Dynamical attractors in Bjorken flow

Now as pointed out by Heller, Spalinski: the set of evolution equations

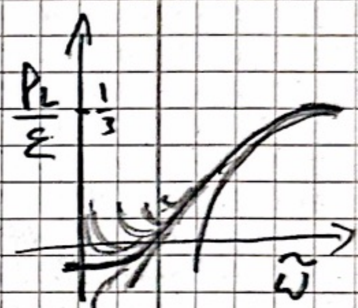
$$\partial_\tau \varepsilon = -\frac{\varepsilon + P_L}{\tau} \qquad P_L = P + \pi^\eta{}_\eta$$

$$\partial_\tau \pi^\eta{}_\eta = -\frac{1}{\tau_\pi}\left(\pi^\eta{}_\eta + \frac{4}{3}\frac{\eta}{\tau}\right)$$

shows a remarkable feature, regarding its solutions.

By introducing a scaled time variable

$$\tilde{w} = \frac{T\tau}{4\pi\,\eta/s} \sim \frac{\tau}{\tau_\pi^R}$$



the solutions exhibit stable fixed points at both early and late times.

Hydro attractor ($\tilde{w} \gtrsim 1$): $\quad \frac{\pi^\eta{}_\eta}{\varepsilon} = -\frac{4}{9\pi\tilde{w}} + O(\tilde{w}^{-2})$

Early time attractor ($\tilde{w} \ll 1$): $\quad \frac{\pi^\eta{}_\eta}{\varepsilon} = -\frac{2}{3} - \sqrt{\frac{4}{9} + \frac{16}{9}\frac{\eta/s}{T\tau_\pi}}$

While the late time behavior should be expected, as eventually the system is expected to converge to Navier-Stokes hydrodynamics and eventually thermalize, the existence of an early time attractor is surprising, and a result of the rapid longitudinal expansion.

Hydrodynamics, in the form of MIS equations showcases the phenomena that occur due to the interplay of anisotropic expansion and interactions in the system.

Namely the emergence of universal late time behavior and the possibility of partial information loss at early times, when the longitudinal expansion is dominant.

However, we don't expect hydrodynamics to accurately describe the early time dynamics, where system is highly anisotropic and subject to strong gradients due to expansion, so we need to resort to a more microscopic description.

Kinetic theory

Description of weakly interacting QFT in terms of phase-space density of quasi particles

$$f(x,p) \qquad x = x^\mu \qquad p = p^i$$

which satisfies a Boltzmann equation

$$p^\mu \partial_\mu f(x,p) - \Gamma^i_{\mu\nu} p^\mu p^\nu \frac{\partial f}{\partial p^i} = C[f](x,p)$$

where the microphysics is encoded in the properties of the collision integral.

By solving this integro-differential equation we can then compute the evolution of the energy momentum tensor

$$T^{\mu\nu}(x) = \nu \int \frac{d^3p}{(2\pi)^3} \sqrt{-g}\, \frac{p^\mu p^\nu}{E_p^0}\, f(x,p)$$

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Now in order to do this we need to specify the microscopic interaction, and we will first consider the simplest possible model

Conformal RTA:

$$C[f] = +\frac{p^\mu u_\mu}{\tau_R}\left(f - f_{eq}\right)$$

where f_{eq} for Bosons

$$f_{eq} = f_{Bose-Einstein}\left(\frac{p_\mu u^\mu}{T}\right)$$

with u^μ and T determined self-consistently from Landau matching condition

$$T^{\mu}{}_{\nu} = -\varepsilon u^\mu \,, \quad \varepsilon = \frac{\pi^2}{30}\nu T^4$$

which is necessary to satisfy energy momentum conservation.

