Hydrodynamics and Kinetic Theory: Attractors and Thermalization
Exercise sheet 2

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1 Dynamical attractor in generalized Bjorken flow

Consider a system of massless particles ($p_0 = E_p = p$) described by the conformal kinetic equation

$$\frac{\partial f}{\partial t} = \frac{\dot{a}}{a} p_z \frac{\partial f}{\partial p_z} - \frac{1}{\tau_R}[f - f_{\text{eq}}(p/T[f])] \, ,$$

that is subject to a weak ($\alpha \ll 1$) harmonic perturbation of the metric

$$a(t) = 1 + \alpha e^{-i\omega t} \, .$$

The system is close to equilibrium and thrown out of equilibrium via the perturbation. We consider correction to equilibrium quantities X to linear order in α and write them as $\delta X_\omega e^{i\omega t} \sim \mathcal{O}(\alpha)$ in anticipation of all perturbations coming with a phase factor $e^{i\omega t}$. For example, we will express the phase space distribution as

$$f(t, p) = f_{\text{eq}}(p/T) + \delta f_\omega(p) e^{-i\omega t} \, .$$

i) Show that at linear order in α the solution to the kinetic equation can be expressed as

$$\delta f_\omega = \left[\frac{-i\omega\tau_R \alpha \frac{p_z^2}{p} - \frac{\delta T_\omega}{T} p}{1 - i\omega\tau_R} \right] \frac{\partial f_{\text{eq}}}{\partial p}$$

Hint: Keep in mind that in the evolution equation, f_{eq} is a function of f and will receive corrections due to how δf_ω changes the temperature T of the system.

We first compute the metric factor to linear order in α.

$$\frac{\dot{a}}{a} = \frac{-i\omega\alpha e^{-i\omega t}}{1 + \alpha e^{-i\omega t}} \approx -i\omega\alpha e^{-i\omega t}$$

Here, the linear term is the leading order term. Thus, if we look at the evolution equation at $\mathcal{O}(\alpha)$, we get

$$\alpha \frac{\partial(\delta f_\omega e^{-i\omega t})}{\partial t} = -i\omega\alpha\delta f_\omega e^{-i\omega t} = -i\omega\alpha e^{-i\omega t} p_z \frac{\partial p_z}{\partial p} \frac{\partial f_{\text{eq}}}{\partial p} - \frac{1}{\tau_R} \left(\delta f_\omega e^{-i\omega t} - \delta f_{\text{eq}} \right)$$

For the change in the equilibrium distribution, we get

$$\delta f_{\text{eq}} = \delta T \frac{\partial f_{\text{eq}}}{\partial T} = \delta T \left(-\frac{p}{T^2} \right) \frac{\partial f_{\text{eq}}}{\partial (p/T)} = \delta T \left(-\frac{p}{T} \right) \frac{\partial f_{\text{eq}}}{\partial p} = -\delta T_\omega e^{-i\omega t} \frac{p}{T} \frac{\partial f_{\text{eq}}}{\partial p}$$

Plugging this in together with $\partial p / \partial p_z = p_z / p$ and multiplying by $\alpha^{-1} \tau_R e^{i\omega t}$, we find

$$(1 - i\omega\tau_R)\delta f_\omega = \left(-i\omega\alpha\tau_R \frac{p_z^2}{p} - \frac{\delta T_\omega}{T} p \right) \frac{\partial f_{\text{eq}}}{\partial p}$$

and the desired equality follows.

ii) Calculate the change of the energy density δT^{00}_ω and use the identity $\delta T_\omega/T = \delta T^{00}/4T^{00}$ to self-consistently determine the induced variation of the temperature $\delta T_\omega/T = -\alpha/3$.
Hint: The energy-momentum tensor is defined as

$$T^{\mu\nu} = \int \frac{\mathrm{d}^3 p}{(2\pi)^3 p_0} p^\mu p^\nu f$$

and for our system in equilibrium, due to isotropy, one will find

$$T^{00} = \frac{1}{2\pi^2} \int_0^\infty \mathrm{d}p\ p^3 f_{\text{eq}}\,.$$

It is a good idea to work in spherical coordinates.
The change in T^{00} (with the phase factor taken out) is given as

$$\delta T^{00}_\omega = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} p\ \delta f_\omega\,.$$

We want to plug in our result for δf_{eq}, but immediately switch to spherical coordinates $\mathrm{d}^3 p = \mathrm{d}p\ p^2 \mathrm{d}\cos\theta \mathrm{d}\phi$, $p_z = p\cos\theta$.

$$\begin{aligned}
\delta T^{00}_\omega &= \frac{1}{(2\pi)^2} \int_0^\infty \mathrm{d}p \int_{-1}^1 \mathrm{d}\cos\theta\ p^3 \alpha \left(\frac{-i\omega\tau_R p \cos^2\theta - \frac{\delta T_\omega}{T} p}{1 - i\omega\tau_R} \right) \frac{\partial f_{\text{eq}}}{\partial p} \\
&= \alpha \frac{1}{2\pi^2} \int_0^\infty \mathrm{d}p\ p^4 \left(\frac{-\frac{1}{3} i\omega\tau_R - \frac{\delta T_\omega}{T}}{1 - i\omega\tau_R} \right) \frac{\partial f_{\text{eq}}}{\partial p} \\
&= \alpha \left(\frac{\frac{1}{3} i\omega\tau_R - \frac{\delta T_\omega}{T}}{1 - i\omega\tau_R} \right) \frac{1}{2\pi^2} \left\{ \underbrace{[p^4 f_{\text{eq}}]_0^\infty}_{=0} - 4 \int_0^\infty \mathrm{d}p\ p^3 f_{\text{eq}} \right\} \\
&= -4\alpha \left(\frac{-\frac{1}{3} i\omega\tau_R - \frac{\delta T_\omega}{T}}{1 - i\omega\tau_R} \right) T^{00}
\end{aligned}$$

