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THE ESR MODEL: A NONCONTEXTUAL HILBERT SPACE FRAMEWORK FOR QM

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INTRODUCTION I

Objectivity of physical properties.

Intuitive definition. A physical property (*e.g.*, the value of an observable) is objective if it is possessed or not possessed by a physical system independently of any measurement.

Formal operational definition. A physical property E is objective for a given state S of a physical system Ω if for every individual example of Ω (*physical object*) in the state S the result of an ideal measurement of E does not depend on the measurement context.

Bell-Kochen-Specker's theorem ⇒ contextuality of quantum mechanics (QM).

Bell's theorem ⇒ nonlocality of QM.

Contextuality ⇒ nonobjectivity of physical properties.

Nonlocality ⇒ contextuality ⇒ nonobjectivity of physical properties.

INTRODUCTION II

Consequences of nonobjectivity.

(i) *Logical*. The classical notion of truth as correspondence cannot be maintained in QM. Quantum Logic.

(ii) *Probabilistic*. The usual *epistemic* interpretation of probabilities cannot be maintained in the case of quantum probabilities, which are necessarily nonepistemic (sometimes called ontic).

(iii) *Quantum measurement theory*. *Objectification problem* (how a macroscopic measurement apparatus may exhibit objective properties), *paradoxes* (Schrödinger's cat, Wigner's friend, EPR paradox).

(iv) *Comprehensibility*. Intuitive geometrical models cannot be constructed for QM (wave-particle duality). Feynman writes (1965, The character of physical laws):

> "There was a time when the newspapers said that only twelve men understood the theory of relativity. I do not believe there ever was such a time. There might have been a time when only one man did, because he was the only guy who caught on before he wrote his paper. But after people read the paper a lot of people understood the theory of relativity in some way or other, certainly more than twelve. On the other hand I think I can safely say that nobody understands quantum mechanics".

INTRODUCTION III

Trying to avoid these problems the Lecce group has proven that the Bell and Bell-Kochen-Specker theorems rest on an implicit epistemological assumption (*metatheoretical classical principle*, or *MCP*), which is problematical from a quantum point of view. If one weakens this assumption (*metatheoretical generalized principle*, or *MGP*), the proofs of the foregoing theorems cannot be completed. This is an important result, for it implies that the possibility of providing an interpretation of the mathematical formalism of QM which is noncontextual (hence local) cannot be excluded, at variance with standard beliefs. By adopting MGP in place of MCP we have thus provided an interpretation of this kind (*semantic realism, or SR, interpretation*).

To show the consistency of the SR interpretation various models have been provided. The last of these, named *extended semantic realism* (ESR) model is a new kind of noncontextual hidden variables theory, which introduces, besides hidden variables, a reinterpretation of standard quantum probabilities. The ESR model modifies and extends the original SR interpretation, but it preserves its basic features, that is, semantic realism and the substitution of MCP with the weaker principle MGP.

INTRODUCTION IV

As every hidden variables (h.v.) theory, the ESR model presupposes that "something is happening" at a microscopic level which underlies the standard quantum picture of physical systems and does not reduce to it. Hence it proposes a set-theoretical description of the microscopic world in which *physical objects* (intuitively interpreted as individual examples of the physical system that is considered), *microscopic properties* (the h.v. of the model) and *microscopic states* are introduced. But, then, this microscopic part is related to the macroscopic (observational) part in an innovatory way, which justifies the introduction at a macroscopic level of a series of assumptions that produce a new theory, different from QM.

As a final result, the macroscopic part of the ESR model, though justified by the features of the microscopic part, can be presented as a self-consistent theoretical proposal without mentioning hv and the metatheoretical principles MCP and MGP (indeed the former does not hold and the latter holds because of the basic assumptions of the model). Its main features can be resumed as follows.

INTRODUCTION V

(i) It brings into every measurement a *no-registration outcome* which is interpreted as providing physical information, as well as any other possible outcome, hence it substitutes the observables of QM with *generalized observables* with enlarged sets of possible values.

(ii) It embodies the basic mathematical formalism of standard Hilbert space QM into a general *noncontextual* framework (which avoids the objectification problem and some quantum paradoxes).

(iii) It reinterprets quantum probabilities as *conditional on detection* instead of *absolute*.

(iv) It provides some predictions that are *formally* identical to those of QM but have a *different* physical interpretation and *further predictions* that differ also formally from those of QM (hence the ESR model can be empirically checked).

By formulating the foregoing theoretical proposal in mathematical terms, we have obtained the following results.

- Each generalized observable is represented by a *pair*, consisting of the standard quantum representation and a (commutative) family of positive operator valued (POV) measures parametrized by the set of all pure states of the physical system that is considered.
- A *generalized projection postulate* (GPP) rules the transformations of pure states induced by nondestructive idealized measurements.

INTRODUCTION VI

- The *Bell-Clauser-Horne-Shimony-Holt* (*BCHSH*) *inequality*, a *modified BCHSH inequality* and *quantum predictions* hold together in the ESR model because they refer to different parts of the picture of the physical world supplied by the model.

- Each *proper mixture* is represented by a *family of pairs*, each pair consisting of a density operator and a convex combination of *detection probabilities*, parametrized by the set of all macroscopic properties characterizing the physical system that is considered.
- Each *improper mixture* is represented by a *single* density operator, as in QM.
- The different representations of proper and improper mixtures avoid some deep interpretative problems that arise in QM.
- A *generalized Lüders postulate* (*GLP*) that generalizes GPP rules the general transformations of proper mixtures induced by nondestructive idealized measurements.

- An experiment with proper mixtures can be envisaged in which the predictions of the ESR model are different from the predictions of QM, thus discriminating empirically between the two theories.

- GPP can be justified by describing a measurement as a dynamical process in which a nonlinear evolution occurs of the composite system made up of the (microscopic) measured object plus the (macroscopic) measuring apparatus.

THE ESR MODEL

According to the ESR model, a *physical system* Ω is operationally defined by a pair $(\Pi, \mathcal{R})$, with Π a set of *preparing devices* and $\mathcal{R}$ a set of *measuring apparatuses*. Every preparing device, when activated, prepares an *individual example* of Ω (which can be identified with the preparation act itself if one wants to avoid any ontological commitment). Every measuring device, if activated after a preparing device, yields an *outcome*, that we assume to be a real number.

In the theoretical description a physical system Ω is characterized by a set $\mathcal{U}$ of *physical objects*, a set $\mathcal{E}$ of *microscopic properties*, a set $\mathcal{S}$ of *macroscopic states* and a set $\mathcal{O}_0$ of *macroscopic generalized observables*.

Physical objects are operationally interpreted as individual examples of Ω, while microscopic properties are purely theoretical entities (the h.v. of the model). Every physical object $x \in \mathcal{U}$ is associated with a set of microscopic properties (the microscopic properties *possessed* by x) which is called the *microscopic state* of x and also is a theoretical entity.

MACROSCOPIC OBSERVATIONAL ENTITIES

Each macroscopic state $S \in \mathcal{S}$ is operationally defined as a class of probabilistically equivalent preparing devices. Every device $\pi \in S$, when constructed and activated, prepares an individual example of Ω, hence a physical object x, and one briefly says that "x is (prepared) in the state S".

Every generalized observable $A_0 \in \mathcal{O}_0$ is operationally defined as a class of probabilistically equivalent measuring apparatuses, and it is obtained by considering an observable A of QM with set of possible values Ξ on the real line $\Re$ and adding a further outcome $a_0 \in \Re \setminus \Xi$ (*no-registration outcome* of A_0), so that the set of all possible values of A_0 is $\Xi_0 = \{a_0\} \cup \Xi$.

Remark. One assumes here, for the sake of simplicity, that $\Re \setminus \Xi$ is non-void. This assumption is not restrictive. Indeed, if $\Xi = \Re$, one can choose a bijective Borel function f: $\Re \rightarrow \Xi' \subset \Re$ (*e.g.*, $\Xi' = \Re^+$) and replace A by f(A).

A MEASUREMENT SCHEME I

Let $\mathbb{B}(\Re)$ be the σ-algebra of all Borel sets on the real line $\Re$.

The set $\mathcal{F}_0$ of all (*macroscopic*) *properties* of Ω is defined by

$$\mathcal{F}_0 = \{ (A_0, X), A_0 \in \mathcal{O}_0, X \in \mathbb{B}(\Re)\},$$

and the subset $\mathcal{F} \subset \mathcal{F}_0$ of all properties associated with observables of QM is defined by

$$\mathcal{F} = \{ (A_0, X), A_0 \in \mathcal{O}_0, X \in \mathbb{B}(\Re), a_0 \notin X\}.$$

A measurement of a property $F=(A_0,X)$ on a physical object x in the state S is described as a *registration* performed by means of a *dichotomic registering device* whose outcomes are denoted by *yes* and *no*. The measurement yields the outcome yes/no (equivalently, x *displays*/*does not display* F) if and only if the value of A_0 *belongs*/*does not belong* to X.

A MEASUREMENT SCHEME II

The connection between the microscopic and the macroscopic part of the ESR model is established by introducing the following assumptions.

(i) A bijective mapping $\varphi: \mathcal{E} \rightarrow \mathcal{F} \subset \mathcal{F}_0$ exists.

(ii) If a physical object x is in the microscopic state S^i and an *idealized measurement* of a macroscopic property $F=\varphi(f)$ is performed on x, then S^i determines a probability $p_{S^i}^d(F)$ that x be detected, and x displays F if it is detected and $f \in S^i$, does not display F if it is not detected or $f \notin S^i$.