If we are interested in a conformal system, we set

$$\tau_R = 5\,\frac{\eta/s}{T} \quad \text{with } \eta/s = \text{const}$$

Now let us adapt this description to Bjorken flow, and to do so, we first note that we are interested in

$$P_L(\tau) = T^{\eta}{}_{\eta}(\tau) = \nu \int \frac{d^2 p_\perp}{(2\pi)^2} \int \frac{dp^\eta}{2\pi}\, a(\tau)\, \frac{p^\eta a^2(\tau) p^\eta}{E_p}\, f(\tau, p_\perp, p^\eta)$$

we see at this point, that it is more advantageous to use a different longitudinal momentum variable

$$p_{\parallel} = a(\tau)\, p^\eta \qquad\qquad p_{\parallel}^2 = p_\eta p^\eta$$

which has dimension of momentum, and the energy-momentum tensor is given by

$$P_L(\tau) = T^{\eta}{}_{\eta}(\tau) = \nu \int \frac{d^2 p_\perp}{(2\pi)^2} \int \frac{dp_{\parallel}}{2\pi}\, \frac{p_{\parallel}^2}{E_p}\, f(\tau, p_\perp, p_{\parallel})$$

Now lets look at the Boltzmann equation, where we need to evaluate

$$\Gamma^{i}_{\mu\nu}\, p^\mu p^\nu \frac{\partial f}{\partial p^i} \underset{\substack{\text{only } i=\eta \\ \text{contributes}}}{=} \left(\Gamma^{\eta}_{\tau\eta} + \Gamma^{\eta}_{\eta\tau}\right) p^\eta p^\tau \frac{\partial f}{\partial p^\eta} = 2\frac{\dot a}{a}\, p^\tau p^\eta \frac{\partial f}{\partial p^\eta}$$

now if we want to change coordinates from p^η to $p_{\parallel} = a(\tau) p^\eta$ we also need to consider that time derivatives at constant p^η are different than time derivatives at constant $p_{\parallel}$

$$p^\mu \partial_\mu f = p^\tau \frac{\partial f}{\partial \tau}\Big|_{p^\eta} = p^\tau \frac{\partial f}{\partial \tau}\Big|_{p_{\parallel}} + p^\tau \frac{\partial p_{\parallel}}{\partial \tau}\Big|_{p^\eta} \frac{\partial f}{\partial p_{\parallel}}\Big|_{\tau} = p^\tau \frac{\partial f}{\partial \tau}\Big|_{p_{\parallel}} + \frac{\dot a}{a}\, p^\tau p_{\parallel} \frac{\partial f}{\partial p_{\parallel}}$$

which partially cancels against the other terms

Collecting everything, we have for generalized Bjorken flow

$$\partial_\tau f = \underbrace{\frac{\dot a}{a} p_\parallel \partial_{p_\parallel} f}_{\text{longi expansion}} + \underbrace{C[f]}_{\text{interactions}} \overset{RTA}{=} \frac{1}{\tau} p_\parallel \frac{\partial f}{\partial p_\parallel} - \frac{1}{\tau_R}\left(f - f_{eq}\right)$$

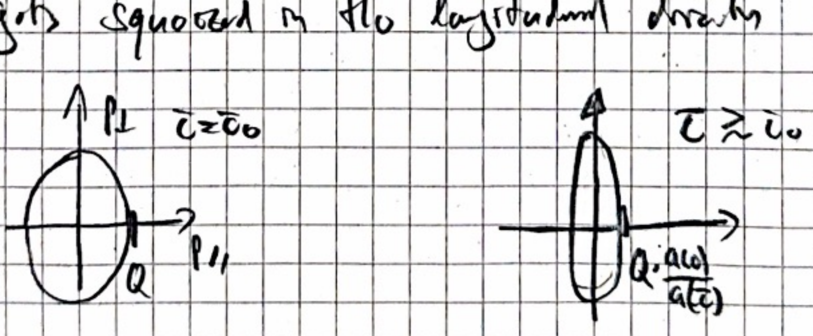
Now for Bjorken flow $\frac{\dot a}{a} = \frac{1}{\tau}$ and the expansion dominates at early times. ($\tau \ll \tau_R$), if for the moment we neglect the interaction terms, we can obtain an analytic solution to free-streaming

Free-Streaming: $\partial_\tau f(\tau, p_\perp, p_\parallel) = \frac{\dot a}{a} p_\parallel \frac{\partial}{\partial p_\parallel} f(\tau, p_\perp, p_\parallel)$

We can solve this by the method of characteristics to yielding

$$f(\tau, p_\perp, p_\parallel) = f_0\left(p_\perp, \frac{p_\parallel \cdot a(\tau)}{a(0)}\right)$$

which indicates, that the distribution remains unchanged wrt to $p_\perp$ as there are no interactions, but gets squeezed in the longitudinal direction



