

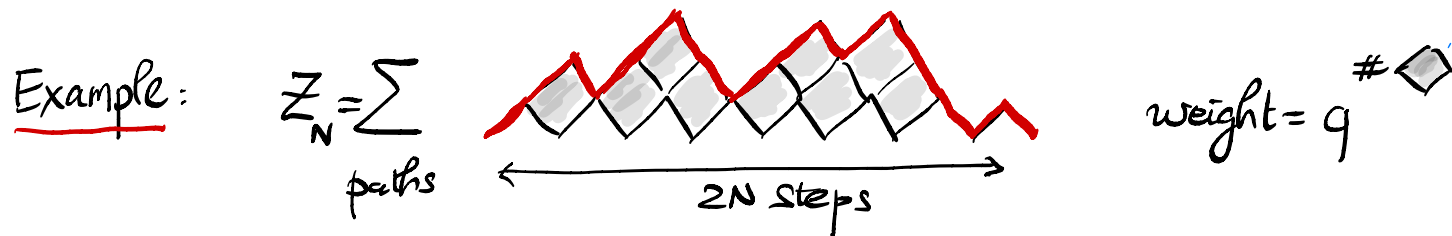
INTEGRABLE COMBINATORICS

(P. Di Francesco IPHT Saclay / UIUC Math)

0. Integrability in combinatorics Structures / Tools
1. Planar Maps geodesics Trees / discrete integrability
2. Cluster Algebra and Dimers Paths / Flat connections
3. Lorentzian Gravity Trees / Commuting transfer matrices
4. GV, ASM, TSSCPP, DPP... loops/qKZ eqn / quantum integrability

0. INTEGRABILITY IN COMBINATORICS

- enumerative combinatorics = count objects in classes
- algebraic combinatorics = use algebraic structures



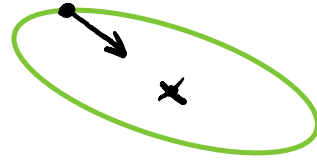
- Weyl algebra $\begin{cases} u d = q d u \\ \diamond \quad \checkmark \end{cases}$ u, d : Weyl generators

- Representation: $\text{Vect} \{ |i\rangle, i \in \mathbb{Z}_+ \}$ $\begin{cases} u |n\rangle = |n+1\rangle \\ d |n\rangle = q^{n-1} |n-1\rangle, d |0\rangle = 0 \end{cases}$

$$Z_N = \langle 0 | (d+u)^{2N} |0\rangle = q\text{-Catalan number}$$

0. INTEGRABILITY IN COMBINATORICS

- An example: central force motion
 $\vec{r} \wedge \vec{p} = \text{const}$

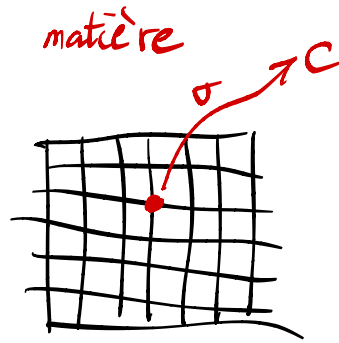


- Integrability = existence of sufficiently many symmetries

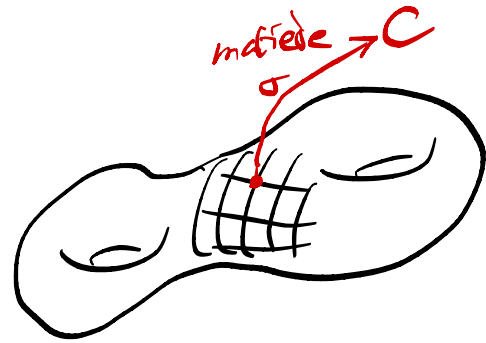
- conservation laws that simplify the problem
 - "flat connections"
 - commuting operators to be simultaneously diagonalized.
- } discrete
- } continuous

1. DISCRETE INTEGRABILITY

A. Planar Maps and geodesic distance



2D Quantum Gravity



2D Lattice Model

← space →

Random Tessellation of a Riemann Surface

map $\sigma: S \rightarrow C$
vertices Target

← matter →

map $\sigma: S \rightarrow C$
vertices target

[Gross-Migdal, Brezin-Kazakov, David, Distler-Hawai 90']

Questions / Answers

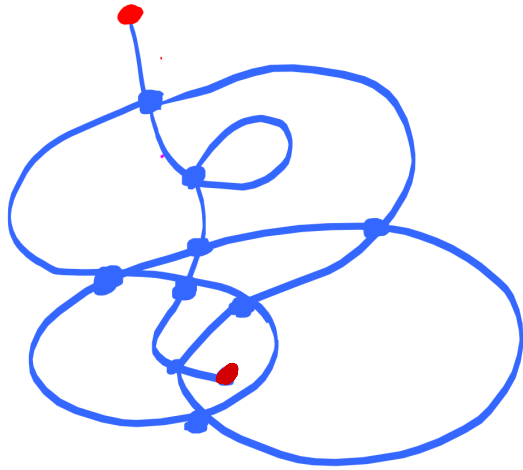
• Q = generate random tessellations + matter

• A = matrix integrals \rightarrow weighted enumeration

Classical integrability = PDE, commuting flows

• Q = intrinsic geometry? generate random tessellations with marked points at prescribed geodesic distance?

• A = bijections with decorated trees \rightarrow discrete integrability



Dual of a random quadrangulation
 \equiv tetravalent planar map with
 two marked univalent vertices
 at a prescribed geodesic distance

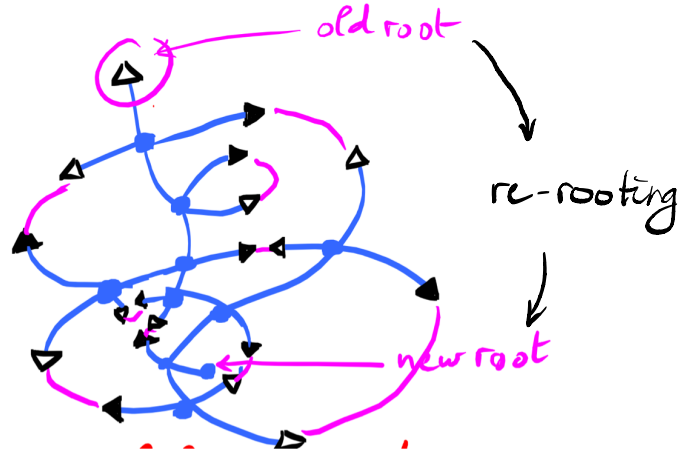
Problem = count these objects,
 connected, w/weight g /vertex

$$g \quad \text{Area(quadrangulation)} \quad |V(\text{Map})| \\ = g$$



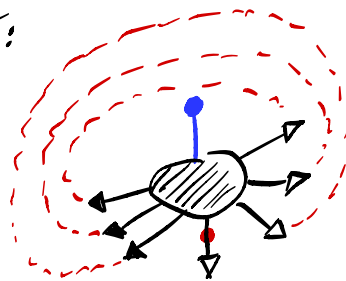
Planar map

bijection
 \longleftrightarrow
 [Schaeffer 98]



Blossom Tree w/ $\uparrow \uparrow$

- Conservation Law: $\# \downarrow - \# \uparrow = 1$
- View from the new root:



\updownarrow # "onion peels"
 = geodesic distance

$R_n(g) = \#$ Planar 4-valent maps with 2 marked pts at distance $\leq n$, weight g/vertex

$$R_n(g) = 1 + g R_n(g) [R_{n+1}(g) + R_n(g) + R_{n-1}(g)]$$

[Bouttier, DF, Guitten '04]

Recurrence! Boundary Conditions ?

$R_n(g) = \#$ Planar 4-valent maps with 2 marked pts at distance $\leq n$, weight g/vertex

$$R_n(g) = 1 + g R_n(g) [R_{n+1}(g) + R_n(g) + R_{n-1}(g)]$$

① $R_{-1}(g) = 0$ ② $R_{\infty}(g) = R(g)$ solⁿ of:

$$R = 1 + 3g R^2 \Rightarrow R = \frac{1 - \sqrt{1 - 12g^2}}{6g}$$

INTEGRABILITY

$$0 = R_n - 1 - g R_n (R_{n+1} + R_n + R_{n-1}) \quad (*) \quad (R_{-1} = 0, R_{\infty} = R)$$

Discrete Integral: $\varphi(x, y) = xy(1 - g(x+y)) - x - y$

$$\varphi(R_n, R_{n+1}) - \varphi(R_{n-1}, R_n) = (R_{n+1} - R_{n-1}) (*) = 0!$$

INTEGRABILITY

$$0 = R_n - 1 - g R_n (R_{n+1} + R_n + R_{n-1}) \quad (*) \quad (R_{-1} = 0, R_{\infty} = R)$$

Discrete Integral: $\varphi(x, y) = xy(1 - g(x+y)) - x - y$

$$\boxed{\varphi(R_n, R_{n+1}) - \varphi(R_{n-1}, R_n) = (R_{n+1} - R_{n-1}) (*) = 0!}$$

EXACT SOLUTION ; DISCRETE SOLITON

$$\boxed{R_n = R(g) \frac{1 - x^{n+1}}{1 - x^{n+2}} \frac{1 - x^{n+4}}{1 - x^{n+3}}} \quad [\text{Boutier DF} \\ \text{Guitier 05-07}]$$

$$x + \frac{1}{x} + 4 = \frac{1}{g R(g)} \quad ; \quad R(g) = 1 + 3gR^2(g)$$

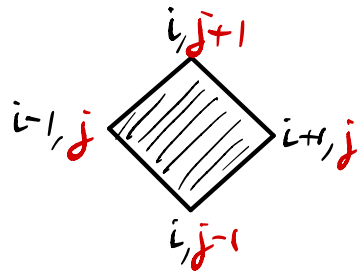
$$X(g) = g(1 + o(g))$$

continuum limit, proba, etc.
[Le Gall, Micromat, BDG, ...]

B. FRISES and FLAT CONNECTION

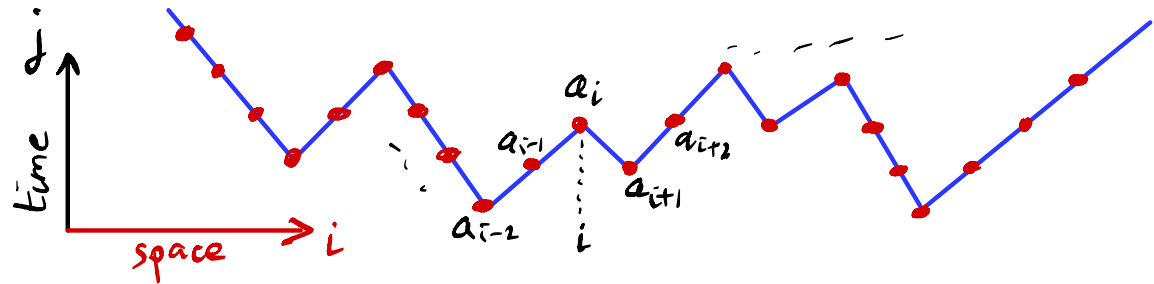
$$T_{i,j+1} T_{i,j-1} = T_{i+1,j} T_{i-1,j} + 1 \quad \left\{ \begin{array}{l} (i,j) \in \mathbb{Z}^2 \\ i+j = 0 \pmod{2} \end{array} \right.$$

$i = \text{espace}$
 $j = \text{temps}$



$$\begin{vmatrix} T_{i,j+1} & T_{i+1,j} \\ T_{i-1,j} & T_{i,j-1} \end{vmatrix} = 1$$

INITIAL CONDITIONS: $T_{i,k_i} = a_i$ ($|k_{i+1} - k_i| = 1$)
 (ligne brisée)
 • $T_{i,j} = ?$ ↑
donnée