The ESR model is *deterministic* if $p_{S^i}^d(F) \in \{0,1\}$, *probabilistic* otherwise. In the former case it is necessarily noncontextual because the outcome of the measurement of a macroscopic property on a physical objects x depends only on the microscopic properties possessed by x and not on the measurement context. In the latter case one can recover noncontextuality by adding suitable assumptions on microscopic states and probabilities of macroscopic properties.

THE FUNDAMENTAL EQUATION I

By using the connection between the microscopic and the macroscopic part of the ESR model one can show that, whenever the property F =(A_0,X)∈ $\mathcal{F}$ (hence $a_0 \notin$ X) is measured on a physical object x in the macroscopic state S, the overall probability $p_S^t(F)$ that x display F is given by

$$p_S^t(F) = p_S^d(F)\, p_S(F) \qquad (1)$$

The symbol $p_S^d(F)$ denotes the probability that x be detected if it is in the state S (*detection probability*), and it is not fixed for a given observable A_0 but it may depend on the property F, hence on the Borel set X.

The symbol $p_S(F)$ is interpreted as the *conditional* probability that a physical object x in the state S display the property F when it is detected.

Since the ESR model is noncontextual, the above connection also implies that an *idealized measurement* of F is such that $p_S^d(F)$ depends only on the features of the physical objects in the state S, hence it does not occur because of flaws or lack of efficiency of the apparatus measuring F.

THE FUNDAMENTAL EQUATION II

Let us consider now a property $F=(A_0, X) \in \mathcal{F}_0 \setminus \mathcal{F}$, hence $a_0 \in X$. Putting $F^c = (A_0, \Re \setminus X)$, we introduce the reasonable physical assumption that, for every state S, the overall probability $p_S^t(F)$ that x display F is given by

$$p_S^t(F) = 1 - p_S^t(F^c) = 1 - p_S^d(F^c)\, p_S(F^c) \qquad (2)$$

which provides the overall probability $p_S^t(F)$ that a physical object x in the state S display F in terms of the overall probability that x display F^c when F^c is measured in place of F.

Because of this assumption we will mainly deal with properties in $\mathcal{F}$ in the following.

Eqs. (1) and (2) imply that three basic probabilities occur in the ESR model. We have as yet no theory which allows us to predict the value of $p_S^d(F)$. But we can consider $p_S^d(F)$ as an unknown parameter to be determined empirically, and introduce theoretical assumptions that connect the ESR model with standard QM, enabling us to provide mathematical representations of the physical entities introduced in the ESR model together with explicit expressions of $p_S^t(F)$ and $p_S(F)$.

THE MAIN ASSUMPTION

Let us begin with $p_S(F)$. The following statement then expresses the main assumption of the ESR model.

AX. If S is a pure state and $F \in \mathcal{F}$ the probability $p_S(F)$ can be evaluated by using the same rules that yield the probability of F in the state S according to QM.

Assumption AX allows one to recover the formalism of QM in the framework of the ESR model, but modifies the standard interpretation of quantum probabilities. Indeed, according to QM, whenever an ideal measurement of a property F is performed, all physical objects that are prepared in a state S are detected, hence the quantum rules for calculating probabilities are intuitively interpreted as yielding the probability that a physical object x display the property F whenever it is selected in the set of all objects in the state S (*absolute probabilities*). According to assumption AX, instead, if S is pure, the same rules yield the probability that a physical object x display the property F whenever it is selected in the subset of all objects in the state S that are detected (*conditional probabilities*).

THE CONDITIONAL PROBABILITY IN THE PURE CASE

Because of the above reinterpretation of quantum probabilities the predictions of the ESR model are different from those of QM. But the detection probabilities can hardly be distinguished from the efficiencies of actual measuring devices, which explains why QM ignores them. Nevertheless, we will show in the following that in some cases there are substantial differences between the two theories that can be experimentally checked.

Assumption AX implies that, as far as $p_S(F)$ is concerned:

(i) the pure state S of the physical system Ω can be represented by a vector $|\psi\rangle$ in the set $\mathcal{V}$ of all unit vectors of the (separable) complex Hilbert space $\mathcal{H}$ associated with Ω, or by the one-dimensional projection operator $\rho_\psi=|\psi\rangle\langle\psi|$, as in standard QM;

(ii) if pure states only are considered, a generalized observable A_0 can be represented by the self-adjoint operator Â which represents the observable A of QM from which A_0 is obtained;

(iii) every property $F=(A_0,X)\in\mathcal{F}$ can be represented by the projection operator $P^{\hat{A}}(X)$ where $P^{\hat{A}}$ is the spectral projection valued (PV) measure associated with Â;

(iv) the probability $p_S(F)$ can be evaluated by using the standard quantum rule

$$p_S(F)=\langle\psi|P^{\hat{A}}(X)|\psi\rangle=Tr[\rho_\psi P^{\hat{A}}(X)]$$

THE OVERALL PROBABILITY IN THE PURE CASE

Let us consider now the overall probability $p_S^t((A_0,X))=p_S^t(F)$, with $F\in\mathcal{F}$ and S a pure state represented in QM by the unit vector $|\psi\rangle$, or by the density operator $\rho_\psi = |\psi\rangle\langle\psi|$. Then we get

$$p_S^t((A_0,X)) = p_S^t(F) = \langle\psi|T_\psi^{\hat{A}}(X)|\psi\rangle = Tr[\rho_\psi T_\psi^{\hat{A}}(X)] \qquad (3)$$

The operator $T_\psi^{\hat{A}}(X)$ is defined by

$$T_\psi^{\hat{A}}(X) = \int_X p_\psi^d(\hat{A},\lambda)dP_\lambda^{\hat{A}} \qquad (a_0 \notin X) \qquad (4)$$

where $P^{\hat{A}}$ is the spectral PV measure associated with $\hat{A}$ and $p_\psi^d(\hat{A},\lambda)$ is such that, for every $|\psi\rangle\in\mathcal{V}$, $<\psi|p_\psi^d(\hat{A},\lambda)dP_\lambda^{\hat{A}}/d\lambda|\psi>$ is a measurable function on $\Re$. Therefore, as far as $p_S^t(F)$ is concerned, the pure state S can still be represented by $|\psi\rangle$ or ρ_ψ.

Eqs. (3) and (4) show that, as far as $p_S^t(F)$ is concerned, the macroscopic property $F=(A_0,X)\in\mathcal{F}$ is represented by the family $\{T_\psi^{\hat{A}}(X)\}_{|\psi\rangle\in\mathcal{V}}$ of bounded positive operators (*effects*). This representation, together with Eqs. (3) and (4), can be easily extended to a macroscopic property $F=(A_0, X)\in\mathcal{F}_0\setminus\mathcal{F}$ by using Eq. (2) and setting

$$T_\psi^{\hat{A}}(X) = I - \int_{\Re\setminus X} p_\psi^d(\hat{A},\lambda)dP_\lambda^{\hat{A}} \qquad (a_0 \in X) \qquad (5)$$

REPRESENTATION OF GENERALIZED OBSERVABLES AND PROPERTIES

By using the above representation one can show that, as far as $p_S^t(F)$ is concerned, the generalized observable $A_0 \in \mathcal{O}_0$ is represented by the family of *commutative positive operator valued* (*POV*) *measures*

$$\mathcal{T}^{\hat{A}}=\{T_\psi^{\hat{A}}: X \in \mathbb{B}(\Re) \longrightarrow T_\psi^{\hat{A}}(X) \in \mathcal{B}(\mathcal{H})\}_{|\psi\rangle \in \mathcal{V}} \quad (6)$$

where $\mathcal{B}(\mathcal{H})$ is the set of all bounded operators on $\mathcal{H}$.

Summing up, we conclude that complete mathematical representations of $F=(A_0, X) \in \mathcal{F}$ and A_0 are provided in the ESR model by the pairs $(P^{\hat{A}}(X), \{T_\psi^{\hat{A}}(X)\}_{|\psi\rangle \in \mathcal{V}})$ and $(\hat{A}, \mathcal{T}^{\hat{A}})$, respectively.

Finally, one gets from assumption AX, Eq. (1) and Eq. (3) that, for every $|\psi\rangle \in \mathcal{V}$, and $F=(A_0,X) \in \mathcal{F}$,

$$p_S^d(F) = \frac{Tr[\rho_\psi T_\psi^{\hat{A}}(X)]}{Tr[\rho_\psi P^{\hat{A}}(X)]} \quad (7)$$

which yields a condition that must be fulfilled by $p_S^d(F)$.

