Nonlinear Systems: Theory and Applications

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Sezioni

Lecce, Perugia

Componenti

- Lecce
  - *M. Boiti*
  - *M. Gianfreda*
  - *B. Konopelchenko* (*rappr.naz.*)
  - *G. Landolfi*
  - *L. Martina*
  - *F. Pempinelli*
  - *B.Prinari*
  - *L. Renna*
- Perugia
  - *S. De Lillo*
  - *G. Lupo*

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2. L.D. Landau Inst. Theor. Phys., Moscow, Russia (L. Bogdanov, M. Pavlov)
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5. Department of Mathematics, Montclair State University, NJ, USA (A. Trubatch)
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4. Universita‘ di Milano Bicocca, Italy (F. Magri, G. De Matteis)
5. Dip. Fisica, Univ. Roma III (Roma, Italia) (D. Levi)
6. Dip. Matematica, Universita‘ di Cagliari, Italy (C. Van der Mee, F. Demontis)
7. Dip. Fisica, Universita‘ del Salento, Lecce Italy (S. Zykov)
8. Dip. Matematica, Universita‘ del Salento, Lecce Italy (R. Vitolo)

Extended resolvent and applications

Boiti, Pempinelli

1. Extended Resolvent generalizes the classical resolvent theory of differential operators.
2. E-R can be used to study the **nonlinear integrable evolution equations**, as the Kadomtsev-Petviashvili I and II equations
3. N solitons on a generic background for KPI
4. Progress for KPII (pure N solitons case)
5. Extended Resolvent for the Heat Operator
6. Asymptotic Analysis of the Jost solutions for the Heat Operator

$$\left(u_t + 6uu_x + u_{xxx}\right)_x + 3\sigma^2 u_{yy} = 0, \quad \sigma = 1 \rightarrow KPII, \quad \sigma = \imath \rightarrow KPI$$

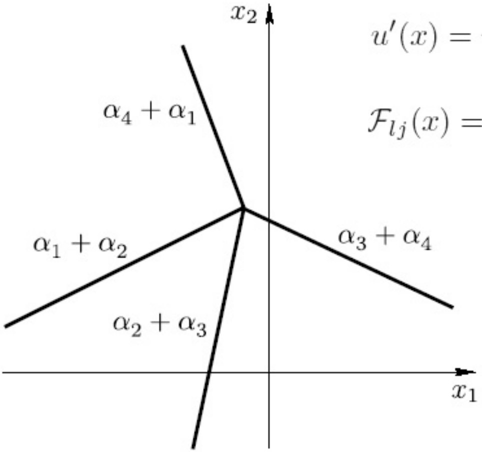
Inverse Spectral Transform

$$\mathcal{L}\left(\vec{x}, \partial_{\vec{x}}\right)\psi \;\; = \left(-\sigma\partial_y + \partial_x^2 - u\left(\vec{x}\right)\right)\psi = 0$$

- M. Boiti, F. Pempinelli and A. K. Pogrebkov: Properties of the solitonic potentials of the heat operator, Theor. and Math. Physics 168, 865-874 (2011)
- M. Boiti *et al.*: Heat operator with pure soliton potential: properties of Jost and dual Jost solutions, Journal of Math. Phys. 52, 083506-1-22 (2011)
- M. Boiti *et al.*: Green's function of heat operator whit pure soliton potential , arXiv :1201.0152
- M. Boiti *et al.*: Extended resolvent of heat operator with multisoliton potential , arXiv :1203.4665 - preprint 2012, to appear in Theor. and Math. Physics

$$\mathcal{L}(x, \partial_x + q) = -\partial_{x_2} - q_2 + (\partial_{x_1} + q_1)^2 - u(x),$$

$$\mathcal{L}(x, \partial_x + q)Mx, x'; q) = \mathcal{L}^{\mathrm{d}}(x', \partial_{x'} + q)M(x, x'; q) = \delta(x - x')$$



$$u'(x) = u(x) - 2\partial_{x_1}^2 \log\det(E_{N_b} + \mathcal{F}c$$

$$\mathcal{F}_{lj}(x) = \frac{e^{(a_j - b_l)(x_1 + (a_j + b_l)x_2)}}{a_j - b_l}$$

