Bjorken flow

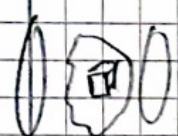
We will now adapt the hydrodynamic evolution equations to the situation realized in HICs.

We are particularly interested in the early stage immediately after the collision, where the system is initially far from equilibrium and eventually approaches hydrodynamic behavior and a local equilibrium.

During this initial phase the system is subject to a rapid longitudinal expansion along the beam direction, while the transverse expansion only develops on longer time scales and is often neglected for the pre-equilibrium phase.

Central objective is then to predict how energy deposited in actual collision is converted to thermal energy at onset of hydrodynamic expansion.

If we neglect transverse dynamics during pre-eq phase can consider transversely homogeneous system

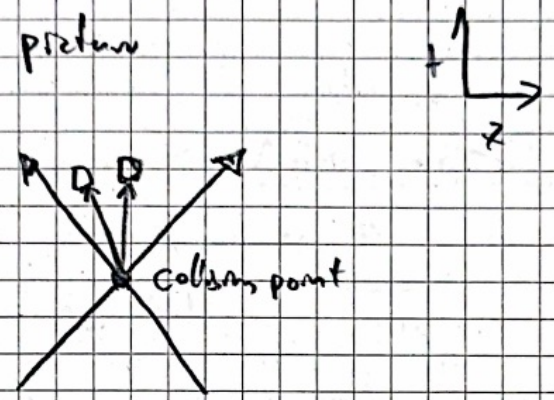


$$T^{\mu\nu}(t, x_\perp, z) \rightarrow T^{\mu\nu}(t, z)$$

$\rightarrow$ translation invariance in transverse plane

System is not translation invariant in long. direction but if we consider high energy HICs, there is an approximate symmetry under boosts along z-direction

Space-time picture



matter at (t, z) in the forward light cone will move with collective velocity

$$v_z = \frac{z}{t} \qquad \gamma = \frac{1}{\sqrt{1-\frac{z^2}{t^2}}}$$

natural rest-frame of the system is

$$u^{\mu_z} \begin{pmatrix} \gamma \\ \gamma v_z \end{pmatrix} = \begin{pmatrix} \frac{t}{\sqrt{t^2-z^2}} \\ \frac{z}{\sqrt{t^2-z^2}} \end{pmatrix} \text{ in Minkowski coordinates}$$

Since the situation in Minkowski space is rather involved, it is better to use coordinates adapted to invariance under long. boosts, Milne coordinates

proper time: $\tau = \sqrt{t^2 - z^2}$ invariant under long. boosts

space-time rapidity $\eta = \operatorname{atanh}\left(\frac{z}{t}\right)$ additive under long. boosts

inverse transformation given by

$$t = \tau \cosh(\eta) \qquad z = \tau \sinh(\eta)$$

such that

$$u^\mu = \begin{pmatrix} \cosh(\eta) \\ \sinh(\eta) \end{pmatrix} = \begin{pmatrix} \cosh(\eta) & \sinh(\eta) \\ \sinh(\eta) & \cosh(\eta) \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{in Minkowski coord.}$$

which corresponds to a static system boosted by η.

We can describe the system even more efficiently, if we consider the <u>Milne coordinates</u> as our new coordinate system in the sense of general relativity, such that

$$\bar{u}^\mu = (\bar{u}^\tau, \bar{u}^x, \bar{u}^y, \bar{u}^\eta)$$

where the vector transforms as

$$\bar{u}^\mu = \frac{\partial \bar{x}^\mu}{\partial x^\nu} u^\nu$$

such that with

$$\frac{\partial \bar{x}^\mu}{\partial x^\nu} = \begin{pmatrix} \frac{\partial \tau}{\partial t} & \frac{\partial \tau}{\partial z} \\ \frac{\partial \eta}{\partial t} & \frac{\partial \eta}{\partial z} \end{pmatrix} = \begin{pmatrix} \cosh(\eta) & -\sinh(\eta) \\ -\frac{\sinh(\eta)}{\tau} & \frac{\cosh(\eta)}{\tau} \end{pmatrix}$$

the rest frame becomes

$$\bar{u}^\mu = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{in Milne coordinates}$$

and the system is at rest in these comoving coordinates.