DISCRETE GENERALIZED OBSERVABLES

Let A_0 be a discrete generalized observable with set of possible values $\Xi_0=\{a_0\}\cup\{a_1, a_2, \ldots\}$ and let us put $p^d_{\psi n}(\hat{A}) =: p^d_{\psi}(\hat{A}, a_n)$ and $P^{\hat{A}}_n =: P^{\hat{A}}(\{a_n\})$. Then, Eqs. (4) and (5) yield

$$\mathrm{T}^{\hat{A}}_{\psi}(X) = \begin{cases} \sum_{n, a_n \in X} p^d_{\psi n}(\hat{A}) P^{\hat{A}}_n & (a_0 \notin X) \\ I - \sum_{n, a_n \in \Re \setminus X} p^d_{\psi n}(\hat{A}) P^{\hat{A}}_n & (a_0 \in X) \end{cases} \quad (8)$$

Let us consider the special case $X=\{a_n\}$, $n \in \aleph_0$, and put $F_n=(A_0,\{a_n\})$. We get from Eqs. (6) and (8)

$$\mathrm{T}^{\hat{A}}_{\psi}(\{a_n\}) = \begin{cases} p^d_{\psi n}(\hat{A}) P^{\hat{A}}_n & (n \neq 0) \\ \sum_{m \in \aleph} (1 - p^d_{\psi m}(\hat{A})) P^{\hat{A}}_m & (n = 0) \end{cases} \quad (9)$$

$$\mathrm{p}^{\mathrm{t}}_{\mathrm{S}}(F_n) = \begin{cases} p^d_{\psi n}(\hat{A}) \langle \psi | P^{\hat{A}}_n | \psi \rangle = Tr[\rho_{\psi} p^d_{\psi n}(\hat{A}) P^{\hat{A}}_n] & (n \neq 0) \\ \sum_{m \in \aleph} (1 - p^d_{\psi m}(\hat{A})) \langle \psi | P^{\hat{A}}_m | \psi \rangle = Tr\left[\rho_{\psi} \sum_{m \in \aleph} (1 - p^d_{\psi m}(\hat{A})) P^{\hat{A}}_m \right] & (n = 0) \end{cases} \quad (10)$$

THE GENERALIZED PROJECTION POSTULATE

If one considers a *nondestructive* idealized measurement of a discrete generalized observable A_0 on a physical object x in a pure state S, consistency with assumption AX suggests that, if a value a_n, with $n \neq 0$, of A_0 is obtained, then S is modified according to standard QM rules. This requirement, together with our representation of generalized observables, supports the introduction of the following *generalized projection postulate* (*GPP*).

GPP. Let S be a pure state represented by the unit vector $|\psi\rangle$, or by the density operator $\rho_\psi = |\psi\rangle\langle\psi|$, and let a *nondestructive* idealized measurement of a property $F=(A_0, X) \in \mathcal{F}_0$ be performed on a physical object x in the state S.

Let the measurement yield the yes outcome. Then, the state S_F of x after the measurement is a pure state represented by the unit vector

$$|\psi_F\rangle = \frac{T_\psi^{\hat{A}}(X)|\psi\rangle}{\sqrt{\langle\psi|T_\psi^{\hat{A}}(X) T_\psi^{\hat{A}\dagger}(X)|\psi\rangle}}$$

or by the density operator

$$\rho_{\psi_F} = \frac{T_\psi^{\hat{A}}(X)\rho_\psi T_\psi^{\hat{A}\dagger}(X)}{Tr[T_\psi^{\hat{A}}(X)\rho_\psi T_\psi^{\hat{A}\dagger}(X)]} \quad (11a)$$

Let the measurement yield the no outcome. Then, the state S'_F of x after the measurement is a pure state represented by the unit vector

$$|\psi'_F\rangle = \frac{T_\psi^{\hat{A}}(\Re \setminus X)|\psi\rangle}{\sqrt{\langle\psi|T_\psi^{\hat{A}}(\Re \setminus X) T_\psi^{\hat{A}\dagger}(\Re \setminus X)|\psi\rangle}}$$

or by the density operator

$$\rho_{\psi'_F} = \frac{T_\psi^{\hat{A}}(\Re \setminus X)\rho_\psi T_\psi^{\hat{A}\dagger}(\Re \setminus X)}{Tr[T_\psi^{\hat{A}}(\Re \setminus X)\rho_\psi T_\psi^{\hat{A}\dagger}(\Re \setminus X)]} \quad (11b)$$

SOME REMARKS ON GPP

GPP replaces the projection postulate for pure states introducing two basic changes.

(i) The positive operator $T_{\psi}^{\hat{A}}(X)$ that depends on $|\psi\rangle$ replaces the projection operator that appears in the projection postulate.

(ii) The terms in the denominators in Eqs. (11) and (12) do not coincide with the probabilities of the yes and no outcomes, respectively.

Moreover, if a measurement of the property $F_n=(A_0,\{a_n\})$ is performed on a physical object x in the state S represented by the unit vector $|\psi\rangle$ and the yes outcome is obtained, then

Projection postulate in the standard form, as required.

$$|\psi_{F_n}\rangle = \begin{cases} \dfrac{P_n^{\hat{A}}|\psi\rangle}{\sqrt{\langle\psi|P_n^{\hat{A}}|\psi\rangle}} & (n\neq 0) \\ \dfrac{\sum_{m\in\aleph}(1-p_{\psi m}^{d}(\hat{A}))P_m^{\hat{A}}|\psi\rangle}{\sqrt{\sum_{m\in\aleph}(1-p_{\psi m}^{d}(\hat{A}))^2\left\|P_m^{\hat{A}}|\psi\rangle\right\|^2}} & (n=0) \end{cases} \qquad (12)$$

EXPECTATION VALUES

Whenever the discrete generalized observable A_0 instead of the property $F=(A_0,X)$ is measured on the physical object x in the pure state S, one obtains one of the outcomes a_0, a_1, a_2, ... If the measurement is idealized, the overall probability of the outcome a_n is given by Eq. (10). Moreover, as a natural extension of our picture of the measurement process, we assume that, if the measurement is nondestructive and the outcome a_n is obtained, the final state of the object is given by Eq. (12). It follows in particular that the *expectation value* of A_0 in the pure state S represented by the unit vector $|\psi\rangle$ is given by

$$\langle A_0 \rangle_S = \sum_{n\in\aleph_0} a_n p_S^t(F_n) = \sum_{n\in\aleph_0} a_n \langle\psi | T_\psi^{\hat{A}}(\{a_n\}) | \psi\rangle = a_0 + \sum_{n\in\aleph} (a_n - a_0) p_{\psi n}^d(\hat{A}) \langle\psi | P_n^{\hat{A}} | \psi\rangle \quad (13)$$

SEQUENTIAL MEASUREMENTS

Let B be a discrete observable of QM represented by the self-adjoint operator $\hat{B}$ with set of possible outcomes $\{b_1, b_2, \ldots\}$, let B_0 be the generalized observable obtained from B, with set of possible outcomes $\{b_0\} \cup \{b_1, b_2, \ldots\}$, and let us assume that idealized nondestructive measurements of A_0 and B_0 are performed. By using GPP we can calculate the probability $p_S^t(a_n, b_p)$ (with $n, p \in \aleph_0$) of obtaining the pair of outcomes (a_n, b_p) when firstly measuring A_0 e then B_0 on a physical object x in the state S. We get

$$p_S^t(a_n, b_p) = \langle \psi | T_\psi^{\hat{A}}(\{a_n\}) | \psi \rangle \langle \psi_{F_n} | T_{\psi_{F_n}}^{\hat{B}}(\{b_p\}) | \psi_{F_n} \rangle \quad (14)$$

Whenever $n \neq 0 \neq p$, Eq. (14) yields

$$p_S^t(a_n, b_p) = p_{\psi n}^d(\hat{A})\, p_{\psi_{F_n} p}^d(\hat{B}) \langle \psi | P_n^{\hat{A}} P_p^{\hat{B}} P_n^{\hat{A}} | \psi \rangle \quad (15)$$

MEASUREMENTS ON PARTS OF A COMPOSITE SYSTEM I

Let Ω be a composite system made up of the subsystems Ω_1 and Ω_2, associated in QM with the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, so that Ω is associated with the Hilbert space $\mathcal{H}=\mathcal{H}_1\otimes\mathcal{H}_2$.

Let A(1) (B(2)) be a discrete observable of Ω_1 (Ω_2), with set of possible outcomes $\Xi_1=\{a_1, a_2, \ldots\}$ ($\Xi_2=\{b_1, b_2, \ldots\}$), represented by the self-adjoint operator $\hat{A}(1)$ ($\hat{B}(2)$) on $\mathcal{H}_1$ ($\mathcal{H}_2$) in QM. When considered as an observable of Ω, A(1) (B(2)) is represented in QM by the self-adjoint operator $\hat{A}(1)\otimes I(2)$ ($I(1)\otimes\hat{B}(2)$), where I(2) (I(1)) is the identity operator on $\mathcal{H}_2$ ($\mathcal{H}_1$).

Let $A_0(1)$ ($B_0(2)$) be a generalized observable obtained from A(1) (B(2)) by adding the no-registration outcome a_0 (b_0) to Ξ_1 (Ξ_2). Let S be a pure state of Ω such that Ω_1 and Ω_2 are spatially separated. Whenever simultaneous measurements of $A_0(1)$ and $B_0(2)$ are performed on a physical object x (individual example of Ω) in the state S, noncontextuality implies that the transformation of S induced by a measurement of $A_0(1)$ must not affect the detection probability associated with the measurement of $B_0(2)$. If S is represented by the unit vector $|\Psi\rangle$, one gets

$$p^d_{\Psi_{F_n}p}(\hat{B}(2)) = p^d_{\Psi_p}(\hat{B}(2)) \qquad (16)$$

MEASUREMENTS ON PARTS OF A COMPOSITE SYSTEM II

It follows from Eq. (15) and (16) that

$$p_S^t(a_n, b_p) = p_{\Psi n}^d(\hat{A}(1))\, p_{\Psi p}^d(\hat{B}(2)) \langle \Psi | P_n^{\hat{A}(1)} \otimes P_p^{\hat{B}(2)} | \Psi \rangle \qquad (17)$$

We can now define the expectation value of the product of the generalized observables $A_0(1)$ and $B_0(2)$ in the state S, as follows,

$$E(A_0(1), B_0(2)) = \sum_{n,p \in \aleph_0} a_n b_p\, p_S^t(a_n, b_p) \qquad (18)$$

By using Eq. (17) and considering only generalized observables such that $a_0 = b_0 = 0$ (hence, for every n,p $\in \aleph$, $a_n \neq 0 \neq b_p$), which is not restrictive, we get

$$E(A_0(1), B_0(2)) = \sum_{n,p \in \aleph} a_n b_p\, p_{\Psi n}^d(\hat{A}(1))\, p_{\Psi p}^d(\hat{B}(2)) \langle \Psi | P_n^{\hat{A}(1)} \otimes P_p^{\hat{B}(2)} | \Psi \rangle \qquad (19)$$

THE LOCAL REALISM ISSUE

The term local realism has been traditionally used to denote the join of the assumptions of "realism",

R: the values of all observables of a physical system in a given state are predetermined for any measurement context,

and "locality",

LOC: if measurements are made at places remote from one another on parts of a physical system which no longer interact, the specific features of one of the measurements do not influence the results obtained with the others.