Konopelchenko

- B. Konopelchenko, L.Martinez Alonso and E. Medina, On the singular sector of the Hermitian random matrix model in the large N limit, *Physics Letters A*, 375, 867-872 (2011)
- B. Konopelchenko and G.Ortenzi, Birkhoff strata in Grassmannian GR^2. Algebraic curves, *Theor. Math. Phys.*, 167 (3), 448-464, (2011)
- B. Konopelchenko and G.Ortenzi, Gradient catastrophe and flutter in vortex filament dynamics, *J. Phys. A: Math. Theor.*, 44, 432001 (2011)
- B. Konopelchenko and G.Ortenzi, Algebraic varieties in the Birkhoff strata of Grassmannan Gr^2: Harrison cohomology and integrable systems, *J. Phys. A: Math. Theor.*, 44, 465201 (2011)
- B. Konopelchenko, L.Martinez Alonso and E.Medina, Singular sectors of the one-layer Benney and dispersionless Toda systems and their interrelations, *Theor. Math.Phys.*, 168 (1), 963-973 (2011)

Unfolding of singularities and differential equations

Konopelchenko

1. Singular sector of the classical one-layer Benney system and Euler-Poisson-Darboux equation
2. dispersionless Toda equation and large N limit Hermitian Random Matrix Model
3. dispersionless KdV and Hermitian Random Matrix Model
4. Hermitian Random Matrix Model and Euler-Poisson-Darboux equation
5. Gradient Catastrophe and Thom's Catastrophe
6. Versal Unfoldings of critical points for singularities of A,D,E type described by Hamilton-Jacobi type equations
7. Non Versal Unfoldings are equivalent to integrable two-component hydrodynamic systems, like classical shallow water equation and dispersionless Toda system.
8. The corresponding hierarchies are related to the Euler-Poisson-Darboux equation

Algebro-Geometric structure in Sato-Grassmannians

Konopelchenko

1. Maps between algebraic curves and subspaces in Sato Grassmannian
2. Algebraic curves inside Sato Grassmannian
3. Algebraic varieties and curves in Birkhoff strata of Sato Grassmannian
4. Isomorphism among ∞-dim associative algebras and algebraic curves in Birkhoff strata
5. Local properties of families of algebraic subsets W_g in Birkhoff strata S_{2g} of $Gr_1, 2$ containing hyperelliptic curves of genus g
6. Particular subsets in W_g are described by the integrable dispersionless coupled KdV systems
7. Harrison's cohomology of algebraic varieties.
8. Deformations of hyperelliptic curves and the dispersionless KP hyerarchy.
9. Relation among blows-ups of 2-cocycles and 2-coboundaries and gradient catastrophes for associated integrable systems.

Gradient catastrophe and flutter in vortex filament dynamics

Da Rios system

$$\vec{X}_t(s,t) \equiv \vec{v}(s,t) = \vec{X}_s \wedge \vec{X}_{ss} = K(s,t)\vec{b}$$

$$K_t = -2K_s\tau - K\tau_s,$$
$$\tau_t = KK_s - 2\tau\tau_s + \left(\frac{K_{ss}}{K}\right)_s.$$

dispersionless Da Rios system

$$K_y = -2K_x\tau - K\tau_x,$$
$$\tau_y = KK_x - 2\tau\tau_x.$$

1. This phenomenon is in the gradient catastrophe for the dispersionless Da Rios system.
2. At the catastrophe the curvature remains finite while the of osculating curvature blows up to infinity
3. It is an elliptic umbilic singularity
4. It is described by the Painlevé - I equation.

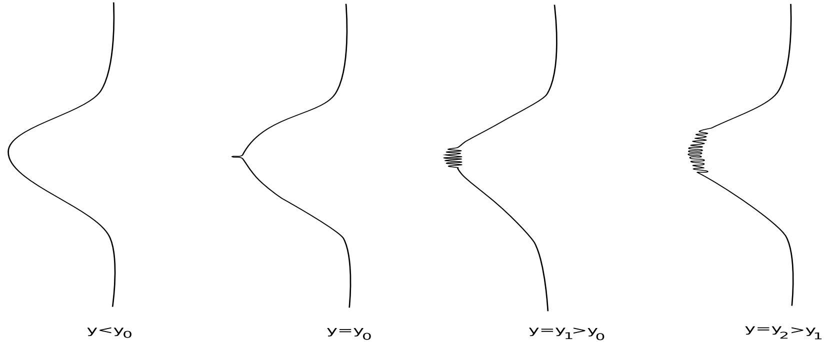


Figure 3: Typical behavior of a flutter vortex line.