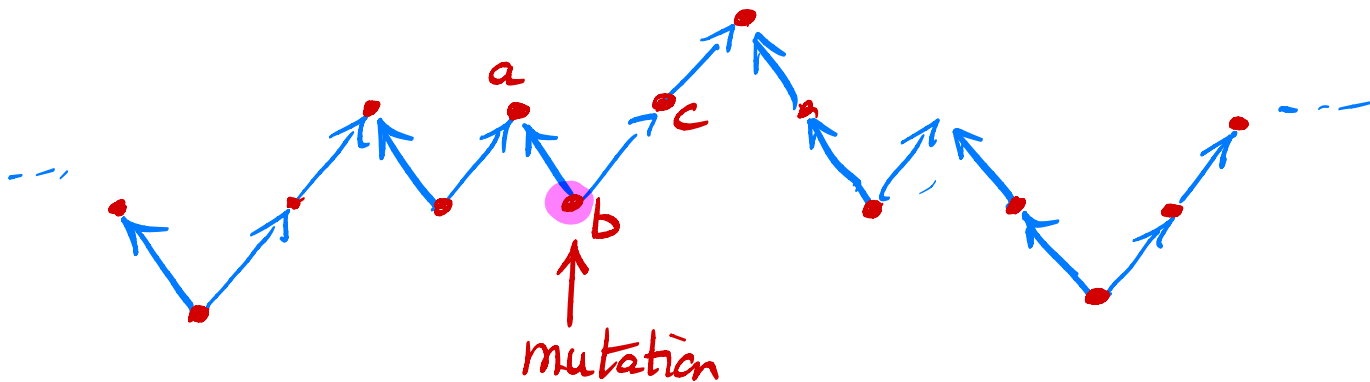


CLUSTER ALGEBRA FORMULATION

- quiver

[Fomin Zelevinsky '00]

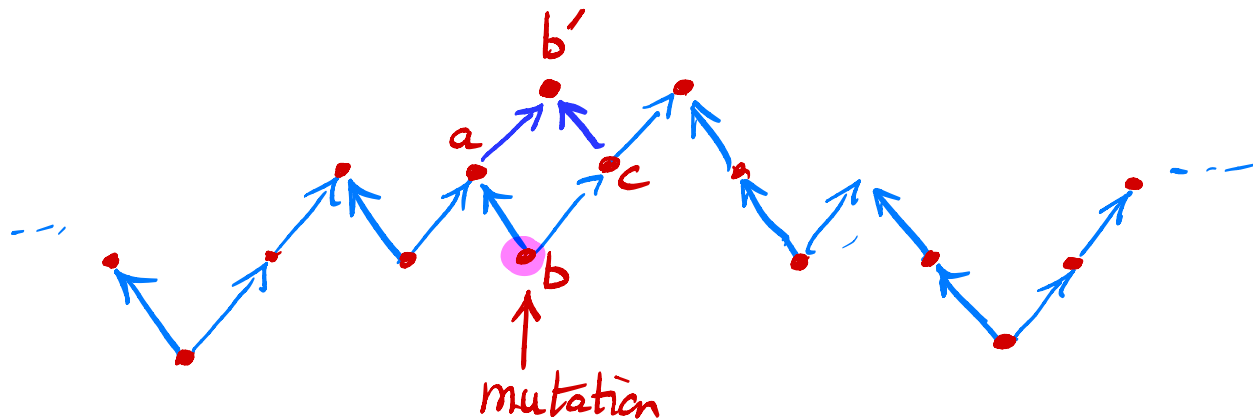
- vertex variables



CLUSTER ALGEBRA FORMULATION

• quiver

• vertex variables

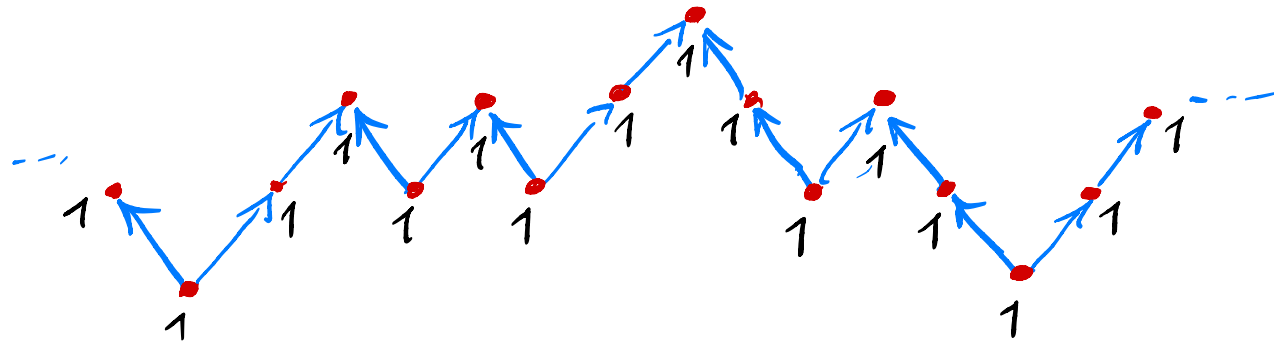


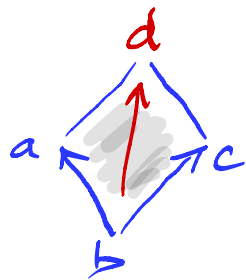
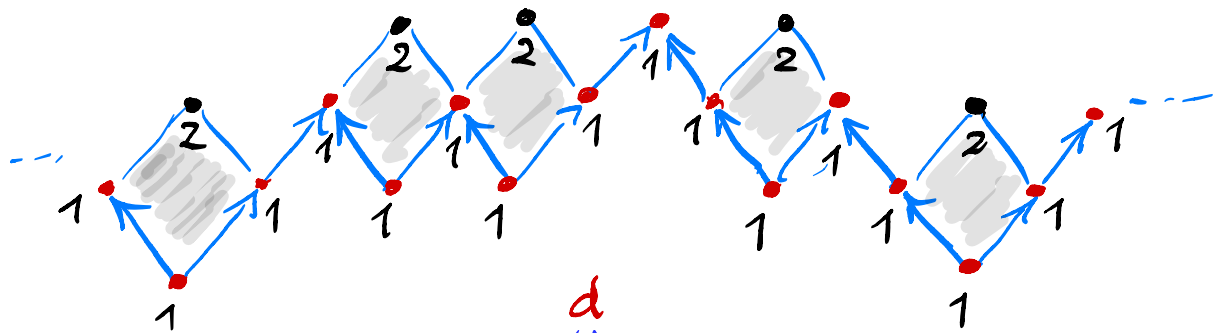
$$b' b = ac + 1$$



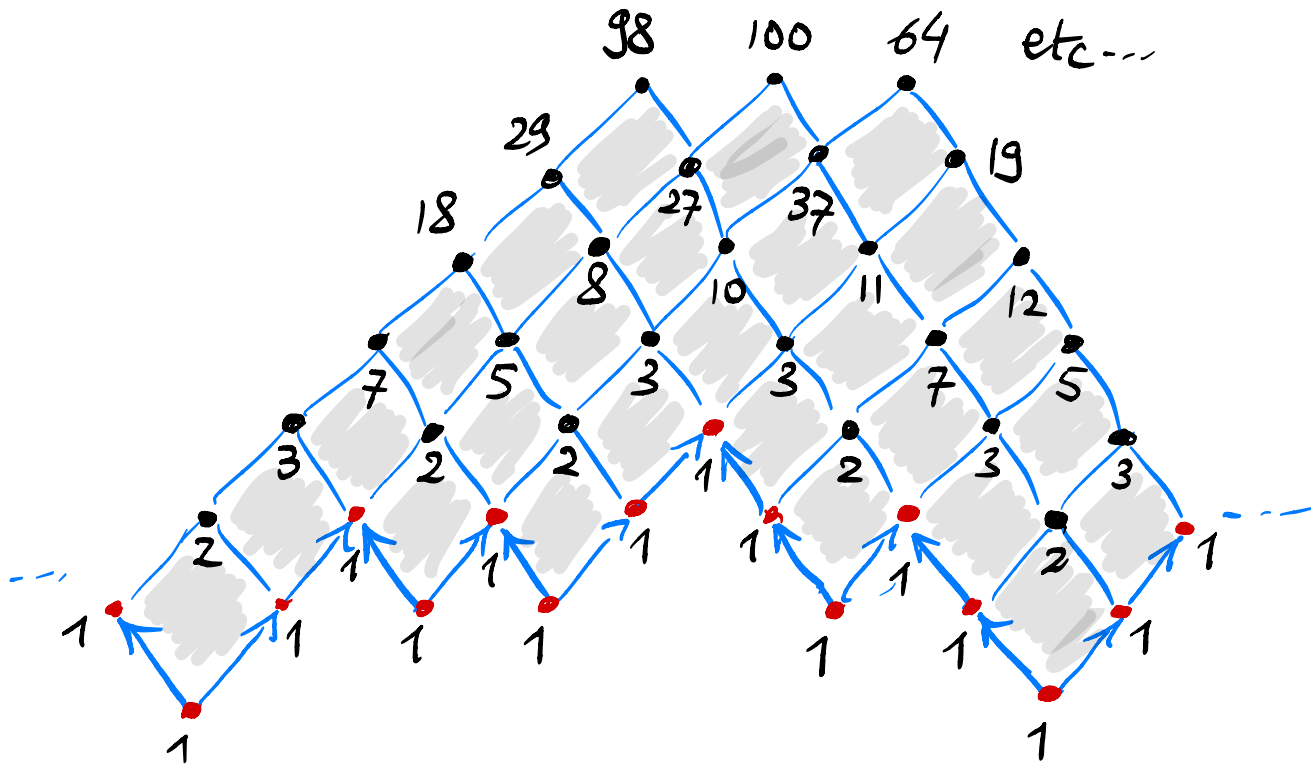
→ iteration until we reach $(i, j) \rightarrow T_{ij}(\{a_i\})$

COMBINATORIAL SOLUTION ($a_i = 1$)





