\bar{d}\to\pi^+} \quad D_1^{\bar{u}\to\pi^+} = D_1^{d\to\pi^+} \quad D_1^{s\to\pi^+} = D_1^{\bar{s}\to\pi^+} \quad D_1^{c\to\pi^+} = D_1^{\bar{c}\to\pi^+} \quad D_1^{b\to\pi^+} = D_1^{\bar{b}\to\pi^+} \quad D_1^{g\to\pi^+}$$

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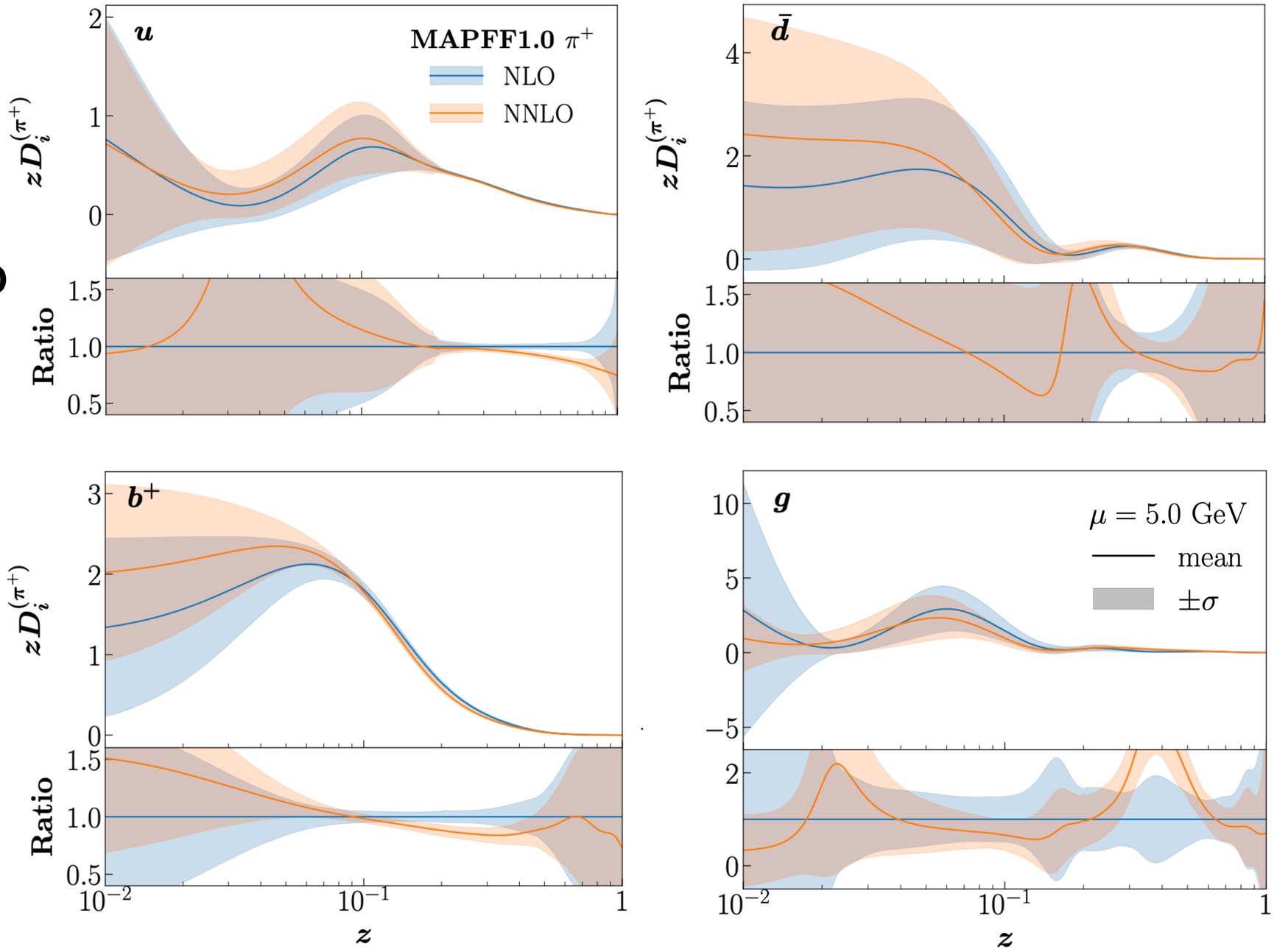
From these 7 functions, we get the ones for π^- **by Charge Conjugation.**

For the K^+ **we only need to switch** $d \leftrightarrow s$.
Then, by Charge Conjugation, we get the ones for K^-.

$$Q_0 = 5\,\mathrm{GeV}$$

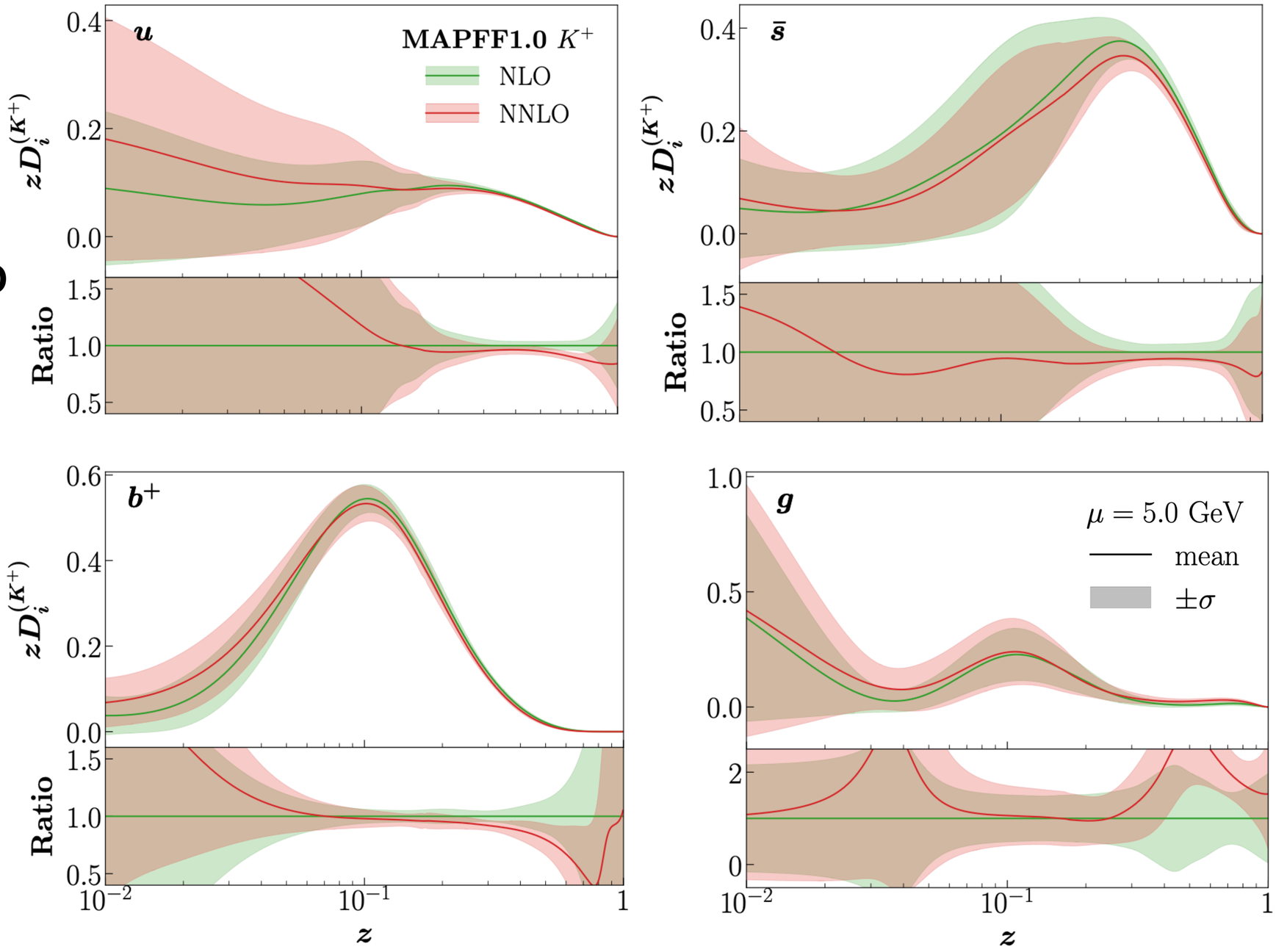
$D_1^{\pi^+}$

Approximated NNLO SIDIS corrections



$D_1^{K^+}$

Approximated NNLO SIDIS corrections



Goals of the new extraction

- Set the initial scale at $Q_0 = 1\,\mathrm{GeV}$:
  - Avoid backward evolution to get D_1 under $5\,\mathrm{GeV}$
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- See the impact of the complete NNLO SIDIS calculations
- Include more data to the fit, like ν data

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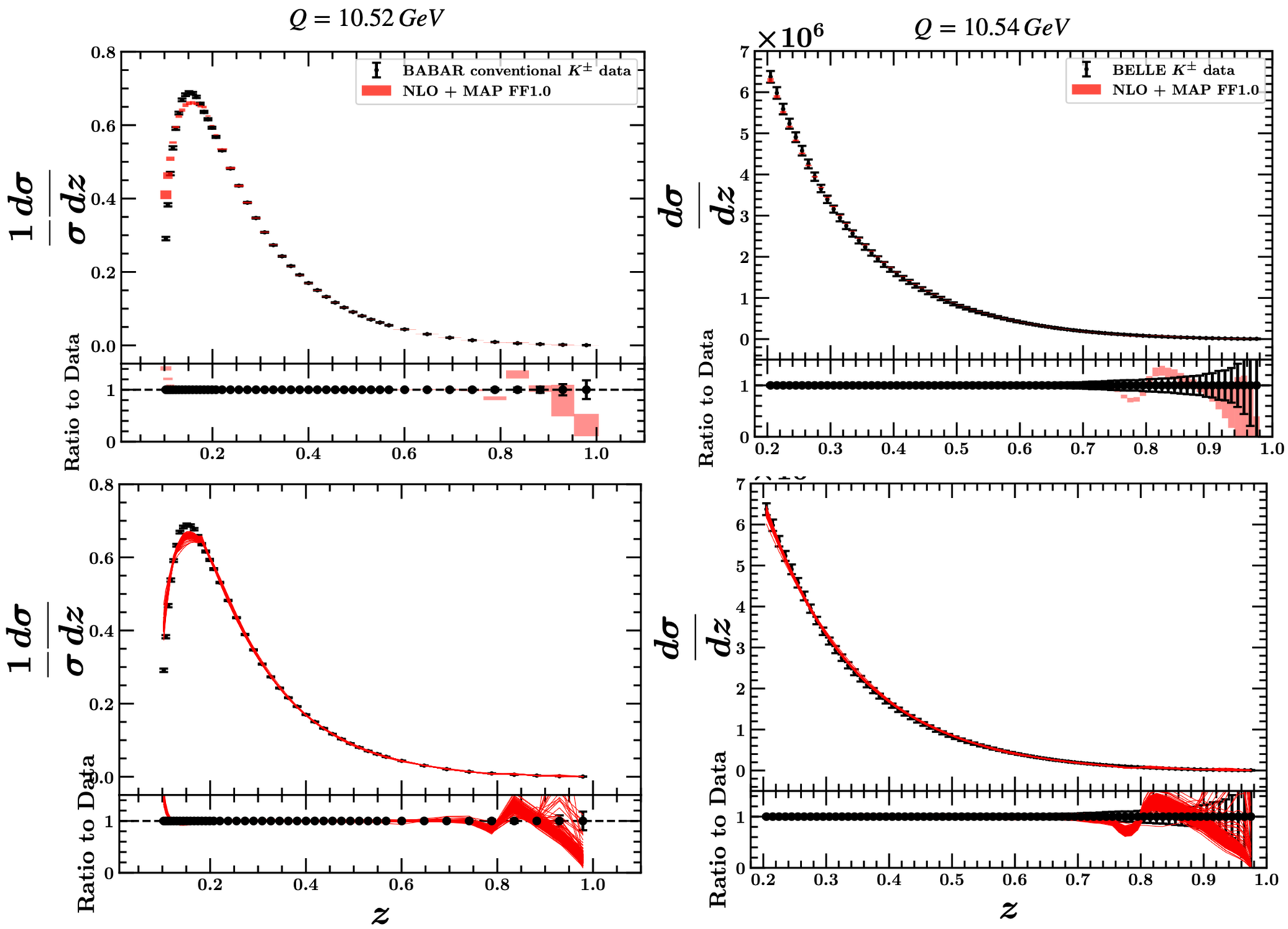
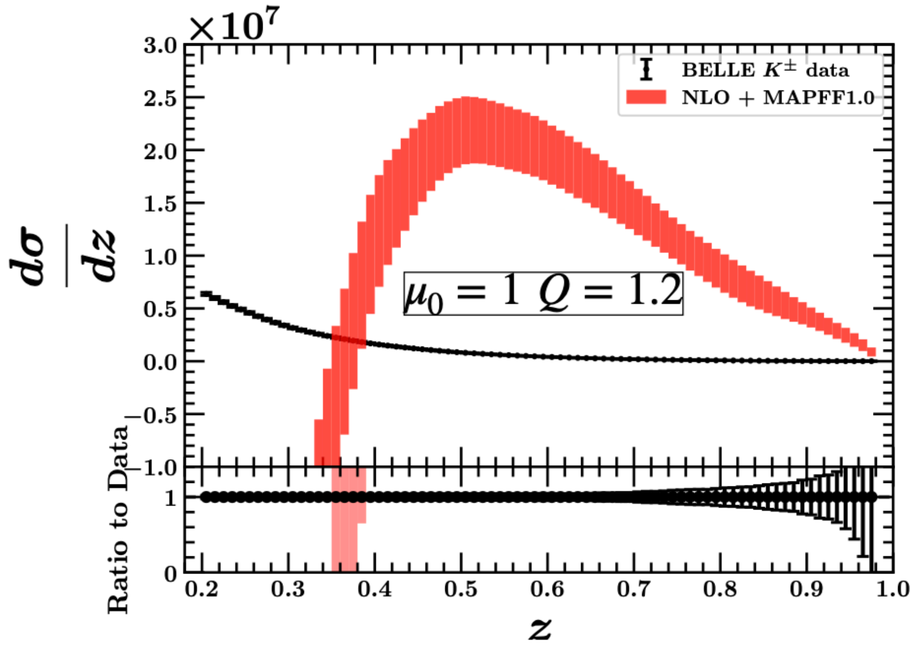
NLO predictions for BELLE and BABAR datasets