The standard procedures leading to the Bell-Clauser-Horne-Shimony-Holt (BCHSH) inequality can be resumed as follows.

Ω: composite system made up of two far away subsystems Ω_1 and Ω_2.

$A(\vec{a})$ $(B(\vec{b}))$: dichotomic observable of Ω_1 (Ω_2) depending on the parameter $\vec{a}$ $(\vec{b})$ and taking either value -1 or 1.

THE BCHSH INEQUALITY

The expectation value of the product of the observables $A(\vec{a})$ and $B(\vec{b})$ in the state S is

$$E(\vec{a},\vec{b}) = \int_{\Lambda} d\lambda \rho(\lambda) A(\lambda,\vec{a}) B(\lambda,\vec{b}) \qquad (20)$$

λ: deterministic hidden variable whose value ranges over the domain Λ.

ρ(λ): probability density on Λ.

$A(\lambda,\vec{a}), B(\lambda,\vec{b}) = \pm 1$: values of the dichotomic observables $A(\vec{a}), B(\vec{b})$.

One gets, by assuming R and LOC,

$$\left|E(\vec{a},\vec{b}) - E(\vec{a},\vec{b}')\right| + \left|E(\vec{a}',\vec{b}) + E(\vec{a}',\vec{b}')\right| \leq 2 \qquad (21)$$

BCHSH inequality

THE MODIFIED BCHSH INEQUALITY I

The proof of Eq. (21) requires the assumption, usually left implicit, that ideal measurements are performed in which all physical objects that are prepared are detected. This condition does not hold in the ESR model, where the dichotomic observables $A(\vec{a}),\ B(\vec{b}),\ A(\vec{a}'),\ B(\vec{b}')$ must be substituted by the trichotomic observables $A_0(\vec{a}),\ B_0(\vec{b}),\ A_0(\vec{a}'),\ B_0(\vec{b}')$, respectively, in each of which a no-registration outcome is adjoined to the outcomes -1 and 1. Hence, the reasonings leading to the BCHSH inequality must be suitably modified if the perspective introduced by the ESR model is adopted.

By using the microscopic part of the ESR model and restricting to generalized observables whose no-registration outcomes are 0, one can show that the following **modified BCHSH inequality** holds.

$$\left|E(A_0(\vec{a}),B_0(\vec{b}))-E(A_0(\vec{a}),B_0(\vec{b}'))\right|+\left|E(A_0(\vec{a}'),B_0(\vec{b}))+E(A_0(\vec{a}'),B_0(\vec{b}'))\right|\leq 2 \qquad (22)$$

THE MODIFIED BCHSH INEQUALITY II

The modified BCHSH inequality replaces the BCHSH inequality within the ESR model.

To grasp the physical meaning of the modified BCHSH inequality, let us restrict to a set $\mathcal{O}_R$ of generalized observables such that, for every $A_0 \in \mathcal{O}_R$, the detection probability in a given state depends on A_0 but not on its specific value. Hence we can drop the dependence on n and p of the detection probabilities in Eq. (19). Since the generalized observables that we are considering can take only values -1, 0, 1, we get from Eq. (19)

$$E(A_0(\vec{a}),B_0(\vec{b})) = p_\Psi^d(\hat{A}(\vec{a}))p_\Psi^d(\hat{B}(\vec{b}))[\langle\Psi|P_1^{\hat{A}(\vec{a})}\otimes P_1^{\hat{B}(\vec{b})}|\Psi\rangle - \langle\Psi|P_1^{\hat{A}(\vec{a})}\otimes P_{-1}^{\hat{B}(\vec{b})}|\Psi\rangle + \langle\Psi|P_{-1}^{\hat{A}(\vec{a})}\otimes P_1^{\hat{B}(\vec{b})}|\Psi\rangle + \langle\Psi|P_{-1}^{\hat{A}(\vec{a})}\otimes P_{-1}^{\hat{B}(\vec{b})}|\Psi\rangle] = p_\Psi^d(\hat{A}(\vec{a}))p_\Psi^d(\hat{B}(\vec{b}))\langle\hat{A}(\vec{a})\hat{B}(\vec{b})\rangle_\Psi \quad (23)$$

Quantum expectation value of the product of $A(\vec{a})$ and $B(\vec{b})$, or *conditional expectation value* of the product of $A_0(\vec{a})$ and $B_0(\vec{b})$, in the state S represented by the unit vector $|\Psi\rangle$.

THE MODIFIED BCHSH INEQUALITY III

Similar equations hold if one considers the other expectation values that appear in Eq. (22). We thus obtain from Eq. (22)

$$p_\Psi^d(\hat{A}(\vec{a}))\left| p_\Psi^d(\hat{B}(\vec{b}))\left\langle \hat{A}(\vec{a})\hat{B}(\vec{b})\right\rangle_\Psi - p_\Psi^d(\hat{B}(\vec{b}'))\left\langle \hat{A}(\vec{a})\hat{B}(\vec{b}')\right\rangle_\Psi \right| + \\ + p_\Psi^d(\hat{A}(\vec{a}'))\left| p_\Psi^d(\hat{B}(\vec{b}))\left\langle \hat{A}(\vec{a}')\hat{B}(\vec{b})\right\rangle_\Psi + p_\Psi^d(\hat{B}(\vec{b}'))\left\langle \hat{A}(\vec{a}')\hat{B}(\vec{b}')\right\rangle_\Psi \right| \leq 2 \qquad (24)$$

Eq. (24) contains four detection probabilities and four conditional expectation values. The latter can be calculated by using the rules of QM (assumption AX), and formally coincide with expectation values of QM. If one puts them into Eq. (24) the inequality must be interpreted as a consistency condition that must be fulfilled by the detection probabilities in the ESR model.

We have as yet no theory allowing us to calculate the detection probabilities. Nevertheless, should one be able to perform measurements that are close to ideality, the detection probabilities could be determined experimentally and then inserted into the modified BCHSH inequality.

IMPLICATIONS OF THE MODIFIED BCHSH INEQUALITY

Two possibilities occur.

- There exist states and observables such that the conditional expectation values violate the modified BCHSH inequality. In this case, the ESR model and/or the additional assumptions introduced to obtain Eq. (24) are refuted.
- For every choice of states and observables the conditional expectation values fit in with the modified BCHSH inequality. In this case, the ESR model is confirmed, hence no conflict emerges between R and LOC, which hold in the model, and the reinterpreted quantum probabilities, which are embodied in it.

The above results show that the ESR model is, at least in principle, falsifiable.

Determining a detection probability: **experimental difficulties**.

- Counting the number of physical objects that are actually prepared.
- Distinguishing the detection probability from the efficiency of the experimental apparatus.

A "CONCILIATORY" RESULT

The ESR model can be further elaborated by introducing, in particular, microscopic observables and their expectation values. One can then show that the BCHSH inequality holds at a microscopic level (which is purely theoretical and cannot be experimentally checked).

On the other side, we have just proven that the modified BCHSH inequality holds at a macroscopic level whenever all physical objects that are actually produced are considered (which can be experimentally checked).

Finally, it follows from assumption AX that the quantum predictions deduced by using standard QM rules hold at a macroscopic level whenever only detected physical objects are considered (which can be experimentally checked).

It follows that the **BCHSH inequality**, the **modified BCHSH inequality** and the **quantum inequalities** do not conflict, but rather pertain to different parts of the picture provided by the ESR model.

BOUNDS ON DETECTION PROBABILITIES: A SPECIAL CASE I

Let Ω be a composite physical system made up of two far apart spin-1/2 quantum particles 1 and 2, in the singlet spin state S represented by the unit vector $|\eta\rangle = \frac{1}{\sqrt{2}}\left(|+,-\rangle - |-,+\rangle\right)$

$A(\vec{a})$ $(A(\vec{a}'))$: observable "spin of particle 1 along $\vec{a}$ $(\vec{a}')$", represented by $\vec{\sigma}(1)\cdot\vec{a} = \sigma_a(1)$ $(\vec{\sigma}(1)\cdot\vec{a}' = \sigma_{a'}(1))$ in QM.

$B(\vec{b})$ $(B(\vec{b}'))$: observable "spin of particle 2 along $\vec{b}$ $(\vec{b}')$", represented by $\vec{\sigma}(2)\cdot\vec{b} = \sigma_b(2)$ $(\vec{\sigma}(2)\cdot\vec{b}' = \sigma_{b'}(2))$ in QM.

By applying standard quantum rules for probabilities one gets from Eq. (23), by considering the pair $(A_0(\vec{a}), B_0(\vec{b}))$,

$$E(A_0(\vec{a}), B_0(\vec{b})) = p^d_\eta(\sigma_a(1))\, p^d_\eta(\sigma_b(2))(-\vec{a}\cdot\vec{b}) \qquad (25)$$

Similar equations hold for the remaining pairs.