Note that we also need to transform the metric

$$\bar{g}_{\rho\sigma} = \frac{\partial x^\mu}{\partial \bar{x}^\rho} g_{\mu\nu} \frac{\partial x^\nu}{\partial \bar{x}^\sigma} =$$

we got

$$\bar{g}_{\mu\nu} = \mathrm{diag}\left(-1, 1, 1, \overset{\overset{a^2(\tau)}{\|}}{\tau^2}\right) \quad \text{in Milne coord.}$$

as well as its inverse

$$\bar{g}^{\mu\nu} = \mathrm{diag}\left(-1, 1, 1, \overset{\overset{1/a^2(\tau)}{\|}}{1/\tau^2}\right) \quad \text{in Milne coord.}$$

We see at this stage that the pre-equilibrium QGP is locally at rest in a spacetime that anisotropically expands at a rate $\frac{\dot{a}(\tau)}{a(\tau)} = \frac{1}{\tau}$ i.e. very rapidly at early times and more slowly at later times.

Due to the expansion the energy density in the system will be diluted, i.e. the system will cool. Moreover, as the expansion is anisotropic, this is creating an anisotropy in the system, which can only be overcome once the interaction rate becomes comparable or larger to the expansion rate.

$\rightarrow$ Interplay of anisotropic expansion & interactions determine evolution of pre-equilibrium QGP

Genuine equilibrium can only be reached in a regime where long. expansion becomes negligibly slow

Evolution of energy momentum tensor

Evolution of the energy momentum tensor in Bjorken flow is governed by conservation equations

$$\nabla_\mu T^{\mu\nu}=0$$

where derivative has been promoted to covariant derivative in non-trivial space time

$$\nabla_\mu T^{\mu\nu}=\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\,T^{\mu\nu}\right)+\Gamma^{\nu}_{\mu\rho}T^{\mu\rho}$$

where

Metric determinant: $\sqrt{-g}=\sqrt{-\det(g_{\mu\nu})}=a(\tau)$

Christoffel symbols:

$$\Gamma^{\alpha}_{\beta\gamma}=\frac{1}{2}g^{\alpha\mu}\left(\frac{\partial g_{\mu\beta}}{\partial x^\gamma}+\frac{\partial g_{\mu\gamma}}{\partial x^\beta}-\frac{\partial g_{\beta\gamma}}{\partial x^\mu}\right)$$

Specifically for anisotropically expanding system

$$\Gamma^{\eta}_{\eta\tau}=\frac{1}{2}g^{\eta\eta}\frac{\partial g_{\eta\eta}}{\partial\tau}=\frac{1}{2}\frac{1}{a^2}2\dot{a}a=\frac{\dot{a}}{a}$$

$$\Gamma^{\eta}_{\tau\eta}=\frac{\dot{a}}{a}$$

$$\Gamma^{\tau}_{\eta\eta}=\frac{1}{2}g^{\tau\tau}\left(-\frac{\partial g_{\eta\eta}}{\partial\tau}\right)=\frac{1}{2}(-1)(-2\dot{a}a)=\dot{a}a$$

Now if we consider the conservation equations, due to the assumed symmetries of transversely homogeneous and long. boost invariant systems, the only nontrivial equation is the time like one $\nu=\tau$

Since $\partial_x, \partial_y, \partial_\eta$ do not contribute, we get

$$\frac{1}{a(\tau)} \partial_\tau \left(a(\tau) T^{\tau\tau} \right) = - \dot{a}(\tau) a(\tau) T^{\eta\eta}$$

so upon re-arranging, and identifying

$$T^{\eta}{}_{\eta} = a^2(\tau) T^{\eta\eta}$$

we have

$$\partial_\tau T^{\tau\tau} = - \frac{\dot{a}(\tau)}{a(\tau)} \left(T^{\tau\tau} + T^{\eta}{}_{\eta} \right)$$

Now to interpret this result, we should first identify the different components of the energy momentum tensor. If we consider the Bjorken flow symmetries, i.e.