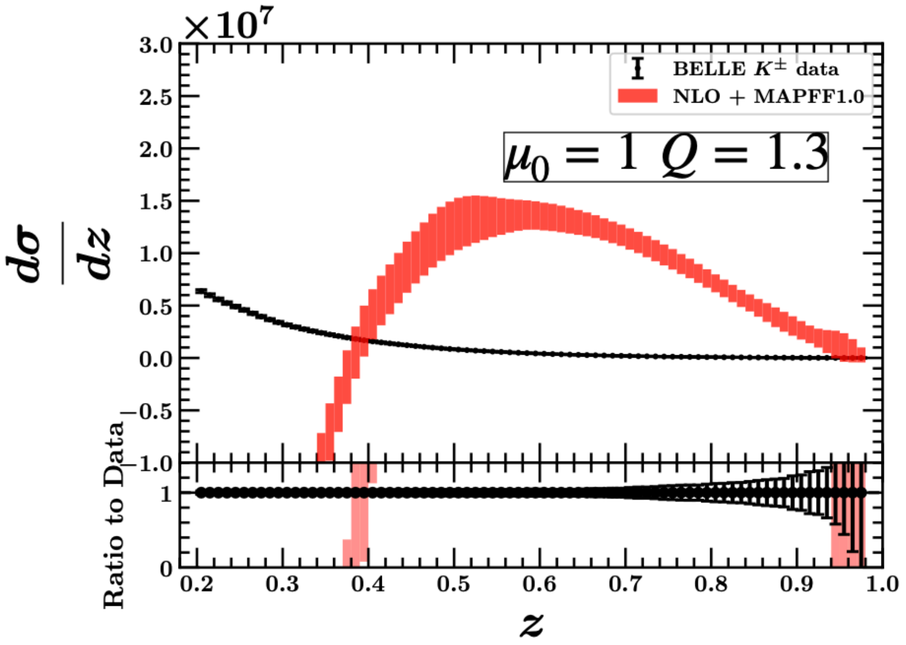
Single replicas

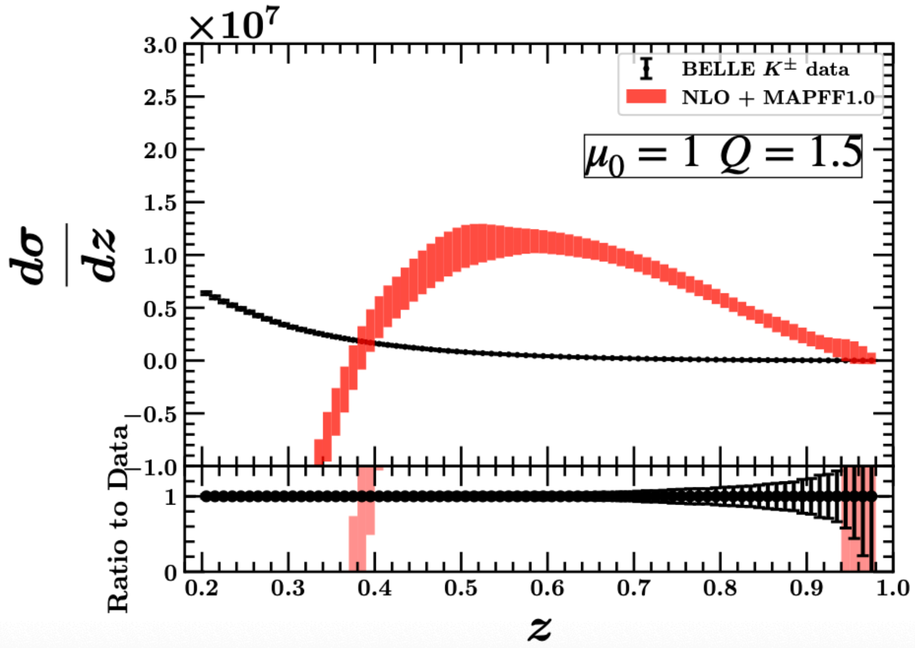


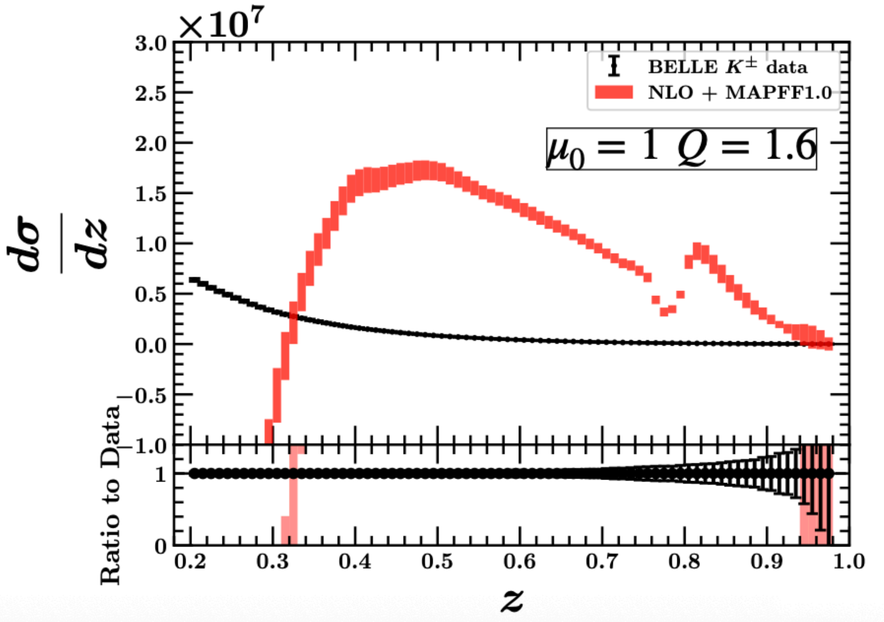
Q^2 test for BELLE predictions

$$m_{\text{charm}} = 1.51\,\text{GeV}$$

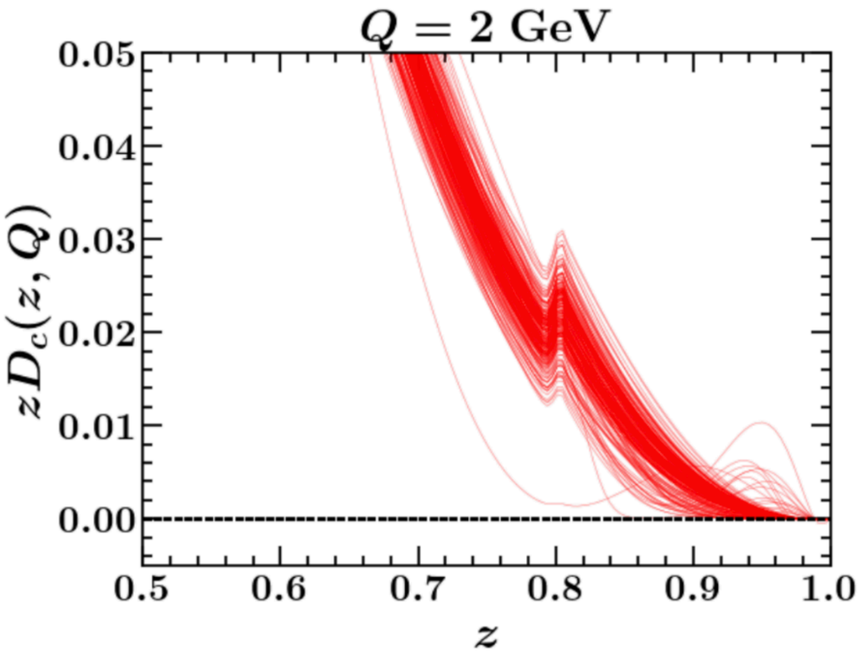


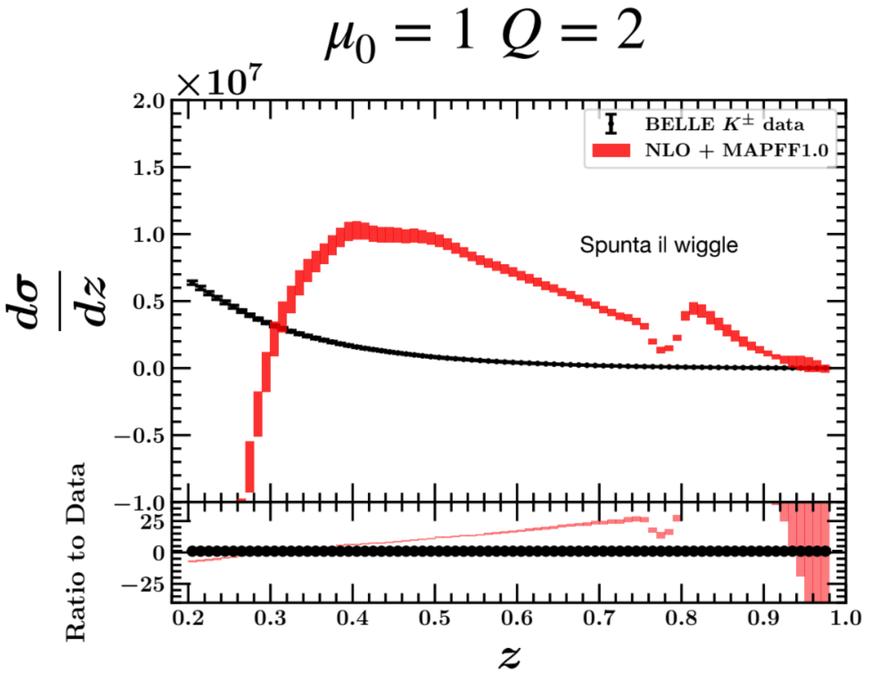






Heavy quark fragmentation functions





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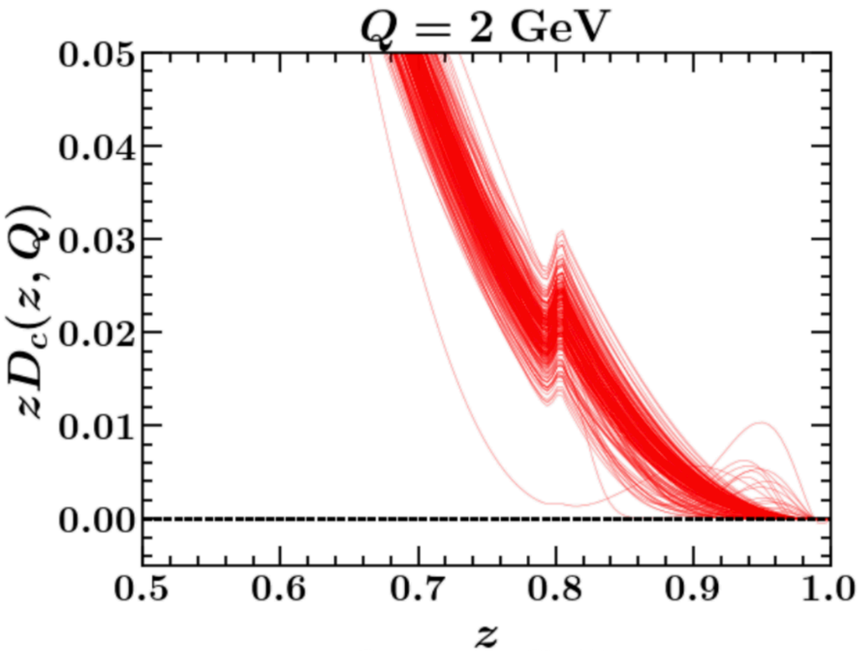
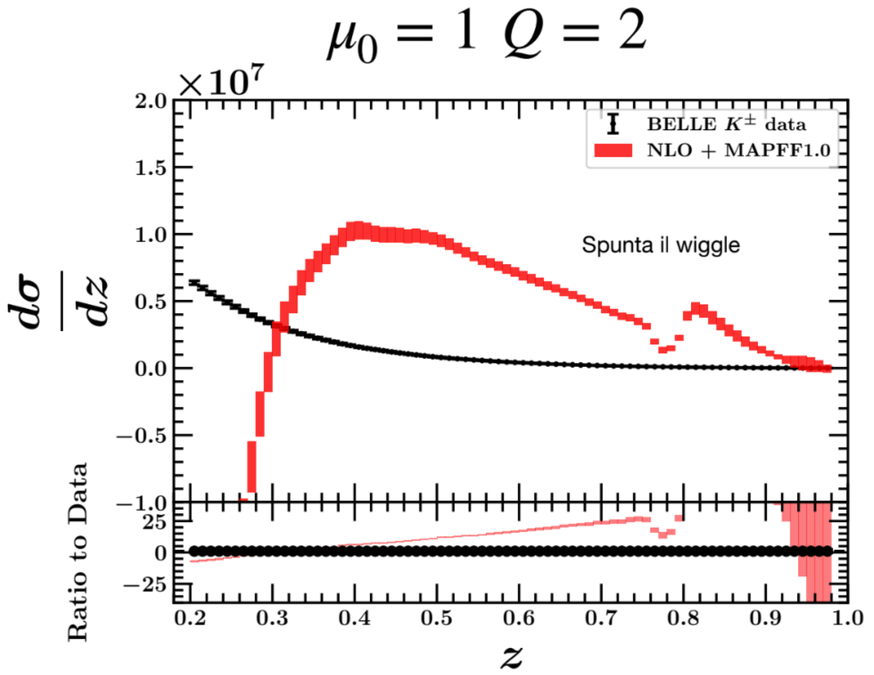
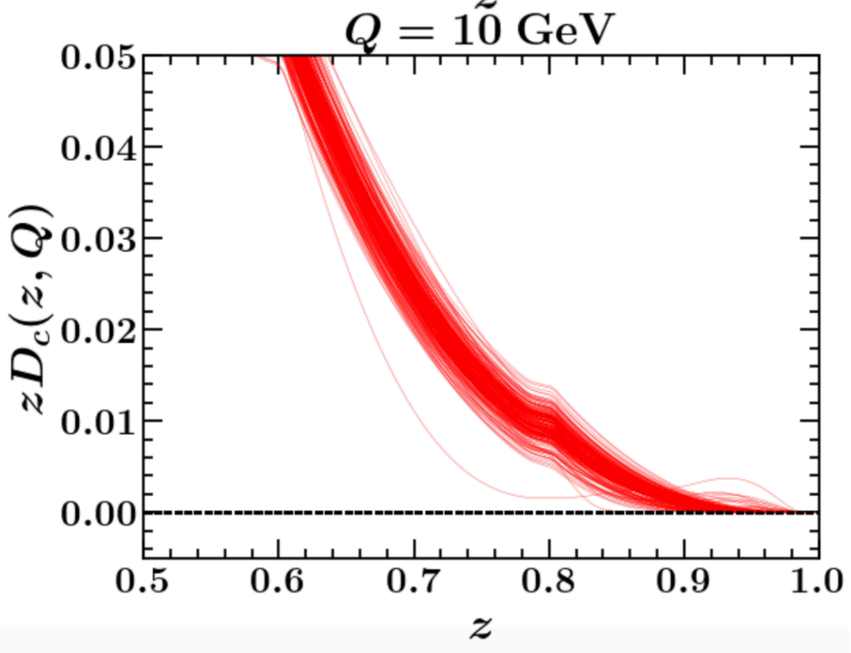
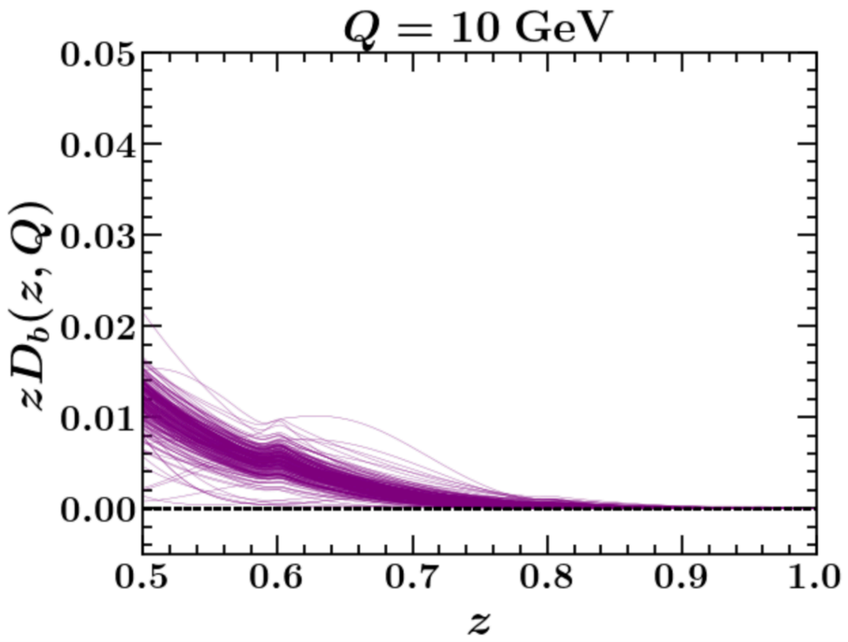


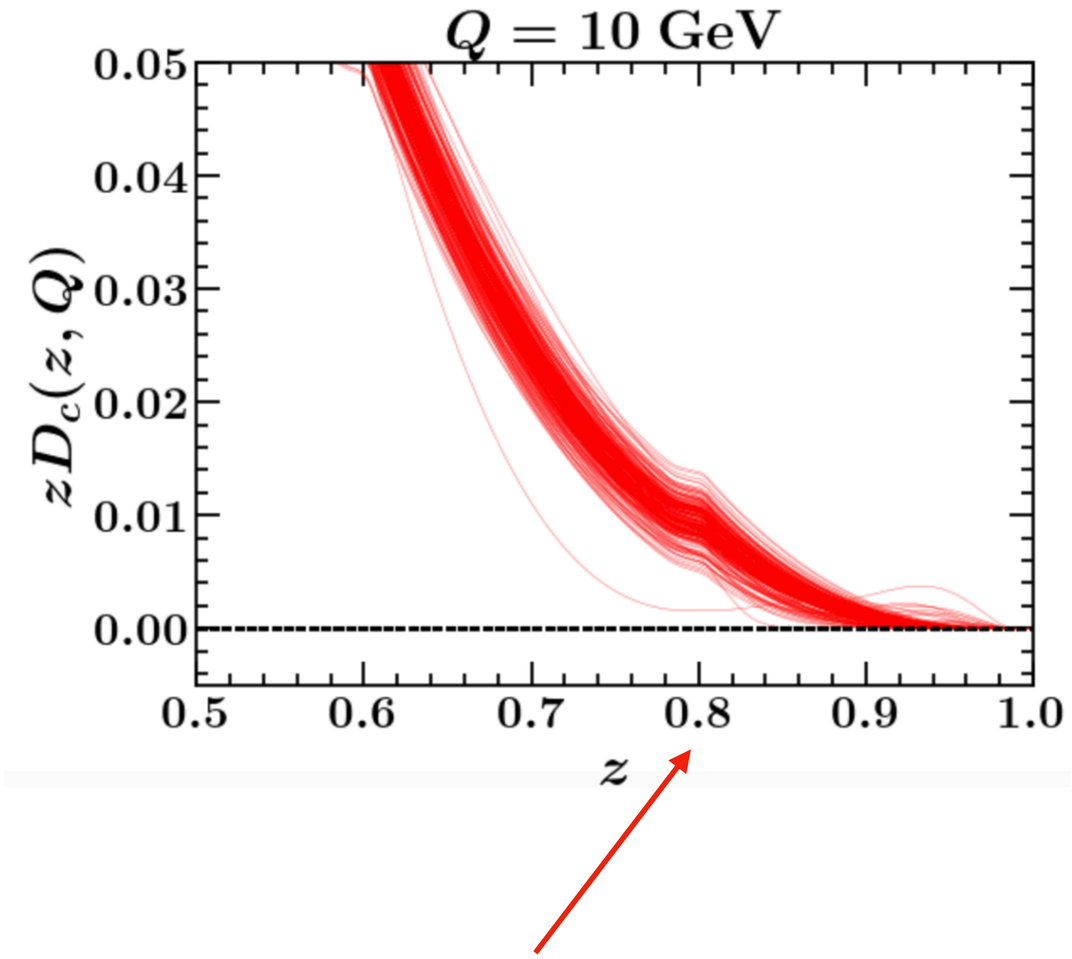


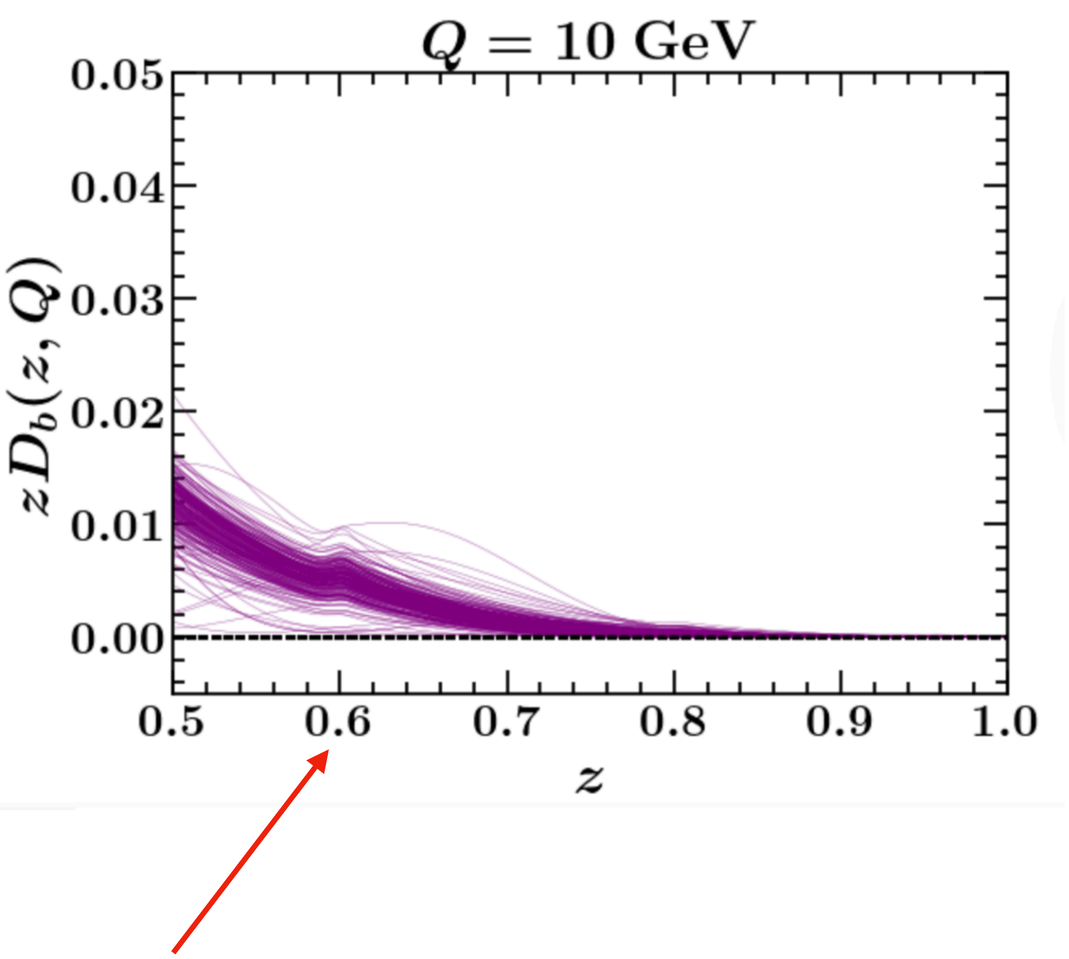




Heavy quark fragmentation functions

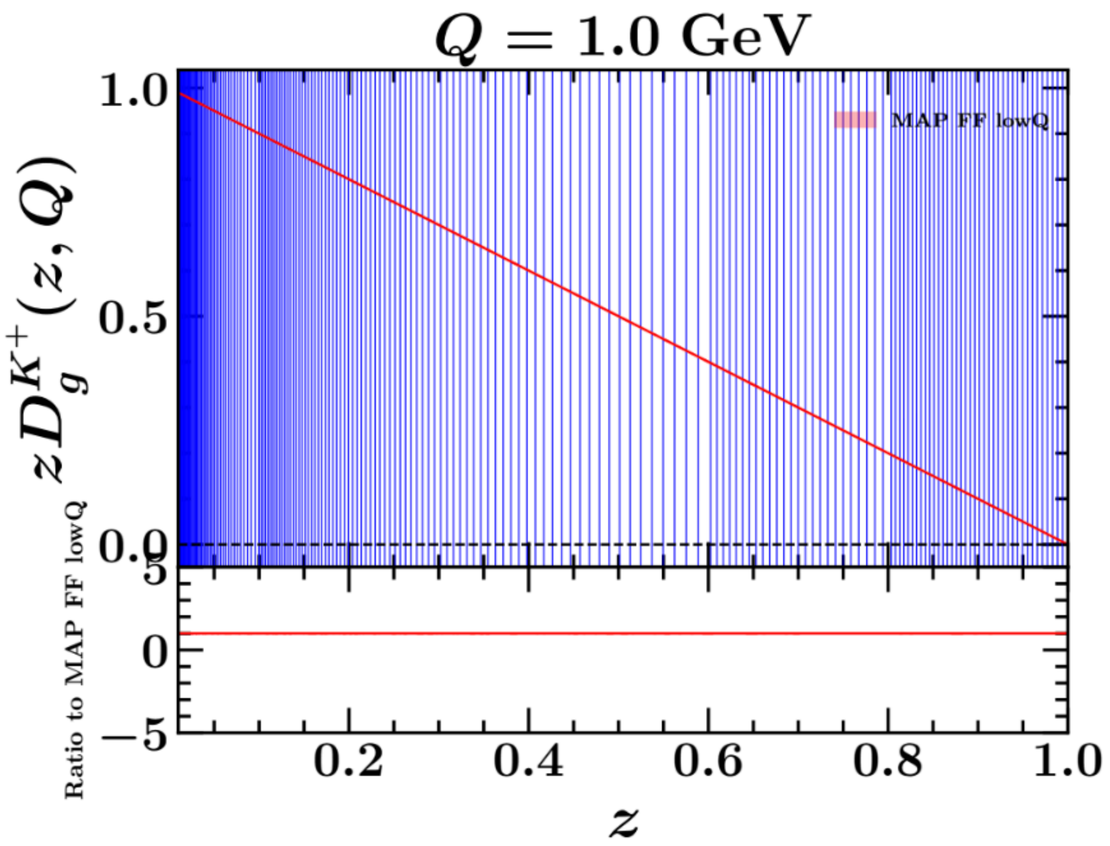
es.GridParameters = {{100, 1e-2, 3}, {100, 1e-1, 3}, {50, 6e-1, 3}, {50, 8e-1, 3}};

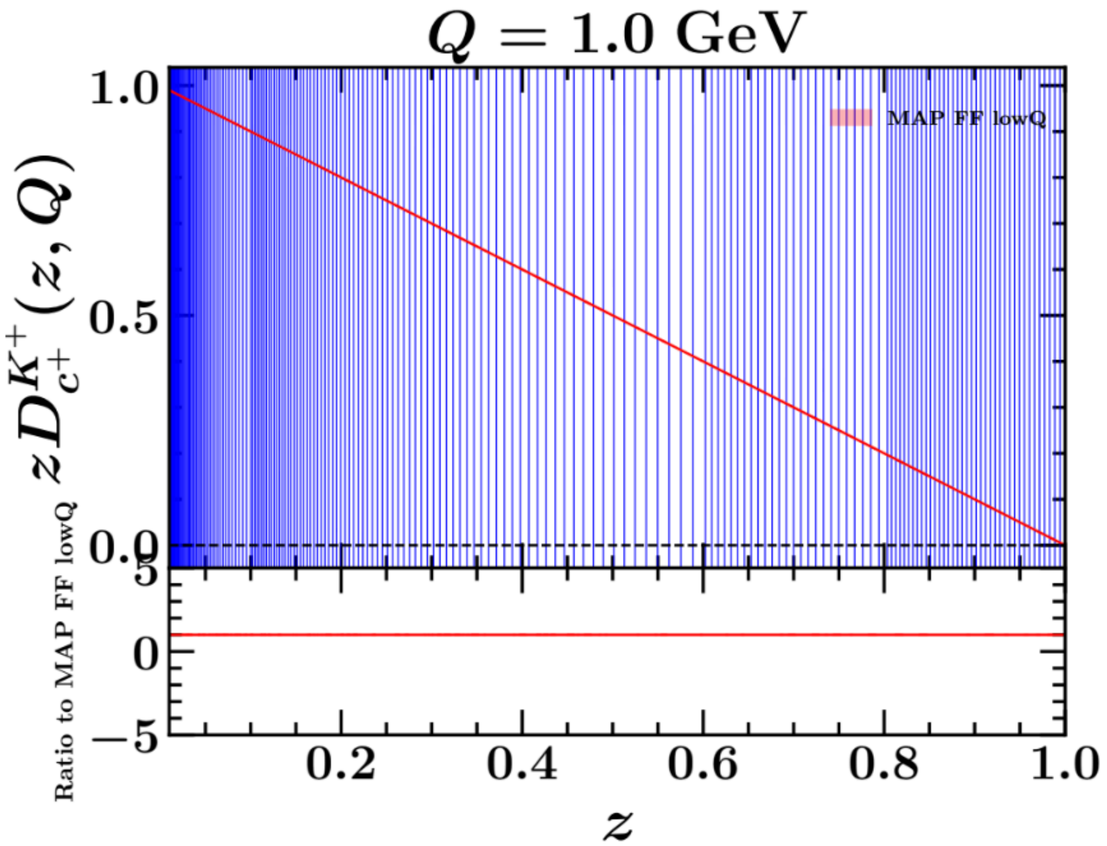




Apfell++ grid studies

$z\,D_1^q(z;Q^2) = 1 - z$ **for all flavors**





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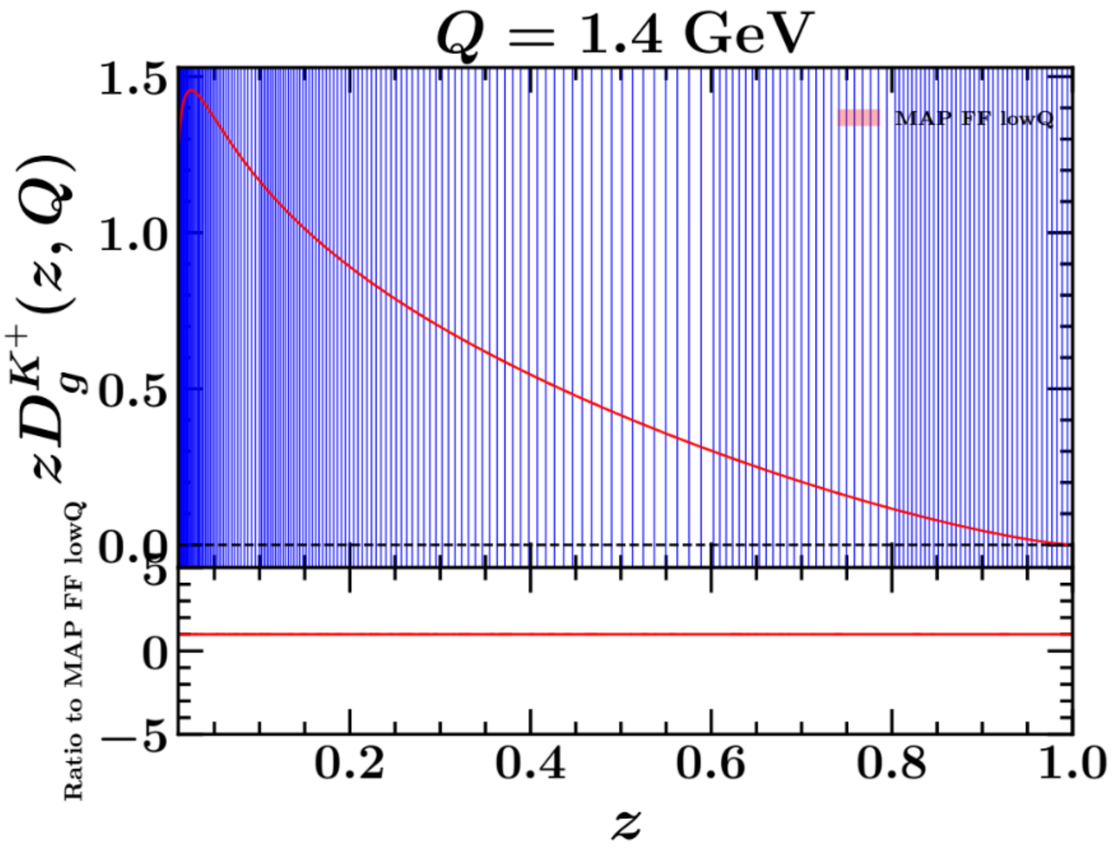

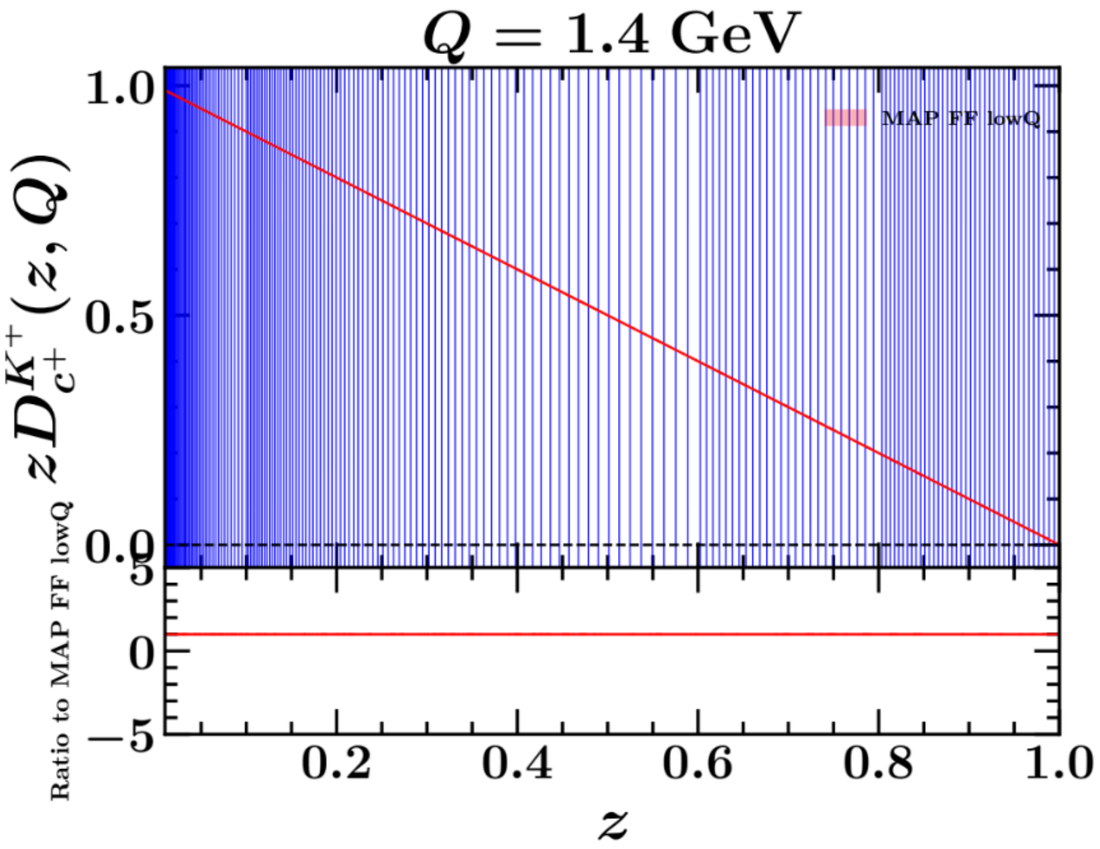

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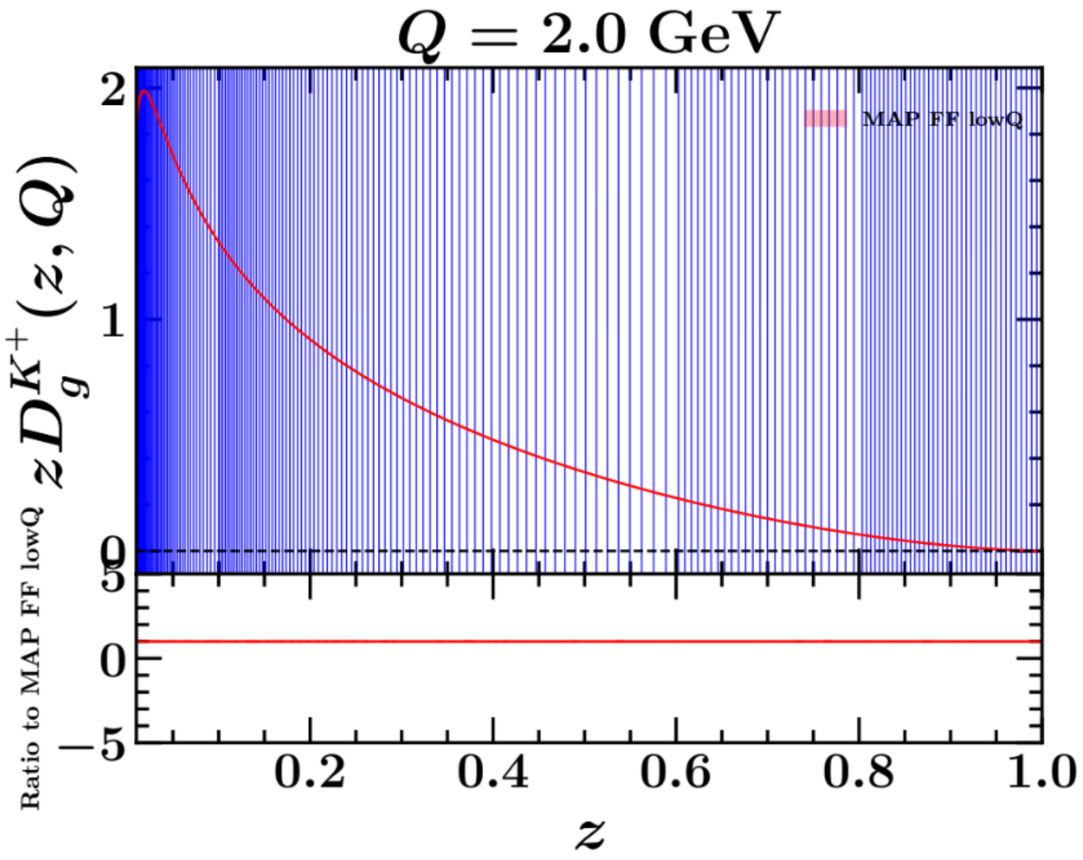