Integrability and Symmetries in Quantum Mechanics

Landolfi Gianfreda : Motivations

- Growing attention attracted by QM, both in its standard and its non-standard fourmulations, such as PT-symmetric or SUSY-symmetric.
- Problem of constructing solutions to operator equations for observables of interest (e.g., the formal integration of the operator equation for the "C" operator in PT-QM theoriee or of the Heisenberg evolution equation).
- Problem of the description of dynamical features (e.g. statistical/entanglement effects).

1. C.M. Bender and M. Gianfreda, *Matrix representation of the time operator*, J. Math. Phys. 53, 062102 (2012);
2. M. Gianfreda and G. Landolfi, *Spectral problems for the Weyl-ordered form of operators* $\frac{1}{\hat{p}^n} q^n$, J. Math. Phys. 52, 122104 (2011)

Statistical and entanglement properties of quantum systems

M. Gianfreda, G. Landolfi, L. Martina

- Entanglement quantification and witness for systems of infinite-dimensional Hilbert spaces.
- Entanglement Witness and Correlation energy for completely solvable models [1]
- Statistical properties (correlations, Wigner functions, ...) of non autonomous systems in standard and SUSY-QM. In particular:
  - features of coherent type states for SUSY partners of Paul trap Hamiltonians [2]
  - features of states with constant position-momentum correlations for non autonomous quadratic systems [3]

1. L. Martina *et al.* , Int. J. Quant. Inf. **6**, n. 3 (2011), 766
2. M. Gianfreda and G. Landolfi, *Wave packets and statistics concerned with SUSY-QM partners of Paul trap Hamiltonians*, Theor. Math. Phys., 168, 924 (2011).
3. M. Gianfreda and G. Landolfi, *O*n the Existence and Robustness of Steady $p - q$ Correlations for Time-Dependent Quadratic Systems , Advan. Math. Phys, vol 2012, 731602 (2012); doi:10.1155/2012/731602.

Inverse Scattering Transform: extensions and Applications

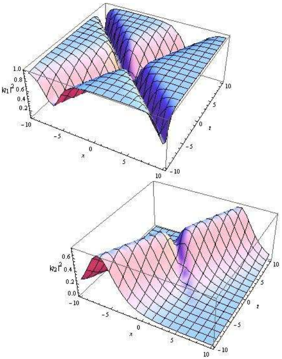
Prinari

B Prinari, G Biondini and A D Trubatch: Inverse scattering transform for the multicomponent nonlinear Schrödinger equation with nonzero boundary conditions at infinity , Stud. Appl. Math. 126, 245-302 (2011).

G. Dean, T. Klotz, B. Prinari, F. Vitale: Dark-dark and dark-bright soliton interactions in the two-component defocusing nonlinear Schrödinger equation, Applic. Anal. (2012), doi: 10.1080/00036811.2011.618126

1. IST for defocusing V-NLS equation with nonvanishing boundary conditions.
2. Dark-dark and dark-bright soliton interaction for 2-NLS.
3. Asymptotic states for solitons of the 2-NLS and N-NLS equation
4. NLS in non euclidean spaces and discretized time
5. Dispersive shock waves and NLS with discontinuous initial data.
6. IST for coupled Maxwell - Bloch systems

$$i\mathbf{q}_t = \mathbf{q}_{xx} - 2\sigma \|\mathbf{q}\|^2 \mathbf{q}$$

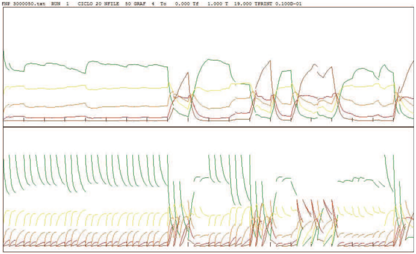


Analysing quality with Generalized Kinetic Methods

M. Lo Schiavo, B. Prinari and A.V. Serio: Mathematical modeling of quality in a medical structure: a case study, *Math. Comp. Model.* 54 (2011) 2087Ð2103.