Since the longitudinal pressure measures the width of the distribution in $p_\parallel$, this leads to a rapid reduction of P_L

If we consider for Bjorken flow $a(\tau)=\tau$

$$P_L(\tau)=T^\eta{}_\eta(\tau)=\nu\int\frac{d^2p_\perp}{(2\pi)^2}\int\frac{dp_\parallel}{2\pi}\frac{p_\parallel^2}{\sqrt{p_\perp^2+p_\parallel^2}}\,f_0\left(p_\perp,p_\parallel\frac{\tau}{\tau_0}\right)$$

we can change variables to $p_\parallel^0=p_\parallel\frac{\tau}{\tau_0}$

$$=\left(\frac{\tau_0}{\tau}\right)^3\nu\int\frac{d^2p_\perp}{(2\pi)^2}\int\frac{dp_\parallel^0}{2\pi}\frac{{p_\parallel^0}^2}{\sqrt{p_\perp^2+\frac{\tau_0}{\tau}{p_\parallel^0}^2}}\,f_0(p_\perp,p_\parallel^0)$$

$$\simeq\left(\frac{\tau_0}{\tau}\right)^3\nu\int\frac{d^2p_\perp}{(2\pi)^2}\int\frac{dp_\parallel^0}{2\pi}\frac{{p_\parallel^0}^2}{p_\perp}\,f_0(p_\perp,p_\parallel^0)$$

If already initially $p_\parallel^0\ll p_\perp$ for typical excitations then

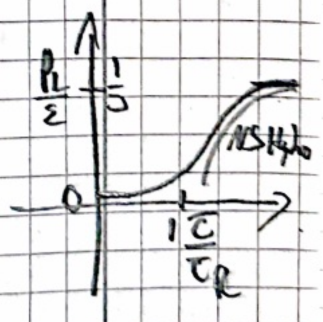
$$\simeq\left(\frac{\tau_0}{\tau}\right)^3P_L(\tau_0)$$

irrespective of this behaviour, the longitudinal pressure decays rapidly and

<u>early time attractor:</u> $P_L=0+\mathcal{O}\left(\frac{\tau}{\tau_R}\right)$

Conversely for $\tau\gtrsim\tau_\pi$ interactions start to play an important role, and for $\tau\gg\tau_\pi$ dominate the dynamics of the system.

Since interactions drive $f\to f_{eq}$, in this regime, we find



late time attractor $P_L=P_L^{eq}+\mathcal{O}\left(\frac{\tau_R}{\tau}\right)$

where the corrections correspond to the usual hydrodynamic expansion

Dynamical attractors beyond Bjorken flow

Now to understand better the concept of attractors consider a situation, where instead of Bjorken expansion the system is periodically driven

$$a(t) = 1 + \alpha \sin(\Omega t)\,\theta(t)$$

such that at $t=0$ space starts to periodically expand and compress, and we can independently control

α magnitude of perturbation

Ω rate of expansion

Since the expansion will drive the system out of equilibrium, the near equilibrium regime, where hydrodynamics is applicable and accurate is given by

Hydro regime: $\alpha \ll 1$, $\Omega \tau_\pi \ll 1$

i.e. response to small perturbations on long time scales compared to dynamics of the system.

Nevertheless, we can obtain analytic solutions for $\alpha \ll 1$ and any $\Omega \tau_\pi$.

Solution in MIS Hydro

We have for the longitudinal component $\pi = \pi^{\eta}{}_{\eta}$ of the shear stress

$$\partial_t \pi = -\frac{1}{\tau_\pi}\left(\pi - \pi_{NS}\right)$$

where $\pi_{NS} = -\frac{4}{3}\eta \frac{\dot a}{a}$.

Such that for weak perturbations

$$\frac{\dot a}{a} \overset{decl}{\simeq} \dot a = \alpha \Omega \cos(\Omega t)\, \theta(t)$$

so we need to solve the ODE

$$\left(1 + \tau_\pi \partial_t\right) \pi = -\frac{4}{3}\eta \alpha \Omega \cos(\Omega t)$$

In principle this is straightforward to solve, e.g. by Laplace transforming however to illustrate the structure of the solutions, it is more insightful to consider the perturbation as a collection of Fourier modes

$$a(t) = \int \frac{d\omega}{2\pi}\, a(\omega)\, e^{-i\omega t}$$

where $a(\omega) = \frac{\alpha}{2}\left[\frac{1}{\omega+\Omega+i\varepsilon} - \frac{1}{\omega-\Omega+i\varepsilon}\right]$

with the $i\varepsilon$ prescription to obtain the $\theta(t)$ dependence.