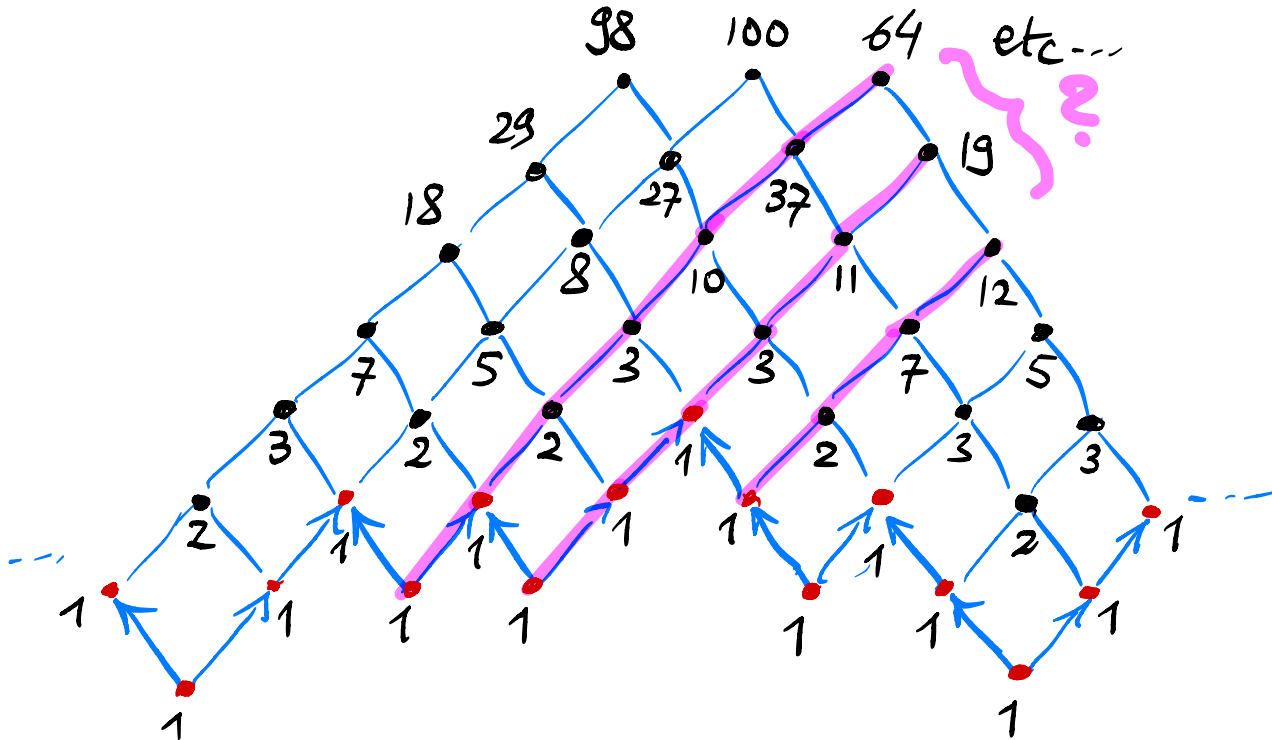
$$bd - ac = 1$$



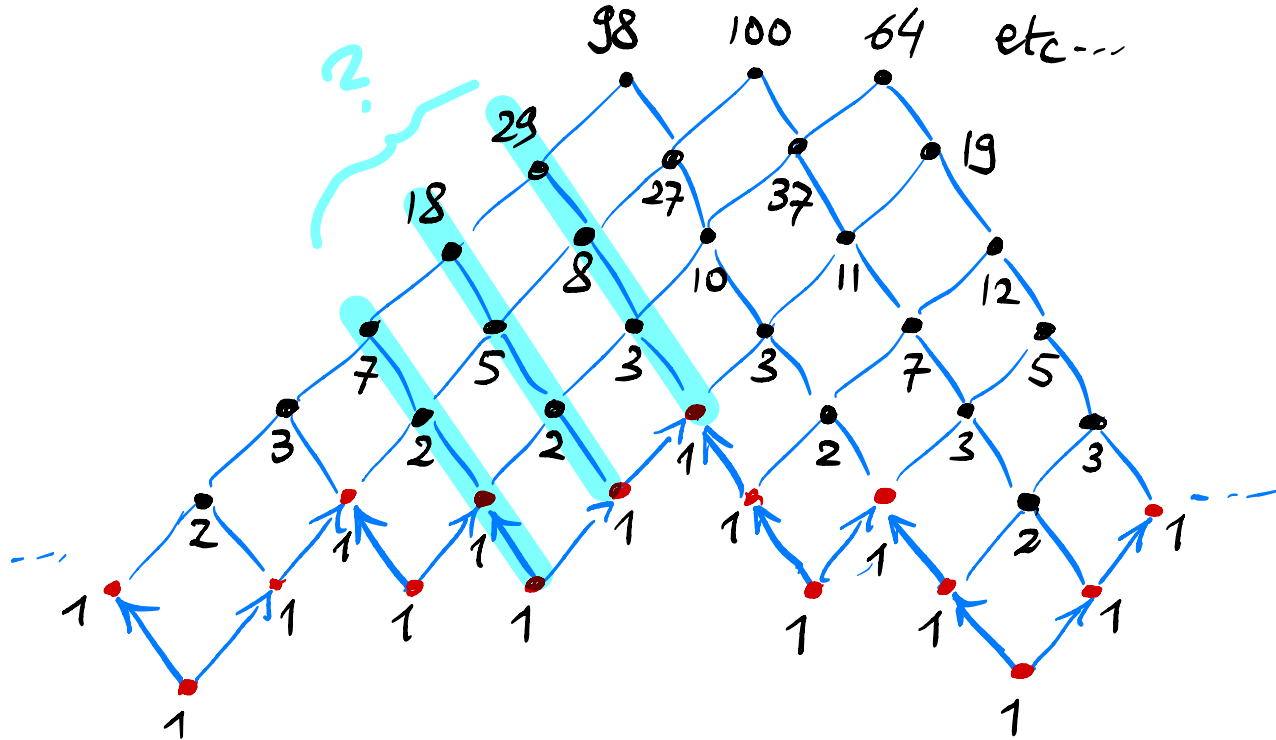
Integers (despite divisions!) \leftrightarrow Property of Cluster Algebra

"Coxeter - Conway Friezes"

INTEGRABILITY = CONSERVATION LAWS



INTEGRABILITY = CONSERVATION LAWS

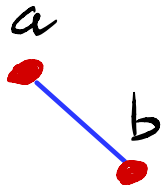


Infinitely many conservation laws!

SOLUTION: CONSTRUCT A FLAT CONNECTION

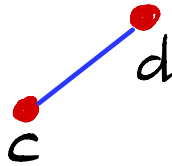
[DF, Ketem'12]

initial broken line



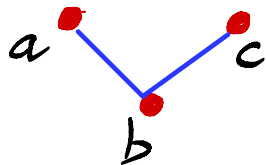
\longrightarrow 2×2 matrices ($\in GL_2$)

$$\longrightarrow D(a, b) = \begin{pmatrix} a & 1 \\ b & 1 \end{pmatrix} \quad (\text{triang. sup.})$$

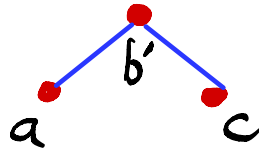


$$\longrightarrow U(c, d) = \begin{pmatrix} 1 & 0 \\ 1 & c/d \end{pmatrix} \quad (\text{triang. inf.})$$

$$D(a, b) U(b, c)$$



$$U(a, b') D(b', c)$$

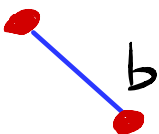


SOLUTION: CONSTRUCT A FLAT CONNECTION

[DF, Kedem'12]

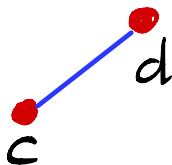
initial broken line

a



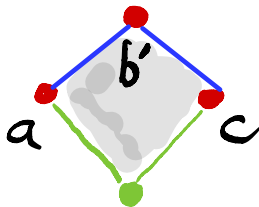
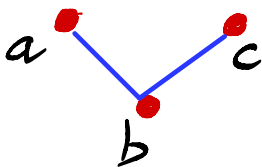
→ 2x2 matrices ($\in GL_2$)

$$\rightarrow D(a,b) = \begin{pmatrix} a & 1 \\ b & 1 \end{pmatrix} \text{ (triang. sup.)}$$



$$\rightarrow U(c,d) = \begin{pmatrix} 1 & 0 \\ d & c \end{pmatrix} \text{ (triang. inf.)}$$

$$D(a,b) U(b,c) = U(a,b') D(b',c) \Leftrightarrow \boxed{bb' = ac + 1}$$



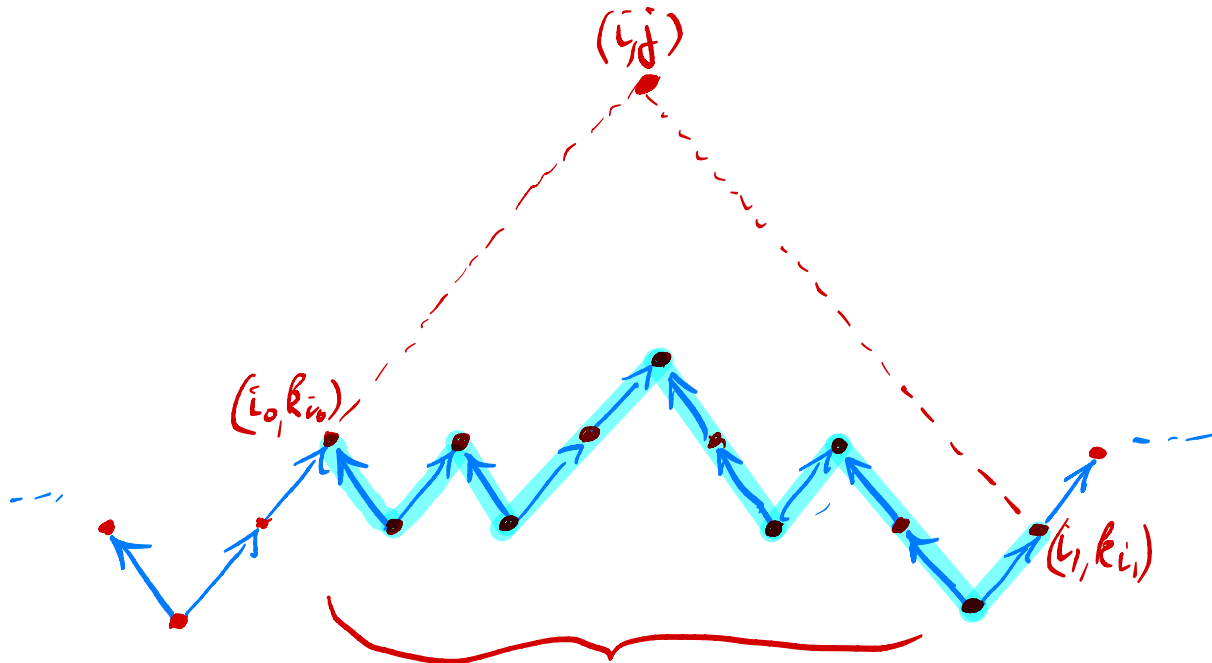
mutation!

FLAT CONNECTION

The product along a line only depends on starting and ending points!

Proof: browse through paths by mutations

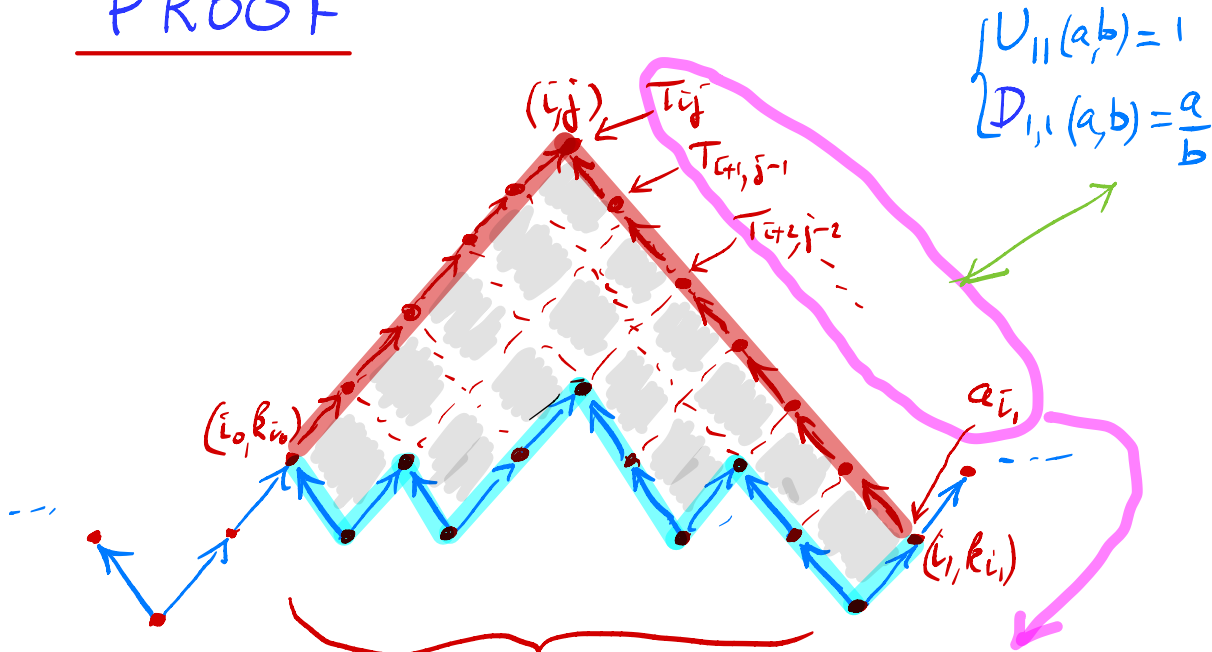
SOLUTION



Thm

$$T_{ij} = \left(\prod_{i=i_0}^{i_j-1} \begin{Bmatrix} D(a_i, a_{i+1}) \\ U(a_i, a_{i+1}) \end{Bmatrix} \right)_{1,1} \cdot a_{i_j}$$

PROOF



$$\begin{cases} U_{1,1}(a,b) = 1 \\ D_{1,1}(a,b) = \frac{a}{b} \end{cases}$$

$$\prod_{i=i_0}^{i-1} \left\{ \begin{array}{l} D(a_i, a_{i+1}) \\ U(a_i, a_{i+1}) \end{array} \right\}_{1,1} \cdot a_{i_0} = \left\{ \left(\prod_{i_0 \uparrow}^{i-1} U_{1,1} \right) \left(\prod_{i \uparrow}^{i_0-1} D_{1,1} \right) \right\}_{1,1} \cdot a_{i_0} = T_{ij}!$$

Triang. inf. (\triangle) Triang. sup. (∇)