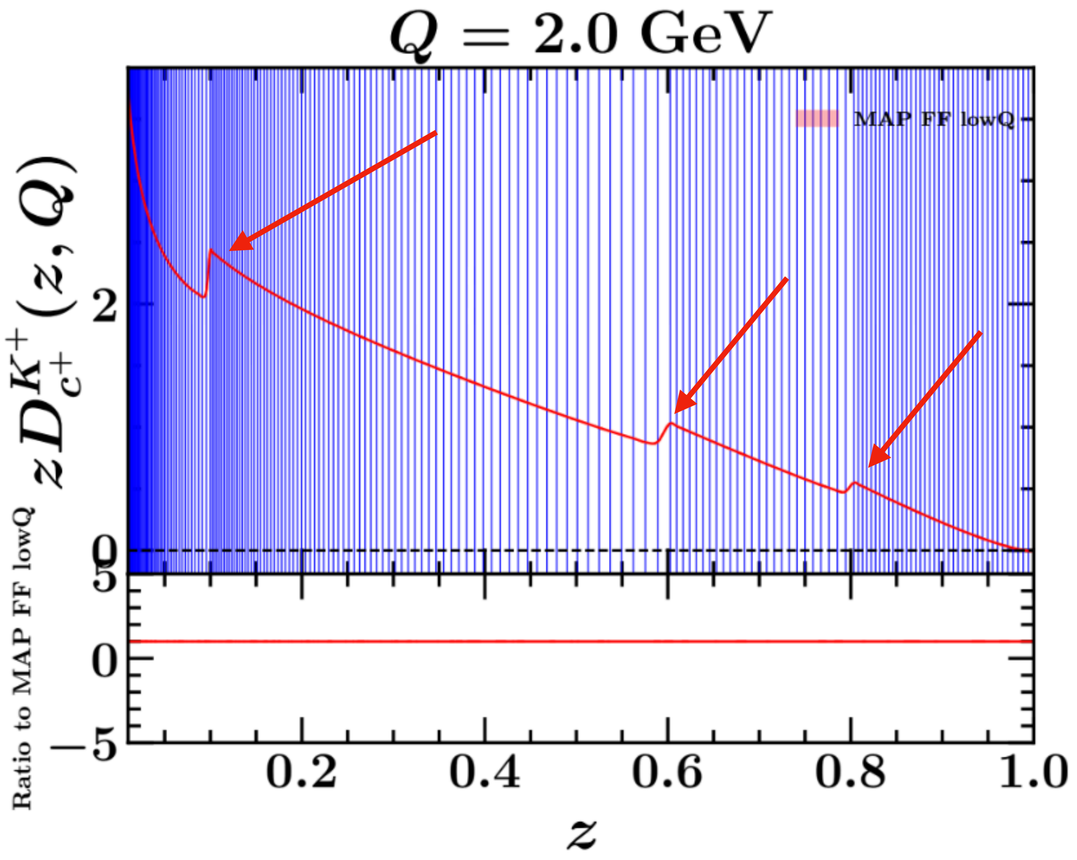

Apfell++ different grids, z-profiles

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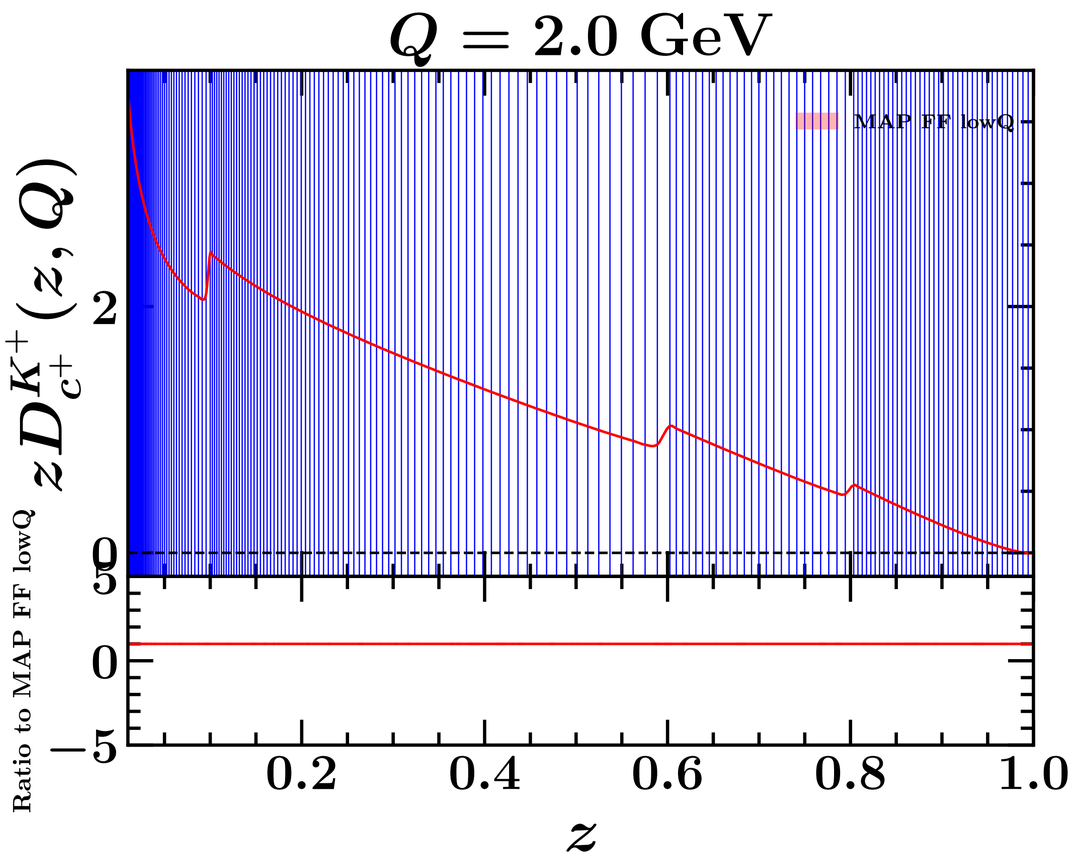
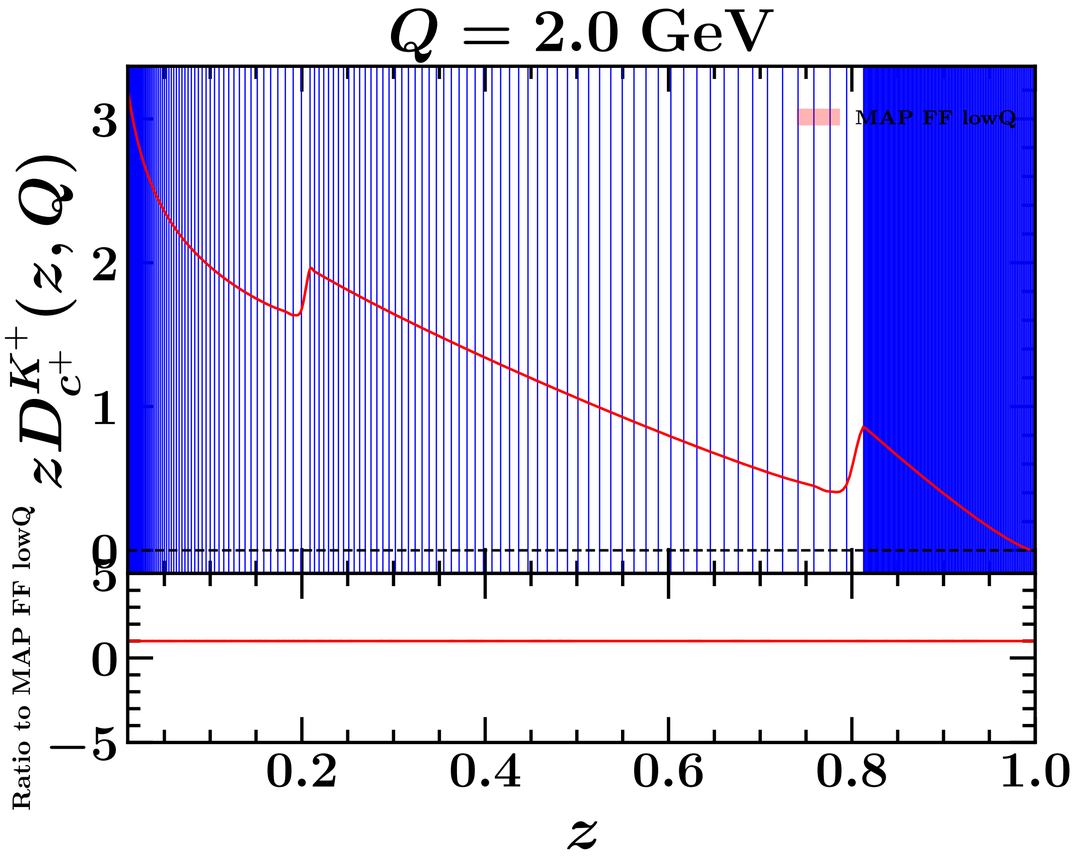
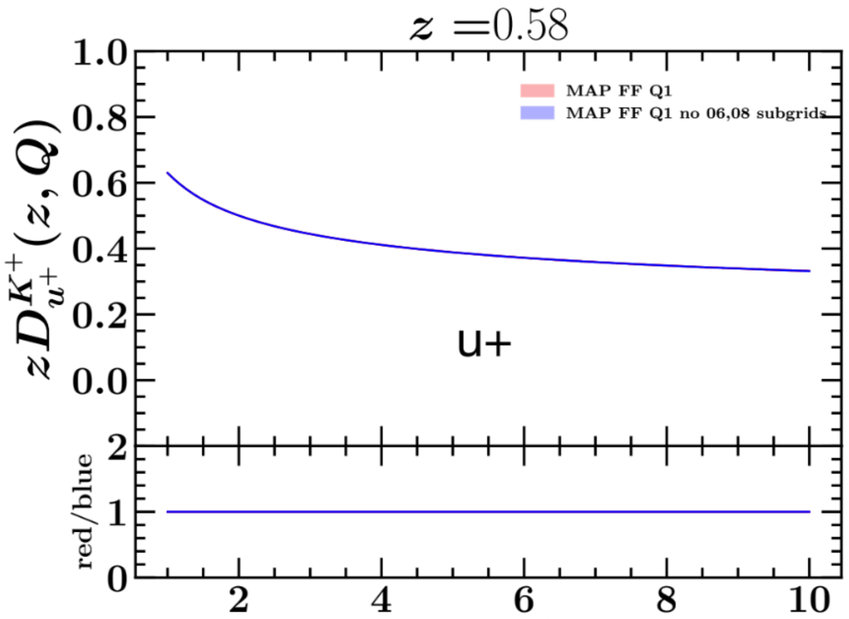
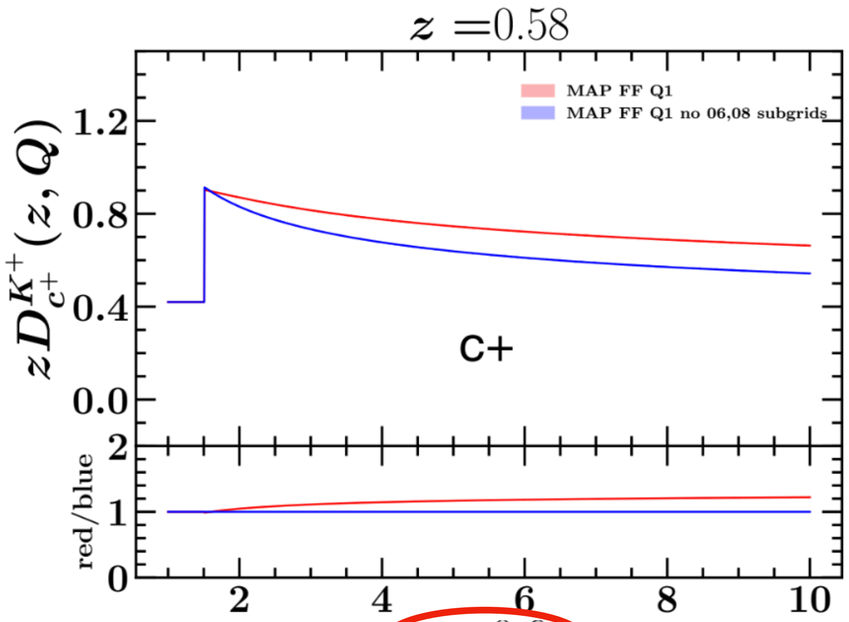


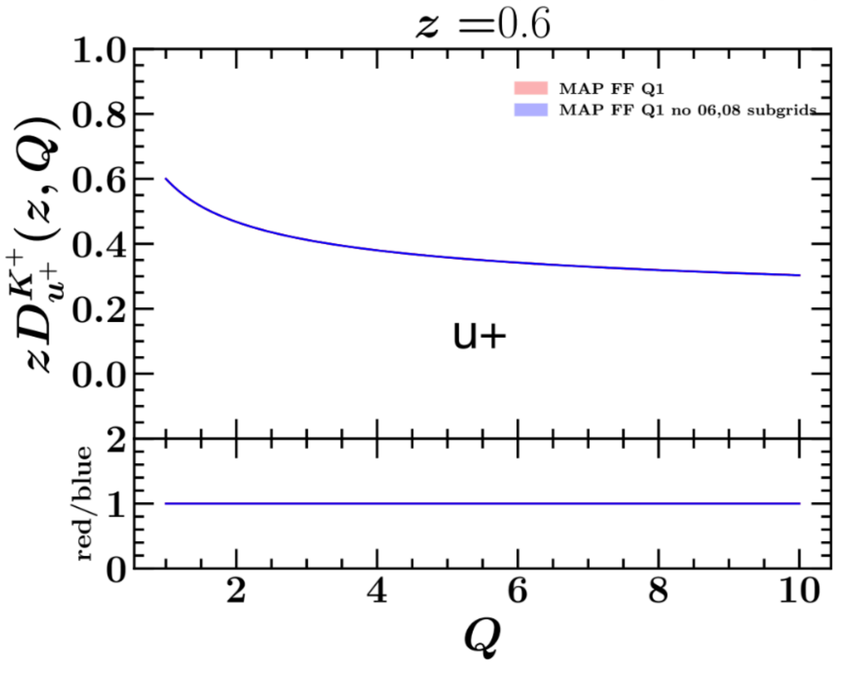
{{100, 1e-2, 3}, {100, 1e-1, 3}, {150, 8e-1, 3}}

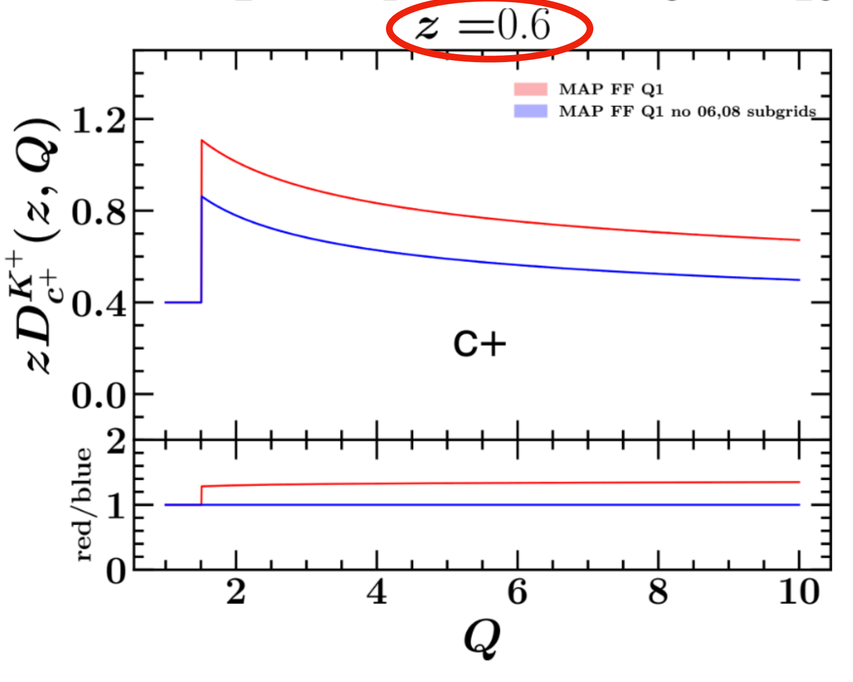


Apfell++ different grids, Q-profiles









matchincondition_tl.cc

Relevant part of the code:

ATS1Hg_0:

```
double ATS1Hg_0::Regular(double const& x) const
{ return 2 * CF * ( 1 + ( 1 - x ) * ( 1 - x ) ) * ( - 1 - 2 * log(x) ) / x;}
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}
double ATS1HH_0::Local(double const& x) const
{
  return 4 * CF * x - 2 * CF * log(1 - x) * ( - 1 + x * ( 2 + x ) + 2 * log(1 - x) );
}
```
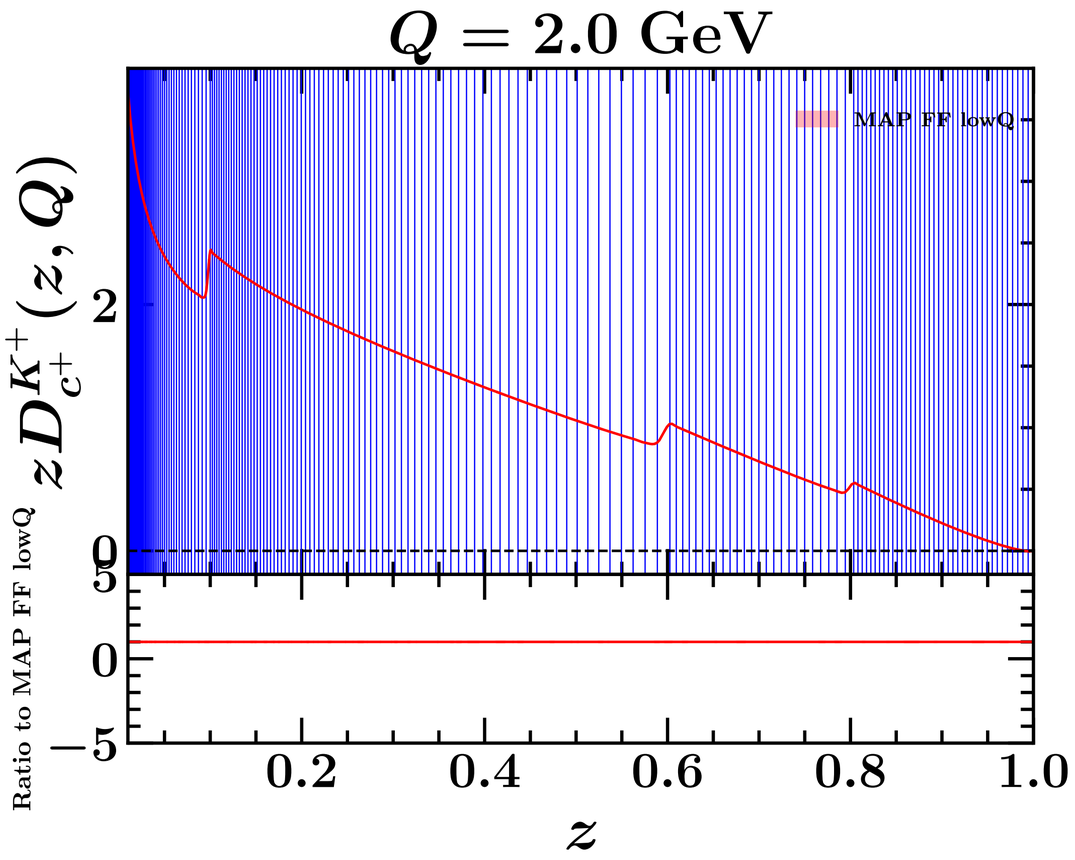
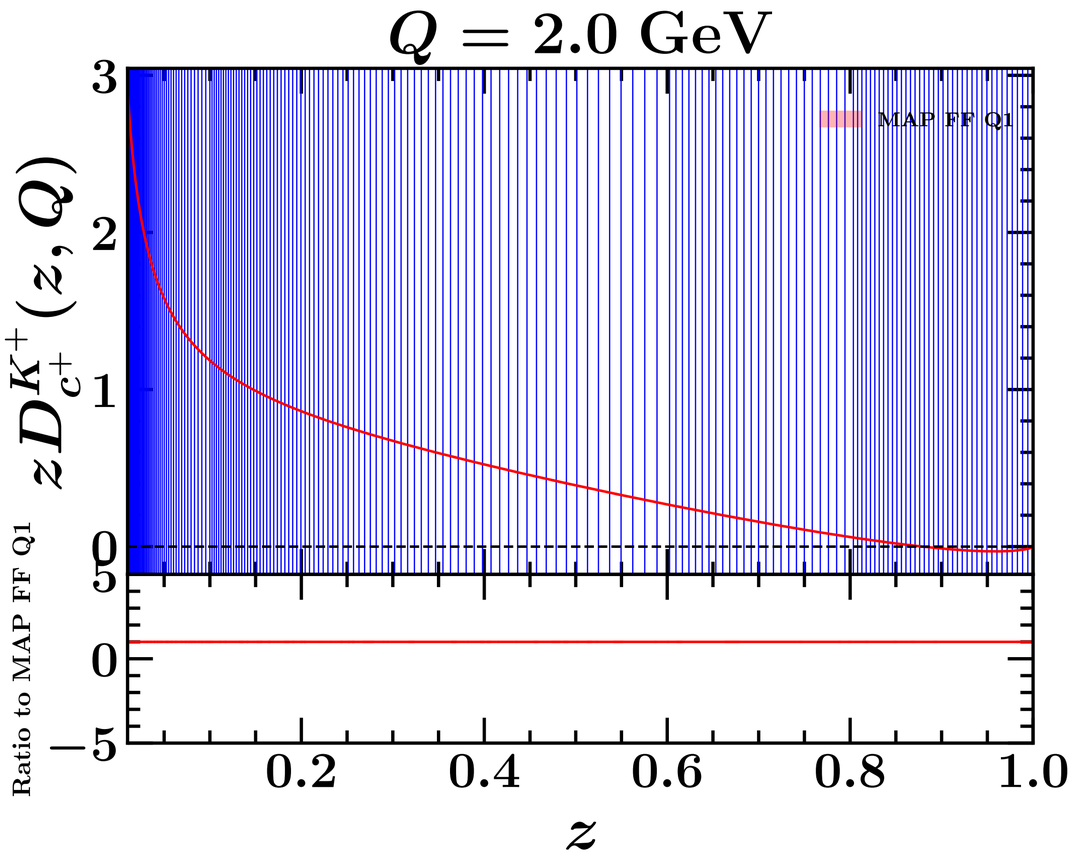

Comparison with and without Apfell++ bug

With Apfel++ bug



Fixed Apfel++ bug



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New fit of $D_1^{\pi^+}$ and $D_1^{K^+}$ at NLO and NNLO

NLO

Report of D_1 with fixed Apfell++ bug

I will show results obtained starting both from $Q_0 = 1$ GeV and $Q_0 = 5$ GeV and compare them with the released D_1.
For the uncertainty band, I will consider both $\mu \pm \sigma$ and the $68\,\%$ band.

I show the results for $Q = 1$ GeV and for $Q = 5$ GeV.