So we find for the change in temperature

$$\begin{aligned}
\frac{\delta T_\omega}{T} = \frac{\delta T^{00}}{4T^{00}} = -\alpha \left(\frac{-\frac{1}{3} i\omega\tau_R - \frac{\delta T_\omega}{T}}{1 - i\omega\tau_R} \right) \\
\Leftrightarrow \underbrace{\left(1 - \frac{1}{1 - i\omega\tau_R} \right)}_{= \frac{-i\omega\tau_R}{1 - i\omega\tau_R}} \frac{\delta T_\omega}{T} = \alpha \frac{\frac{1}{3} i\omega\tau_R}{1 - i\omega\tau_R} \quad \Leftrightarrow \frac{\delta T_\omega}{T} = -\frac{\alpha}{3}\,.
\end{aligned}$$

iii) Show that the induced variation of the longitudinal pressure is given as

$$\delta T^{zz}_\omega = \alpha \frac{\frac{4}{5} i\omega\tau_R - \frac{4}{9}}{1 - i\omega\tau_R} T^{00} \,.$$

The change in T^{zz}, again with the phase factor taken out, is given as

$$\delta T^{zz}_\omega = \int \frac{\mathrm{d}^3 p}{(2\pi)^3 p} p_z^2 \, \delta f_\omega$$

We plug in δf_ω with the result for δT_ω from ii) and switch to spherical coordinates.

$$\begin{aligned}
\delta T^{zz} &= \frac{1}{(2\pi)^2} \int_0^\infty \mathrm{d}p \int_{-1}^{1} \mathrm{d}\cos\theta \; p^3 \cos^2\theta \; \alpha \left(\frac{-i\omega\tau_R p \cos^2\theta + \frac{1}{3} p}{1 - i\omega\tau_R} \right) \frac{\partial f_{\mathrm{eq}}}{\partial p} \\
&= \alpha \frac{1}{2\pi^2} \int_0^\infty \mathrm{d}p \; p^4 \left(\frac{-\frac{1}{5} i\omega\tau_R + \frac{1}{9}}{1 - i\omega\tau_R} \right) \frac{\partial f_{\mathrm{eq}}}{\partial p} \\
&= \alpha \left(\frac{-\frac{1}{5} i\omega\tau_R + \frac{1}{9}}{1 - i\omega\tau_R} \right) \frac{1}{2\pi^2} \left\{ \underbrace{[p^4 f_{\mathrm{eq}}]_0^\infty}_{=0} - 4 \int \mathrm{d}p \; p^3 f_{\mathrm{eq}} \right\} \\
&= \alpha \left(\frac{\frac{4}{5} i\omega\tau_R - \frac{4}{9}}{1 - i\omega\tau_R} \right) T^{00}
\end{aligned}$$

iv) Find the variation in the shear $\delta\pi^{zz}_\omega$ (Hint: $\pi^{zz} = T^{zz} - p$ and $\delta p_\omega = \delta T^{00}/3$). Use this to find the Green's function

$$G(\omega) = \frac{\partial \delta\pi^{zz}_\omega}{\partial\alpha}$$

and then use the results $T\tau_R = 5\eta/s$ and $\tau_\pi = \tau_R$ for the transport coefficients to compare your result for $G(\omega)$ to the solution in MIS hydro that was discussed in the lecture.

The variation in pressure evaluates to

$$\frac{\delta p_\omega}{T^{00}} = \frac{1}{3} \frac{\delta T^{00}_\omega}{T^{00}} = \frac{4}{3} \frac{\delta T_\omega}{T} = -\frac{4}{9} \alpha \,.$$

With this, the Green's function for the shear becomes

$$G(\omega) = \frac{\partial T^{zz}_\omega}{\partial\alpha} - \frac{\partial \delta p_\omega}{\partial\alpha} = \left(\frac{\frac{4}{5} i\omega\tau_R - \frac{4}{9}}{1 - i\omega\tau_R} \right) T^{00} - \frac{4}{9} T^{00} = \frac{16}{45} \frac{i\omega\tau_R}{1 - i\omega\tau_R} T^{00} \,.$$

We use $\tau_R/5 = \eta/(\frac{4}{3}e)$ in the numerator and $\tau_R = \tau_\pi$ in the denominator. We also identify $T^{00} = e$. With this, the Green's function becomes

$$G(\omega) = \frac{16}{45} \frac{5 \cdot 3}{4} \frac{i\omega\eta}{1 - i\omega\tau_\pi} = \frac{4}{3} \eta \frac{i\omega}{1 - i\omega\tau_\pi} \,.$$

The result is the same as the one that was found for MIS hydro, which means that MIS hydro with RTA transport coefficients has the same transient dynamics as RTA.