- translation invariance in transverse plane
- isotropy in transverse plane
- boost invariance in longitudinal direction

the energy momentum tensor takes the general form

$$T^{\mu}{}_{\nu} = \mathrm{diag}\left(T^{\tau}{}_{\tau}, T^{x}{}_{x}, T^{y}{}_{y}, T^{\eta}{}_{\eta} \right)$$

where

$T^{\tau}{}_{\tau} = -\varepsilon$ energy density

$T^{x}{}_{x} = T^{y}{}_{y} = p_T$ transverse pressure

$T^{\eta}{}_{\eta} = p_L$ longitudinal pressure

If we consider Bjorken flow $a(\tau) = \tau$, we thus get

$$\partial_\tau \varepsilon = - \frac{\varepsilon + p_L}{\tau}$$

and we can easily interpret the result, where the term $\propto -\frac{\varepsilon}{\tau}$ describes the dilution due to increase in the volume, and the second term describes the work performed against the long. expansion. Indeed if we compare with

$$dU = -p\, dV \qquad U = \varepsilon \cdot V$$

we get

$$d\varepsilon\, V = -(\varepsilon + p)\, dV$$

Such that upon differentiating wrt τ and identifying $\frac{1}{V}\frac{dV}{d\tau} = \frac{\dot{a}(\tau)}{a(\tau)}$ we get $\partial_\tau \varepsilon = - \frac{\dot{a}(\tau)}{a(\tau)}(\varepsilon + p)$

which is similar to the above result, except that in the general derivation, it is the longitudinal pressure p_L that is relevant, which is not necessarily identical to the thermodynamic pressure p.

Evidently the conservation law alone is not sufficient to predict the evolution of the energy density, which is what we want, so to close this equation, we need information about the longitudinal pressure, which can come from

a) Hydrodynamic constitutive equations
b) Microscopic calculations (Kinetic Theory, Holography, ...)

Understanding the applicability of hydrodynamics then reduces to the question if constitutive equations are satisfied.

Hydrodynamic constitutive equations (conformal fluid)

Ideal hydrodynamics: $p_T = p_L = p$

Since for a conformal system $T^{\mu}{}_{\mu} = -\varepsilon + 2p_T + p_L = 0$

$$\Rightarrow \varepsilon = 3p \Rightarrow p = \frac{\varepsilon}{3}$$

The conservation equation then takes the form

$$\partial_\tau \varepsilon = -\frac{4}{3}\frac{\varepsilon}{\tau} \Rightarrow$$

which gives the solution for

Ideal Bjorken flow: $\varepsilon(\tau) = \dfrac{\varepsilon_0 \tau_0^{4/3}}{\tau^{4/3}}$

Navier-Stokes: If we consider first order dissipative corrections, due to symmetries of Bjorken the energy momentum tensor takes diagonal form

$$T^{\mu}{}_{\nu} = (-\varepsilon, p_T, p_T, p_L)$$

but now p_T, p_L receive dissipative corrections due to longitudinal shear induced by expansion, so we have

$$p_L = T^{\eta}{}_{\eta} = p + \pi^{\eta}{}_{\eta}$$

and due to $T^{\mu}{}_{\mu} = 0$ for a conformal system

$$p_T = T^{x}{}_{x} = T^{y}{}_{y} = p + \pi^{x}{}_{x} = p - \frac{1}{2}\pi^{\eta}{}_{\eta}$$