1. Generalized kinetic models represent a descriptive tool in the area of the social sciences.
2. Time evolution of a global variable - ÒatmosphereÓ - related to the quality of a complex system such as a medical ward.
3. Individuals of the same population are identical, and only addressed to by a state variable denoting their activity
4. Activity is assumed to be a scalar random variable over some (hidden) measure space of elementary events
5. The probability density functions $f_i : (t, u) \in [0, T] \times I_i \to f_i(t, x) \in [0, \infty[$ refer on how the individuals of each population i = 1, 2 are distributed with respect to their state variable

$$\partial_t f_i + \partial_u \Phi_i[\mathbf{f}] = \mathcal{G}_i[\mathbf{f}] - \mathcal{L}_i[\mathbf{f}]\,, \qquad i = 1, 2.$$



Chaotic systems and applications

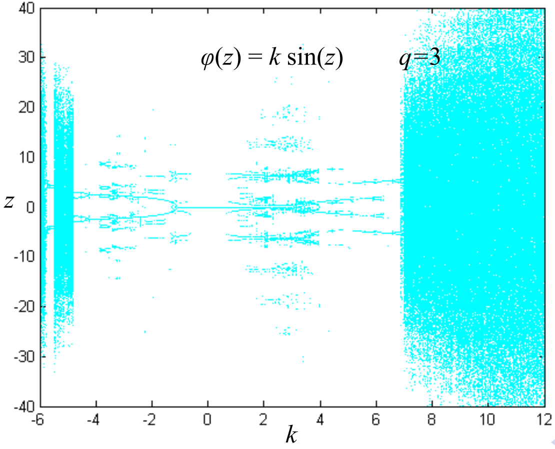
Renna

1. Qualitative behavior of a periodically kicked mechanical oscillator, with damping.
2. Numerical analysis with (i) sinusoidal and (ii) Gaussian pulses
3. Forcing symmetry and resonance symmetry dominance
4. Climate change detection by use of bayesian approaches.

L. Renna, F. Paladini, Theor. Math. Phys. 168 (2011) 1010-1019

$$z_{n+1} = [bw_n + \varphi(z_n)]\sin\alpha + z_n \cos\alpha$$
$$w_{n+1} = [bw_n + \varphi(z_n)]\cos\alpha - z_n \sin\alpha$$

$\phi(z) = \frac{F(z)}{\sin\alpha}$ $\alpha_{ris} = \frac{2\pi}{q}, \quad q \in \mathbb{N}$



Nonlinear propagation and diffusion in rods and polymers

- V.Barone,S. De Lillo and A.Polimeno: Dirichlet-to-Neumann Map for a Nonlinear Diffusion Equation, *Studies in App.Math.* 126, 145-155 (2011)
- D.Burini and S.De Lillo, Nonlinear heat diffusion under impulsive forcing, *Mathematical and Computer Modelling* 55,269-277 (2012)
- M.J.Ablowitz,V.Barone,S.De Lillo and M.Sommacal, Travelling waves in elastic rods featuring arbitrare curvature and torsion, *Journal of Nonlinear Science* (accepted for publication) (2012)

1. A Dirichlet-to-Neumann Map for nonlinear diffusive equation with moving boundaries.
2. Existence and uniqueness of the solution for small times
3. Nonlinear diffusion equation under the influence of an external forcing of distribution type
4. Mapping the initial value problem on the semiline and to the Linear Heat equation with moving boundaries.
5. Mapping into a nonlinear Volterra integral equation and uniqueness of solutions for small intervals of time.
6. A novel special travelling wave solution in thin infinite elastic rod, which can be interpretred as a conformational soliton travelling at constant speed.
7. The square of the velocity of the solitary wave is directly proportional to the bending of stiffness and inversely proportional to the density and to the principal momentum of inertia of the chain.
8. Applications to the polymeric chains.