Blue Path

=

Red Path

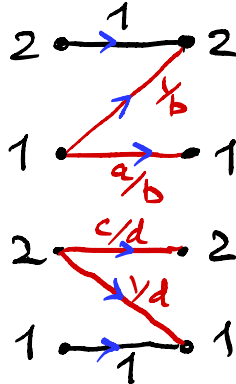
INTERPRETATIONS OF THE FLAT CONNECTION:

NETWORKS

$$D(a,b) = \begin{matrix} & \begin{matrix} 1 & 2 \end{matrix} \\ \begin{matrix} 1 \\ 2 \end{matrix} & \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \end{matrix}$$

$$U(c,d) = \begin{matrix} & \begin{matrix} 1 & 2 \end{matrix} \\ \begin{matrix} 1 \\ 2 \end{matrix} & \begin{pmatrix} 1 & 0 \\ d & c \end{pmatrix} \end{matrix}$$

weighted graph



Matrix Product



Concatenation

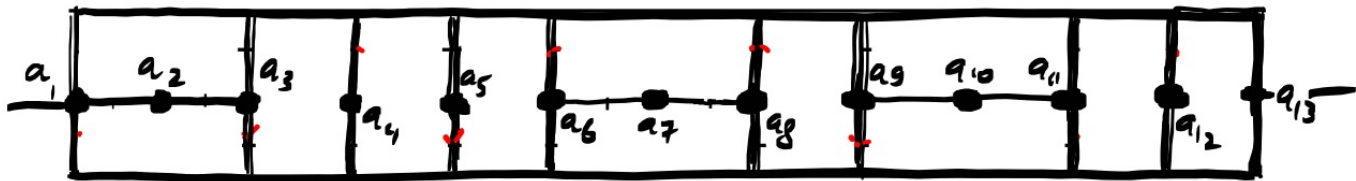
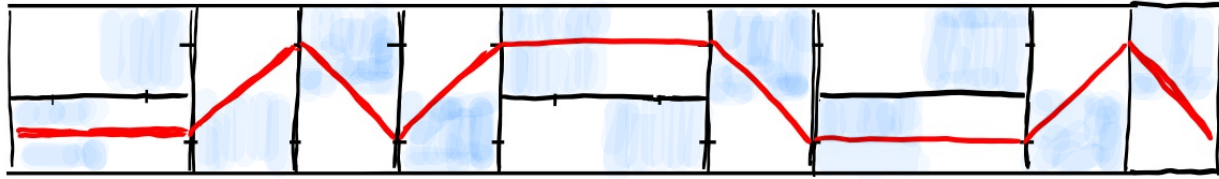
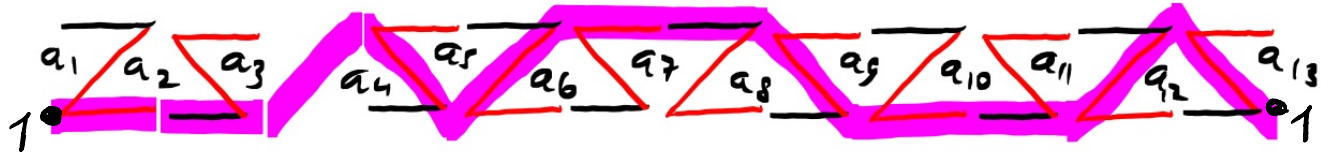
Ex "flat" broken line



$$\Rightarrow T_{ij} = \sum_{\substack{\text{paths} \\ 1 \rightarrow 1}} \left(\begin{matrix} & \begin{matrix} 2 & & & & & & & & & & 2 \end{matrix} \\ \begin{matrix} 1 \\ 1 \end{matrix} & \begin{matrix} i_0 & & & & & & & & & & i_1 \end{matrix} \end{matrix} \right) \times a_{i_1}$$

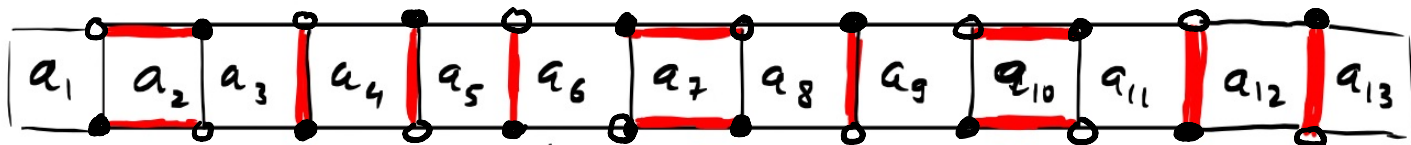
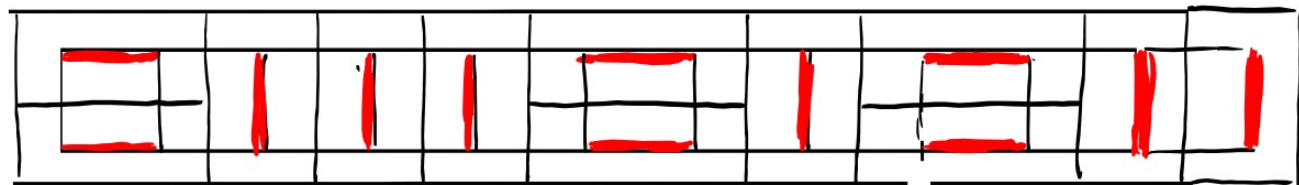
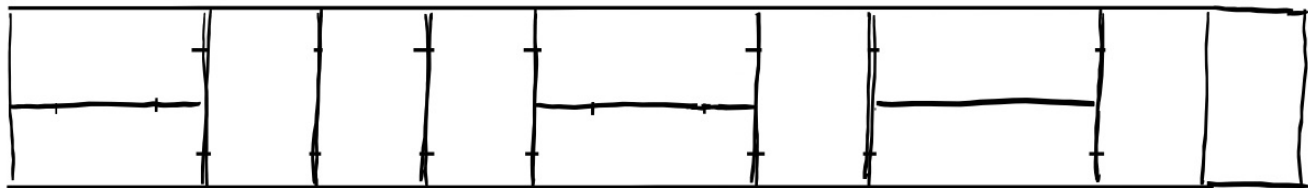
(weight of steps)

FROM PATHS TO DOMINO TILINGS



$$\text{Weight} = \prod a_i^{N_i - 3} \rightarrow \text{degree of vertex } i$$

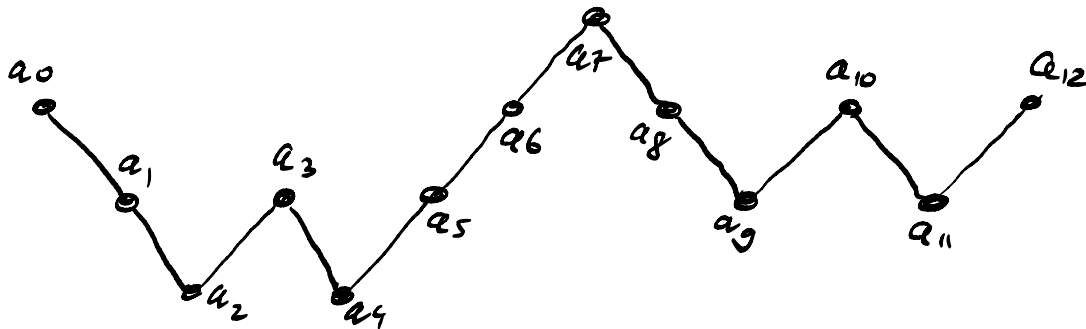
From Dominoes to Dimers



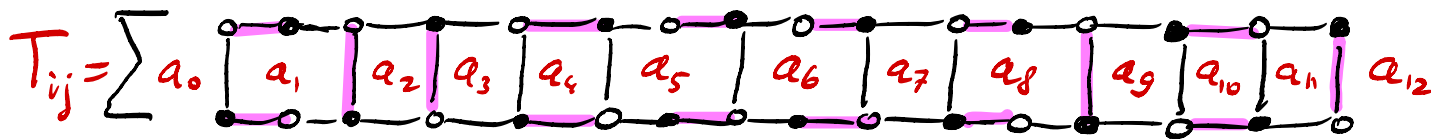
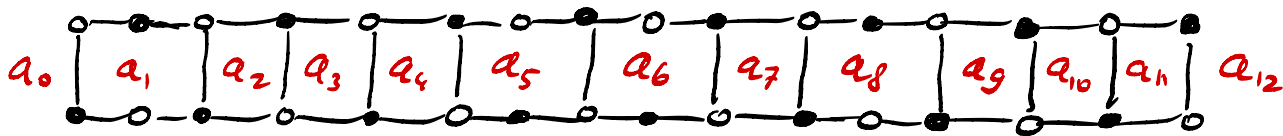
Weight = $\prod a_i^{1-N_i}$

\nwarrow # dimers on square i
 $(N_i \in \{0, 1, 2\})$

GENERAL CASE



D D U D U U U D D U D U



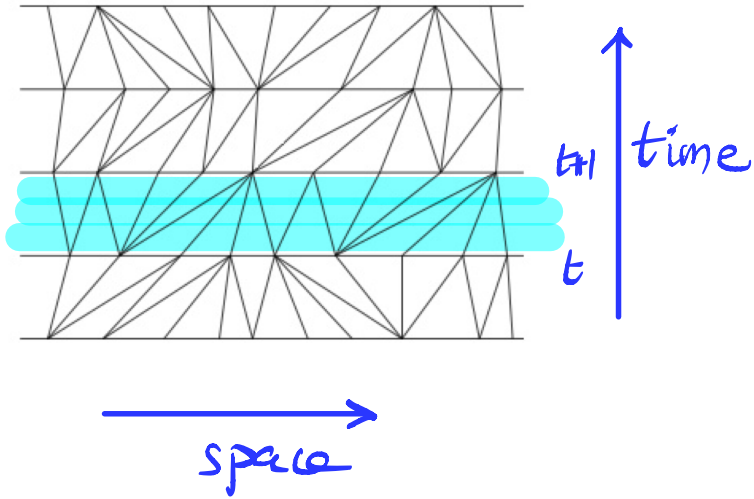
dimer
configs

wt (square) = $a_i^{1-N_i}$

wt (hexagon) = $a_i^{2-N_i}$

2. QUANTUM INTEGRABILITY

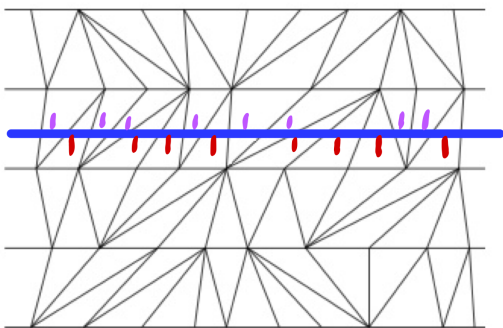
A. Lorentzian Triangulations in 1+1 Dimension



Semi-Random objects

- time = fixed lattice (\mathbb{Z})
- space = random triangulation of time slice

ENUMERATION: TRANSFER MATRIX



$$T_{i,j} = \binom{i+j}{i}$$
$$i, j \geq 0$$

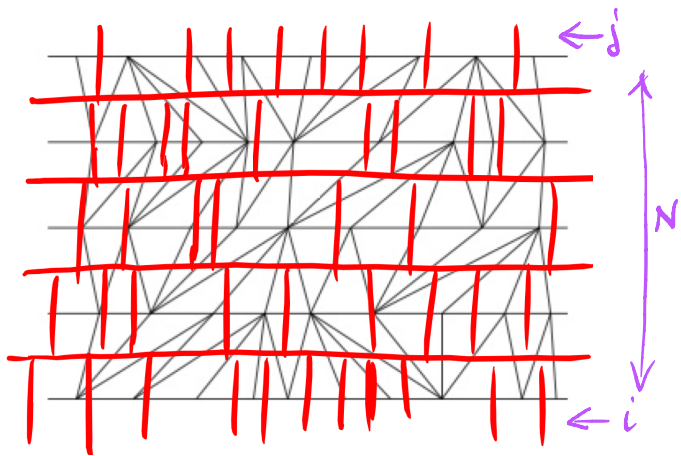
infinite matrix

GENERATING FUNCTION with weight g /triangle

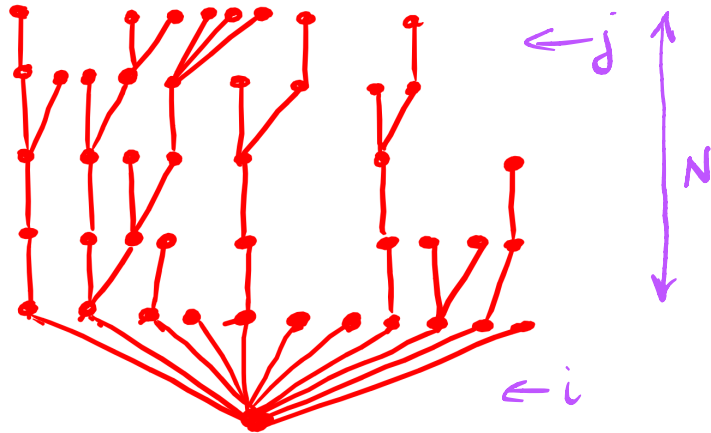
$$f_T(z, w) = \sum_{i, j \geq 0} T_{i,j} z^i w^j = \frac{1}{1 - g(z + w)}$$

Q: Diagonalize T ?