Now in first order hydro

$$\pi^{\mu\nu} = -\eta\sigma^{\mu\nu} = -\eta\left(\Delta^{\mu\alpha}\Delta^{\nu\beta} + \Delta^{\mu\beta}\Delta^{\nu\alpha} - \frac{2}{3}\Delta^{\mu\nu}\Delta^{\alpha\beta}\right)\nabla_\alpha u_\beta$$

where

$$\nabla_\alpha u_\beta = \partial_\alpha u_\beta - \Gamma^{\mu}_{\alpha\beta} u_\mu$$

Since the fluid is at rest in the comoving coordinates, the only non-trivial entry is

$$\nabla_\eta u_\eta = -\Gamma^{\tau}_{\eta\eta} u_\tau = \dot{a}a \qquad \left(\Gamma^{\tau}_{\eta\eta} = \dot{a}a,\ u_\tau = -1\right)$$

so we get $\left(g^{\eta\eta} = \frac{1}{a^2}\right)$

$$\sigma^{\eta\eta} = \left(2 - \frac{2}{3}\right)\frac{1}{a^4}\dot{a}a$$

$$\Rightarrow \sigma^{\eta}{}_{\eta} = \frac{4}{3}\frac{\dot{a}}{a} \quad \Rightarrow \pi^{\eta}{}_{\eta} = -\frac{4}{3}\eta\frac{\dot{a}}{a}$$

So for Bjorken flow

$$P_L^{NS} = P - \frac{4}{3}\frac{\eta}{\tau}$$

we see that the long. expansion leads to reduction of the longitudinal pressure.

Since η is a dimensionful quantity, it is useful to express

$$\eta = \eta \frac{\eta T}{\varepsilon + P}\frac{\varepsilon + P}{T} \underset{Ts = \varepsilon + P}{=} \frac{\eta}{s}\frac{\varepsilon + P}{T} \underset{P = \frac{\varepsilon}{3}}{=} \frac{4}{3}\frac{\eta}{s}\frac{\varepsilon}{T}$$

Such that

$$\frac{P_L^{NS}}{\varepsilon} = \left(\frac{1}{3} - \frac{16}{9}\frac{\eta}{s}\frac{1}{T\tau}\right) \varepsilon$$

Since $T(\tau)\tau$ grows as a function of time, the correction becomes small at late times and the solution converges to ideal hydrodynamics. However, we also see that at early times, the expression can become negative, which is unphysical and signals breakdown of the hydrodynamic description.

Need dynamical description for very early times.

MIS Hydrodynamics:

Clearly the simplest possible description is MIS hydrodynamics where now, $\pi^{\mu\nu}$ satisfies a dynamical equation

$$u^\mu \nabla_\mu \pi^{\alpha\beta} = -\frac{1}{\tau_\pi}\left(\pi^{\alpha\beta} - \pi^{\alpha\beta}_{NS}\right)$$

evaluating $\nabla_\lambda \pi^{\alpha\beta} = \partial_\lambda \pi^{\alpha\beta} + \Gamma^\alpha_{\lambda\sigma}\pi^{\sigma\beta} + \Gamma^\beta_{\lambda\sigma}\pi^{\alpha\sigma}$
we get

$$u^\mu \nabla_\mu \pi^{\eta\eta} = \nabla_\tau \pi^{\eta\eta} = \partial_\tau \pi^{\eta\eta} + 2\frac{\dot{a}}{a}\pi^{\eta\eta} = \frac{1}{a^2}\partial_\tau\left(a^2\pi^{\eta\eta}\right)$$

So using $g_{\eta\eta} = a^2$ to lower the index, we get

$$\boxed{\partial_\tau \pi^\eta{}_\eta = -\frac{1}{\tau_\pi}\left(\pi^\eta{}_\eta + \frac{4}{3}\eta\frac{\dot{a}}{a}\right)}$$

which together with the evolution equation for the energy density and the energy dependence of transport coefficients has interesting physical properties.

→ See exercise sheet 1