Symmetries in Nonlinear models: Vortices and Waves

Martina

- L. M., G.I. Martone and S. Zykov: Symmetry reductions of the Skyrme - Faddeev model, Acta Math. Appl. (2012), 1-12
- G. De Matteis , L. M.: Lie point symmetries and reductions of one-dimensional equations describing perfect Korteweg-type nematic fluids , *J. Math. Phys.* **53**, 033101 (2012)
- L.M. , A. Protogenov, V. Verbus: A chain of strongly correlated $SU(2)_4$ anyons: Hamiltonian and Hilbert space states, *Theor. Math. Phys.* 167(3) (2011), 843-855
- L. M., M.V. Pavlov and S. Zykov: Waves in the Skyrme-Faddeev model and Integrable reductions , in preparation

Physical Origin

1. $^3He - A$ superfluid $(M_L = 1, M_S = 0)$
2. 2-band superconductor (Nb-doped $SrTiO_3$, MgB_2)
3. charged condensates of tightly bounded fermion pairs
4. Spin-Charge Separation of the pure Yang-Mills theory in Infrared background

- Stability of the order parameter configurations
- Knotted and/or linked quasi-1-dimensional configurations
- Coexistence/Competition of short/long (UV/IR) wave modes
- Properties of knots and tangles
- Topological ordering in disordered background

The Skyrme Faddeev- Model

$$E[\vec{n}] = \int_{\mathbb{R}^3} \left\{ (\partial_a \vec{n})^2 + \left(\frac{1}{2} (\vec{n} \cdot \partial_a \vec{n} \times \partial_b \vec{n}) \right)^2 \right\} d^3x,$$

$$\partial_a^2 \vec{n} - (\partial_a \mathcal{F}_{ab}) (\vec{n} \times \partial_b \vec{n}) = (\vec{n} \cdot \partial_a^2 \vec{n}) \vec{n}.$$

$$\lim_{|\vec{x}| \to \infty} \vec{n}(\vec{x}) = \vec{n}_\infty = \pm \vec{z} \Rightarrow \vec{n} : \mathbb{S}^3 \to \mathbb{S}^2, \quad O(3) \hookrightarrow O(2)$$

$$E[\vec{n}] \geq c \, |N[\vec{n}]|^{3/4}, \quad c \approx (3/16)^{3/8}$$

hedgehog solution

$$\vec{n} \cdot \vec{\sigma} = U (\vec{n}_\infty \cdot \vec{\sigma}) U^\dagger$$

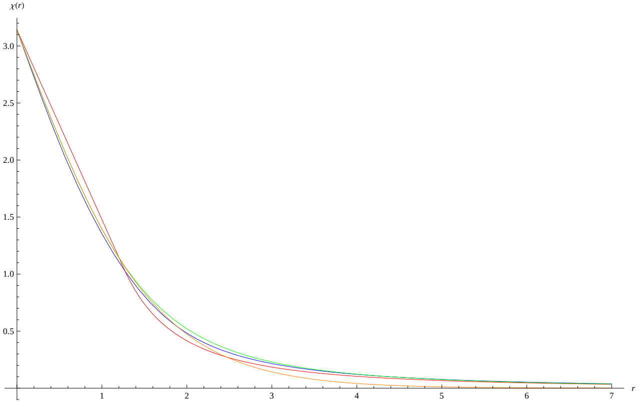
$$U = \exp[i\chi(r) \vec{\nu}(\vartheta, \varphi) \cdot \vec{\sigma}] = \cos\chi(r) I + i \sin\chi(r) \vec{\nu}(\vartheta, \varphi) \cdot \vec{\sigma}$$

Approximated solutions by rational f.

$$g_{rat}(r) = \frac{1 + a_1 r + a_2 r^2}{1 + a_1 r + b_2 r^2 + b_3 r^3 + b_4 r^4},$$

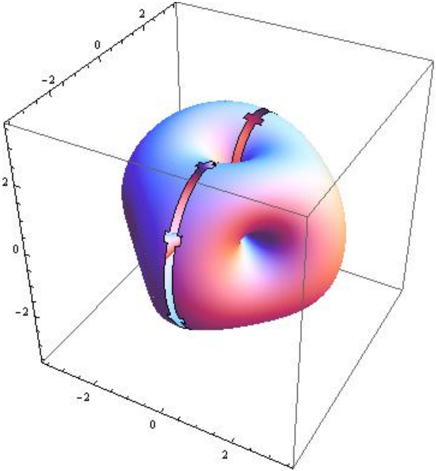
$$a_1 = 0.216, \quad a_2 = 0.230, \quad b_2 = 0.752, \quad b_3 = -0.018, \quad b_4 = 0.302,$$