Set to be compared:

π^+

My fit:

MAPFFPI_Q5 -> from $Q_0 = 5\,GeV$ -> 197 replicas

MAPFFPI_lowQ -> from $Q_0 = 1\,GeV$ -> 192 replicas

Released:

MAPFF10NLOPIp, from $Q_0 = 5\,GeV$ -> 201 replicas

K^+

My fit:

MAPFFKA_Q5 -> from $Q_0 = 5\,GeV$ -> 202 replicas

MAPFFKA_lowQ -> from $Q_0 = 1\,GeV$-> 187 replicas

Released:

MAPFF10NLOKAp, from $Q_0 = 5\,GeV$ -> 201 replicas

χ_2

π^+

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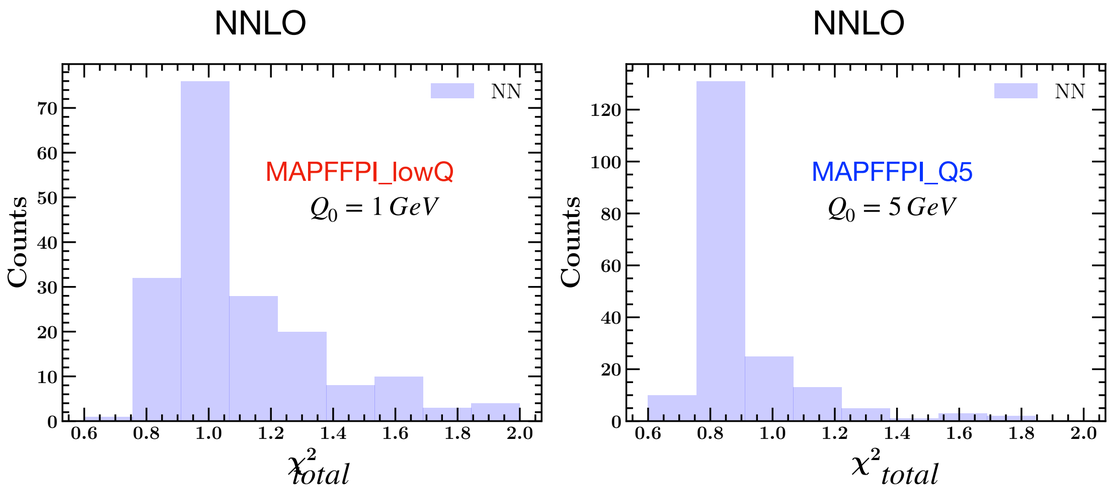

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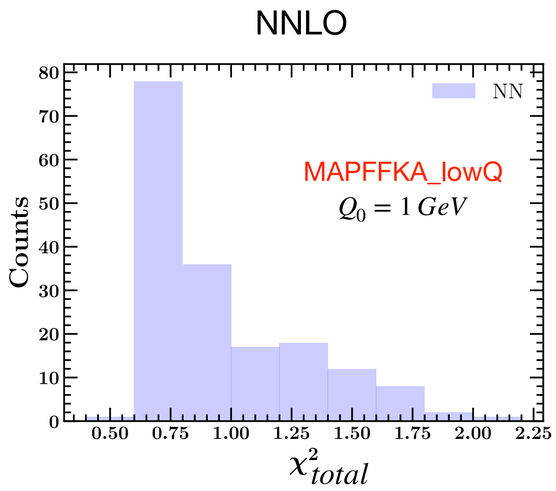
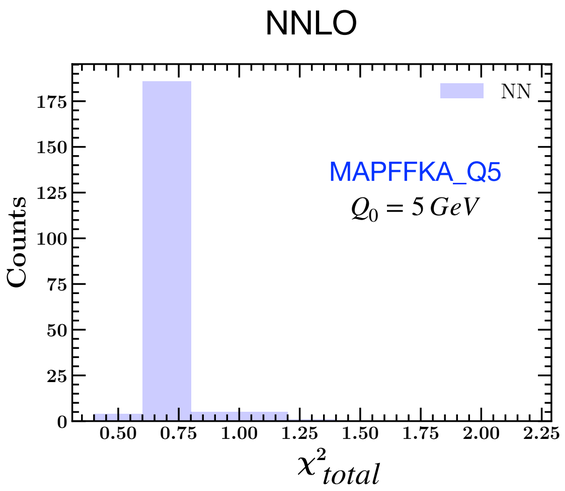
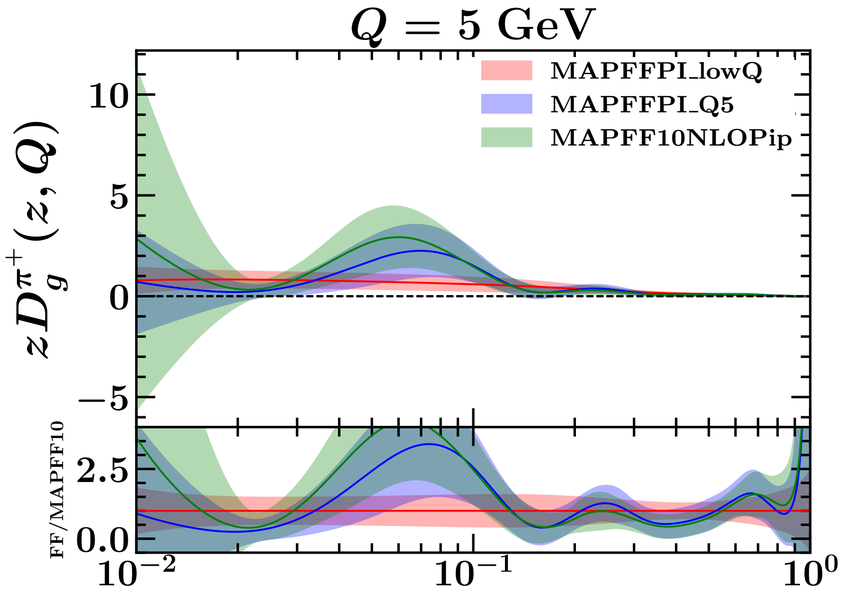