REMARK = THESE ARE TREES!



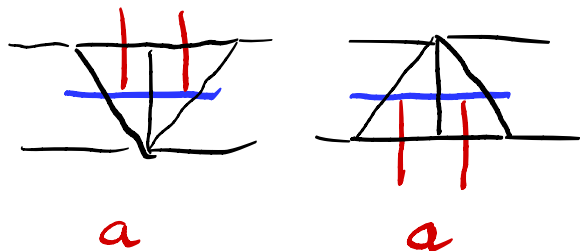
Dual Graph = trivalent
 "random brick wall"



Planar rooted tree

Partition function $Z = \left(T^N \right)_{i,j}$

MODEL WITH CURVATURE



weights = $g / \text{triangle}$

$a / \text{consecutive pair}$
of triangles $\nabla \nabla$ or $\Delta \Delta$

$$T_{ij}(g, a) = g^{l+j} \sum_{k=0}^{\min(i,j)} \binom{l}{k} \binom{j}{k} a^{-2k}$$

$$\oint T(g, a)(z, w) = \frac{1}{1 - ag(z+w) - g^2(1-a^2)zw}$$

INTEGRABILITY

THM (Integrability) [DF, Guitter]

$$[T(g, a), T(g', a')] = 0$$

$$\Leftrightarrow \varphi(g, a) := \frac{1 - g^2(1 - a^2)}{ga} = \varphi(g', a')$$

Proof: product of infinite matrices \Leftrightarrow convolution of their generating functions.

EXACT SOLUTION

New parameters q
 λ

$$\varphi(a, q) = q + q^{-1}; \quad \lambda = \frac{1 - q^{-1} a q}{1 - q a q}; \quad T(q, a) = T_q(\lambda)$$

$$\oint T_q(\lambda)(z, w) = \frac{1 - \lambda q^2}{(1 - qz)(1 - qw) - \lambda(q - z)(q - w)}$$

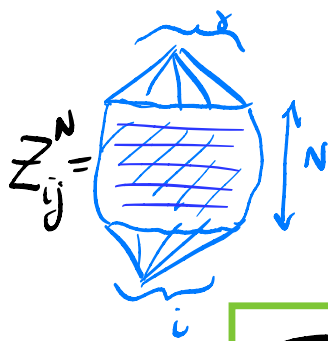
$$= \sum_{m \geq 0} \underbrace{\frac{1 - \lambda q^2}{1 - q^2} \lambda^m}_{\text{eigenvalue}} \underbrace{\sqrt{1 - q^2} \frac{(q - z)^m}{(1 - qz)^{m+1}}}_{\text{left eigenvector}} \underbrace{\sqrt{1 - q^2} \frac{(q - w)^m}{(1 - qw)^{m+1}}}_{\text{right eigenvector}}$$

$$Z = (T^N)_{ij}$$

[DF, Gütter, Kristjansen]

Solution: Generating function of Lorentzian triang. on a time interval

$Z_{ij}^N = f.g.$ (Lorentzian Triangulations between i triangles ∇ & j triangles \triangle with N Time slices)



$$Z_{ij}^N = \sum_{m \geq 0} v_i^{(m)} \left(\frac{1-\lambda q^2}{1-q^2} \lambda^m \right)^N v_j^{(m)}$$

or:

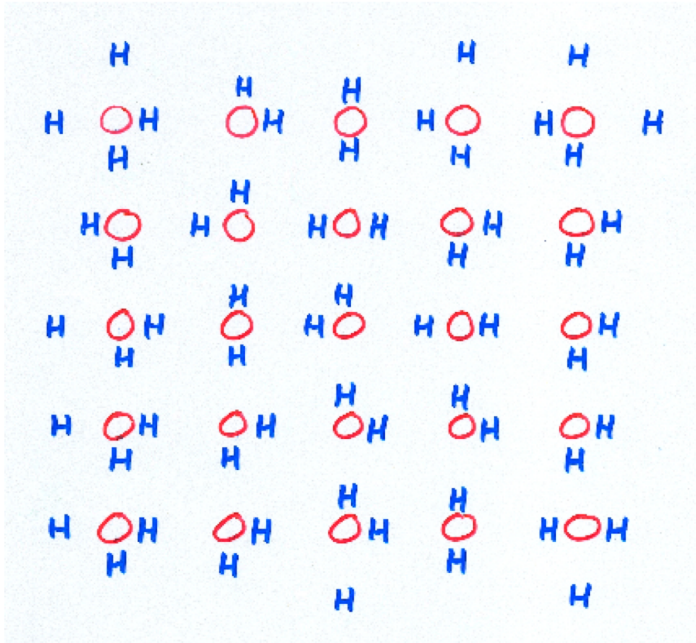
$$\sum_{i,j} Z_{ij}^N z^i w^j = \frac{1-q^2}{(1-qz)(1-qw)} \frac{1}{1 - \left(\frac{1-\lambda q^2}{1-q^2} \lambda \right)^N \left(\frac{q-z}{1-qz} \frac{q-w}{1-qw} \right)}$$

→ continuum limit, limit distribution, etc ... ($N \rightarrow \infty, \lambda \rightarrow \lambda_c = \frac{1}{q^2} - 1$)

$$(g, a) \mapsto \left(\frac{1-g^2(1-a^2)}{ga} = q+q^{-1}, \frac{1-q^{-1}ag}{1-qag} = \lambda \right)$$

B. Vertex Models

ICE IN 2 DIMENSIONS

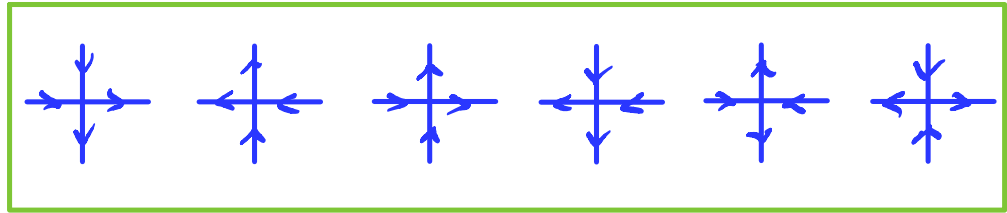
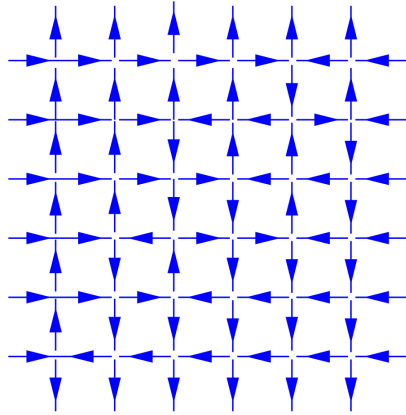


Free energy / site
[Lieb '67]

$$e^{\beta} = \left(\frac{4}{3}\right)^{3/2}$$

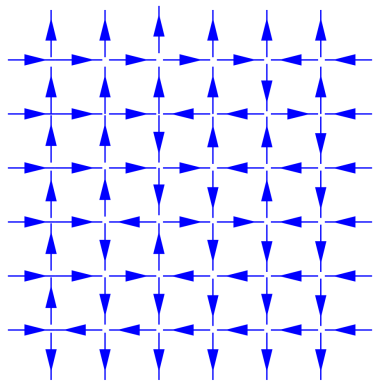
Reformulation = 6 Vertex model

orientations of edges + ice rule
= 2 in + 2 out at each vertex

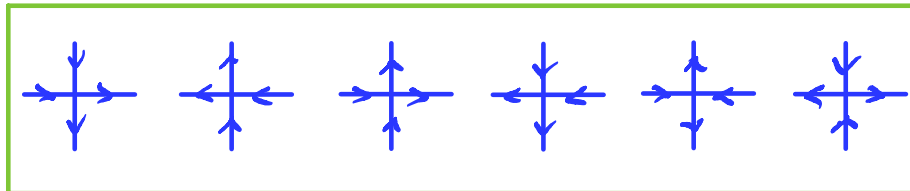


grid $N \times N$
"Domain Wall"
Boundary Conditions

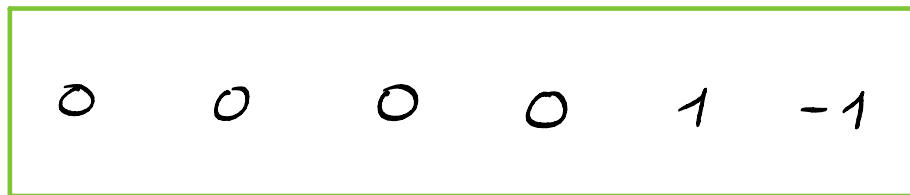
ALTERNATING SIGN MATRICES (ASM)



$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$



↕ bijection

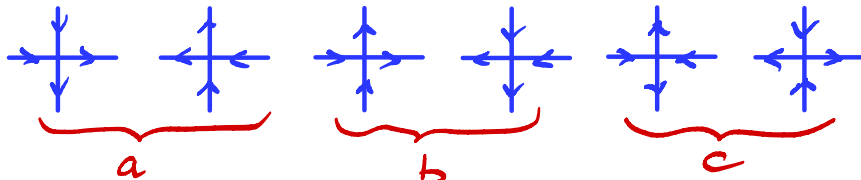


[Mills-Robbins-Rumsey '82]

[Kuperberg '96]

INTEGRABILITY

weights of configurations

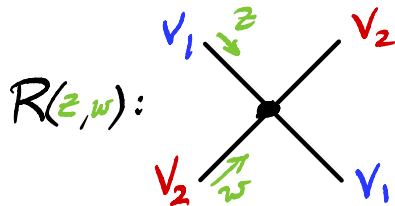


= matrix entries of an operator

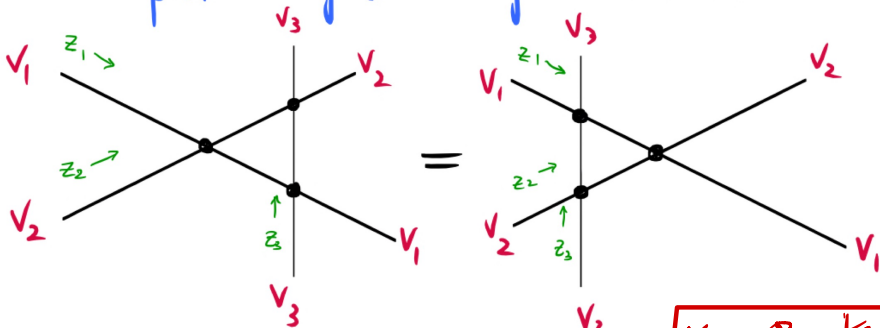
$$V_1 \cong \mathbb{C}^2 = \text{Vect}(\rightarrow, \leftarrow)$$

$$V_2 \cong \mathbb{C}^2 = \text{Vect}(\uparrow, \downarrow)$$

$$R: V_1 \otimes V_2 \rightarrow V_2 \otimes V_1$$



One can pick "integrable weights" such that:



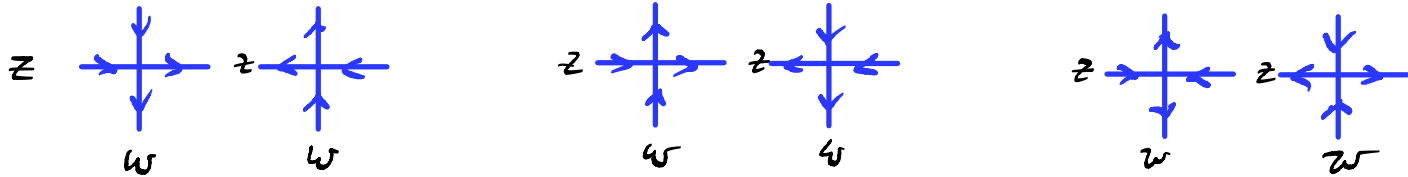
$$V_1 \otimes V_2 \otimes V_3 \rightarrow V_3 \otimes V_2 \otimes V_1$$

Yang-Baxter equation

TRIGONOMETRIC SOLUTION OF THE YANG-BAXTER EQUATION

z, w = spectral parameters

$$a(z, w) = qz - q^{-1}w ; b(z, w) = q^{-1}z - q^1w ; c(z, w) = (q^2 - q^{-2})\sqrt{zw}$$



(= matrix elements of R)

so what?