Hedgehog Profile



Blu : numerical solution. Green: $\chi_{rat} = 2 \arcsin g_{rat}$. Red: test χ_p-function. Orange: Atiyah - Manton test function $\frac{|E[\chi_{num}] - E[\chi_{rat}|]}{|E[\chi_{num}]|} \approx 10^{-3}$

Rational Maps Ansatz



$$R_T = \frac{z^3 - \sqrt{3}\imath z}{\sqrt{3}\imath z^2 - 1}$$

Quasi-Periodic Solutions

$\vec{n} = \left(\sin(\Theta)\sin(\tilde{\Phi}), \sin(\Theta)\cos(\tilde{\Phi}), \cos(\Theta)\right)$

Quasi-1 dimensional reduction

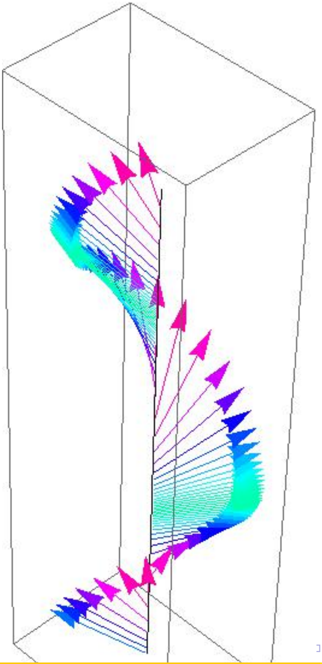
$$\Theta = \Theta\left[\theta\left(\alpha_i x_i\right)\right], \tilde{\Phi} = \Phi\left[\theta\left(\alpha^i x_i\right)\right] + \beta^i x_i$$

θ-Reduced Conserved Energy-Momentum Density

$B_3\alpha_i^2\Theta_\theta^2 + \sin^2(\Phi)\left[C_i + \frac{\lambda}{8}\mathcal{B}\alpha_0\Theta_\theta^2 + 2B_3\beta_i\Phi_\theta + B_3\alpha_i\Phi_\theta^2\right] = \frac{\alpha_i(U_1 + B_2 U_2 \beta_0)}{\alpha_0} - B_2 U_2 \beta_i$

$$\Theta_\theta^2 = \frac{8B_3\left(B_1\sin^2(\Theta) + U_3\right) - 2B_2^2\left(\sin^2(\Theta) + U_2^2\csc^2(\Theta)\right)}{B_3\left(8B_3 - \lambda\mathcal{B}\sin^2(\Theta)\right)}$$

$$\Theta \to \sin^{-1}\left(\sqrt{\psi}\right) \Rightarrow \qquad \psi_\theta^2 = \frac{64(\psi - 1)(\psi - A_1)(\psi - A_2)}{\lambda^2\mathcal{B}\psi_1(\psi_1 - \psi)}$$



Non Canonical Systems in 3D

L. M.
Non-commutative mechanics and Exotic Galilean symmetry
arXiv:1011.3545, Theor. Math. Phys. **167** *(3) (2011), 816-825*

Dynamics with exotic symmetries *J. Phys.: Conf. Ser. 343 (2012) 012072*

Dynamics in Non-Commutative Spaces and Generalizations *Int. J. Geom. M. Mod. Phys. (2012) ,1260012*

Dynamics of a noncommutative monopole *to appear in Theor.Math. Phys.*

Non canonical systems in 3D

$$\dot{\mathbf{r}} = \frac{\partial \epsilon_n(\mathbf{k})}{\partial \mathbf{k}} - \dot{\mathbf{k}} \times \vec{\Theta}(\mathbf{k}), \quad \dot{\mathbf{k}} = -e\mathbf{E} - e\dot{\mathbf{r}} \times \mathbf{B}(\mathbf{r}) \qquad \text{(Bloch electron)}$$

$$\dot{\mathbf{r}} = \frac{\partial E_s(\mathbf{k})}{\partial \mathbf{k}} + \dot{\mathbf{k}} \times \vec{\Theta}_s, \qquad \dot{\mathbf{k}} = -e\vec{E}, \qquad \text{(Spin-Hall effect)}$$

$$E_s(\mathbf{k}) = \frac{\hbar^2}{2m}\left(A - Bs^2\right)k^2, \; \vec{\Theta}_s = s\left(2s^2 - \frac{7}{2}\right)\frac{\vec{k}}{k^3}, \qquad s = \pm\frac{1}{2}, \pm\frac{3}{2}$$