In Fourier-space

$$\dot a(\omega) = -i\omega\, a(\omega)$$

and we obtain

$$(1 - i\omega\tau_\Pi)\,\Pi(\omega) = +\frac{4}{3}\eta\, i\omega\; a(\omega)$$

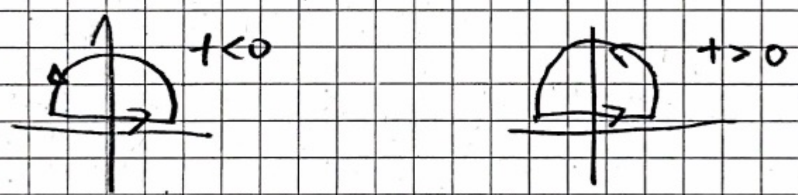
such that, we can obtain the Green's function

$$G(\omega) = \frac{\partial \Pi(\omega)}{\partial a(\omega)} = \frac{4}{3}\eta\, \frac{i\omega}{1 - i\omega\tau_\Pi}$$

which we can use to determine the response to an arbitrary perturbation according to

$$\Pi(t) = \int \frac{d\omega}{2\pi}\, G(\omega)\, a(\omega)\, e^{-i\omega t}$$

When computing the frequency integral, we can use the residue theorem, and close the contour:



the singularities of G and a now determine the structure of the solution, i.e. for $t<0$ no singularity is enclosed, whereas for $t>0$, we get

two singularities at $\omega = \pm\Omega - i\varepsilon$ from a

one singularity at $\omega = -i/\tau_\Pi$ from G

such that the solution is a combination of oscillating and exponentially decaying behavior

One finds for sinusoidal perturbations

$$\Pi = -\frac{4}{3}\eta\Omega\alpha\left[\frac{\cos(\Omega t) + \Omega\tau_\Pi \sin(\Omega t) - e^{-t/\tau_\Pi}}{1+\Omega^2\tau_\Pi^2}\right]$$

If we also consider an initial stress $\Pi(t=0) \neq 0$ we get an extra contribution

$$+\,\Pi(t=0)\, e^{-t/\tau_\Pi}$$

which is the homogeneous solution of the linear ODE.

Now we can observe the characteristic behavior of the solution;

Early times: $t \lesssim \tau_\Pi$ transient decay of e^{-t/τ_Π} terms associated with initial shear stress and non-hydrodynamic modes corresponding to poles of $G(\omega)$

Late times: $t \gg \tau_\Pi$ Universal time dependence of $\Pi(t)$ for different initial conditions

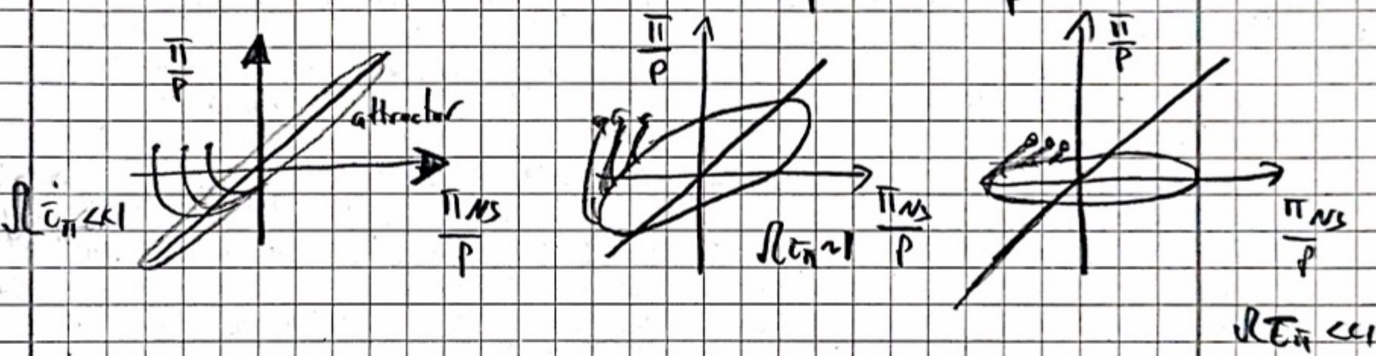
Hydro attractor: $\Pi(t) = -\frac{4}{3}\eta\Omega\alpha\left[\frac{\cos(\Omega t) + \Omega\tau_\Pi \sin(\Omega t)}{1+\Omega^2\tau_\Pi^2}\right]$