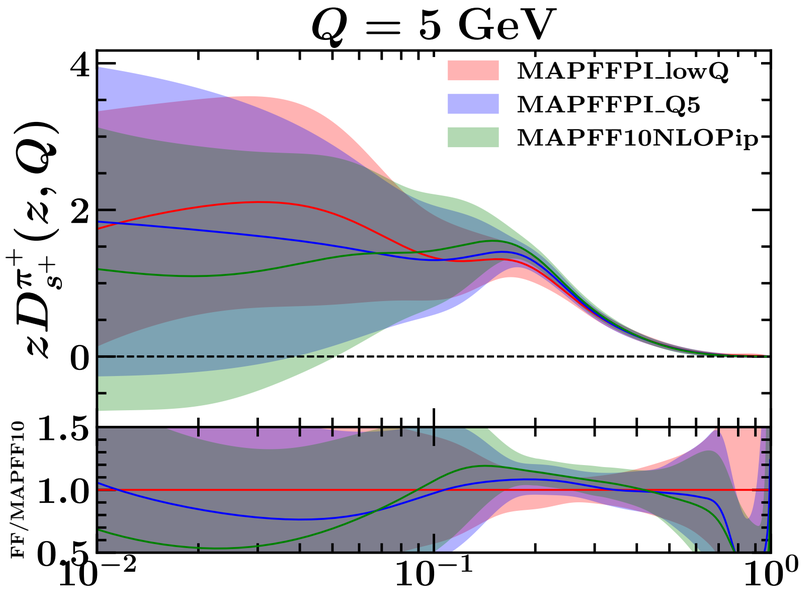


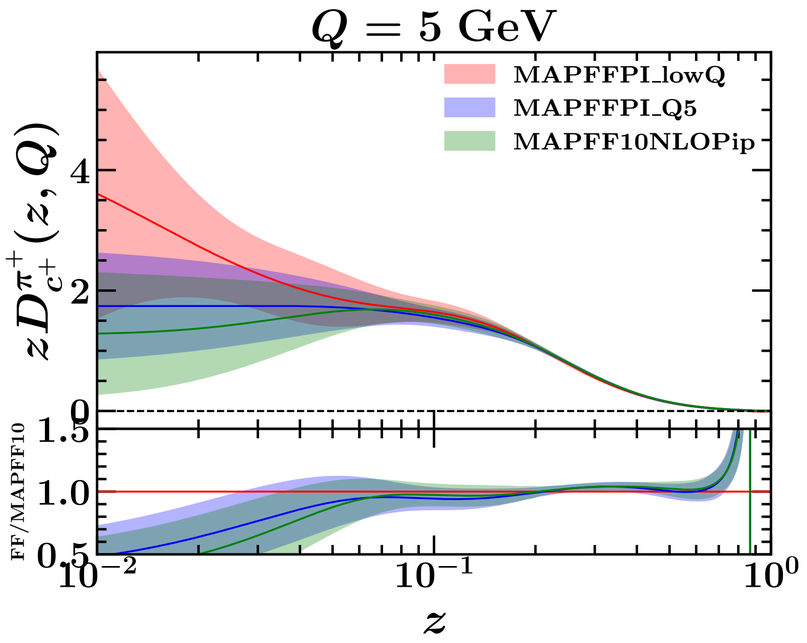
π^+

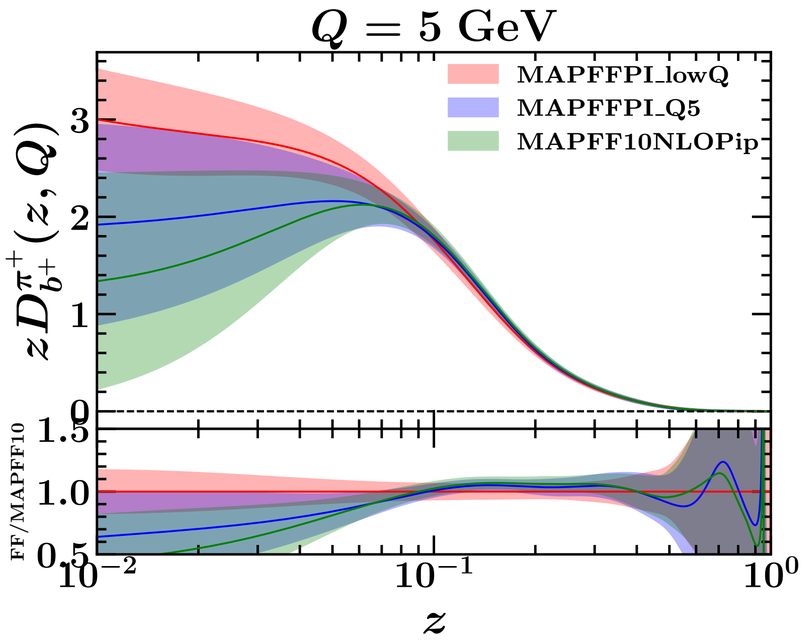
$\mu \pm \sigma$

$Q = 5\,GeV$





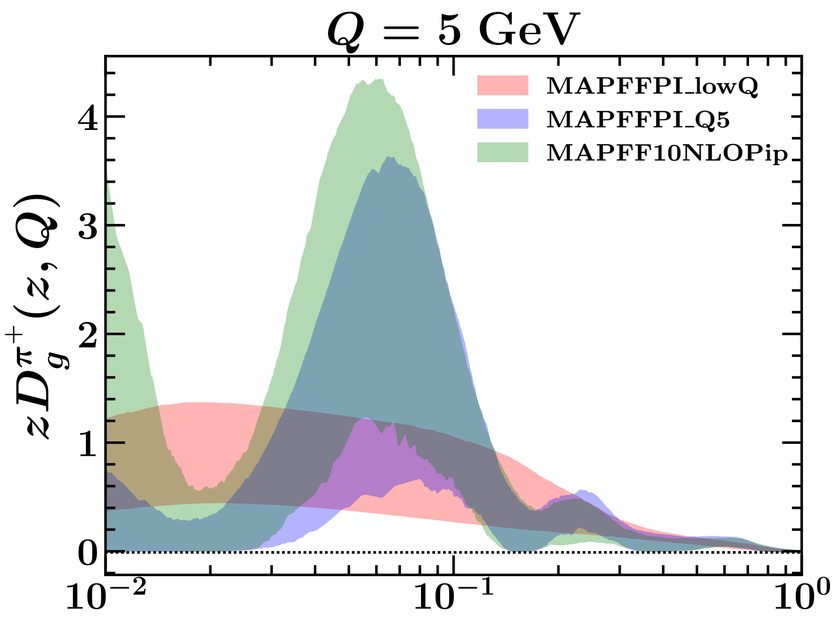


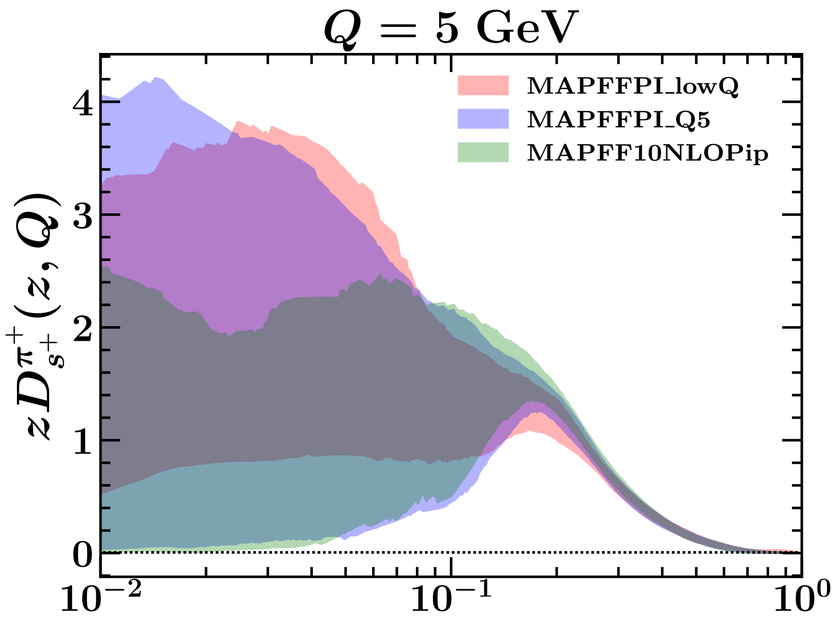


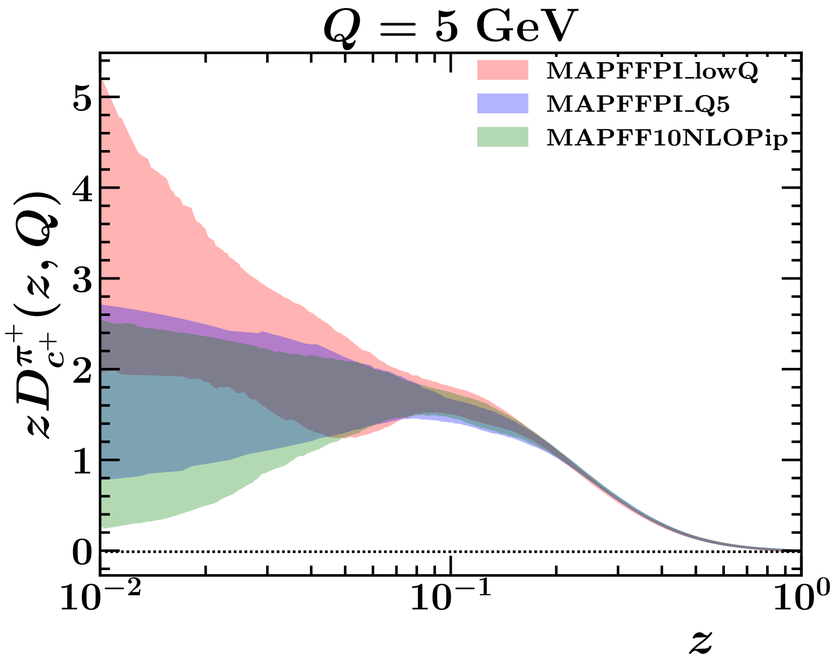
π^+

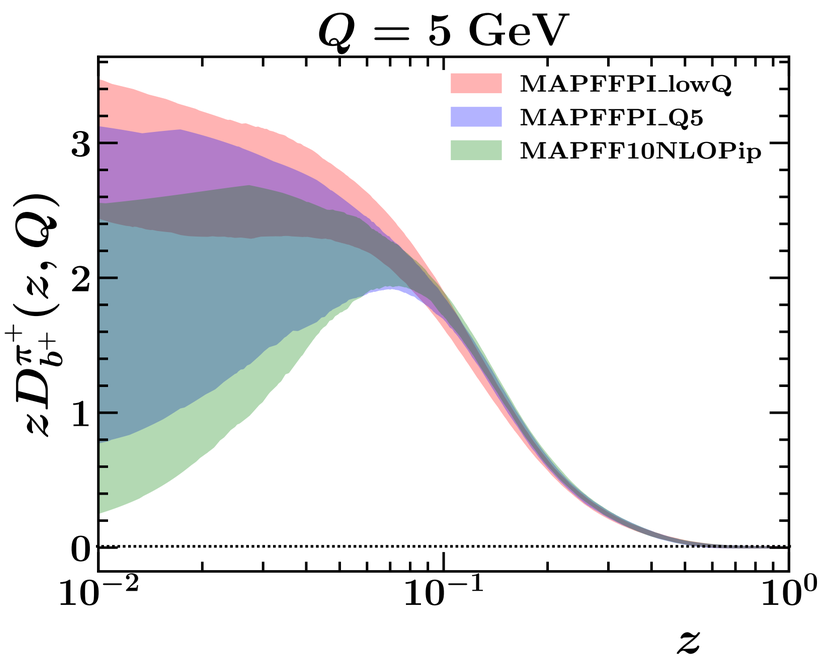
68%

$Q = 5\,GeV$


$Q = 5$ GeV
MAPFFPI_lowQ
MAPFFPI_Q5
MAPFF10NLOPip
$zD_g^{\pi^+}(z,Q)$


$Q = 5$ GeV
MAPFFPI_lowQ
MAPFFPI_Q5
MAPFF10NLOPip
$zD_{s^+}^{\pi^+}(z,Q)$

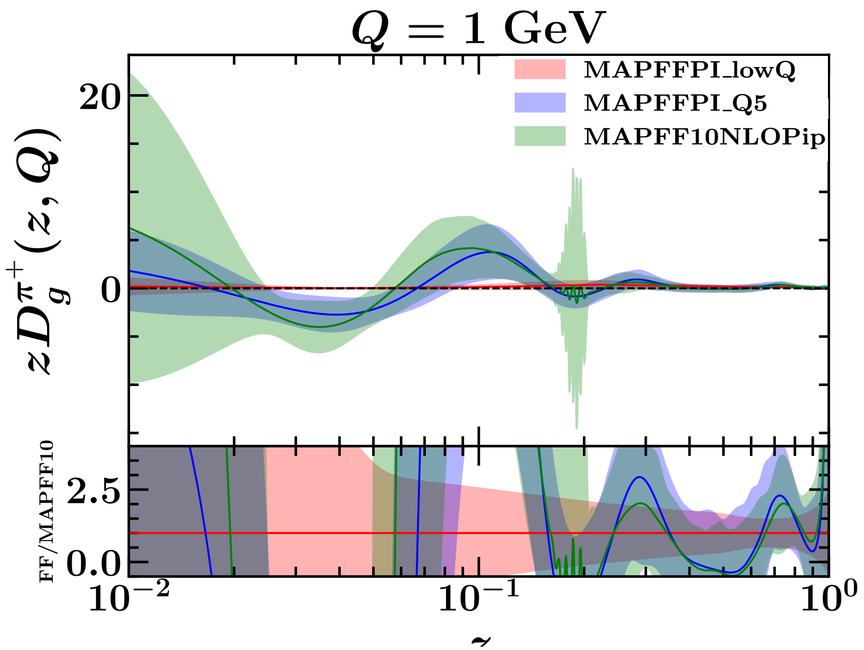

$Q = 5$ GeV
MAPFFPI_lowQ
MAPFFPI_Q5
MAPFF10NLOPip
$zD_{c^+}^{\pi^+}(z,Q)$
z

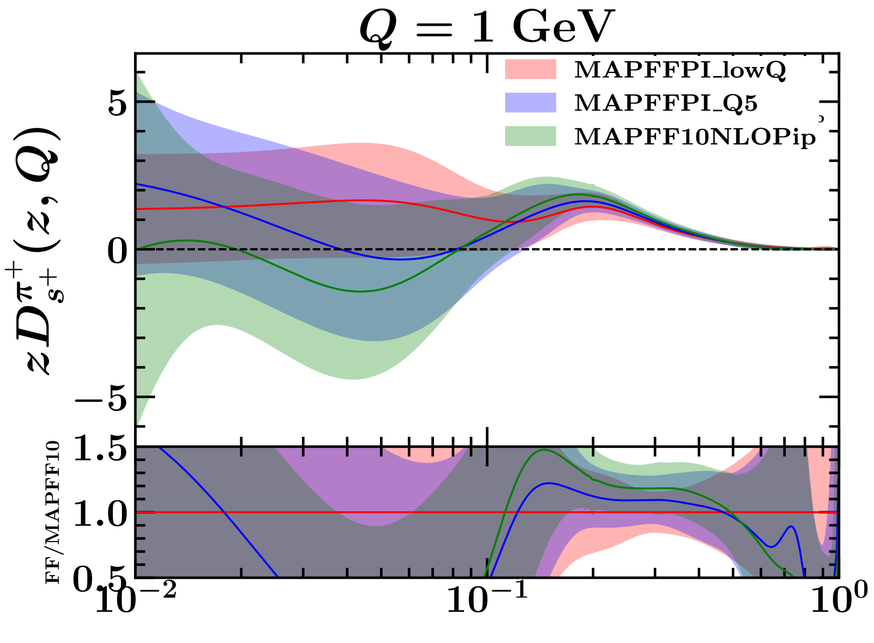

$Q = 5$ GeV
MAPFFPI_lowQ
MAPFFPI_Q5
MAPFF10NLOPip
$zD_{b^+}^{\pi^+}(z,Q)$
z

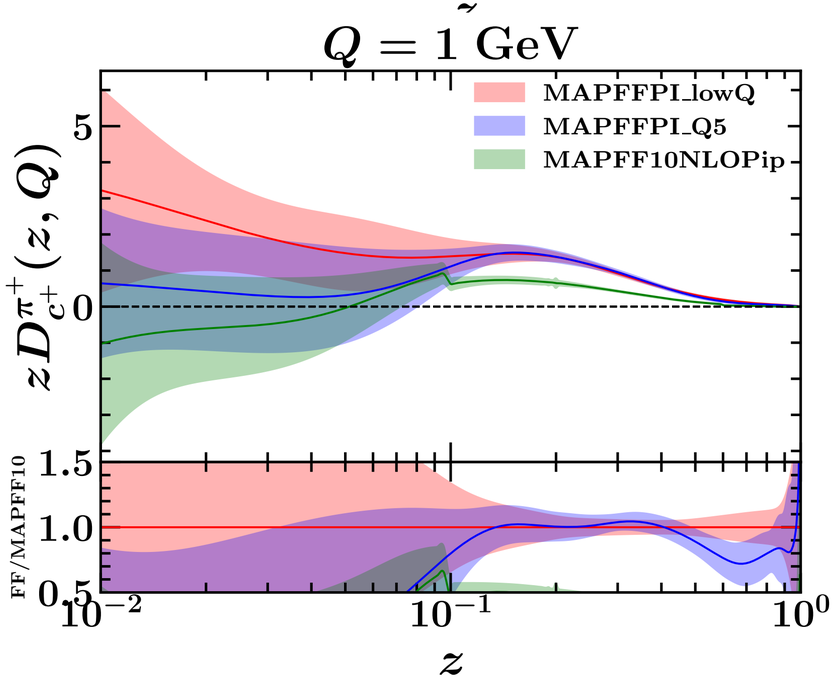
π^+

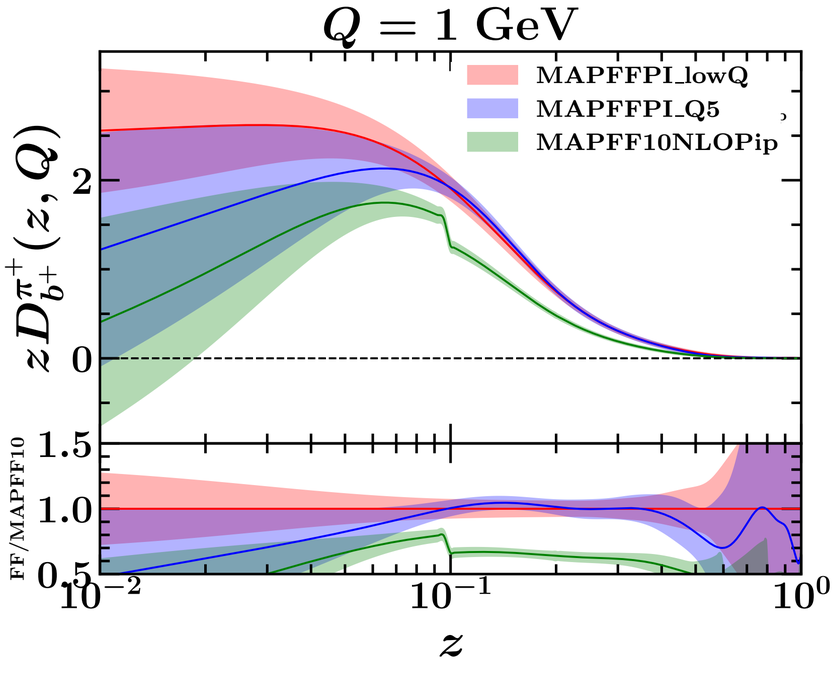
$\mu \pm \sigma$

$Q = 1\,GeV$





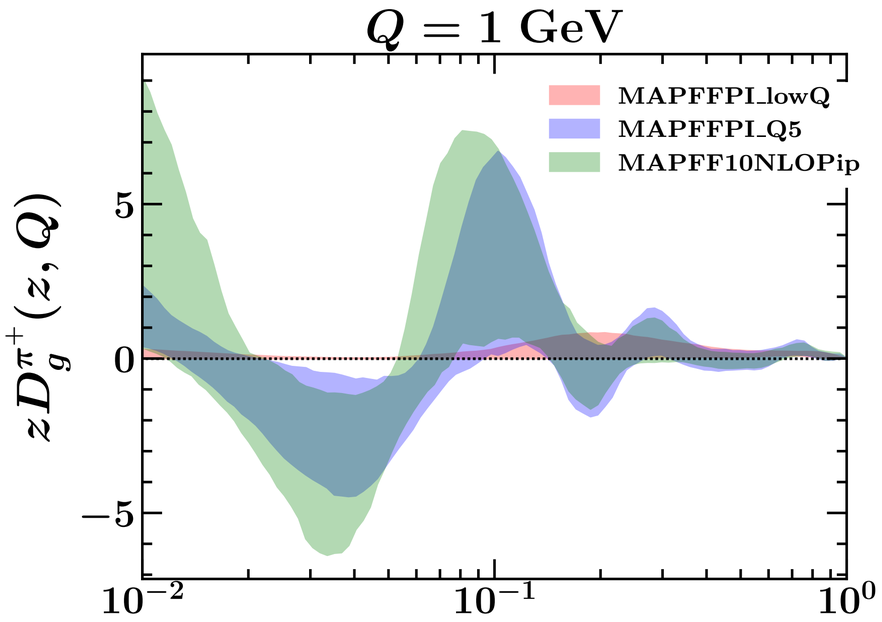


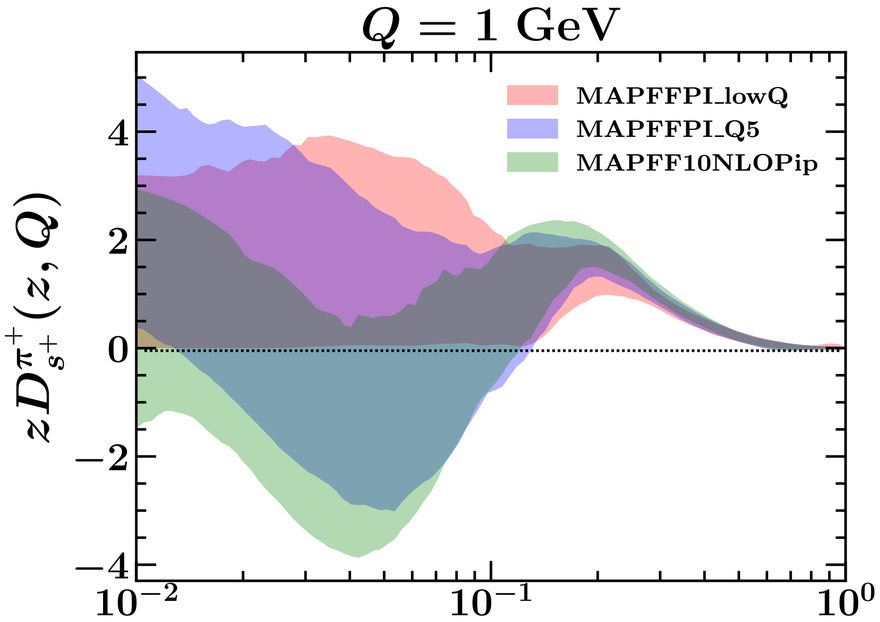


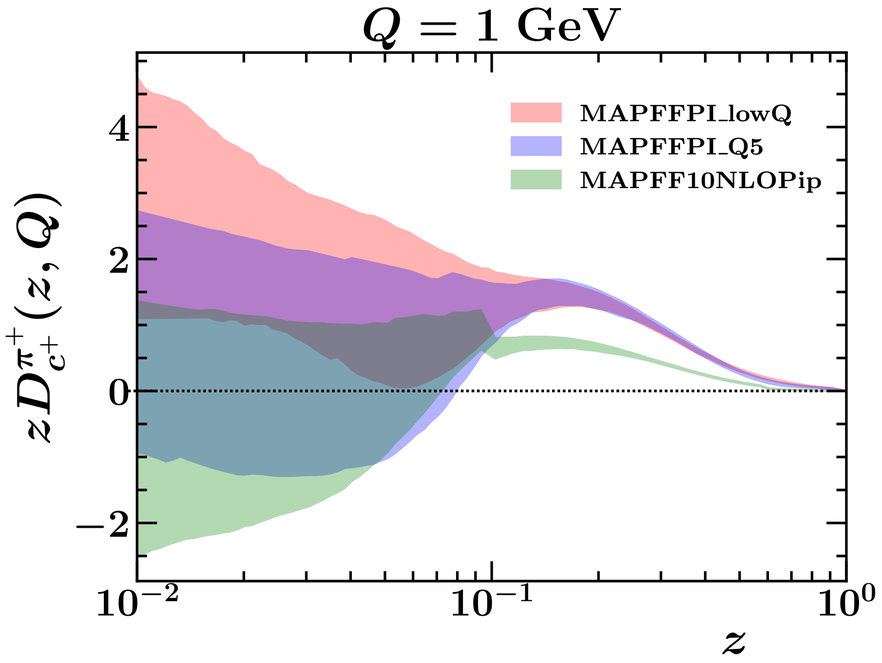
π^+

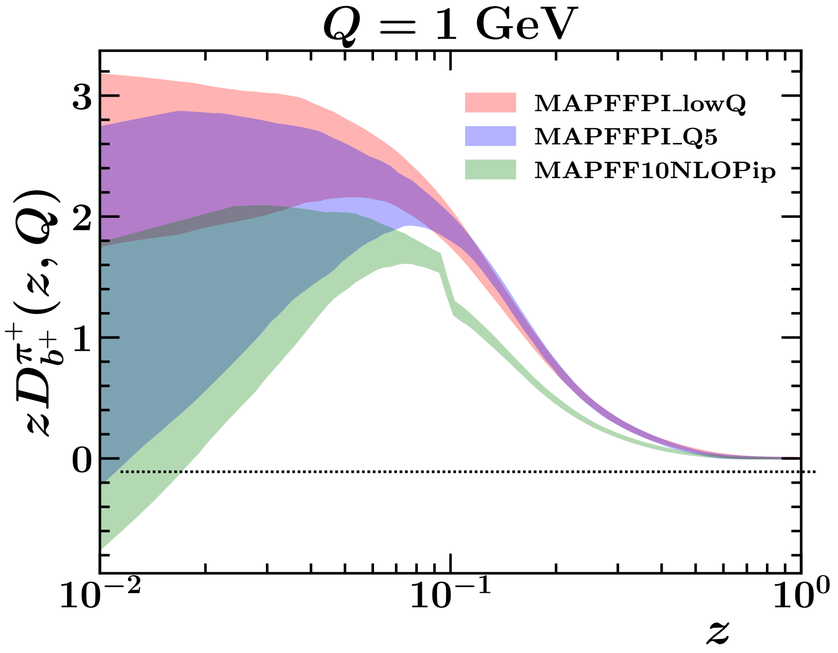
68%

$Q = 1\,GeV$


$Q = 1$ GeV
MAPFFPI_lowQ
MAPFFPI_Q5
MAPFF10NLOPip
$zD_g^{\pi^+}(z,Q)$
5
0
−5
10^{-2}
10^{-1}
10^{0}


$Q = 1$ GeV
MAPFFPI_lowQ
MAPFFPI_Q5
MAPFF10NLOPip
$zD_{s^+}^{\pi^+}(z,Q)$
4
2
0
−2
−4
10^{-2}
10^{-1}
10^{0}

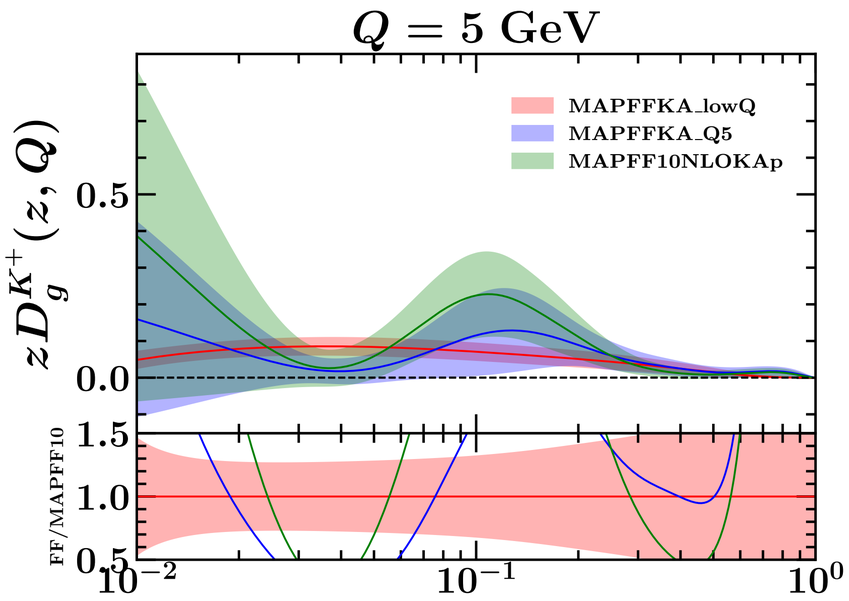

$Q = 1$ GeV
MAPFFPI_lowQ
MAPFFPI_Q5
MAPFF10NLOPip
$zD_{c^+}^{\pi^+}(z,Q)$
4
2
0
−2
10^{-2}
10^{-1}
10^{0}
z

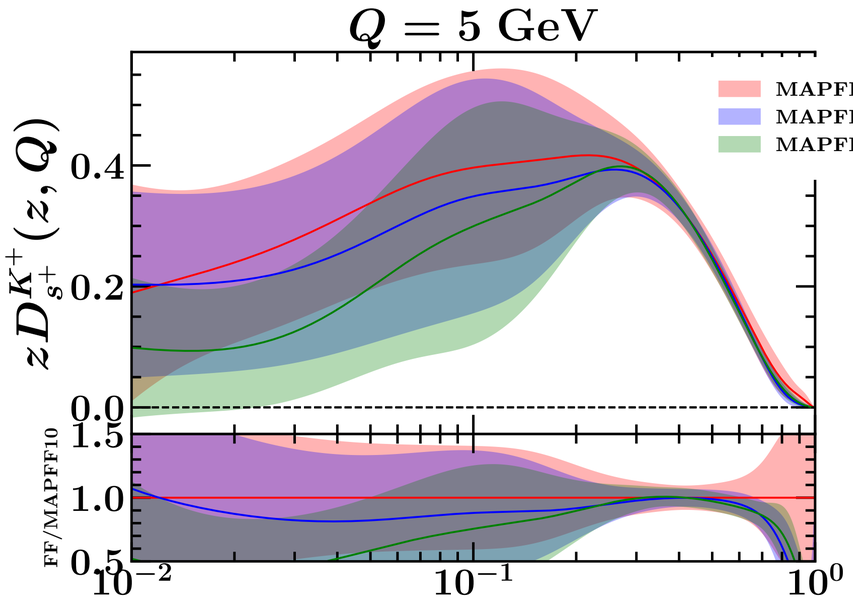

$Q = 1$ GeV
MAPFFPI_lowQ
MAPFFPI_Q5
MAPFF10NLOPip
$zD_{b^+}^{\pi^+}(z,Q)$
3
2
1
0
10^{-2}
10^{-1}
10^{0}
z

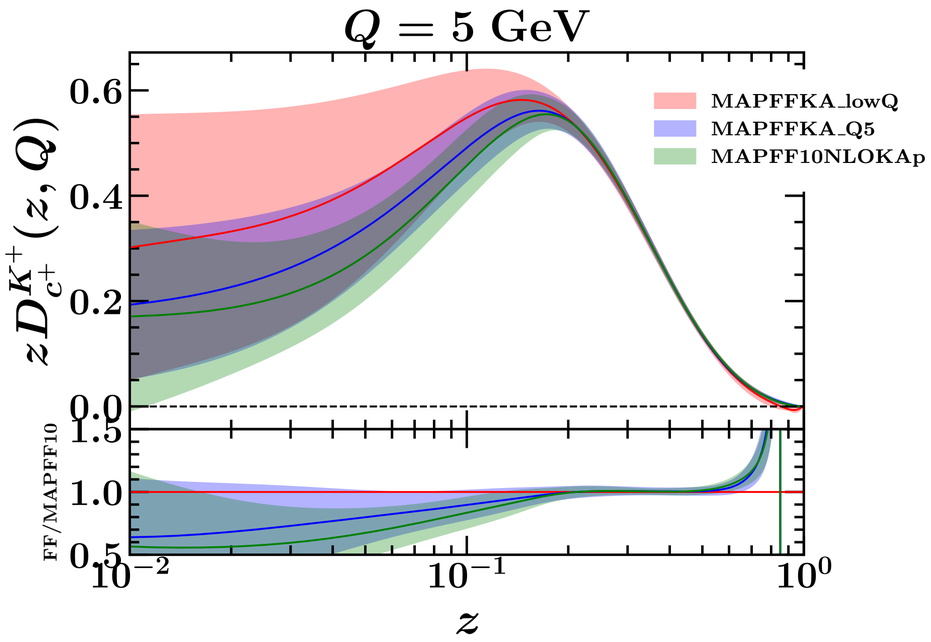
K^+

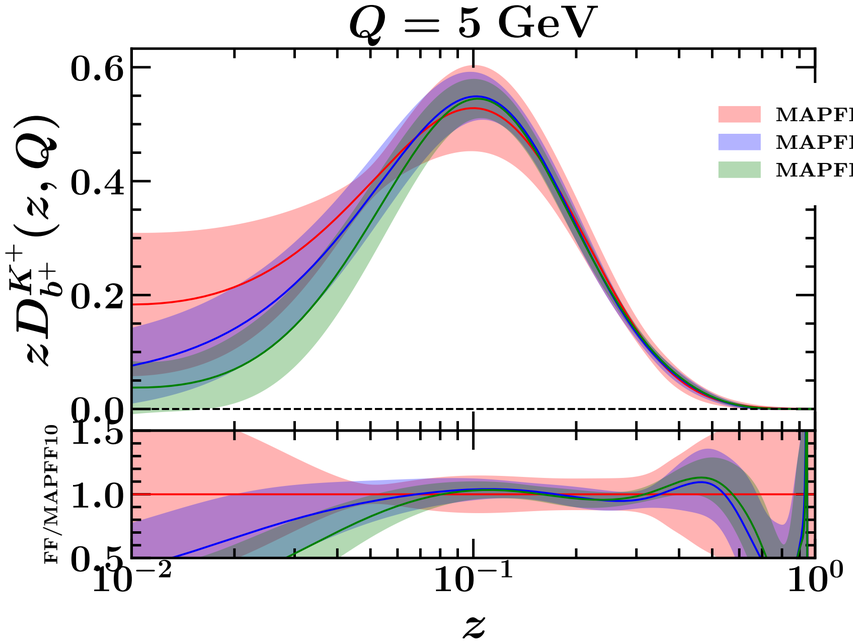
$\mu \pm \sigma$

$Q = 5\,GeV$





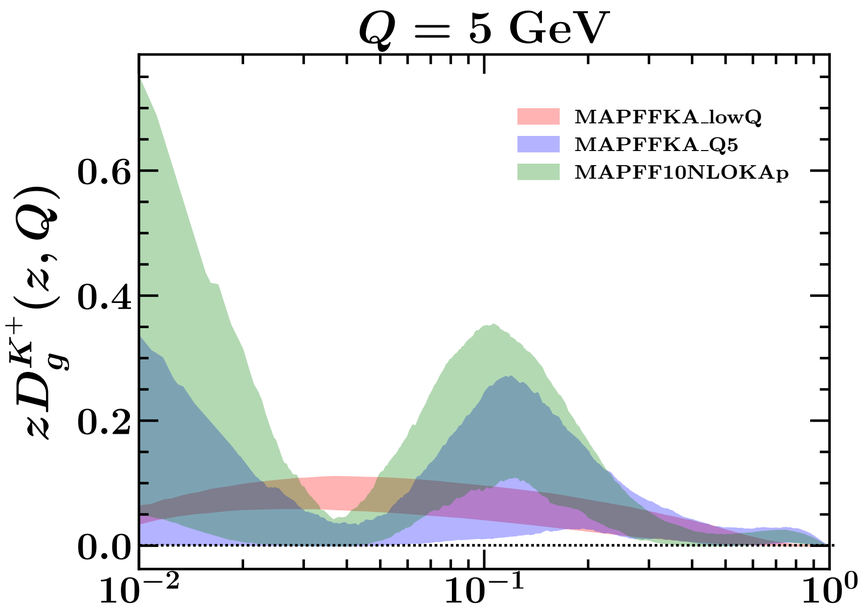


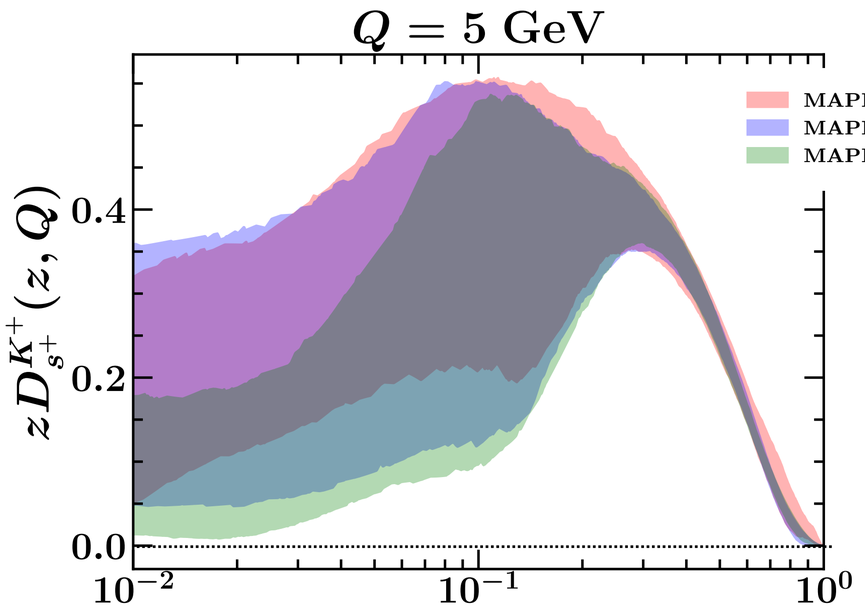


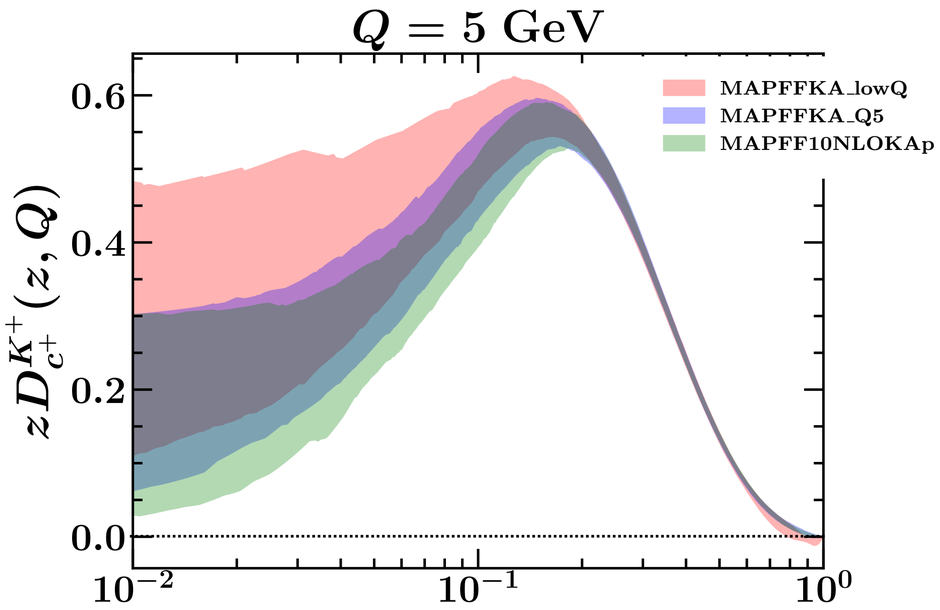
K^+

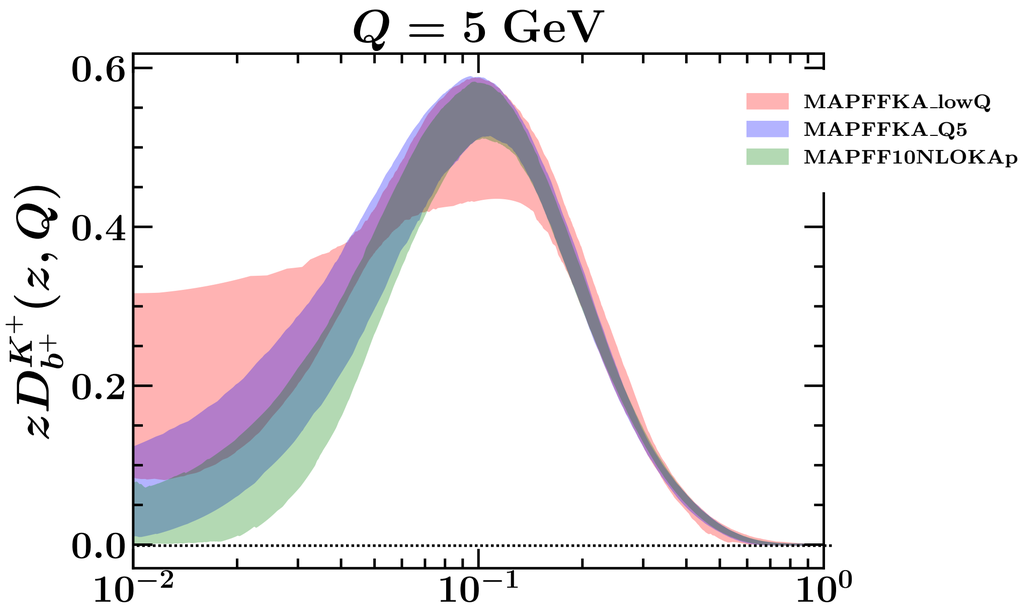
68%

$Q = 5\,GeV$


$Q = 5$ GeV
MAPFFKA_lowQ
MAPFFKA_Q5
MAPFF10NLOKAp
$zD_g^{K^+}(z,Q)$
0.0
0.2
0.4
0.6
10^{-2}
10^{-1}
10^{0}


$Q = 5$ GeV
MAPFFKA_lowQ
MAPFFKA_Q5
MAPFF10NLOKAp
$zD_{s^+}^{K^+}(z,Q)$
0.0
0.2
0.4
10^{-2}
10^{-1}
10^{0}

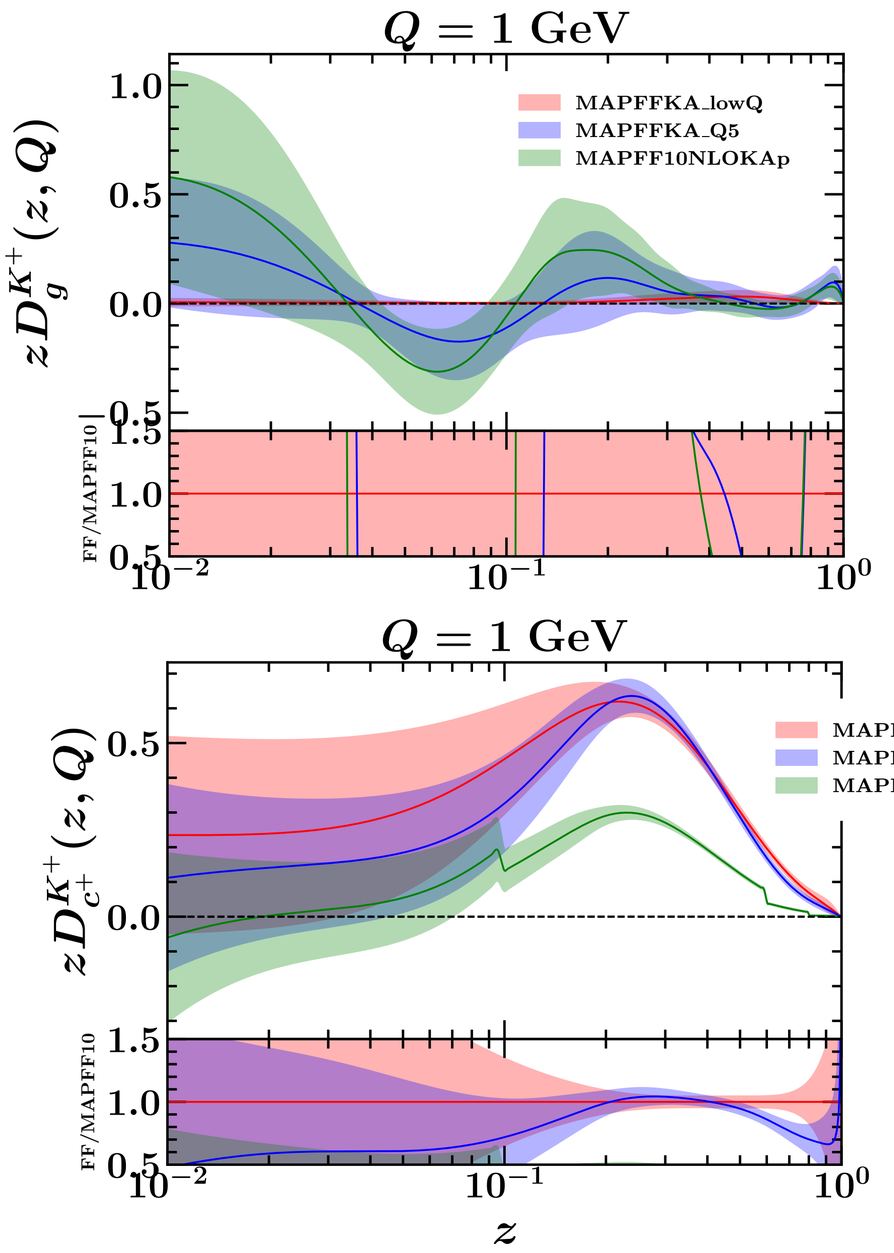

$Q = 5$ GeV
MAPFFKA_lowQ
MAPFFKA_Q5
MAPFF10NLOKAp
$zD_{c^+}^{K^+}(z,Q)$
0.0
0.2
0.4
0.6
10^{-2}
10^{-1}
10^{0}

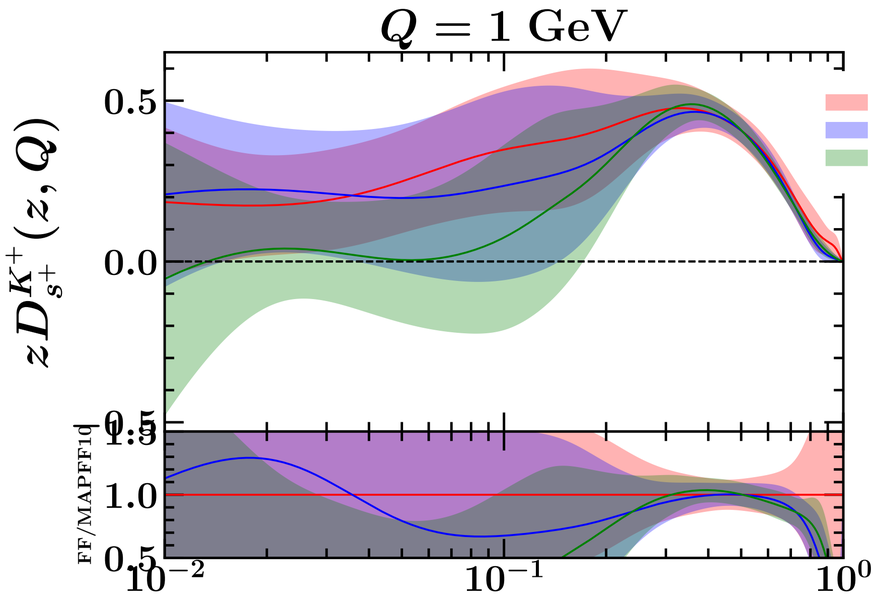

$Q = 5$ GeV
MAPFFKA_lowQ
MAPFFKA_Q5
MAPFF10NLOKAp
$zD_{b^+}^{K^+}(z,Q)$
0.0
0.2
0.4
0.6
10^{-2}
10^{-1}
10^{0}