BOUNDS ON DETECTION PROBABILITIES: A SPECIAL CASE II

The rotational invariance of the vector |η> and the choice of the observables suggest that all detection probabilities have the same value in the singlet spin state, say e_η. Hence, Eq. (24) yields

$$e_\eta^2 \leq \frac{2}{\left|-\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{b}'\right|+\left|-\vec{a}'\cdot\vec{b}-\vec{a}'\cdot\vec{b}'\right|} \qquad (26)$$

Since e_η does not depend on $\vec{a}, \vec{a}', \vec{b}, \vec{b}'$, Eq. (26) provides the upper bound

$$e_\eta \leq \frac{1}{\sqrt[4]{2}} \approx 0.841 \qquad (27)$$

Eq. (27) implies that no spin measurement on particle 1 or 2 can have a detection efficiency greater than 0.841, hence it shows that, notwithstanding the difficulties pointed out above, there are special cases in which the predictions of the ESR model differ in a substantial way from those of QM. Should Eq. (27) be contradicted by experimental data, the ESR model, or the "reasonable" assumptions that we have introduced, or both, would be falsified. If not, one can consider this result as a clue that the ESR model is correct.

THE MODIFIED BELL INEQUALITY AND ITS PHYSICAL PREDICTIONS I

We have recently proved that the original *Bell inequality* can be dealt with in a similar way. Indeed, let us consider the deterministic ESR model, replace the directions $\vec{a}, \vec{a}', \vec{b}, \vec{b}'$ with the directions $\vec{a}, \vec{b}, \vec{c}$ and assume that a perfect correlation law holds for detected objects in the ESR model, that is, whenever an idealized measurement of $A_0(\vec{a})$ on particle 1 yields outcome +1 (-1), then a simultaneous idealized measurement of the same generalized observable on particle 2 yields outcome -1 (+1), and viceversa, if both particles are detected.

Then, by using the same symbols introduced in the case of the BCHSH inequality, one gets the following *modified Bell inequality*:

$$\left| E(A_0(\vec{a}), B_0(\vec{b})) - E(A_0(\vec{a}), B_0(\vec{c})) \right| \leq 1 + E(A_0(\vec{b}), B_0(\vec{c})) \qquad (28)$$

THE MODIFIED BELL INEQUALITY AND ITS PHYSICAL PREDICTIONS II

Let us now consider again the singlet spin state represented by the unit vector |η> and assume again that all detection probabilities have the same value in this state, say e_η. Then, we get

$$e_\eta \leq \sqrt{\frac{2}{3}} \approx 0.8165 \qquad (29)$$

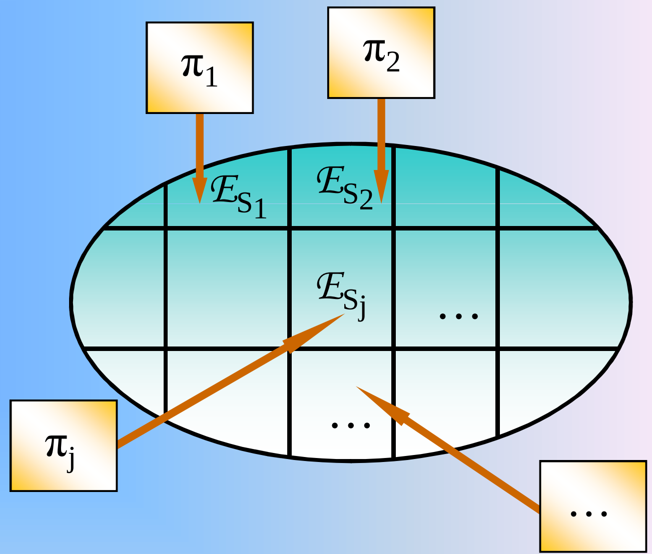
We have thus obtained another prediction which can be in principle be confirmed or falsified by experimental data.

Eq. (29) provides a limit which is similar to the limits obtained in other h.v. theories for QM, but its interpretation is quite different. In fact, most standard h.v. theories maintain that local realism contradicts QM, and establish a lower limit for the efficiency of any real measurement intended to decide whether local realism or QM is correct. Should the efficiency be smaller than this limit, the measurement could not be suitable for distinguishing the two alternatives. On the contrary local realism and reinterpreted QM coexist in the ESR model, and e_η constitutes an upper limit for the efficiency of any (even idealized) measurement.

OPERATIONAL DEFINITIONS OF PROPER MIXTURES

A typical *preparation procedure* of a physical object x in a state M that is a mixture of the pure states S_1, S_2, …, represented by the density operators $\rho_{\psi_1}, \rho_{\psi_2}$, …, with probabilities p_1, p_2, …, respectively, can be described as follows.

Choose a preparing device π_j for every pure state S_j, use each π_j to prepare an ensemble $\mathcal{E}_{S_j}$ of n_j physical objects in the pure state S_j and choose n_j such that $n_j = np_j$, with $n = \Sigma_j n_j$. Then mingle the ensembles $\mathcal{E}_{S_1}, \mathcal{E}_{S_2}$, … to prepare an ensemble $\mathcal{E}_M$ of n physical objects, remove any memory of the way in which the ensembles $\mathcal{E}_{S_1}, \mathcal{E}_{S_2}$, … have been mingled and select a physical object x in $\mathcal{E}_M$.



The class of preparation procedures obtained proceeding as above and selecting the preparing devices in the states S_1, S_2, …, in all possible ways will be denoted by σ_M and called *operational definition* of M in the following.

THE STANDARD INTERPRETATION OF PROPER MIXTURES IN QM I

The operational definition of the mixture M implies that the probabilities p_1, p_2, … are interpreted as *epistemic*, that is, formalizing the loss of memory about the pure state in which each physical object has been actually prepared (*ignorance interpretation*).

In QM states are defined as classes of probabilistically equivalent preparing devices, and every equivalence class is represented by a density operator. But the density operator representing a given mixture can be decomposed in different ways as a convex combination of density operators representing pure states (*nonunique decomposition of quantum mixtures*), which is a source of interpretative problems. Indeed, the operational definition of M implies that M can be represented in QM by the density operator $\rho_M=\Sigma_j p_j \rho_{\psi_j}$. But one-dimensional projection operators ρ_{χ_1}, ρ_{χ_2}, … generally exist, none of which coincides with one of the projection operators ρ_{ψ_1}, ρ_{ψ_2}, …, which are such that $\rho_M=\Sigma_l q_l \rho_{\chi_l}$, with $0 \leq q_l \leq 1$ and $\Sigma_l q_l=1$. If this expression of ρ_M is adopted, the coefficients q_l cannot be interpreted as probabilities bearing an ignorance interpretation.

THE STANDARD INTERPRETATION OF PROPER MIXTURES IN QM II

Let now M′ be a mixture of the pure states T_1, T_2, … represented by the density operators ρ_{χ_1}, ρ_{χ_2}, …, with probabilities q_1, q_2, …, respectively. M′ has an operational definition $\sigma_{M'}$ which is different from σ_M. According to QM, M and M′ must be identified because they are represented by the same density operator. But the probabilities q_1, q_2, … now admit an ignorance interpretation, which contradicts the conclusion obtained when M is considered.

Many scholars therefore maintain that an ignorance interpretation of the probabilities that appear in the various possible expressions of ρ_M must be avoided, rejecting the interpretation of the probabilities p_j and q_l following from the operational definitions σ_M and $\sigma_{M'}$. Other authors instead maintain that the standard representation of proper mixtures does not account for some physically relevant differences, which generates the ambiguities in the interpretation of the formalism of QM.

OPERATIONAL DEFINITIONS OF IMPROPER MIXTURES I

We have considered so far physical objects that are prepared by activating preparing devices that produce examples of a given physical system Ω. But examples of Ω can also be obtained by preparing examples of a composite physical system Γ such that Ω is a subsystem of Γ. A preparation procedure of this kind can be described as follows.

Consider a composite physical system Γ made up of two subsystems Ω and Δ. Choose a preparing device $\pi \in S$, with S a pure state of Γ, prepare a set $\mathcal{E}_S$ of individual examples of Γ, select an element of $\mathcal{E}_S$ and consider the part x of it that constitutes an individual example of Ω.

One can attribute a state N to x which is represented in QM by a density operator ρ_N obtained by tracing over the one-dimensional projection operator representing S. If ρ_N also is a one-dimensional projection operator, N is considered as a pure state in a standard sense. Otherwise N is said to be an *improper mixture*.

OPERATIONAL DEFINITIONS OF IMPROPER MIXTURES II

In the latter case the preparation procedure does not privilege any convex decomposition of ρ_N into one-dimensional projection operators representing pure states of Ω, hence N can be considered as a mixture of pure states in (infinitely) many different ways, with coefficients that can be interpreted as probabilities but never admit an ignorance interpretation (hence probabilities are not epistemic in this case).

The distinction between proper and improper mixtures is often overlooked by physicists because the two kinds of mixtures have the same mathematical representations in QM. But the preparation procedures imply that proper and improper mixtures are empirically distinguishable, as stressed by some authors. Indeed, if one prepares an ensemble $\mathcal{E}_N$ of physical objects in the improper mixture N, every subensemble of $\mathcal{E}_N$ has the same statistical properties possessed by $\mathcal{E}_N$ (that is, it is a fair sample of $\mathcal{E}_N$), which does not occur if one prepares an ensemble $\mathcal{E}_M$ of physical objects in the proper mixture M.