IZERGIN-KOREPIN DETERMINANT

$$Z_{6V} \left[\begin{array}{c} w_1, w_2, \dots, w_N \\ \begin{array}{c} \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \\ z_1 \\ \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \\ z_2 \\ \vdots \\ \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \\ z_N \end{array} \end{array} \right] = \frac{\prod_{i,j=1}^N a(z_i, w_j) b(z_i, w_j)}{\prod_{1 \leq i < j \leq N} (z_i - z_j)(w_i - w_j)} \times$$

$$\times \det \left(\frac{c(z_i, w_j)}{a(z_i, w_j) b(z_i, w_j)} \right)_{1 \leq i, j \leq N}$$

uniform case : $q = e^{i\pi/6}$
 $z_i \rightarrow 1 \quad w_i \rightarrow 1$

$$Z_n = \prod_{i=0}^{n-1} \frac{(\beta i + 1)!}{(n+i)!} = \text{ASM}_n$$

[Kuperberg 96]

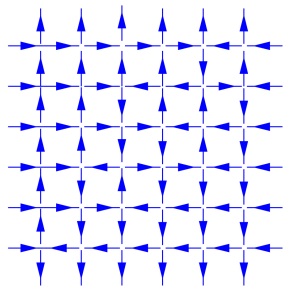
$$= 1, 2, 7, 42, 429, \dots$$

THE SAGA OF ALTERNATING SIGN MATRICES

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

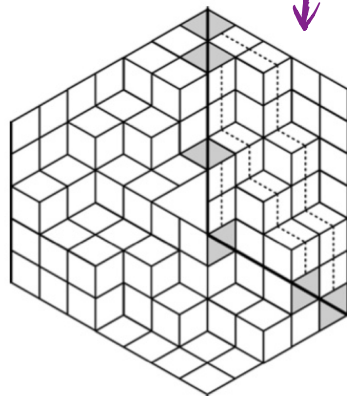
ASM

bijection



6V

ref.

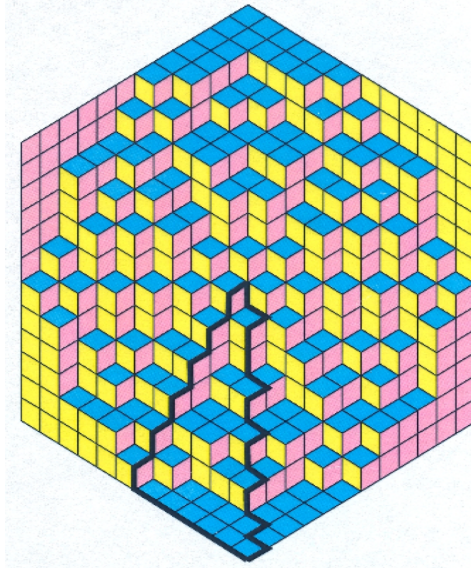


DPP [Andrews 80']

$$A_n = \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}$$

[Zeilberger 96]

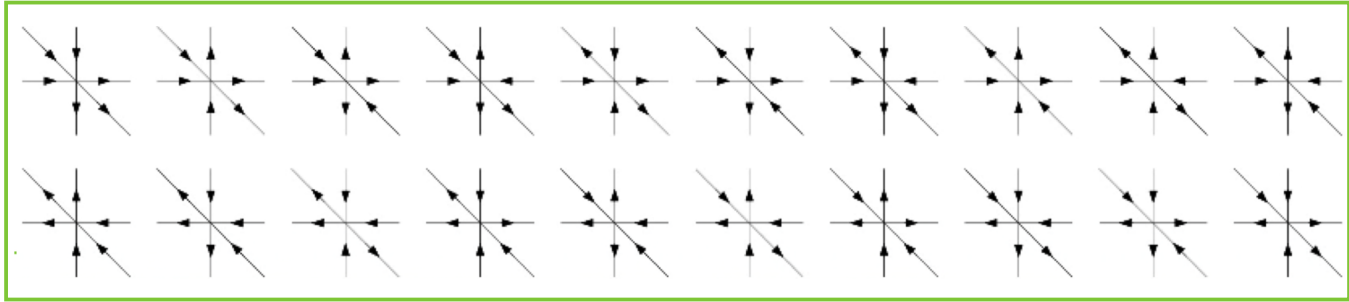
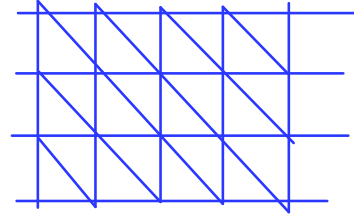
[Rupeberg 96]



TSSCPP
[Andrews 70']

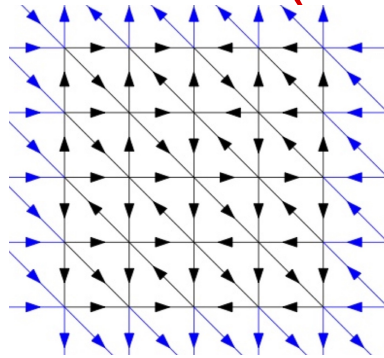
TRIANGULAR ICE

same idea, but triangular lattice

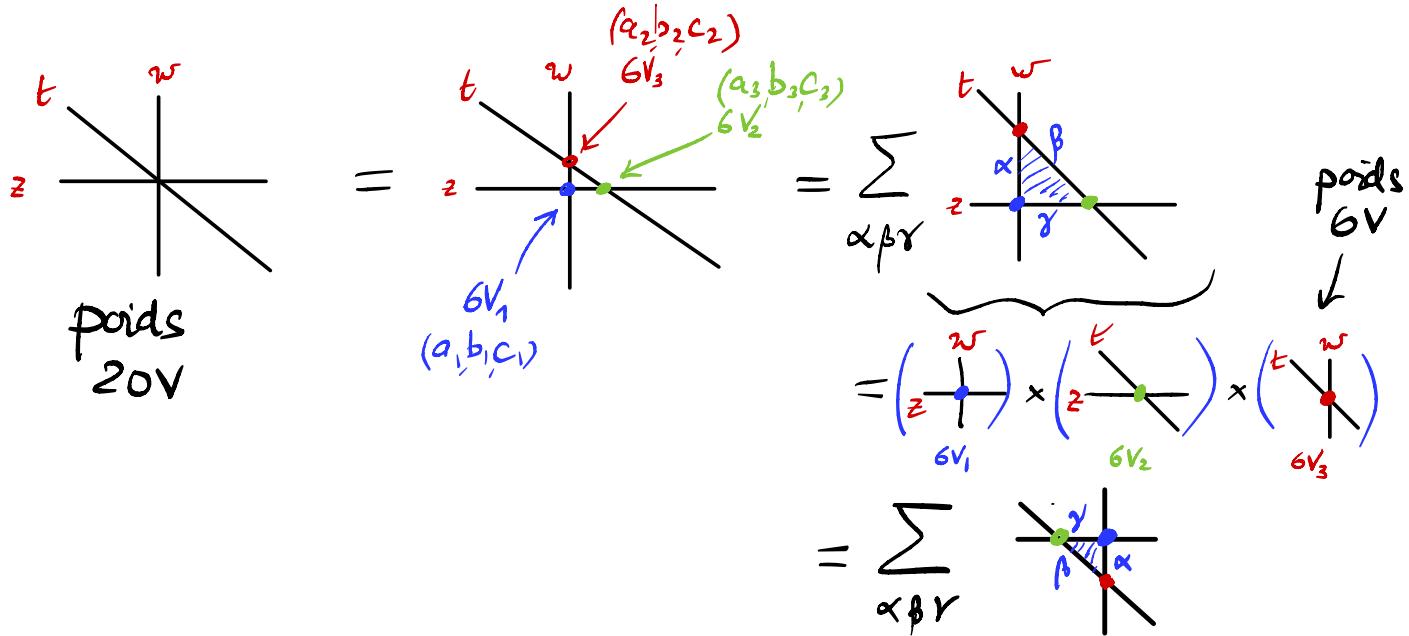


20 VERTEX MODEL $(20 = \binom{6}{3})$

"Domain-Wall" Boundary
Conditions on $N \times N$ grid

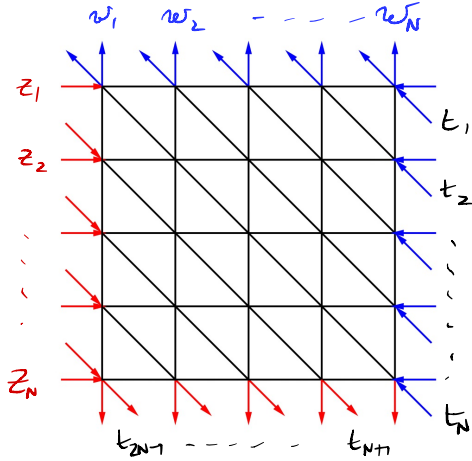


INTEGRABILITY : desingularize!

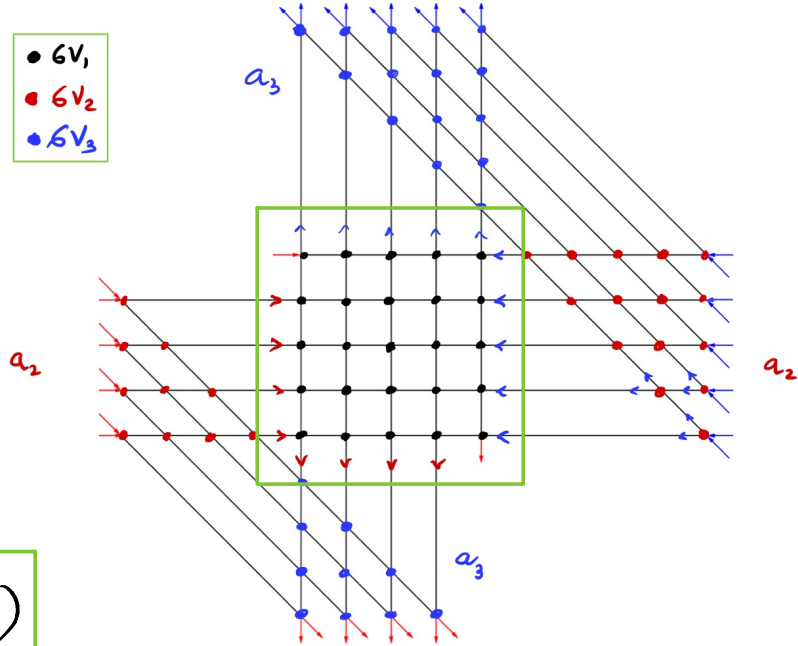


\Rightarrow Integrable weights for the 20 Vertex model
 uniform 20V $\Leftrightarrow (a=1, b=\sqrt{2}, c=1)$ (6V)