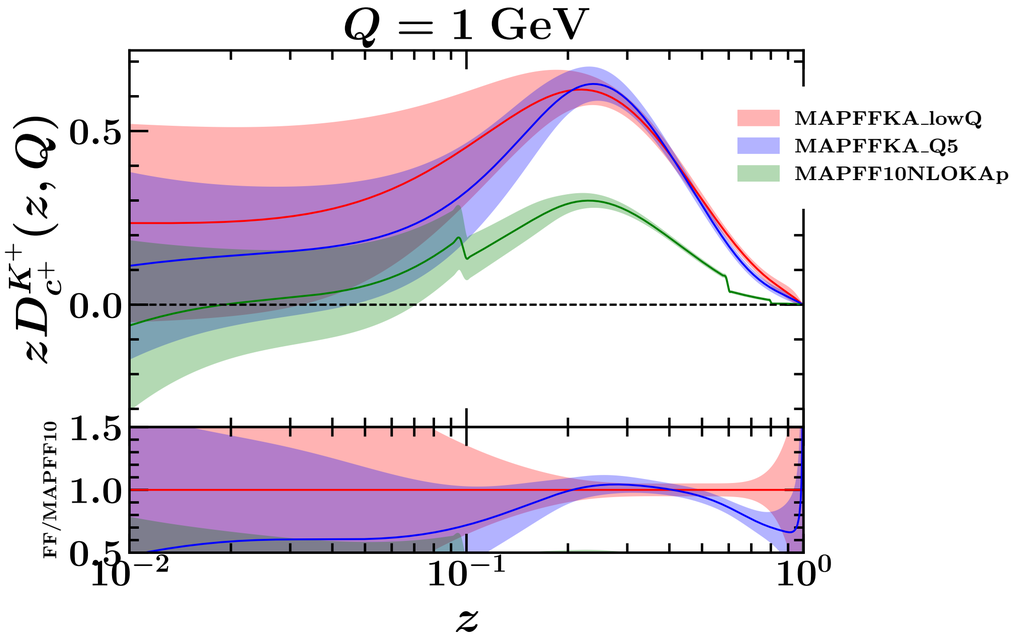
K^+

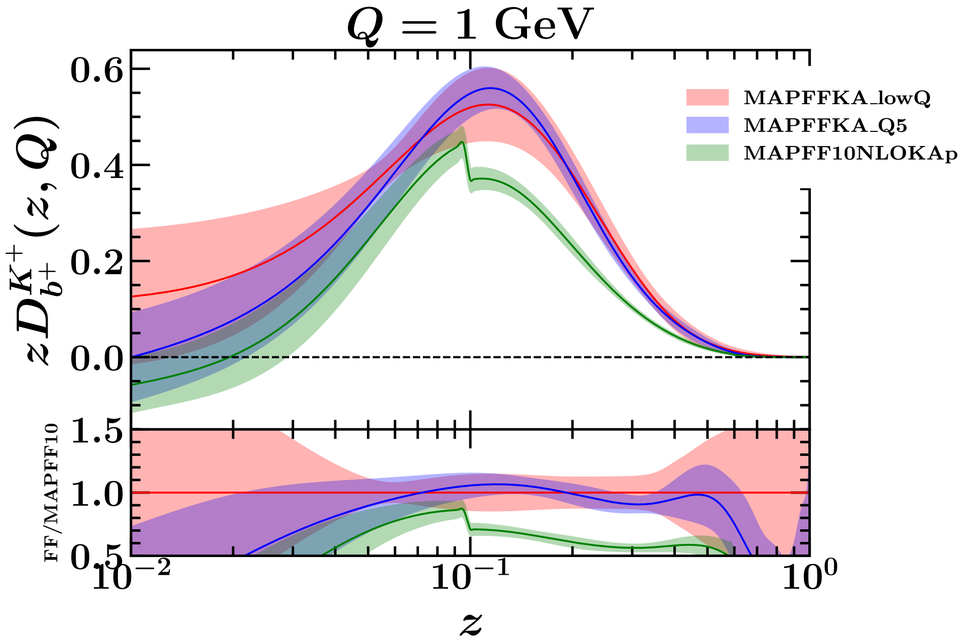
$\mu \pm \sigma$

$Q = 1\,GeV$





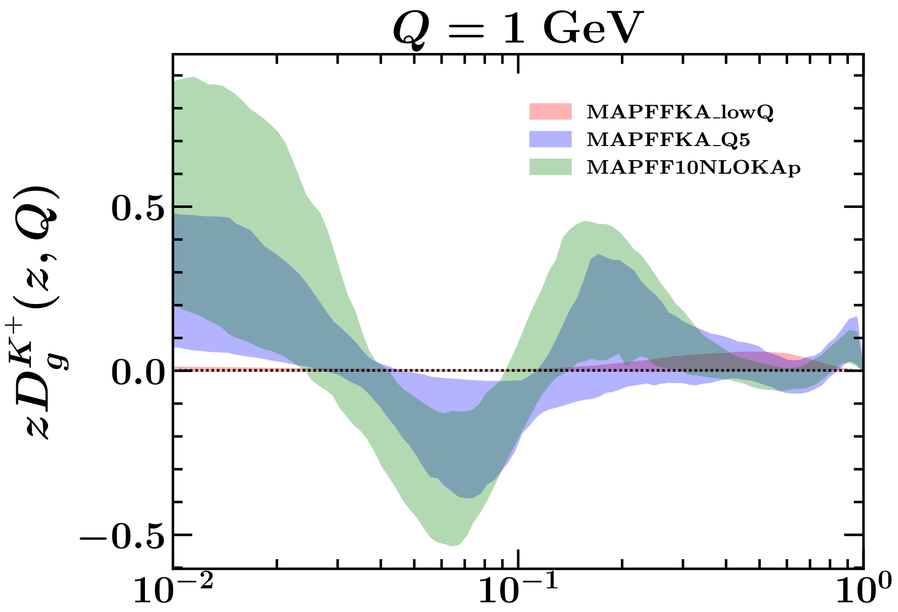


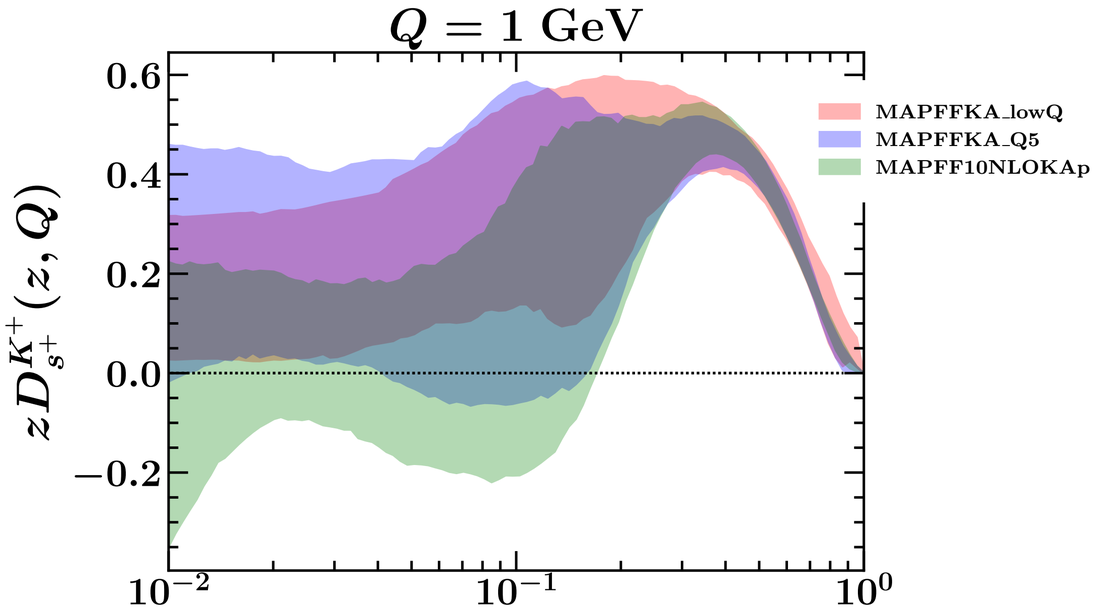


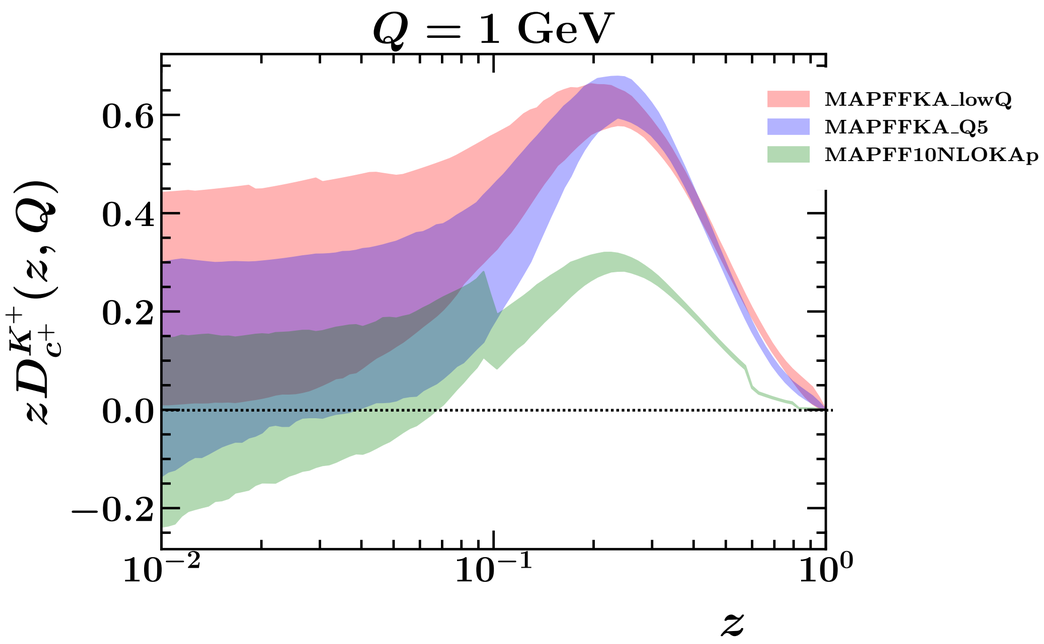
K^+

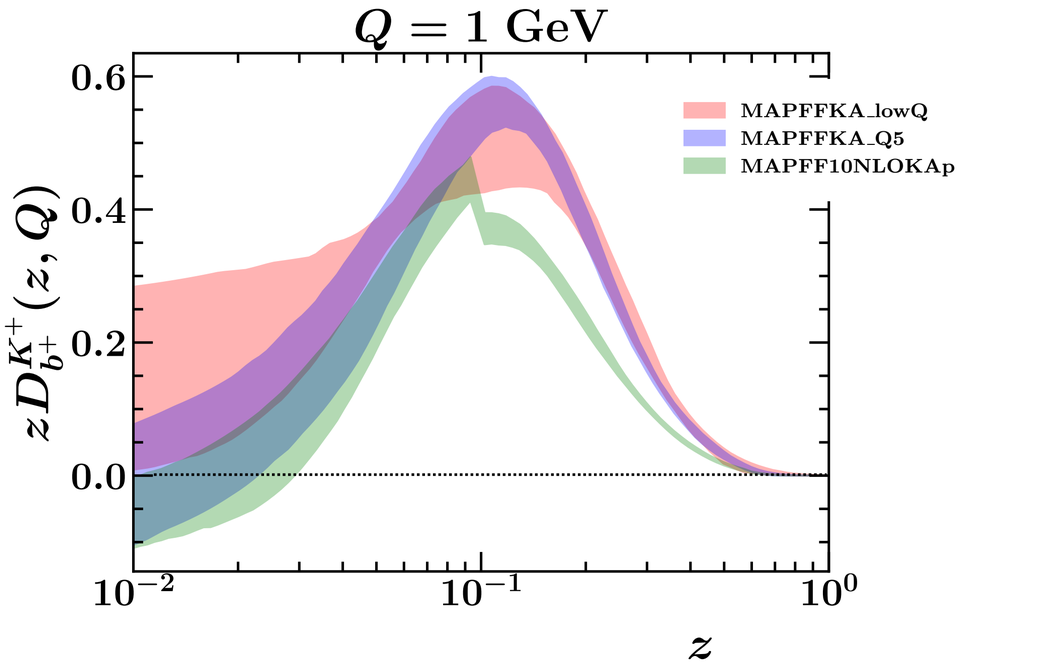
68%

$Q = 1\,GeV$


$Q = 1$ GeV
MAPFFKA_lowQ
MAPFFKA_Q5
MAPFF10NLOKAp
$zD_g^{K^+}(z,Q)$
0.5
0.0
−0.5
10^{-2}
10^{-1}
10^{0}


$Q = 1$ GeV
MAPFFKA_lowQ
MAPFFKA_Q5
MAPFF10NLOKAp
$zD_{s^+}^{K^+}(z,Q)$
0.6
0.4
0.2
0.0
−0.2
10^{-2}
10^{-1}
10^{0}


$Q = 1$ GeV
MAPFFKA_lowQ
MAPFFKA_Q5
MAPFF10NLOKAp
$zD_{c^+}^{K^+}(z,Q)$
0.6
0.4
0.2
0.0
−0.2
10^{-2}
10^{-1}
10^{0}
z


$Q = 1$ GeV
MAPFFKA_lowQ
MAPFFKA_Q5
MAPFF10NLOKAp
$zD_{b^+}^{K^+}(z,Q)$
0.6
0.4
0.2
0.0
10^{-2}
10^{-1}
10^{0}
z

NNLO

Report of D_1 with complete NNLO and with fixed Apfell++ bug

I will show results obtained starting both from $Q_0 = 1$ GeV and $Q_0 = 5$ GeV and compare them with the released D_1.
For the uncertainty band, I will consider both $\mu \pm \sigma$ and the $68\,\%$ band.

I show the results for $Q = 1$ GeV and for $Q = 5$ GeV.

Set to be compared:

π^+

My fit:

MAPFFPI_Q5 -> from $Q_0 = 5\,GeV$ -> 206 replicas

MAPFFPI_lowQ -> from $Q_0 = 1\,GeV$ -> 196 replicas

Released:

MAPFF10NNLOPIp, from $Q_0 = 5\,GeV$ -> 210 replicas

K^+

My fit:

MAPFFKA_Q5 -> from $Q_0 = 5\,GeV$ -> 201 replicas

MAPFFKA_lowQ -> from $Q_0 = 1\,GeV$-> 200 replicas

Released:

MAPFF10NNLOPIp, from $Q_0 = 5\,GeV$ -> 210 replicas

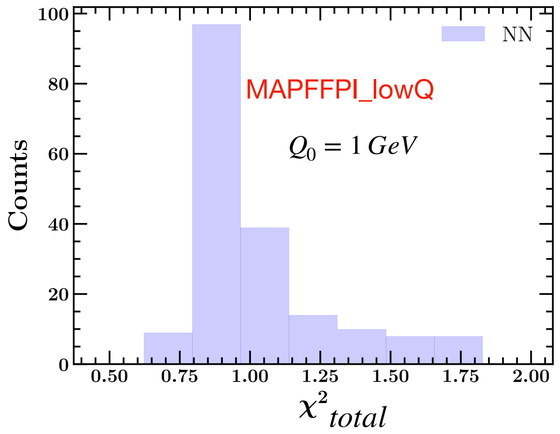
χ_2

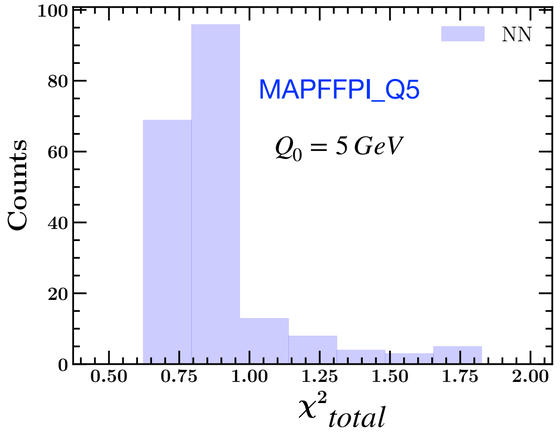
π^+

My fit:

MAPFFPI_Q5 -> from $Q_0 = 5\ GeV$ -> 206 replicas

MAPFFPI_lowQ -> from $Q_0 = 1\ GeV$ -> 196 replicas



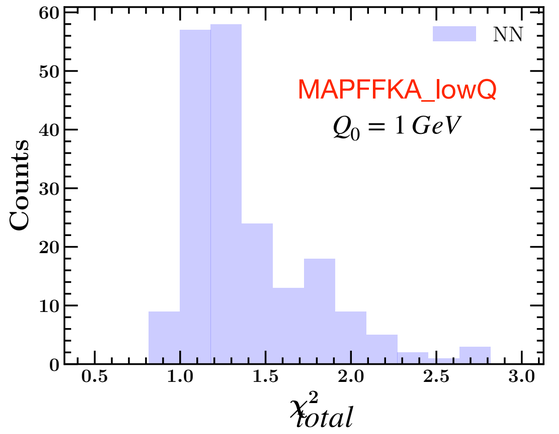


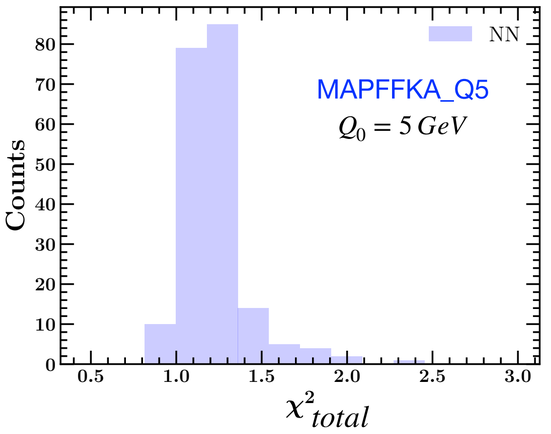
K^+

My fit:

MAPFFKA_Q5 -> from $Q_0 = 5\ GeV$ -> 201 replicas

MAPFFKA_lowQ -> from $Q_0 = 1\ GeV$-> 200 replicas

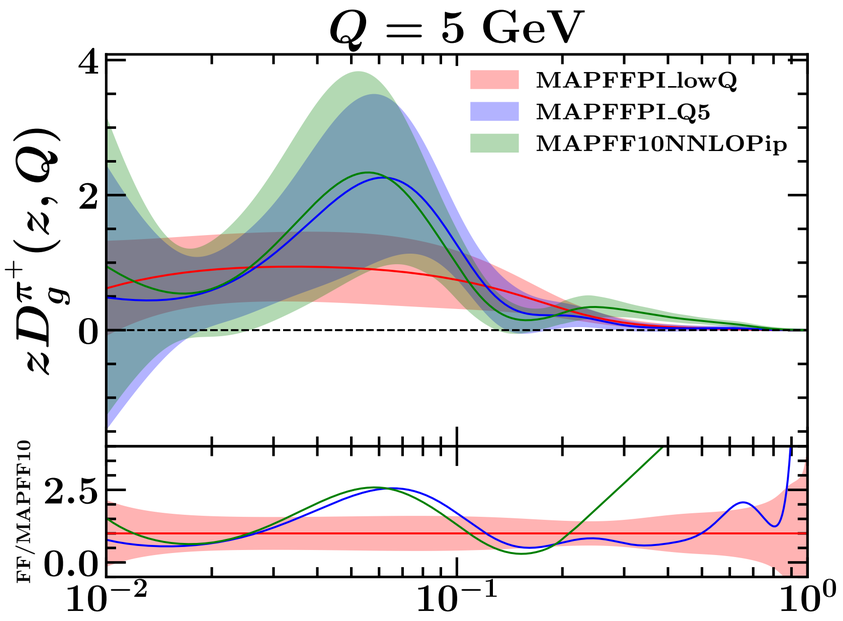


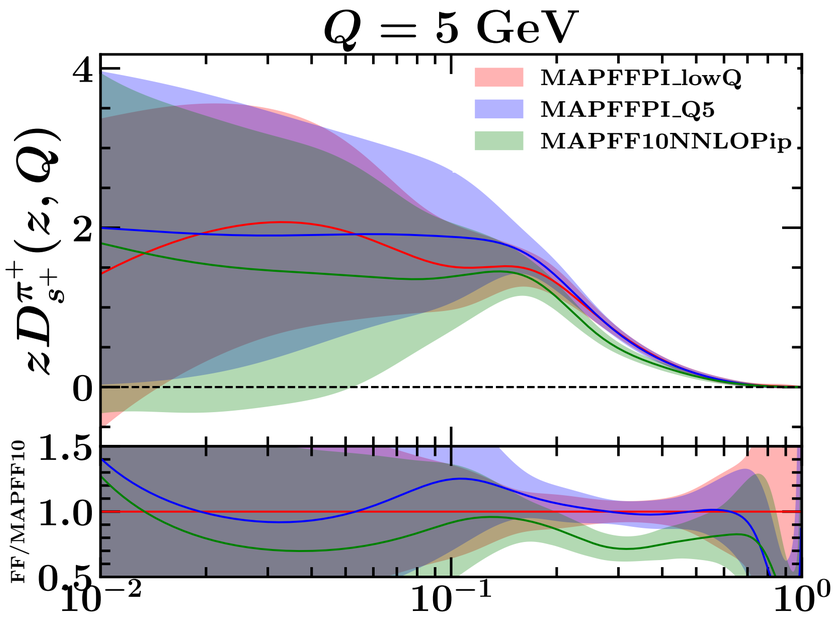


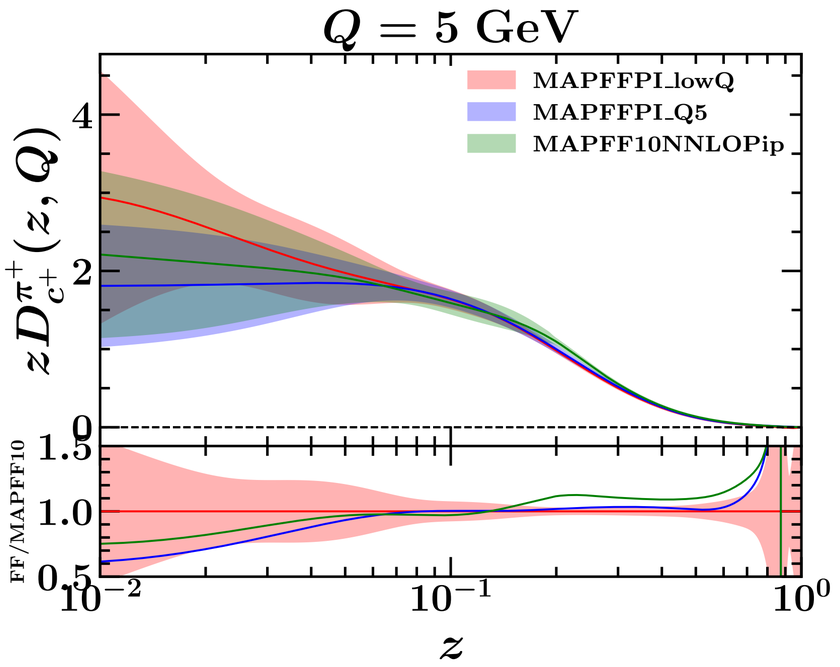
π^+

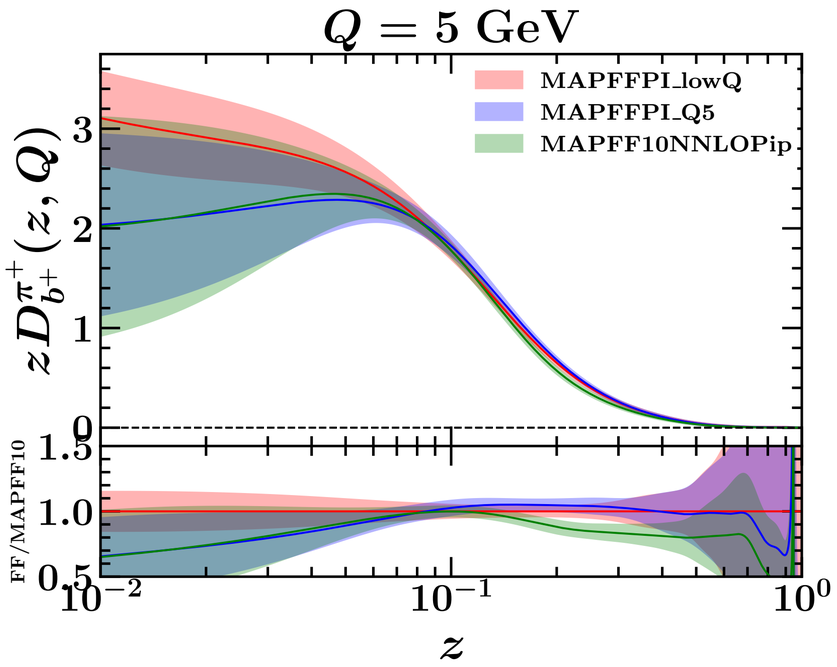
$\mu \pm \sigma$

$Q = 5\,GeV$





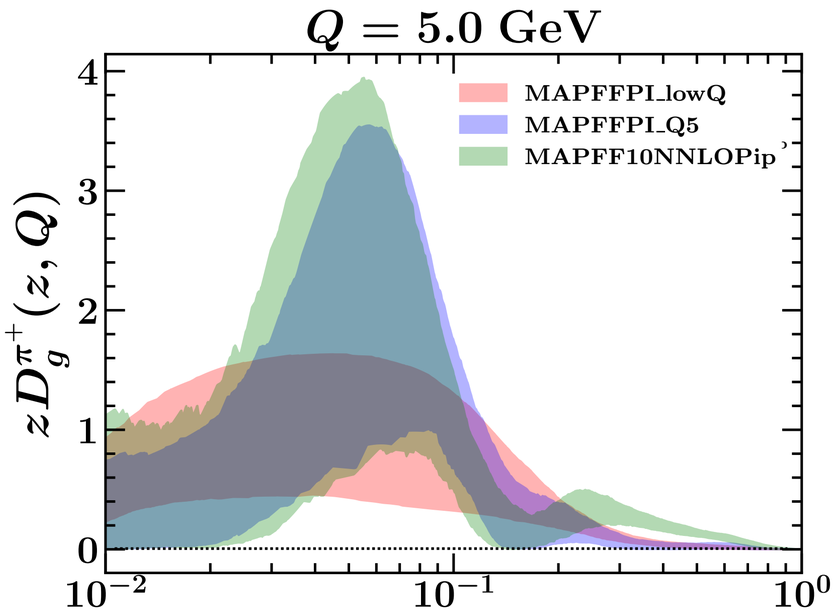


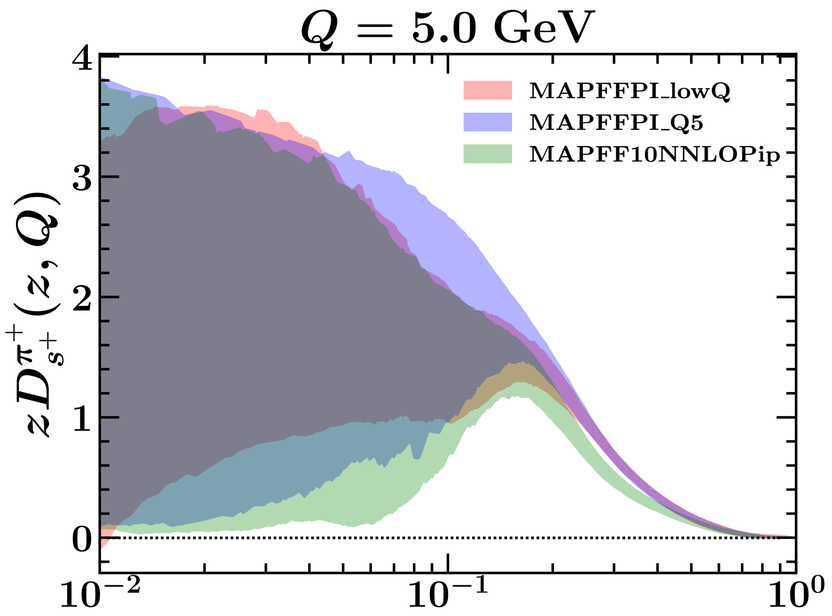


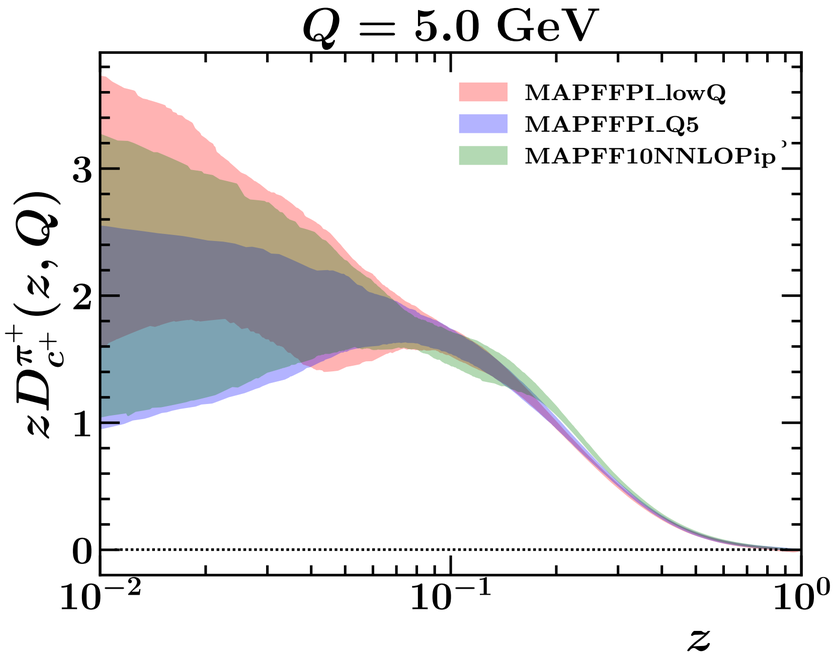
π^+

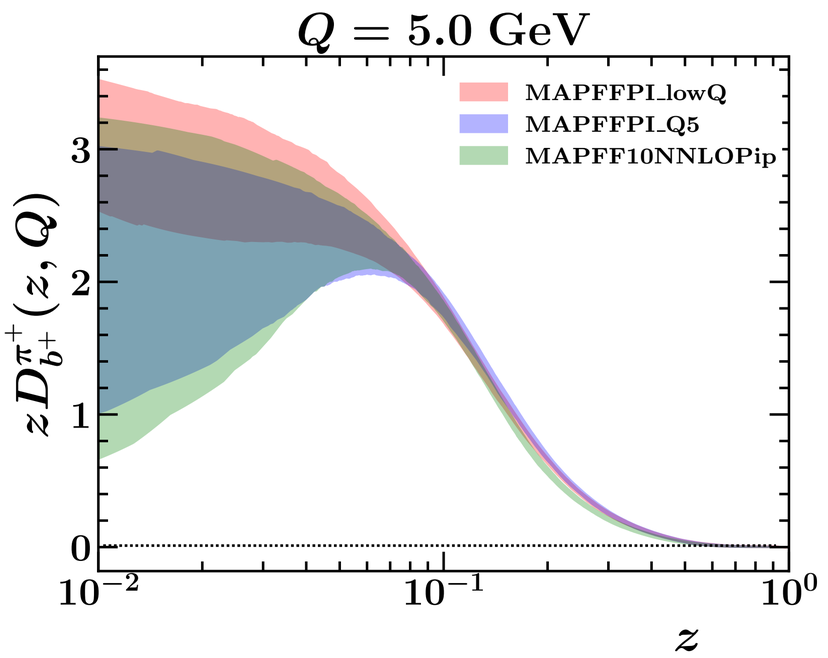
68%

$Q = 5\,GeV$


$Q = 5.0$ GeV
MAPFFPI_lowQ
MAPFFPI_Q5
MAPFF10NNLOPip'
$zD_g^{\pi^+}(z,Q)$


$Q = 5.0$ GeV
MAPFFPI_lowQ
MAPFFPI_Q5
MAPFF10NNLOPip
$zD_{s^+}^{\pi^+}(z,Q)$

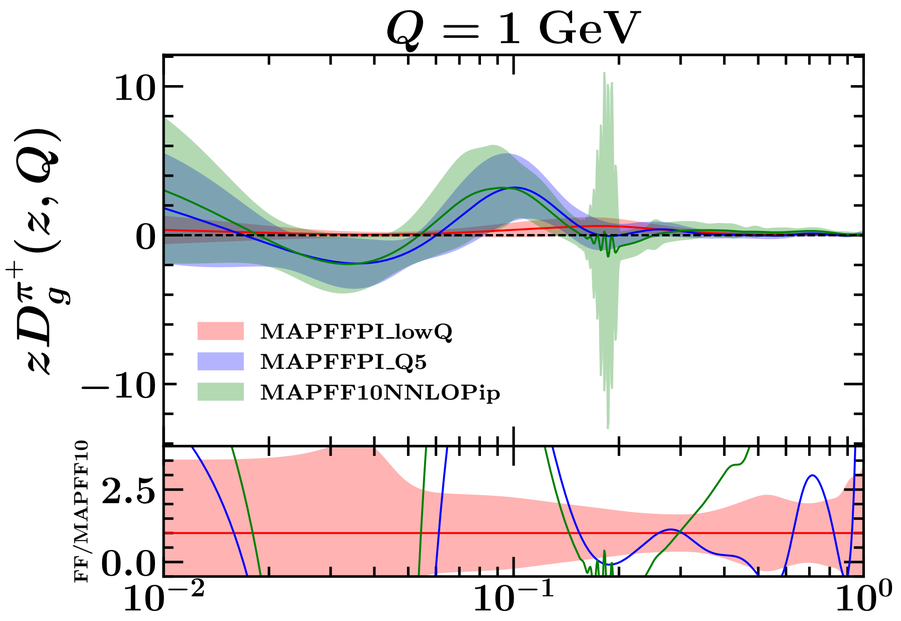

$Q = 5.0$ GeV
MAPFFPI_lowQ
MAPFFPI_Q5
MAPFF10NNLOPip'
$zD_{c^+}^{\pi^+}(z,Q)$
z

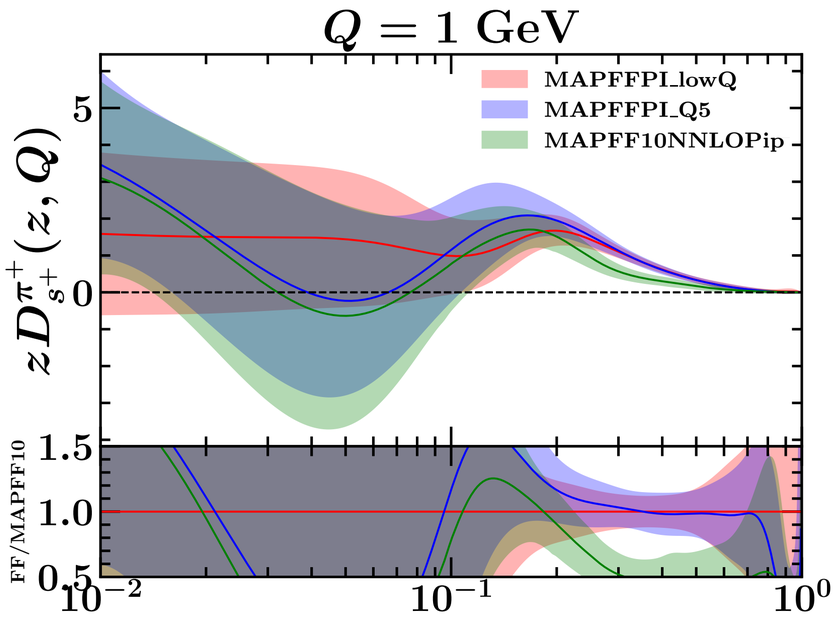

$Q = 5.0$ GeV
MAPFFPI_lowQ
MAPFFPI_Q5
MAPFF10NNLOPip
$zD_{b^+}^{\pi^+}(z,Q)$
z

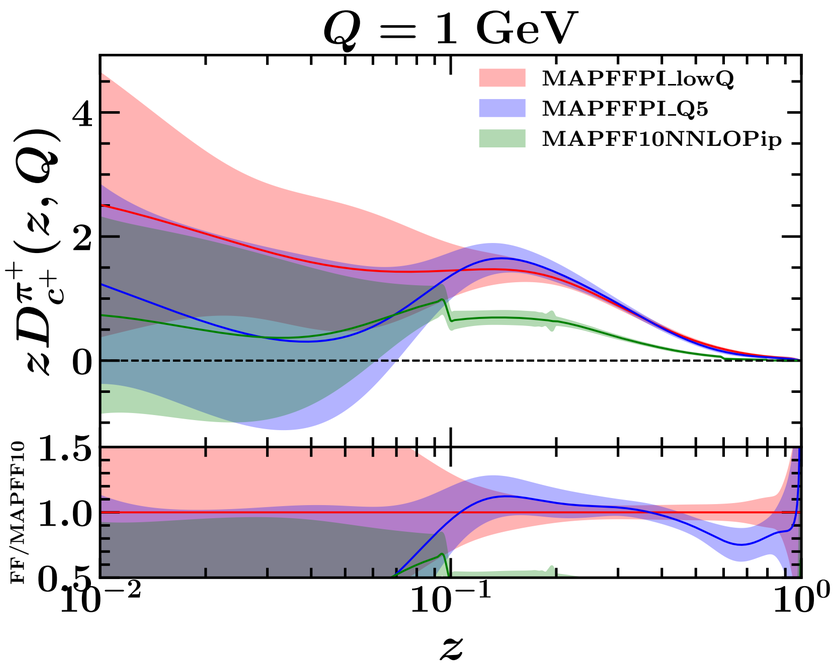
π^+

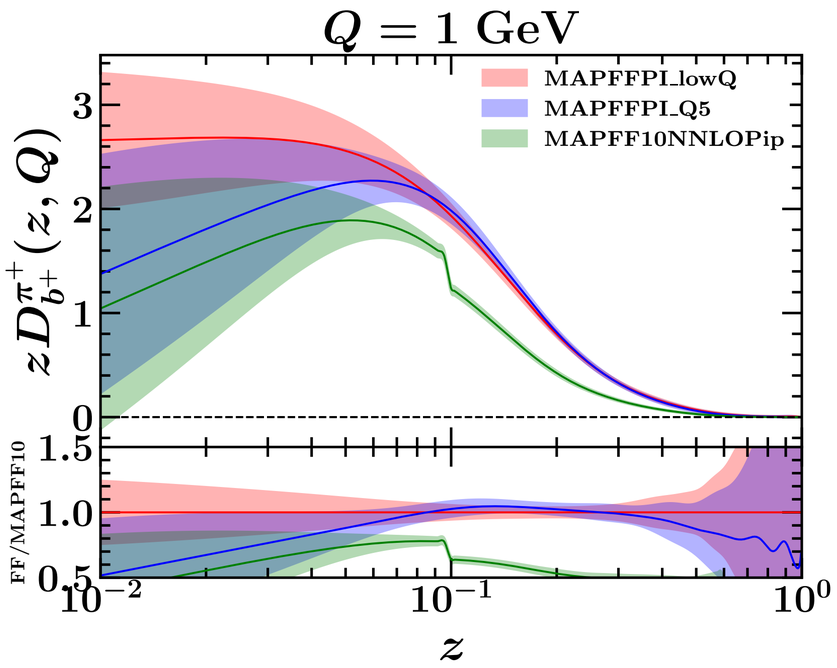
$\mu \pm \sigma$

$Q = 1\,GeV$





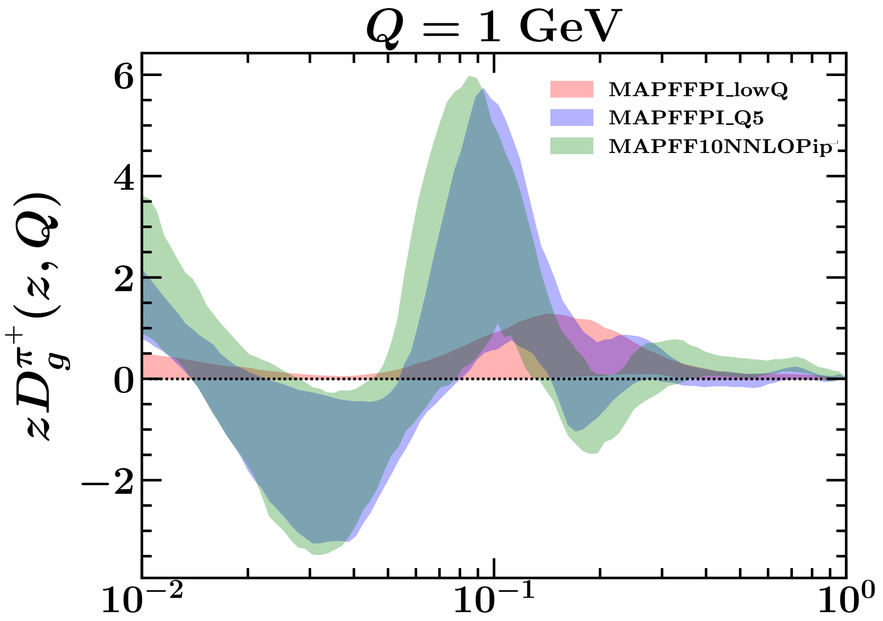


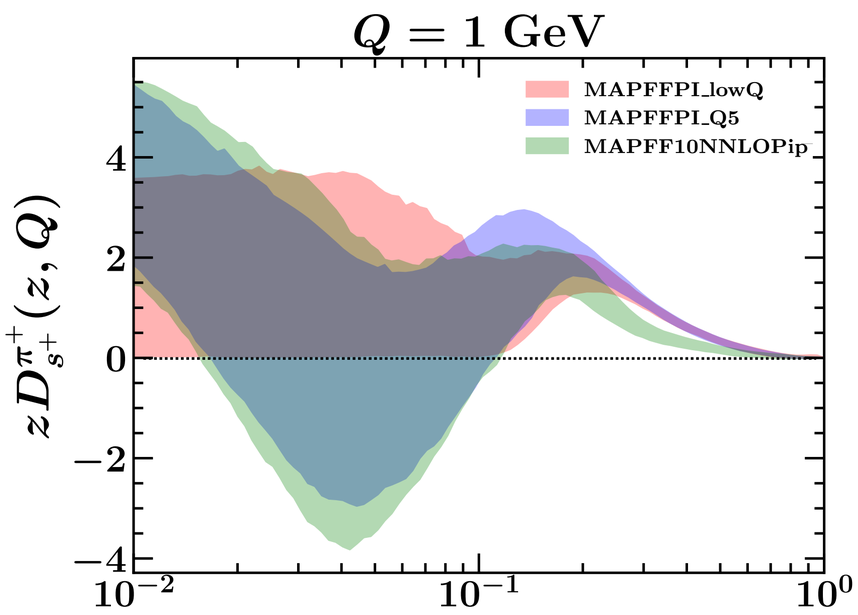


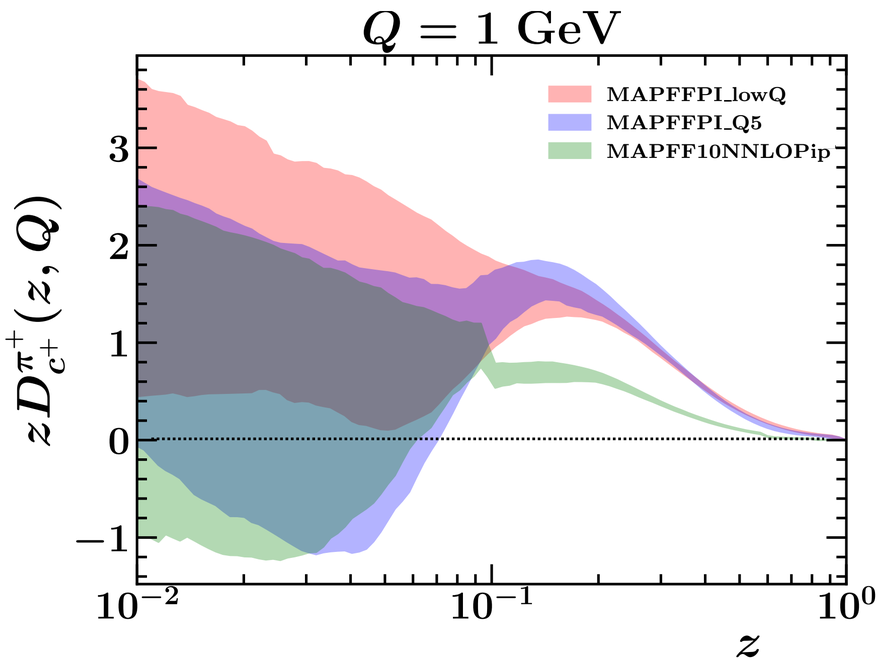
π^+

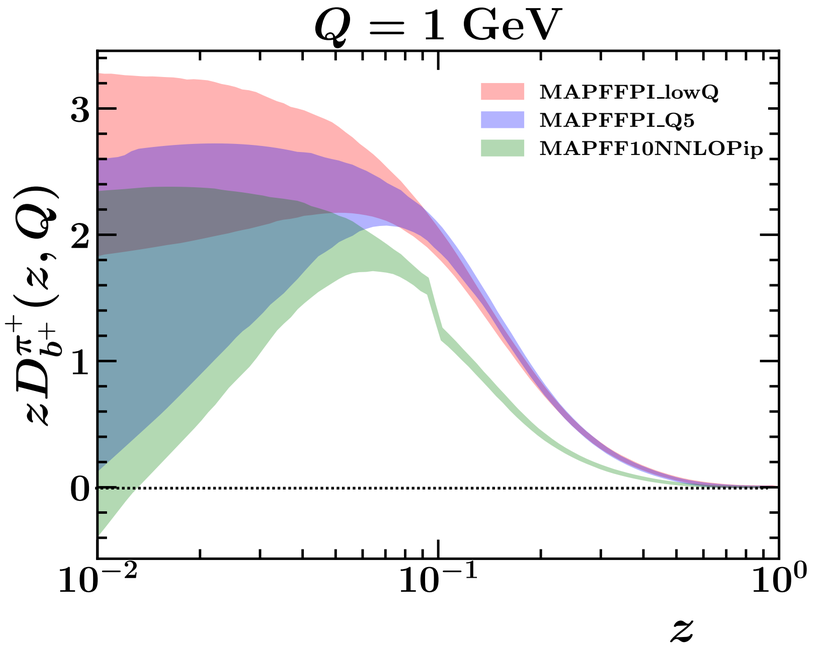
68%

$Q = 1\,GeV$


$Q = 1$ GeV
MAPFFPI_lowQ
MAPFFPI_Q5
MAPFF10NNLOPip
$zD_g^{\pi^+}(z,Q)$


$Q = 1$ GeV
MAPFFPI_lowQ
MAPFFPI_Q5
MAPFF10NNLOPip
$zD_{s^+}^{\pi^+}(z,Q)$

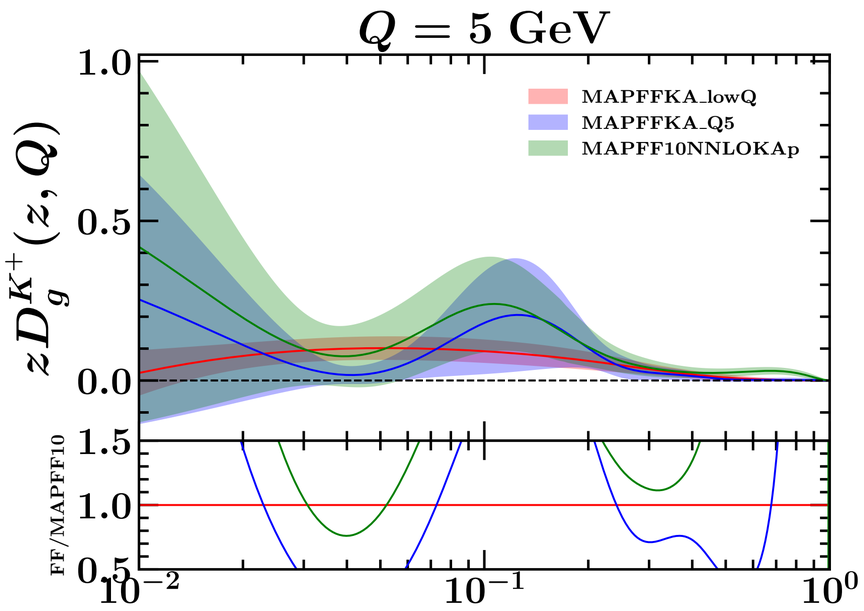

$Q = 1$ GeV
MAPFFPI_lowQ
MAPFFPI_Q5
MAPFF10NNLOPip
$zD_{c^+}^{\pi^+}(z,Q)$
z

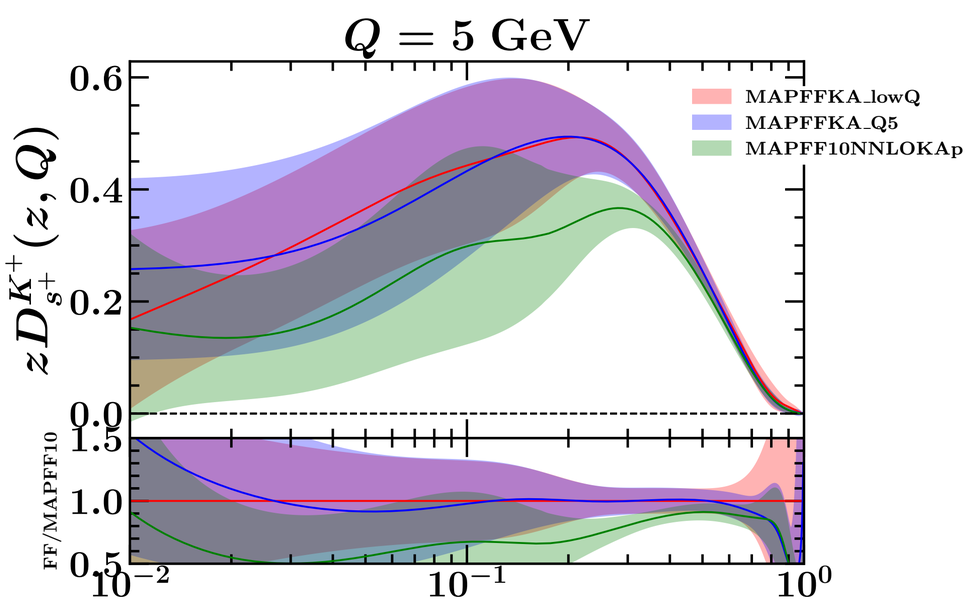

$Q = 1$ GeV
MAPFFPI_lowQ