Now in the hydrodynamic limit $\Omega\tau_\pi \ll 1$

$$\Pi(t) \simeq -\frac{4}{3}\eta\Omega\alpha\cos(\Omega t) = \Pi_{NS}(t)$$

reduces to first order hydrodynamics, and the system is in some sense close to equilibrium. However the dynamical attractor emerges also for $\Omega\tau_\pi \gtrsim 1$, where it shows significant deviations from equilibrium / Navier-Stokes hydrodynamics

We can illustrate this by plotting $\frac{\Pi}{P}$ vs $\frac{\Pi_{NS}}{P}$



while for $\Omega\tau_\pi \ll 1$ the system remains close to equilibrium, for large $\Omega\tau_\pi \gg 1$ it is unable to follow the rapid expansion, the late time behaviour is universal, but quite different from ordinary Navier-Stokes hydro

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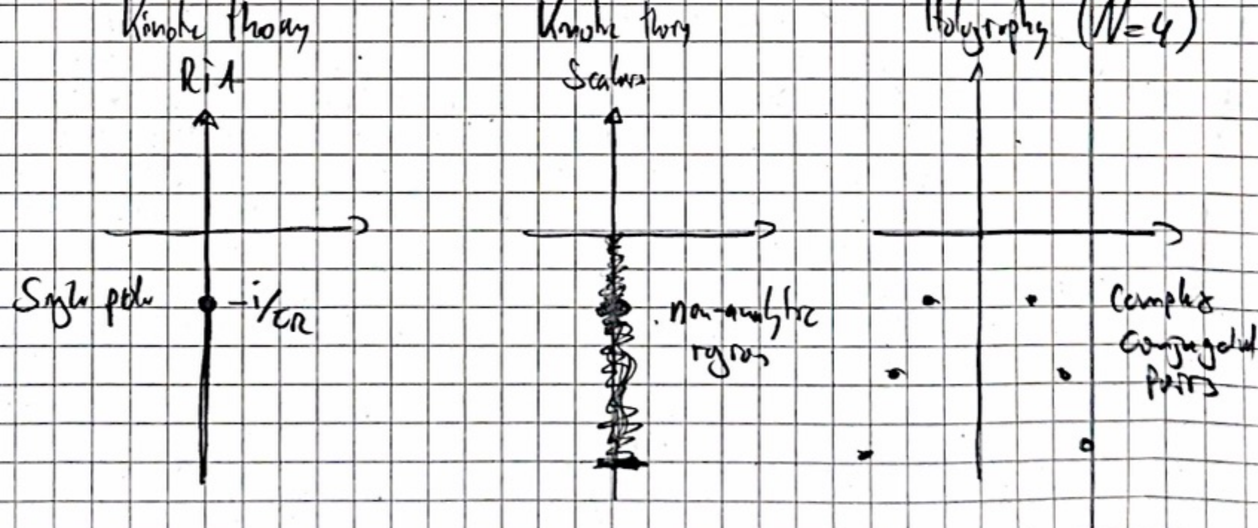
We have seen that hydrodynamic attractors

- describe long time evolution of non-equilibrium system
- emerge after decay of transients ($t \gg \tau_\pi$) as described by poles of retarded Green's function $G(\omega)$ of the energy momentum tensor

While in MIS Hydrodynamics, the structure of $G(\omega)$ is rather simple, with a single pole on the imaginary axis, the investigation of these structures in microscopic theories is an area of active investigation

Generally stability demands singularities to be in the lower complex half plane, however the structure can be different for different microscopic theories

$k=0$



Note that if spatial dependence (e^{ikx}) is also taken into account structure becomes significantly more complex