Domain Wall Boundary Partition function



- $6V_1$
- $6V_2$
- $6V_3$

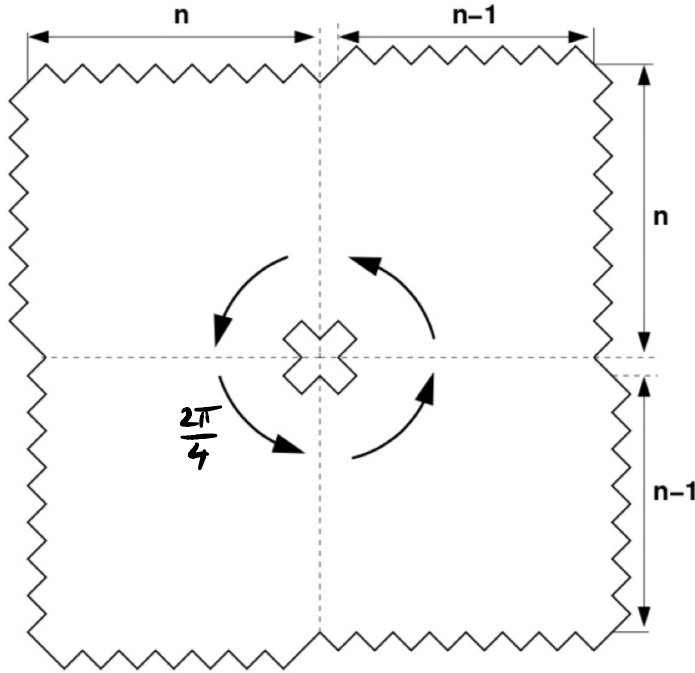


$$Z_{20V} \sim Z_{6V}(a_1, b_1, c_1)$$

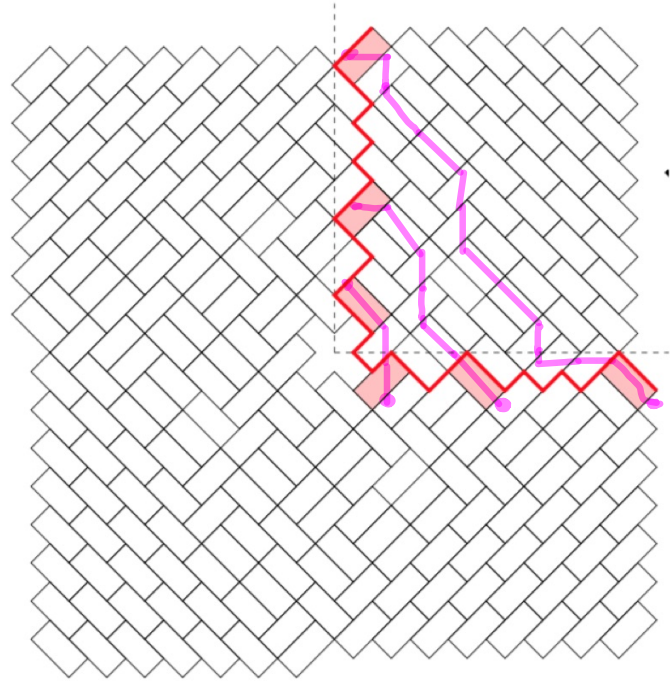
Z uniform $20V \Leftrightarrow Z(1, \sqrt{2}, 1) 6V_1$

limit of Izergin Korepin $\Rightarrow Z_N = 1, 3, 23, 433, 19705 \dots$

A REMARKABLE NON-BIJECTION WITH HOLEY AZTEC SQUARE DOMINO TILINGS W $\frac{1}{4}$ TURN SYMMETRY



S_n

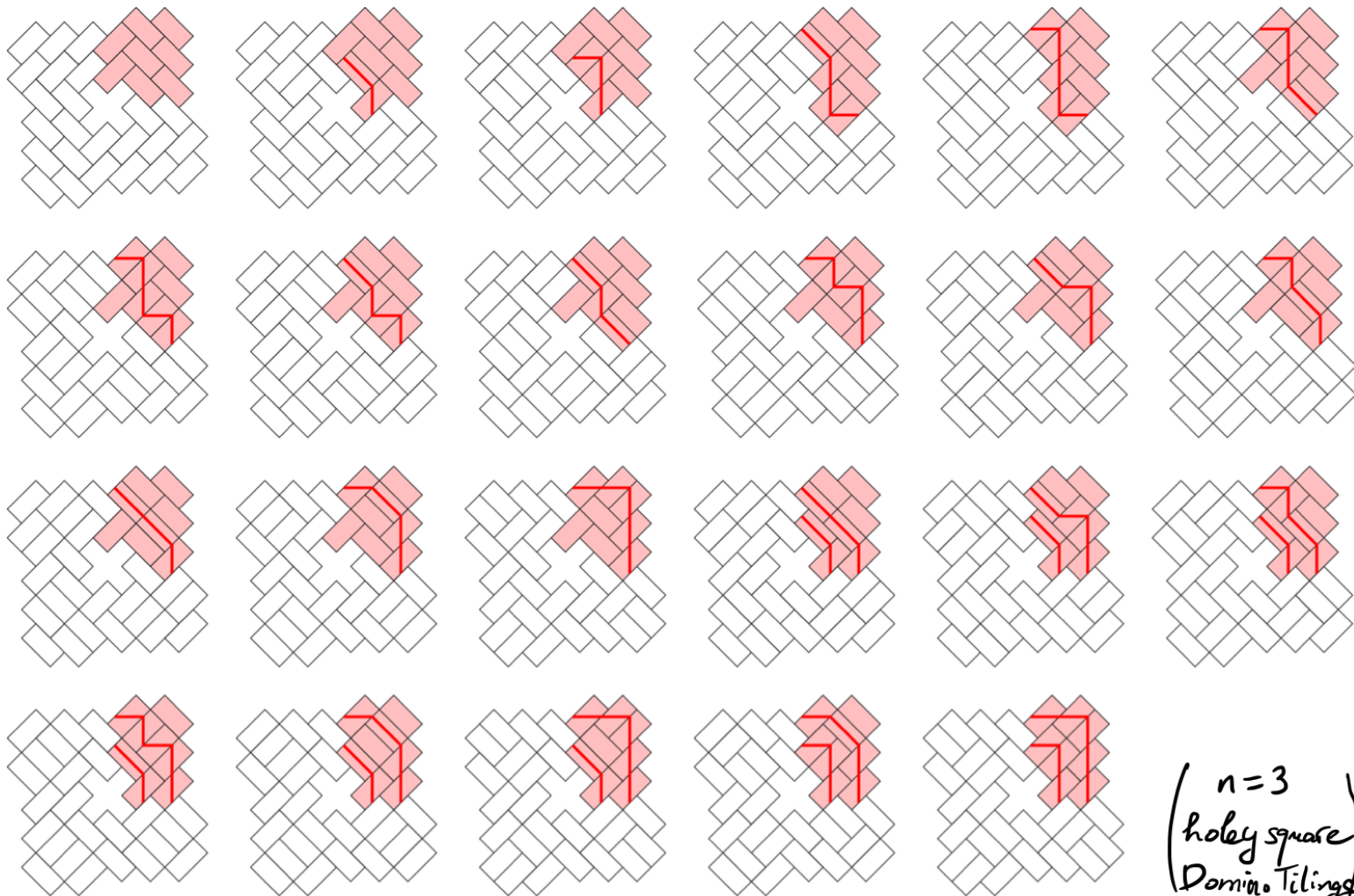


sample tiling

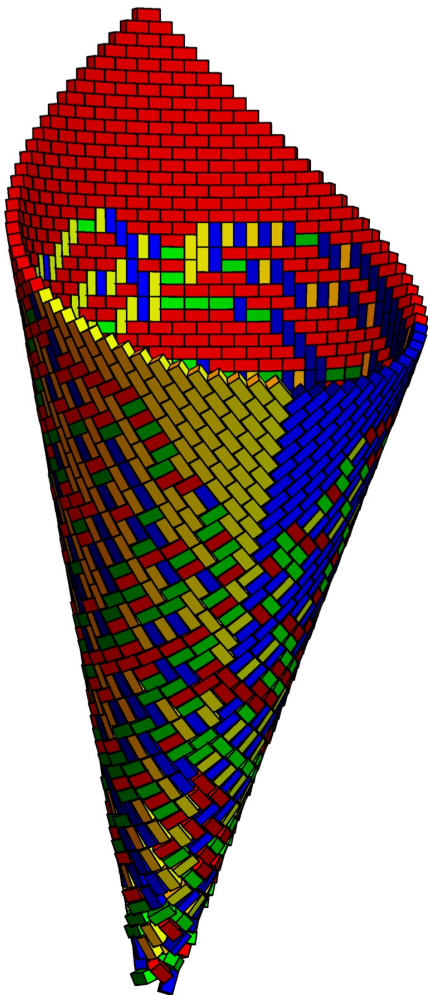
Thm [DF, Guitter '19]

$$S_n = Z_n^{20V}$$

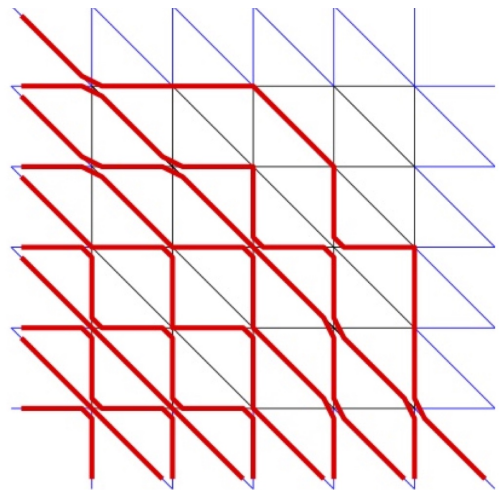
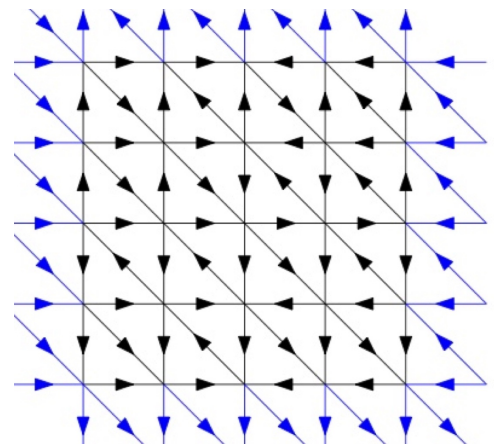
Proof: 2 determinants!



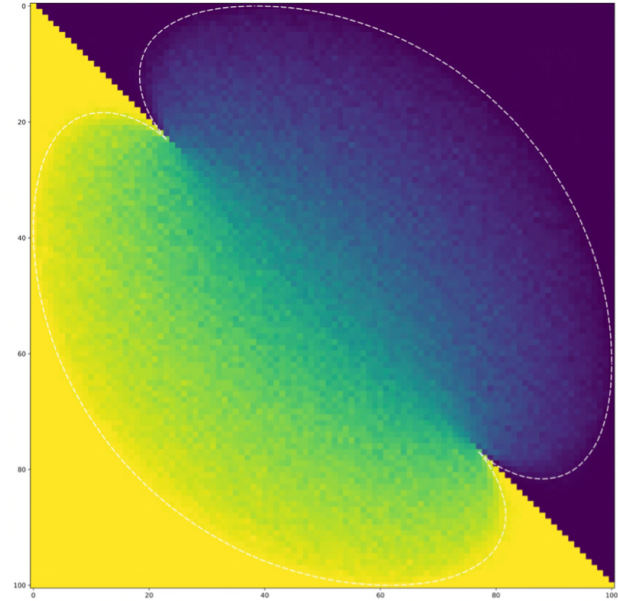
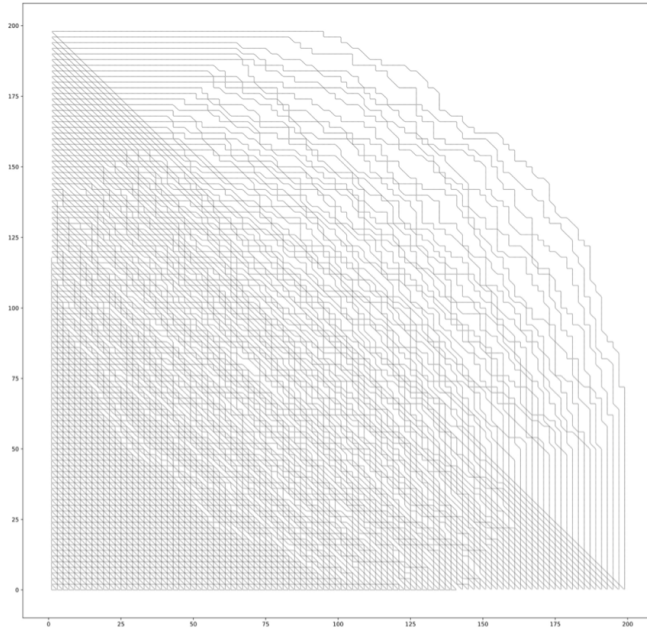
$n=3$
 (holey square
 Domino Tilings)