GENERALIZED OBSERVABLES AND PROPER MIXTURES

Let M be a proper mixture of the pure states S_1, S_2, …, represented by the density operators $\rho_{\psi_1}, \rho_{\psi_2}, \ldots$, with probabilities p_1, p_2, …, respectively. The probability $p_M^t((A_0, X))$ that a measurement of the generalized observable A_0 on a physical object x in the state M yield an outcome in the Borel set $X \in \mathbb{B}(\Re)$, with $a_0 \notin X$, or, equivalently, the probability $p_M^t(F)$ that x in the state M display the macroscopic property $F=(A_0,X) \in \mathcal{F}$, is given by

$$\mathrm{p}_{\mathrm{M}}^{\mathrm{t}}((A_0, X)) = \mathrm{p}_{\mathrm{M}}^{\mathrm{t}}(F) = \sum_j p_j p_{S_j}{}^t(F) = \sum_j p_j p_{S_j}{}^d(F) p_{S_j}(F) \qquad (30)$$

Overall probability that x display F whenever it is in the pure state S_j

Probability that x be detected whenever it is in the pure state S_j and F is measured

Conditional probability that x display F whenever it is in the pure state S_j and it is detected

CONDITIONAL PROBABILITY AND PROPER MIXTURES I

Hence we get, using Eq. (1)

$$p_M(F) = \sum_j p_j \frac{p_{S_j}{}^d(F)}{p_M{}^d(F)} p_{S_j}(F) \qquad (31)$$

This term can be interpreted, because of the Bayes theorem, as the conditional probability that x be in the state S_j whenever F is measured and x is detected.

Eq. (31) can be rewritten by using the mathematical representations of pure states and generalized observables. Indeed, assumption AX yields

$$p_{S_j}(F) = Tr[\rho_{\psi_j} P^{\hat{A}}(X)] \qquad (32)$$

Spectral PV measure associated with the self-adjoint operator Â representing the observable A of QM from which A_0 is obtained.

CONDITIONAL PROBABILITY AND PROPER MIXTURES II

Hence we obtain from Eq. (31)

$$\mathrm{p}_{\mathrm{M}}(F) = Tr[\rho_M(F) P^{\hat{A}}(X)] \quad (33)$$

with

$$\rho_M(F) = \sum_j p_j \frac{p^d_{S_j}(F)}{p^d_M(F)} \rho_{\psi_j} \quad (34)$$

Let us now introduce the obvious assumption

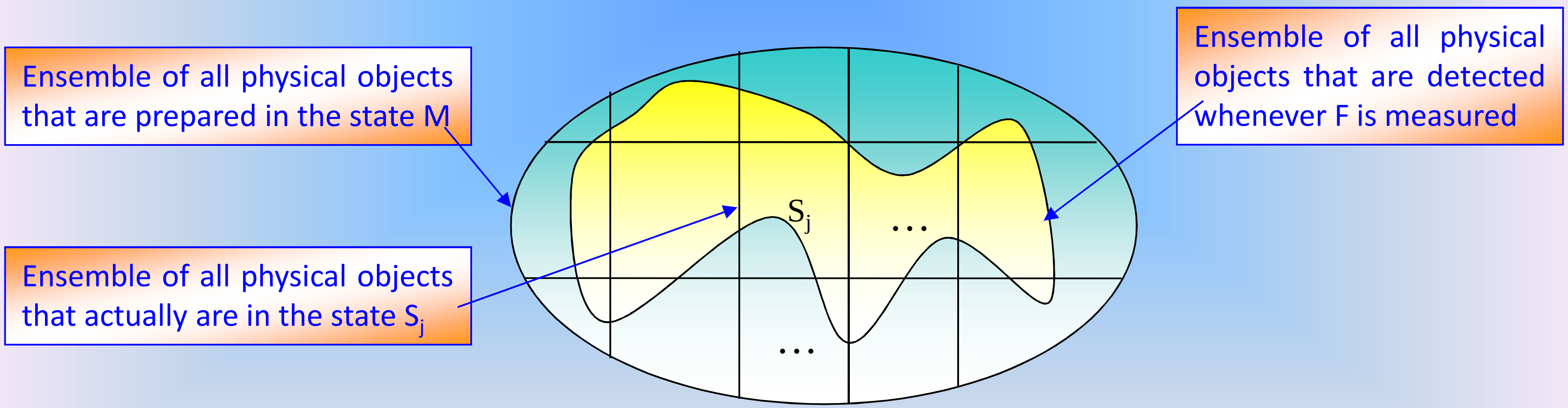
$$\mathrm{p}^{\mathrm{d}}_{\mathrm{M}}(F) = \sum_j p_j \mathrm{p}^{\mathrm{d}}_{\mathrm{S_j}}(F) \quad (35)$$

Then we get from Eqs. (7), (32) and (34)

$$\rho_{\mathrm{M}}(F) = \frac{\sum_j p_j \frac{Tr[\rho_{\psi_j} T^{\hat{A}}_{\psi_j}(X)]}{Tr[\rho_{\psi_j} P^{\hat{A}}(X)]} \rho_{\psi_j}}{\sum_j p_j \frac{Tr[\rho_{\psi_j} T^{\hat{A}}_{\psi_j}(X)]}{Tr[\rho_{\psi_j} P^{\hat{A}}(X)]}} \quad (36)$$

CONDITIONAL PROBABILITY AND PROPER MIXTURES III

Eqs. (33) and (36) show that the conditional probability $p_S(F)$ does not coincide, in general, with the probability obtained by applying standard QM rules, *i.e.*, calculating $Tr[\rho_M P^{\hat{A}}(X)]$, with $\rho_M = \sum_j p_j \rho_{\psi_j}$. This can be explained by observing that, if a macroscopic property F is measured on an ensemble of physical objects prepared in the state M, then the ensemble of detected objects depends on F and generally is not a fair sample of the set of all prepared objects. Hence, as far as $p_M(F)$ is concerned, M must be represented by a density operator that depends on F and coincides with ρ_M only in special cases.



OVERALL PROBABILITY AND PROPER MIXTURES

Let us come to the overall probability $p_M^t(F)$. By using Eq. (3) we get, for every $F=(A_0,X) \in \mathcal{F}$,

$$p^t_{S_j}(F) = Tr[\rho_{\psi_j} T^{\hat{A}}_{\psi_j}(X)] \qquad (37)$$

Hence we obtain from Eq. (30)

$$p^t_M(F) = Tr\left[\sum_j p_j \rho_{\psi_j} T^{\hat{A}}_{\psi_j}(X)\right] \qquad (38)$$

or, equivalently, because of Eqs. (1) and (33),

$$p^t_M(F) = Tr[p^d_M(F)\rho_M(F) P^{\hat{A}}(X)] \qquad (39)$$

THE REPRESENTATION OF PROPER MIXTURES

Eqs. (33) and (39) show that, for every F$\in \mathcal{F}$, one needs $\rho_M(F)$ to calculate $p_M(F)$ and both $p_M^d(F)$ and $\rho_M(F)$ to calculate $p_M^t(F)$. Hence a complete mathematical representation of the proper mixture M is provided in the ESR model by the family of pairs

$$\left\{\left(\rho_M(F), p_M^d(F)\right)\right\}_{F\in\mathcal{F}} = \left\{\left(\frac{\sum_j p_j p_{S_j}^d(F)\rho_{\psi_j}}{\sum_j p_j p_{S_j}^d(F)}, \sum_j p_j p_{S_j}^d(F)\right)\right\}_{F\in\mathcal{F}} \qquad (40)$$

The representation in Eq. (40) depends on the operational definition σ_M of M, hence a state M′ such that $\sigma_{M'} \neq \sigma_M$ is *generally* different from M (in the sense that M and M′ lead to different probabilistic predictions). One can then assume that physics is such that M′ is *necessarily* different from M, so that operational definitions and mathematical representations of proper mixtures are in one-to-one correspondence. Thus, no physical information is lost, and the ambiguities that occur in QM disappear.

THE REPRESENTATION OF IMPROPER MIXTURES I

Let Γ be a composite system made up of the subsystems Ω and Δ associated with the Hilbert spaces $\mathcal{H}$ and $\mathcal{G}$, respectively, so that Γ is associated with the Hilbert space $\mathcal{H}\otimes\mathcal{G}$. Let S be a pure state of Γ represented by the unit vector $|\Psi\rangle$, or by the projection operator $\rho_\Psi=|\Psi\rangle\langle\Psi|$, and let F = ($A_0$, X) ∈ $\mathcal{F}$ be a property of Ω represented by the pair ($P^{\hat{A}}(X)$, $\{T_\psi^{\hat{A}}(X)\}_{|\psi\rangle\in\mathcal{V}}$). Then F corresponds to a property $\tilde{F}$ of Γ, and one gets from assumption AX and standard calculations

$$p_S(\tilde{F}) = Tr[\rho_\Psi P^{\hat{A}}(X)\otimes I] = Tr[\rho_N P^{\hat{A}}(X)] \quad (41)$$

with ρ_N a density operator obtained by tracing ρ_Ψ over Δ. Moreover, one gets

$$p_S^t(\tilde{F}) = Tr[\rho_\Psi T_\Psi^{\hat{A}\otimes I}(X)] \quad (42)$$

$$T_\Psi^{\hat{A}\otimes I}(X) = \int_X p_\Psi^d(\hat{A}\otimes I, \lambda)\, dP_\lambda^{\hat{A}\otimes I} \quad (43)$$

Let us now put

$$p_\Psi^d(\hat{A}\otimes I, \lambda) = p_{\rho_N}^d(\hat{A}, \lambda)$$

$$T_{\rho_N}^{\hat{A}}(X) = \int_X p_{\rho_N}^d(\hat{A}, \lambda)\, dP_\lambda^{\hat{A}}$$

THE REPRESENTATION OF IMPROPER MIXTURES II

It follows

$$T_{\Psi}^{\hat{A}\otimes I}(X) = \int_X p_{\rho_N}^d(\hat{A},\lambda)(dP_\lambda^{\hat{A}} \otimes I) = T_{\rho_N}^{\hat{A}}(X) \otimes I \quad (44)$$

hence, substituting in Eq. (42) and tracing over Δ,

$$p_S^t(\tilde{F}) = Tr[\rho_N T_{\rho_N}^{\hat{A}}(X)] \quad (45)$$

Bearing in mind the operational definition of improper mixtures one can now consider the preparation of an example of Γ as a preparation of an example of Ω in a state N which is an improper mixture, and put $p_S(\tilde{F}) = p_N(F), p_S^t(\tilde{F}) = p_N^t(F)$. Eqs. (41) and (45) then show that the density operator ρ_N provides the mathematical representation of N. This representation coincides with the quantum representation of N and is basically different from the representation of a proper mixture, which entails that the ESR model neatly distinguishes proper from improper mixtures.