MAPFFPI_Q5
MAPFF10NNLOPip
$zD_{b^+}^{\pi^+}(z,Q)$
z

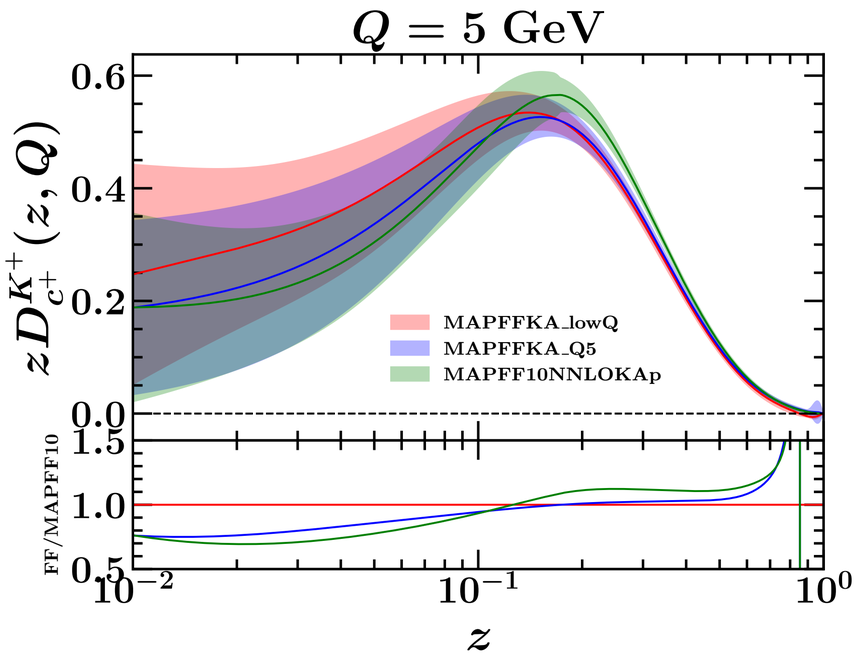
K^+

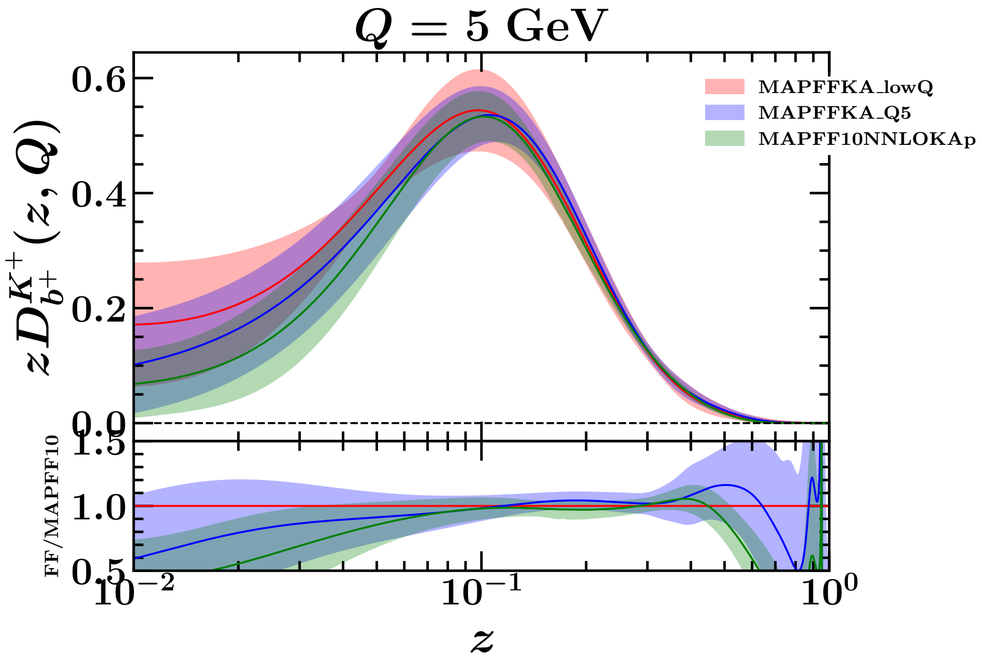
$\mu \pm \sigma$

$Q = 5\,GeV$





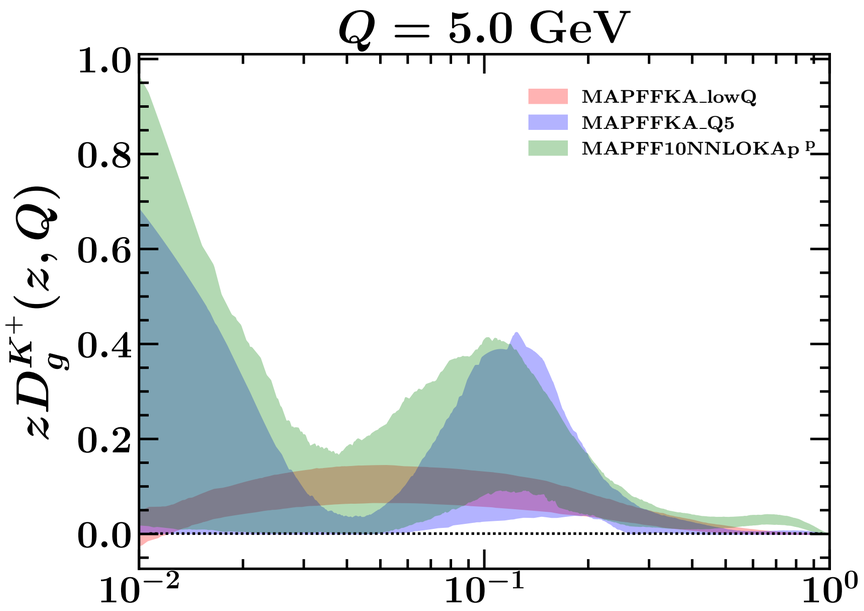


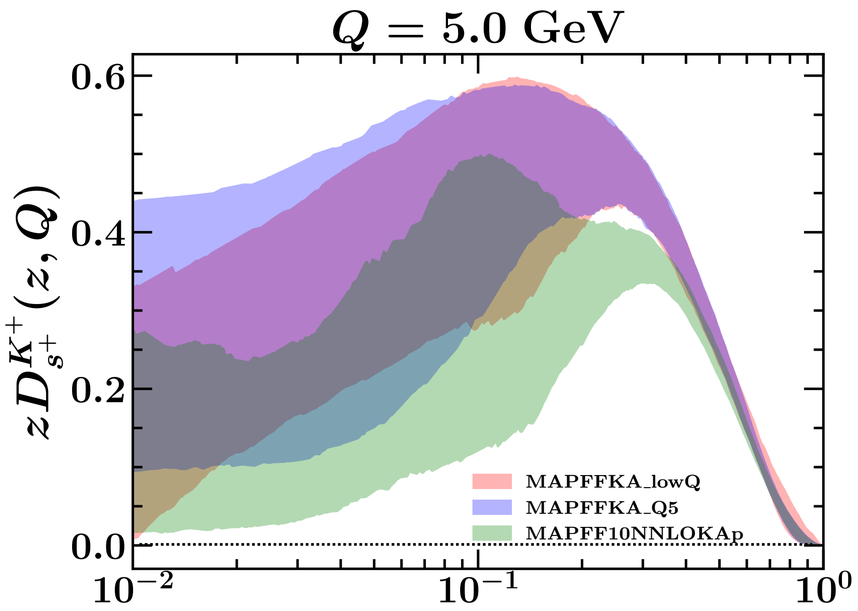


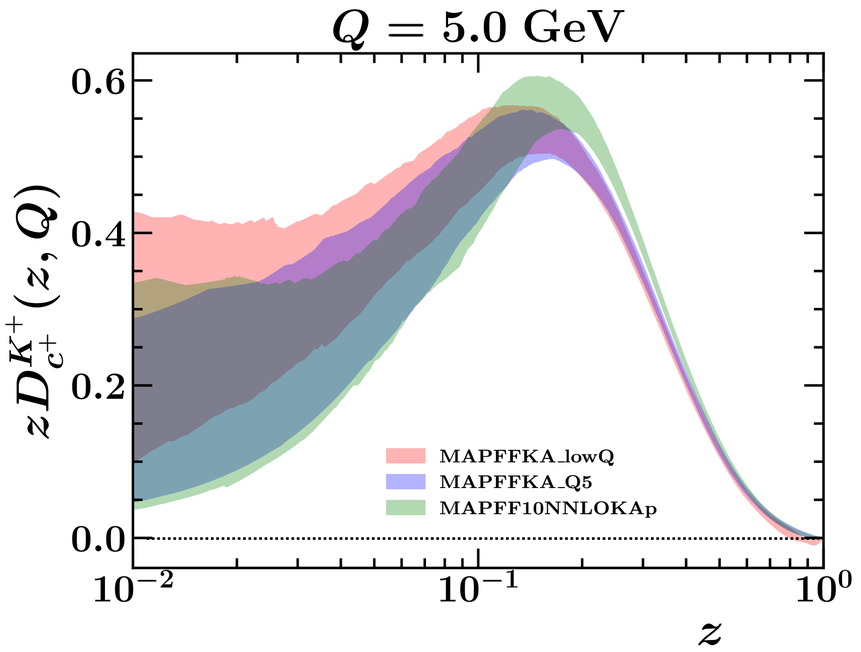
K^+

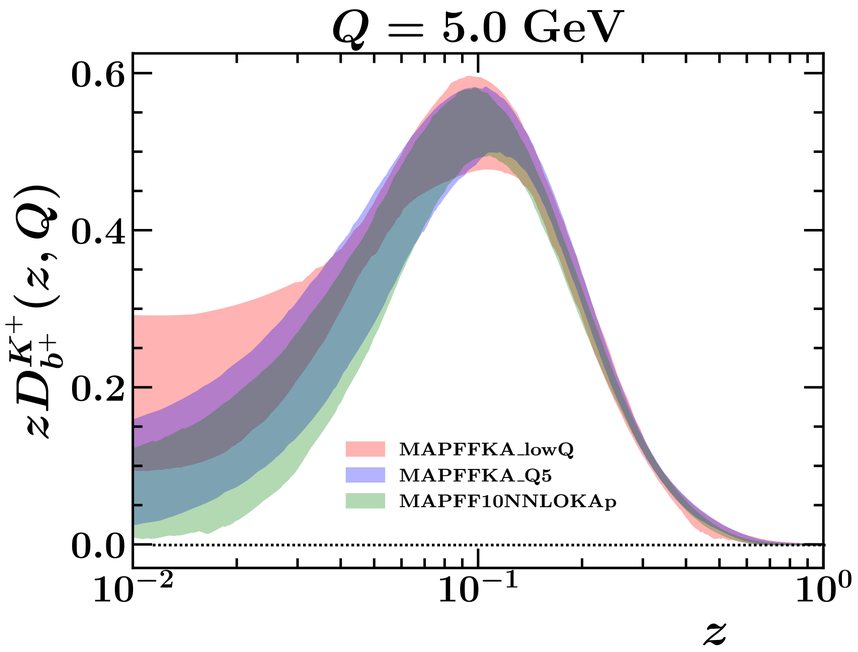
68%

$Q = 5\,GeV$


$Q = 5.0$ GeV
MAPFFKA_lowQ
MAPFFKA_Q5
MAPFF10NNLOKAp
$zD_g^{K^+}(z,Q)$


$Q = 5.0$ GeV
MAPFFKA_lowQ
MAPFFKA_Q5
MAPFF10NNLOKAp
$zD_{s^+}^{K^+}(z,Q)$

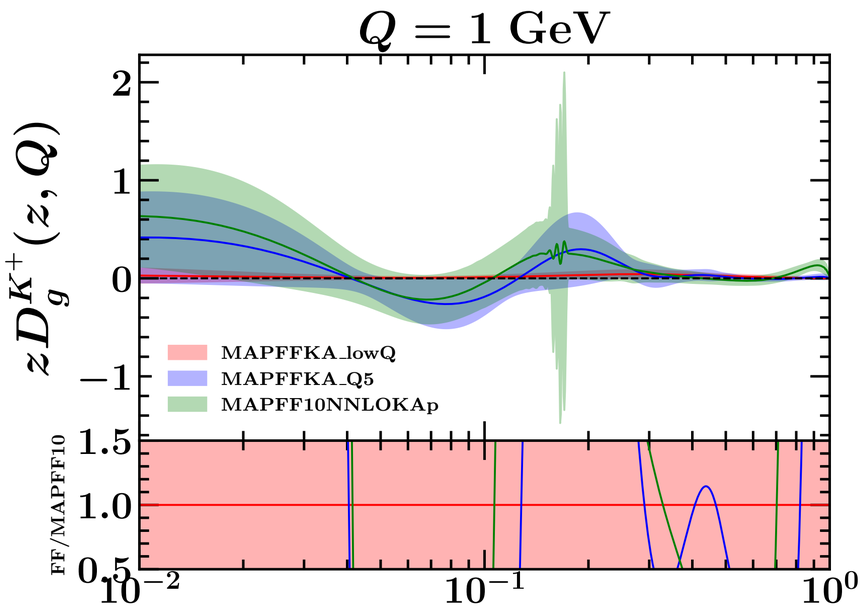

$Q = 5.0$ GeV
MAPFFKA_lowQ
MAPFFKA_Q5
MAPFF10NNLOKAp
$zD_{c^+}^{K^+}(z,Q)$
z

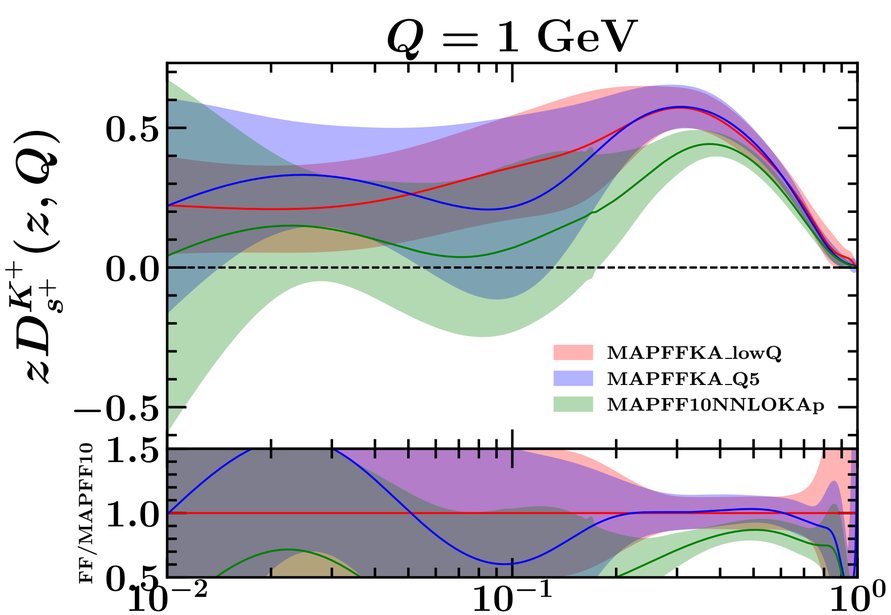

$Q = 5.0$ GeV
MAPFFKA_lowQ
MAPFFKA_Q5
MAPFF10NNLOKAp
$zD_{b^+}^{K^+}(z,Q)$
z

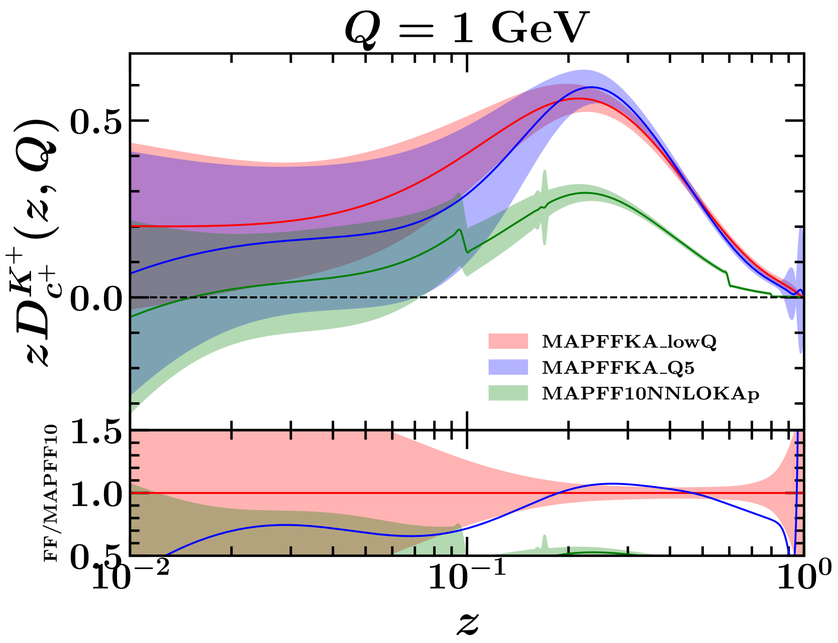
K^+

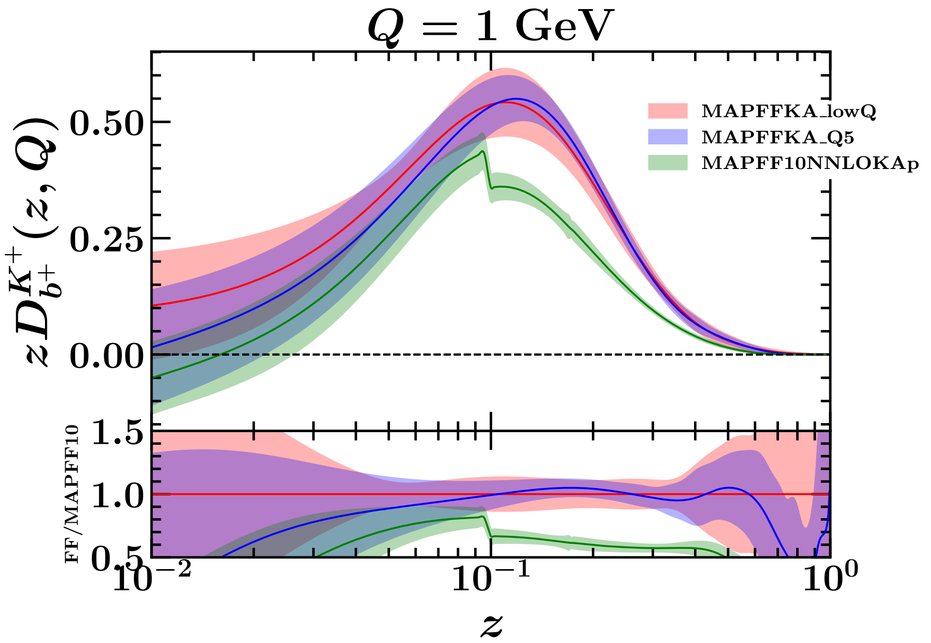
$\mu \pm \sigma$

$Q = 1\,GeV$





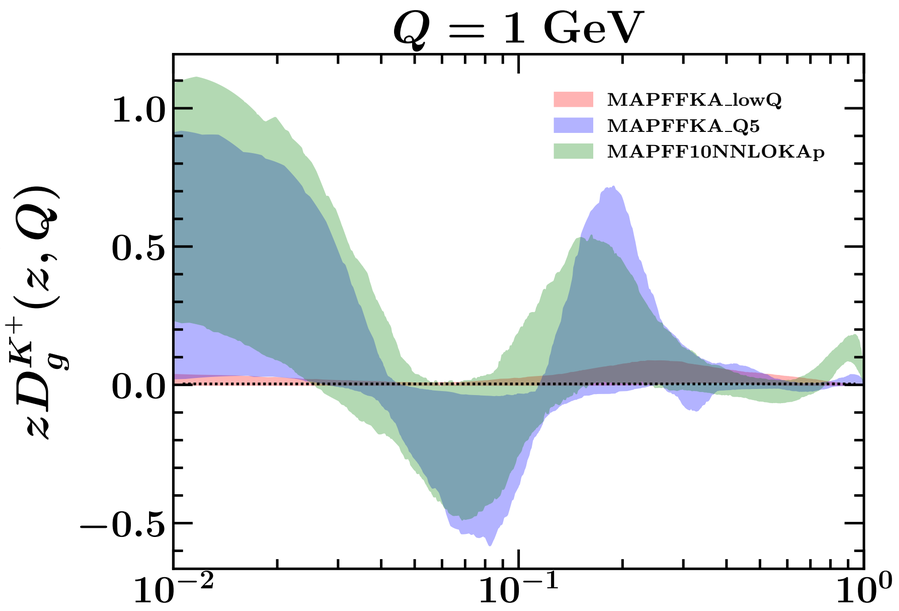


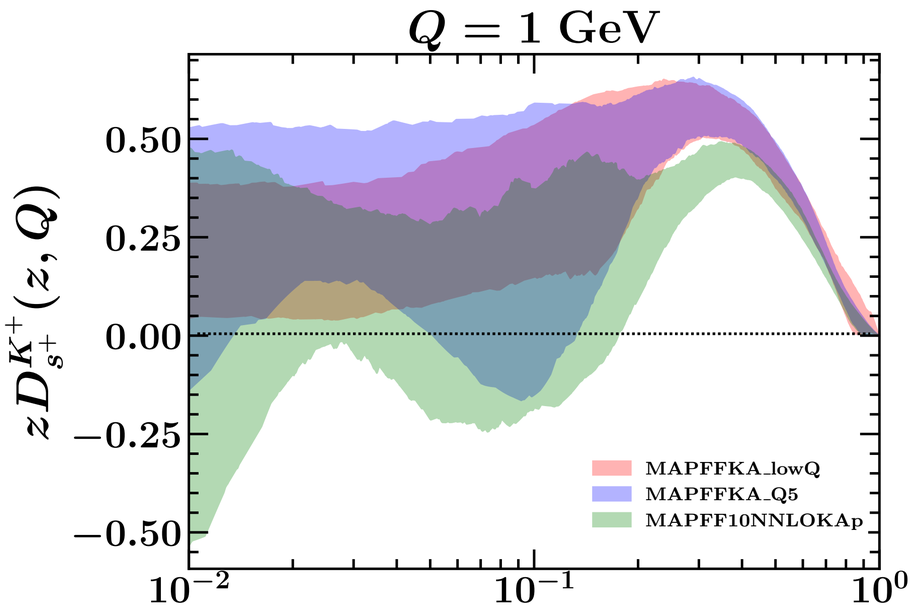


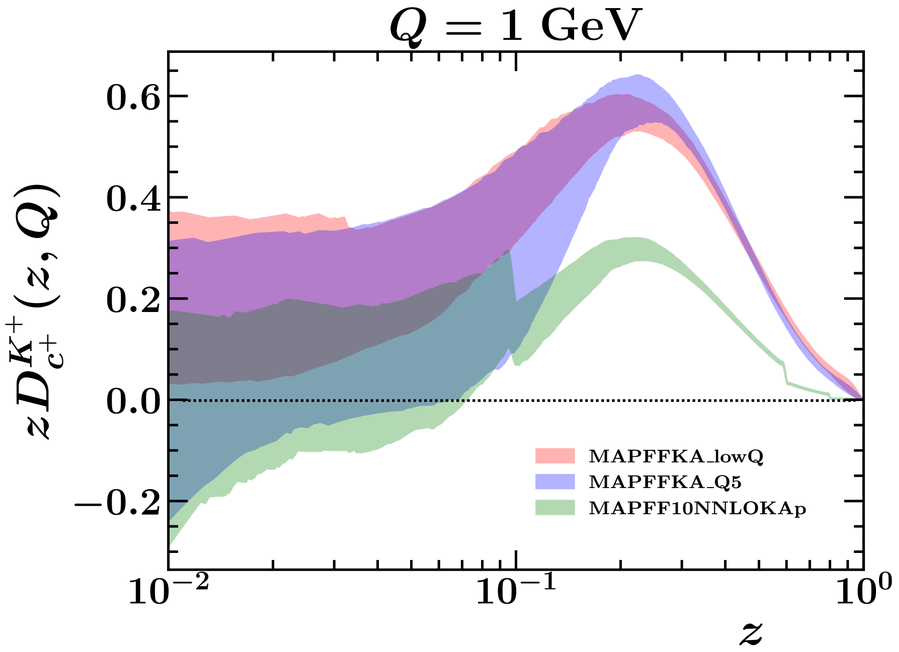
K^+

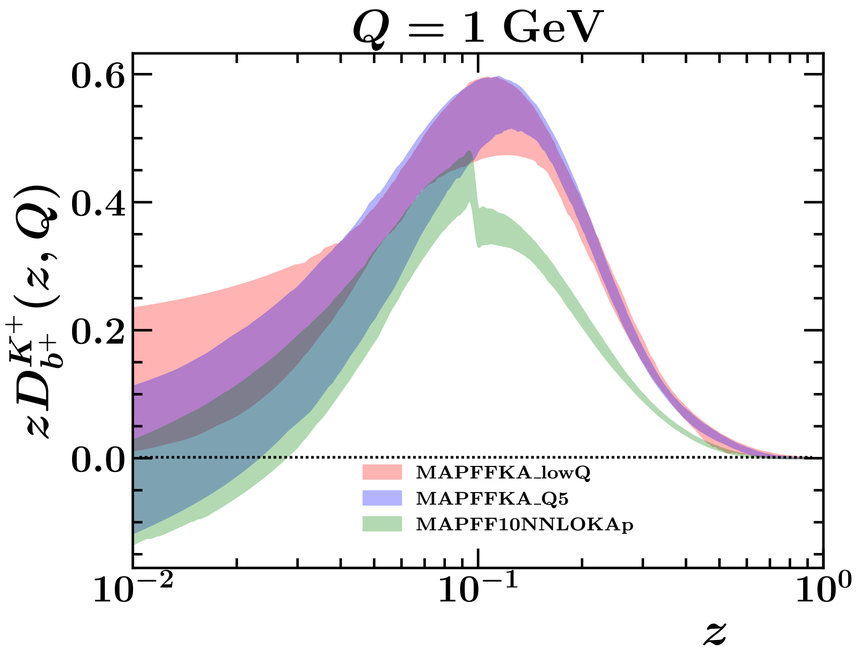
68%

$Q = 1\,GeV$


$Q = 1$ GeV
MAPFFKA_lowQ
MAPFFKA_Q5
MAPFF10NNLOKAp
$zD_g^{K^+}(z,Q)$
1.0
0.5
0.0
−0.5
10^{-2}
10^{-1}
10^{0}


$Q = 1$ GeV
MAPFFKA_lowQ
MAPFFKA_Q5
MAPFF10NNLOKAp
$zD_{s^+}^{K^+}(z,Q)$
0.50
0.25
0.00
−0.25
−0.50
10^{-2}
10^{-1}
10^{0}


$Q = 1$ GeV
MAPFFKA_lowQ
MAPFFKA_Q5
MAPFF10NNLOKAp
$zD_{c^+}^{K^+}(z,Q)$
0.6
0.4
0.2
0.0
−0.2
10^{-2}
10^{-1}
10^{0}
z


$Q = 1$ GeV
MAPFFKA_lowQ
MAPFFKA_Q5
MAPFF10NNLOKAp
$zD_{b^+}^{K^+}(z,Q)$
0.6
0.4
0.2
0.0
10^{-2}
10^{-1}
10^{0}
z