$\updownarrow \sim$



LIMIT SHAPES



2D ∇ uniform (100x100)

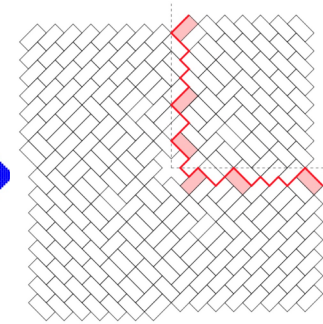
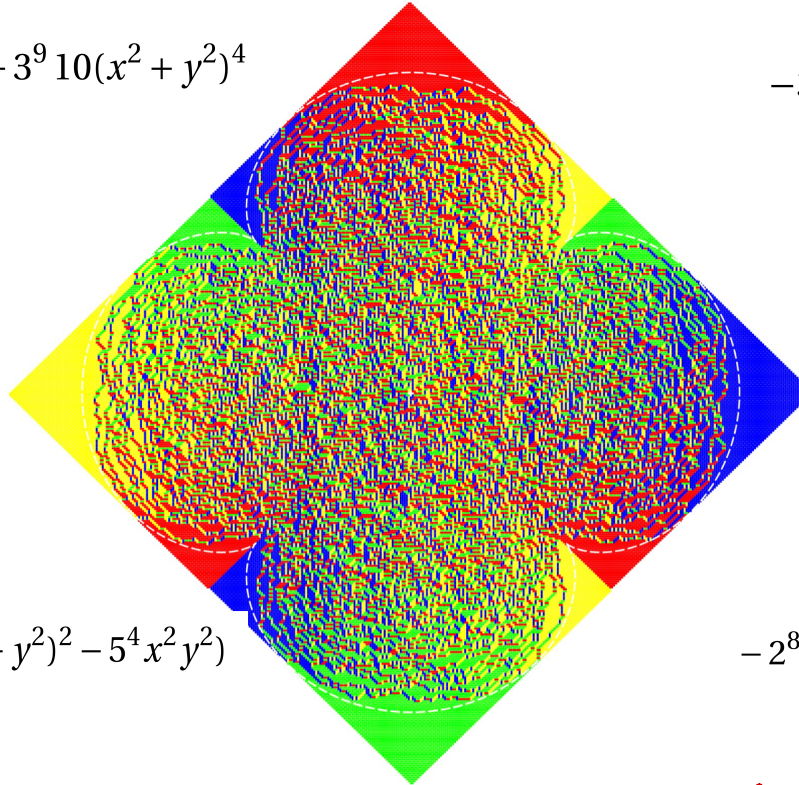
Arctic Curve is Piecewise Algebraic!

[use Izergin-Korepin
+ Tangent Method of
Colomo-Sportiello]

Arctic Curve for the Holley Aztec Domino Tilings w $\frac{1}{4}$ turn sym.

$$3^{11}(x^2 + y^2)^5 + 3^9 10(x^2 + y^2)^4$$

$$-3^6 5(x^2 + y^2)^3$$

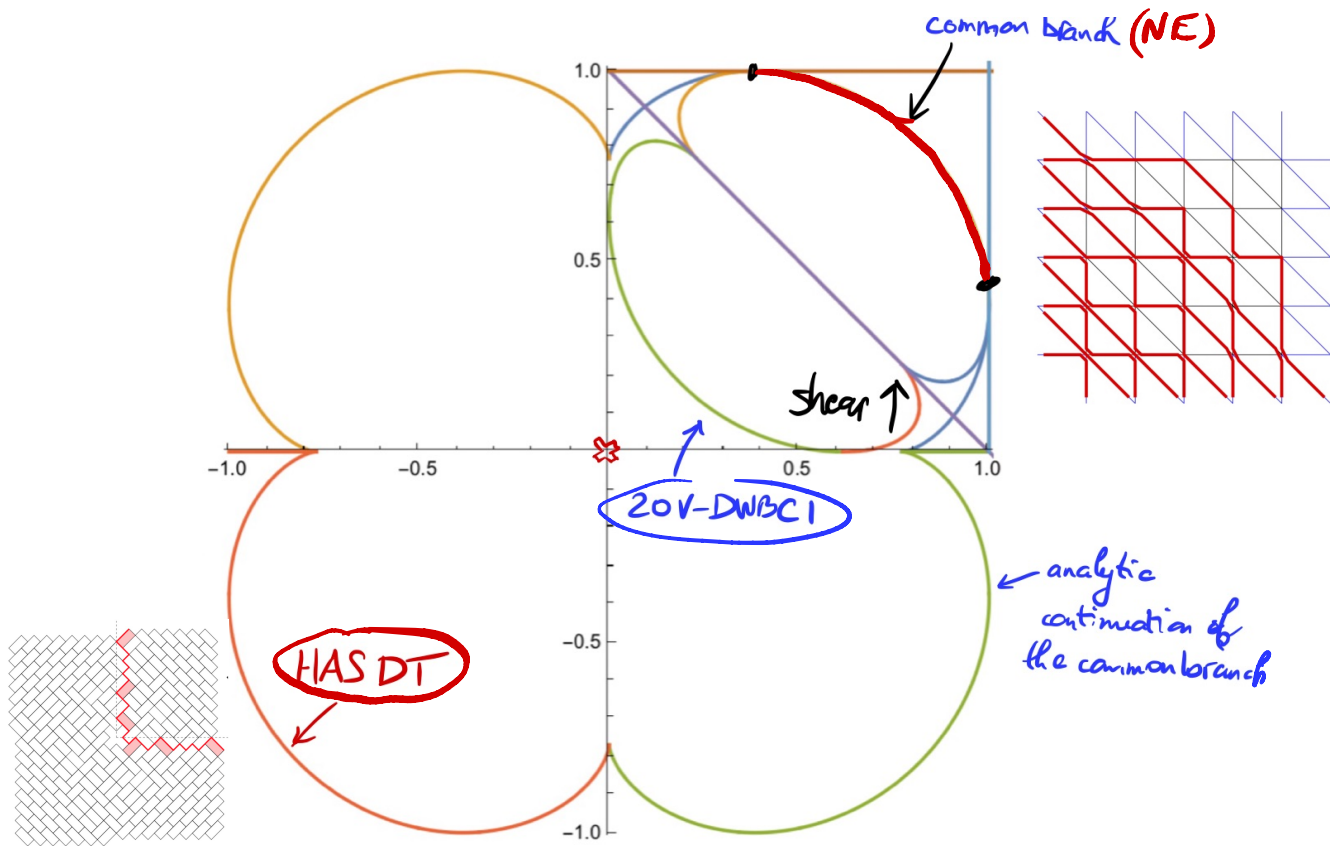


$$+ 6^2 20(73(x^2 + y^2)^2 - 5^4 x^2 y^2)$$

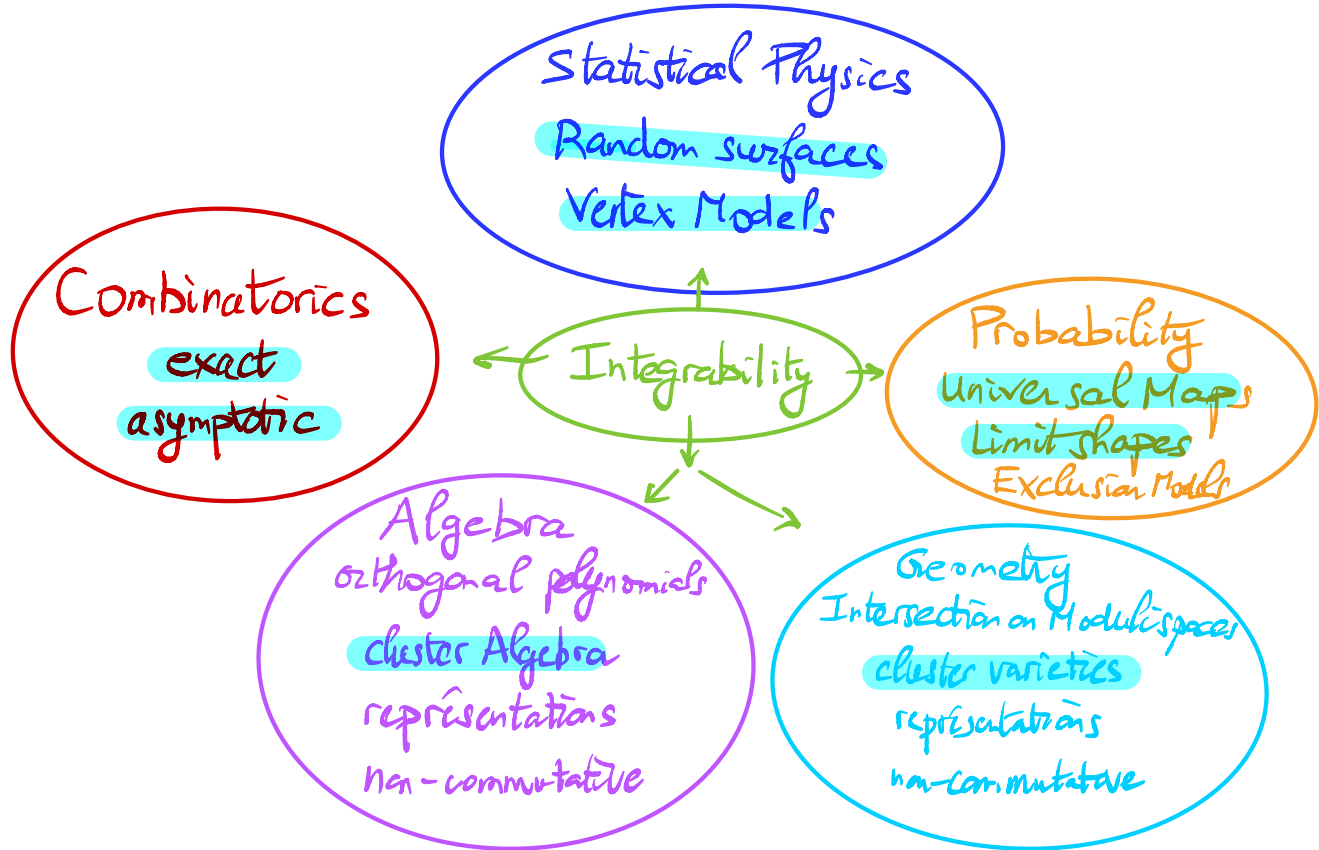
$$- 2^8 15(x^2 + y^2) - 2^{12} = 0.$$

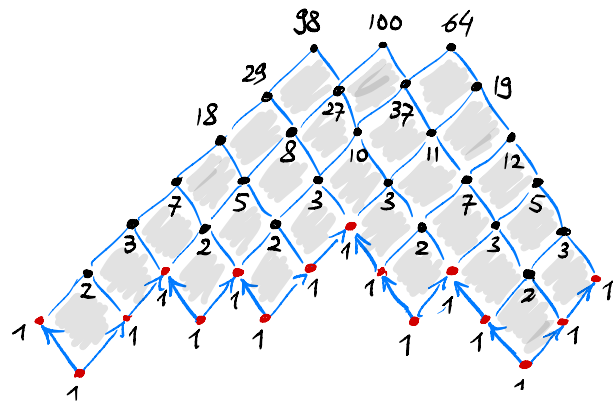
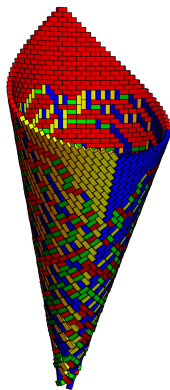
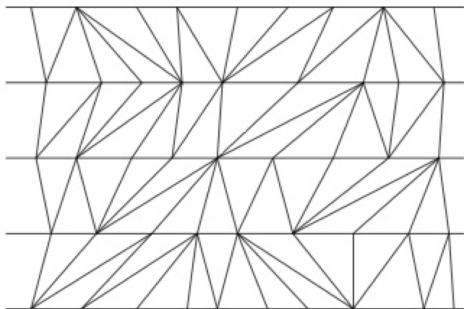
algebraic

CORRESPONDENCE OF LIMIT SHAPES



CONCLUSION / CONNECTIONS





$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Grazie Mille!