Eqs. (41) and (45) suggest that improper mixtures could be considered as generalized pure states in the ESR model, consistently enlarging the mathematical representation of generalized observables by replacing ψ with ρ_N in Eq. (3). Moreover, assumption AX can be extended to improper mixtures, which allows one to recover the quantum formalism for mixtures in the ESR model and suggests that QM actually deals only with improper mixtures.

TESTABLE PREDICTIONS AND PROPER MIXTURES I

Let the physical system Ω be a spin-1/2 quantum particle and let Σ_z be the quantum observable "spin of Ω along the z-axis". Let S_+ and S_- be the eigenstates corresponding to the eigenvalues +1 and -1 of Σ_z, represented by the projection operators $|+\rangle\langle+|$ and $|-\rangle\langle-|$, respectively, and let M be a proper mixture of S_+ and S_- with probabilities p_+ and $p_- = 1-p_+$, respectively.

Then M is represented in the ESR model by the family of pairs

$$\{(p_+ \frac{p^d_{S_+}(F)}{p^d_M(F)}|+\rangle\langle+| + p_- \frac{p^d_{S_-}(F)}{p^d_M(F)}|-\rangle\langle-|, p^d_M(F))\}_{F\in\mathcal{F}} \quad (46)$$

$$p^d_M(F) = p_+ p^d_{S_+}(F) + p_- p^d_{S_-}(F)$$

Let Σ_n be the quantum observable "spin of Ω along the direction n", let S_n be the eigenstate corresponding to the eigenvalue +1 of Σ_n, represented by the projection operator $|+_n\rangle\langle+_n|$, and let the property $F_n=(\Sigma_n, \{+1\})$ be measured on an individual example of Ω in the state M.

TESTABLE PREDICTIONS AND PROPER MIXTURES II

Then

Conditional probability $p_M(F_n)$

$$p_M(F_n) = Tr[(p_+ \frac{p^d_{S_+}(F)}{p_+ p^d_{S_+}(F) + p_- p^d_{S_-}(F)} |+\rangle\langle+| + p_- \frac{p^d_{S_-}(F)}{p_+ p^d_{S_+}(F) + p_- p^d_{S_-}(F)} |-\rangle\langle-|) |+_n\rangle\langle+_n|] \quad (47)$$

Overall probability $p_M^t(F_n)$

$$p^t_M(F_n) = Tr[(p_+ p^d_{S_+}(F) |+\rangle\langle+| + p_- p^d_{S_-}(F) |-\rangle\langle-|) |+_n\rangle\langle+_n|] \quad (48)$$

The probability $p_M^Q(F_n)$ that an example of Ω in the state M yield the yes outcome when an ideal measurement of F_n is performed on it is given in QM by

$$p^Q_M(F_n) = Tr[(p_+ |+\rangle\langle+| + p_- |-\rangle\langle-|) |+_n\rangle\langle+_n|] \quad (49)$$

TESTABLE PREDICTIONS AND PROPER MIXTURES III

$$p_M^Q(F_n) = p_M(F_n) \Leftrightarrow p_{S_+}^d(F_n) = p_{S_-}^d(F_n)$$
$$p_M^Q(F_n) = p_M^t(F_n) \Leftrightarrow p_{S_+}^d(F_n) = p_{S_-}^d(F_n) = 1 \quad (50)$$

but, in general

$$p_M(F_n) \neq p_M^Q(F_n) \neq p_M^t(F_n) \quad (51)$$

The predictions of the ESR model do not coincide with the predictions of QM. One can then check Eq. (51) with different choices of n (one should construct measurements that are very close to idealized measurements). Should the predictions of QM be violated one would get a clue in favor of the ESR model, and try to determine experimentally the unknown parameters $p_{S+}^d(F_n)$ and $p_{S-}^d(F_n)$, then checking Eqs. (47) and (48). Should instead the predictions of QM be fulfilled, one must remind that $p_M^Q(F_n)$ expresses an overall probability. Because of Eq. (50) the obtained result would be compatible with the ESR model only if $p_M^t(F_n) = p_M^Q(F_n)$ for all choices of n, which betrays the spirit of the ESR model and can be seen as a falsification of it.

THE GENERALIZED LÜDERS POSTULATE I

GPP can now be extended to proper mixtures by introducing the following *generalized Lüders postulate*.

GLP. Let M be a proper mixture of the pure states S_1, S_2, …, represented by the density operator ρ_{ψ_1}, ρ_{ψ_2}, …, with probabilities p_1, p_2, …, respectively, and let a nondestructive idealized measurement of a property $F=(A_0,X)\in \mathcal{F}_0$ be performed on a physical object x in the state M.

Let the measurement yield the yes outcome. Then, the state M_F of x after the measurement is a proper mixture of the pure states S_{1F}, S_{2F}, …, represented by the density operators $\rho_{\psi_{1F}}$, $\rho_{\psi_{2F}}$, …, respectively, with

$$\rho_{\psi_{jF}} = \frac{T_{\psi_j}^{\hat{A}}(X)\rho_{\psi_j}T_{\psi_j}^{\hat{A}\dagger}(X)}{\mathrm{Tr}[T_{\psi_j}^{\hat{A}}(X)\rho_{\psi_j}T_{\psi_j}^{\hat{A}\dagger}(X)]} \qquad (j=1,2,\ldots) \tag{52}$$

and probabilities p_{1F}, p_{2F},…, respectively, with

$$p_{jF} = p_j \frac{p_{S_j}^t((A_0,X))}{p_M^t((A_0,X))} = p_j \frac{Tr[\rho_{\psi_j}T_{\psi_j}^{\hat{A}}(X)]}{Tr\left[\sum_j p_j \rho_{\psi_j} T_{\psi_j}^{\hat{A}}(X)\right]} \qquad (j=1,2,\ldots) \tag{53}$$

Conditional probability that x be in the state S_{jF} whenever F is measured and the yes outcome is obtained (Bayes' theorem).

THE GENERALIZED LÜDERS POSTULATE II

Hence M_F is represented by the family of pairs

$$\{(\rho_{M_F}(H), p^d_{M_F}(H))\}_{H\in\mathcal{F}} = \left\{\left(\sum_j p_{jF}\frac{p^d_{S_{jF}}(H)}{p^d_{M_F}(H)}\rho_{\psi_{jF}}, \sum_j p_{jF}\, p^d_{S_{jF}}(H)\right)\right\}_{H\in\mathcal{F}} \qquad (54)$$

Let the measurement yield the no outcome. Then, the state M'_F of x after the measurement is a mixture of the pure states S'_{1F}, S'_{2F}, …, represented by the density operators $\rho_{\psi'_{1F}}$, $\rho_{\psi'_{2F}}$, …, respectively, with

$$\rho_{\psi'_{jF}} = \frac{T^{\hat{A}}_{\psi_j}(\Re\setminus X)\rho_{\psi_j}T^{\hat{A}\dagger}_{\psi_j}(\Re\setminus X)}{\mathrm{Tr}[T^{\hat{A}}_{\psi_j}(\Re\setminus X)\rho_{\psi_j}T^{\hat{A}\dagger}_{\psi_j}(\Re\setminus X)]} \quad (j=1,2,\ldots) \qquad (55)$$

and probabilities p'_{1F}, p'_{2F},…, respectively, with

Conditional probability that x be in the state S'_{jF} whenever F is measured and the no outcome is obtained (Bayes' theorem).

$$p'_{jF} = p_j\frac{p^t_{S_j}((A_0,\Re\setminus X))}{p^t_M((A_0,\Re\setminus X))} = p_j\frac{Tr[\rho_{\psi_j}T^{\hat{A}}_{\psi_j}(\Re\setminus X)]}{Tr\left[\sum_j p_j\rho_{\psi_j}T^{\hat{A}}_{\psi_j}(\Re\setminus X)\right]} \quad (j=1,2,\ldots) \qquad (56)$$

THE GENERALIZED LÜDERS POSTULATE III

Hence M'_F is represented by the family of pairs

$$\{(\rho_{M'_F}(H), p^d_{M'_F}(H))\}_{H\in\mathcal{F}} = \left\{\left(\sum_j p'_{jF} \frac{p^d_{S'_{jF}}(H)}{p^d_{M'_F}(H)} \rho_{\psi'_{jF}}, \sum_j p'_{jF}\, p^d_{S'_{jF}}(H)\right)\right\}_{H\in\mathcal{F}} \qquad (57)$$

It is now interesting to observe that the standard representation of M_F in QM is provided by the density operator

$$\rho_{M_F} = \sum_j p_j \frac{Tr[\rho_{\psi_j} T^{\hat{A}}_{\psi_j}(X)]}{Tr\left[\sum_j p_j \rho_{\psi_j} T^{\hat{A}}_{\psi_j}(X)\right]} \frac{T^{\hat{A}}_{\psi_j}(X)\rho_{\psi_j} T^{\hat{A}\dagger}_{\psi_j}(X)}{\mathrm{Tr}[T^{\hat{A}}_{\psi_j}(X)\rho_{\psi_j} T^{\hat{A}\dagger}_{\psi_j}(X)]} \qquad (58)$$

A similar formula with $\Re\backslash X$ in place of X, holds if one considers the density operator representing M'_F. But we stress that if a further measurement of a property $H\in\mathcal{F}$ is performed, conditional and overall probabilities of H in the state M_F (M'_F) cannot be evaluated by using ρ_{M_F} ($\rho_{M'_F}$). One must calculate instead $\rho_{M_F}(H)$ and $p_{M_F}{}^d(H)$ ($\rho_{M'_F}(H)$ and $p_{M'_F}{}^d(H)$) and then apply Eqs. (33) and (39), respectively.