$$\dot{\vec{r}} = \vec{p} - \frac{s}{\omega}\,\mathrm{grad}(\frac{1}{n}) \times \vec{p}, \qquad \dot{\vec{p}} = -n^3\omega^2\mathrm{grad}(\frac{1}{n}), \quad \text{(Optical Magnus)}$$

$$M\left(\frac{\partial A_j}{\partial q^i}\right)\dot{\vec{q}} + \vec{F} \times \vec{r} = -\frac{\partial h}{\partial \vec{r}}, \qquad M\dot{\vec{r}} = \frac{\partial h}{\partial \vec{q}}, \qquad \text{(Bogoliubov q-particle)}$$

$$\dot{x}_i = \frac{p_i}{m} + \Theta_{ij}\frac{\partial V}{\partial x_j}, \qquad \dot{p}_i = -m\frac{\partial V}{\partial x_j} + m\Theta_{ij}\frac{\partial^2 V}{\partial x_i \partial x_j} \quad \text{(NC Kepler problem)}$$

Lagrange-Souriau 2-form

$$3D)\ \sigma = [(1-\mu_i)\, dp_i - e\, E_i\, dt] \wedge (dr_i - g_i\, dt) + \tfrac{1}{2}\, e\, B_k\, \epsilon_{kij}\, dr_i \wedge dr_j + \tfrac{1}{2}\kappa_k\, \epsilon_{kij}\, dp_i \wedge dp_j + q_k\, \epsilon_{kij}\, dr_i \wedge dp_j$$

only gauge invariant quantities

closure condition $d\sigma = 0$ (Maxwell's principle)

Kernel condition $\sigma\,(\delta y, \cdot) = 0,\ \delta y = (\delta\vec{r}, \delta\vec{p}, \delta t)$

Cartan 1-form $\sigma = d\lambda$

$$\lambda = \left(\vec{p} + \overrightarrow{\mathcal{A}}\right) \cdot d\vec{r} + \vec{\mathcal{R}} \cdot d\vec{p} - \left(\mathcal{E}\,(\vec{p}, t) + \varphi\,(\vec{r}, t)\right) dt$$

$$\partial_t \vec{\mathcal{A}} = \partial_t \vec{\mathcal{R}} \equiv 0 \Rightarrow \sigma = \omega - dH \wedge dt \qquad d\omega = 0,$$

$$\mathcal{H} = \mathcal{E}\,(\vec{p}, t) + \varphi\,(\vec{r}, t)$$

$$\omega = \omega_{\alpha\beta}\, d\xi_\alpha \wedge d\xi_\beta = (\delta_{i,j} + \Xi_{ij})\, dr_i \wedge dp_j + \frac{1}{2}\left[\mathcal{B}_{ij}\, dr_i \wedge dr_j - \Theta_{ij}\, dp_i \wedge dp_j\right]$$

Charge in Magnetic and Dual Monopole

$$\omega = \delta_{i,j} dr_i \wedge dp_j + \frac{\epsilon_{ijk}}{2}\left[e\frac{r_k}{|\vec{r}|^3}\, dr_i \wedge dr_j + \theta\frac{p_k}{|\vec{p}|^3}\, dp_i \wedge dp_j\right]$$

J.-M. Souriau *loc. cit.* ; J. F. Cariñena *et al.*, J. Math. Phys. **16** (1975) 1416; J. F. Cariñena *et al.* in M. Asorey et al. Ed.s, (2009)

D. J. P. Morris et al. *Science* **326**, 411 (2009)

$$M^*\dot{r}_i = \left(p_i - e\theta\frac{r_i}{|\vec{p}||\vec{r}|^3}\right)|\vec{r}|^3|\vec{p}|^3, \qquad M^* = |\vec{r}|^3|\vec{p}|^3 - e\theta\, \vec{r}\cdot\vec{p}$$

$$\sqrt{\det(\omega_{\alpha\beta})} = \left(\frac{M^*}{|\vec{r}|^3|\vec{p}|^3}\right)^2$$

$$M^*\dot{p}_i = e\varepsilon_{ijk} p_j r_k |\vec{p}|^3, .$$