Thank You for

Your Attention!

NONSELECTIVE MEASUREMENTS IN THE DISCRETE CASE I

Whenever the discrete generalized observable A_0 instead of the property $F=(A_0,X)$ is measured on the physical object x in the pure state S, one obtains one of the outcomes a_0, a_1, a_2, ... and a natural extension of GPP consists in assuming that, if the outcome a_n is obtained and the measurement is idealized and nondestructive, the final state of the object is given by Eq. (12) and the probabilities of the properties F_0, F_1, F_2, … by Eq. (9). This assumption will be kept in the following. Hence, in particular, if the measurement is *nonselective*, that is, *the actual outcome of the measurement remains unknown*, the final state of x is the mixture $\tilde{M}$ whose representation in the ESR model is provided by the family of density operators

$$\{\sum_{n\in\aleph_0} p_S^t(F_n)\frac{p^d_{S_{F_n}}(F)}{p^d_{\tilde{S}}(F)}|\psi_{F_n}\rangle\langle\psi_{F_n}|\}_{F\in\mathcal{F}} \qquad (58)$$

NONSELECTIVE MEASUREMENTS IN THE DISCRETE CASE II

The representation of $\tilde{M}$ as an improper mixture is instead provided by the density operator

$$\tilde{\rho} = p_S^t(F_0)|\psi_{F_0}\rangle\langle\psi_{F_0}| + \sum_{n\in\aleph} p_{\psi n}^d(\hat{A})P_n^{\hat{A}}|\psi\rangle\langle\psi|P_n^{\hat{A}} \quad (59)$$

Let $\mathcal{S}_1$, $\mathcal{S}_2$, …, be the eigenspaces associated with the eigenvalues a_1, a_2, …, respectively, of the self-adjoint operator Â representing the observable A of QM from which A_0 is obtained, and let $\{|a_n^\mu\rangle\}_{n,\mu}$ be an orthonormal (ON) basis in $\mathcal{H}$, where $\{|a_n^\mu\rangle\}_\mu$ is an ON basis in $\mathcal{S}_n$. Then

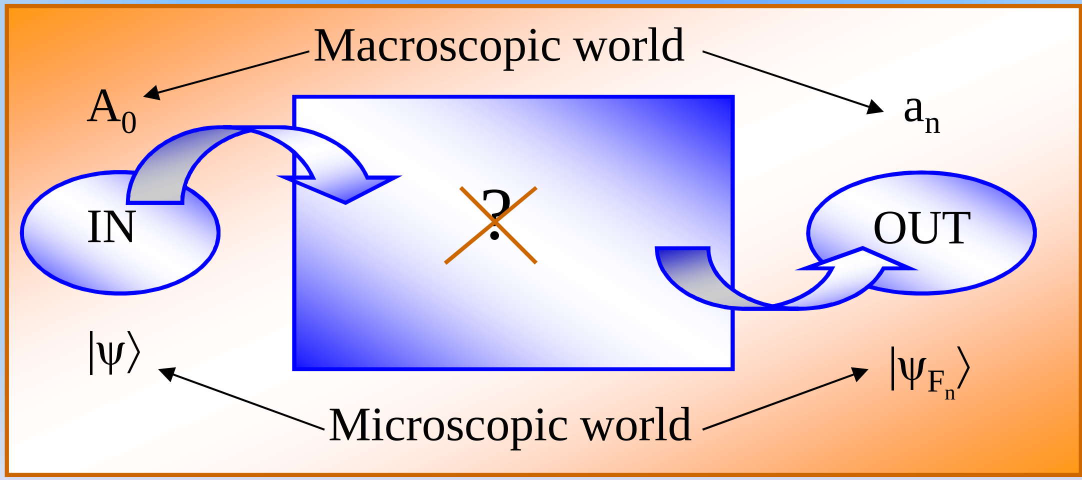
$$|\psi\rangle = \sum_{n\in\aleph}\sum_{\mu} c_n^\mu |a_n^\mu\rangle \quad (60)$$

hence

$$\tilde{\rho} = p_S^t(F_0)|\psi_{F_0}\rangle\langle\psi_{F_0}| + \sum_{n\in\aleph} p_{\psi n}^d(\hat{A})\sum_{\mu,\nu} c_n^\mu c_n^{\nu *}|a_n^\mu\rangle\langle a_n^\nu| \quad (61)$$

A DYNAMICAL DESCRIPTION OF THE MEASUREMENT PROCESS I

If one limits himself to consider pure states and discrete generalized observables only, GPP can be partially justified by introducing a reasonable physical assumption on the evolution of the composite system made up of the (microscopic) measured object plus the (macroscopic) measuring apparatus. From the point of view of the quantum measurement theory this justification rests on a measurement scheme that is exceedingly simple and idealized, but its simplicity will allow us to better grasp the conceptual novelties introduced by our approach.



A DYNAMICAL DESCRIPTION OF THE MEASUREMENT PROCESS II

(i) Let us consider the macroscopic apparatus measuring A_0 as an individual example of a macroscopic physical system Ω_M associated with the Hilbert space $\mathcal{H}_M$.

(ii) Let $|0\rangle$, $|1\rangle$, $|2\rangle$, … be unit vectors of $\mathcal{H}_M$ representing the macroscopic states of the apparatus and corresponding to the outcomes a_0, a_1, a_2, …, respectively (hence $|0\rangle$ represents the macroscopic state of the apparatus when it is ready to perform a measurement or when x is not detected).

(iii) Suppose that $\{|0\rangle, |1\rangle, |2\rangle, \ldots\}$ is an ON basis in $\mathcal{H}_M$.

(iv) Consider the composite system made up of the physical system Ω and the measuring apparatus Ω_M, in an initial state S_0 represented by the unit vector $|\Psi_0\rangle = |\psi\rangle|0\rangle$, and assume that it undergoes the (generally *nonlinear*) time evolution

$$\left|\psi\right\rangle\left|0\right\rangle = \sum_{n\in\aleph}\sum_{\mu} c_n^{\mu}\left|a_n^{\mu}\right\rangle\left|0\right\rangle \rightarrow \left|\Psi\right\rangle = \sum_{n\in\aleph}\alpha_{\psi n}\sum_{\mu} c_n^{\mu}\left|a_n^{\mu}\right\rangle\left|n\right\rangle + \beta_{\psi 0}\left|\psi_{F_0}\right\rangle\left|0\right\rangle \qquad (62)$$

A DYNAMICAL DESCRIPTION OF THE MEASUREMENT PROCESS III

where

$$\alpha_{\psi n} = \sqrt{p^d_{\psi n}(\hat{A})}e^{i\theta_{\psi n}} \quad , \quad \theta_{\psi n} \in \Re$$

$$\beta_{\psi 0} = \sqrt{p^t_S(F_0)}e^{i\varphi_{\psi 0}} \quad , \quad \varphi_{\psi 0} \in \Re$$

$$\langle \Psi | \Psi \rangle = \sum_{n \in \aleph} |\alpha_{\psi n}|^2 \sum_{\mu} |c^\mu_n|^2 + |\beta_{\psi 0}|^2 = 1$$

The final state of the compound system after the interaction is also represented by the density operator

$$\rho_C = |\Psi\rangle\langle\Psi| = (\sum_{n \in \aleph} \alpha_{\psi n} \sum_{\mu} c^\mu_n |a^\mu_n\rangle|n\rangle + \beta_{\psi 0}|\psi_{F0}\rangle|0\rangle) \otimes$$

$$(\sum_{m \in \aleph} \alpha^*_{\psi m} \sum_{\nu} c^{\nu *}_m \langle a^\nu_m|\langle m| + \beta^*_{\psi 0}\langle\psi_{F0}|\langle 0|) \qquad (63)$$

JUSTIFYING THE GENERALIZED PROJECTION POSTULATE

Let us now perform the partial trace of ρ_C with respect to Ω_M . We obtain

$$Tr_M \rho_C = p_S^t(F_0)\left|\psi_{F_0}\right\rangle\left\langle\psi_{F_0}\right| + \sum_{n\in\aleph} p_{\psi n}^d(\hat{A})\sum_{\mu,\nu} c_n^\mu c_n^{\nu *}\left|a_n^\mu\right\rangle\left\langle a_n^\nu\right| \quad (64)$$

By comparing Eq. (64) with Eq. (61) we get

$$\tilde{\rho} = Tr_M \rho_C \quad (65)$$

GPP is thus completely justified for the measured object on the basis of the assumed evolution of the composite system made up of the measured object plus the measuring apparatus. It must be observed that the descriptions provided by Eqs. (61) and (65) coincide also from an interpretative point of view. Indeed, because of objectivity of properties, all probabilities of properties of Ω_m and Ω_M are epistemic.