Hydrodynamics and Kinetic Theory: Attractors and Thermalization
Exercise sheet 1

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1 Pressure anisotropy in conformal Bjorken flow

Consider a conformal system ($e = 3P \propto T^4$ and $sT = e + P$) in Bjorken flow that follows the first order Müller-Israel-Stewart-type hydrodynamic equations

$$\frac{\mathrm{d}e}{\mathrm{d}\tau} = -\frac{e + P_L}{\tau}\,, \qquad \tau_\pi \frac{\mathrm{d}\pi_\eta^\eta}{\mathrm{d}\tau} = -\pi_\eta^\eta - \frac{4\eta}{3\tau}\,,$$

with the longitudinal pressure $P_L = P + \pi_\eta^\eta$, the shear viscosity $\eta = C_\eta s$ and the shear relaxation time $\tau_\pi = C_{\tau\pi} T^{-1}$, where C_η and $C_{\tau\pi}$ are constants. We also introduce the scaled time $\tilde{w} = \frac{T\tau}{4\pi\eta/s}$. Note that we have to assume that $\mathrm{d}\tilde{w}/\mathrm{d}\tau > 0$ at all times for this to make sense.

i) Show that a defined through $\tau\,\mathrm{d}/\mathrm{d}\tau = a(\tilde{w})\tilde{w}\,\mathrm{d}/\mathrm{d}\tilde{w}$ satisfies

$$a = \frac{2}{3} - \frac{f_\pi}{4},$$

where $f_\pi = \pi_\eta^\eta / e$.
By acting with the derivatives on $\tilde{w}$, we get $a = \frac{\tau}{\tilde{w}}\frac{\mathrm{d}\tilde{w}}{\mathrm{d}\tau}$. Evaluating the derivatives using $\tilde{w} \propto \tau T$, we find

$$a = \frac{\tau}{\tilde{w}}\frac{\mathrm{d}\tilde{w}}{\mathrm{d}\tau} = 1 + \frac{\tau}{T}\frac{\mathrm{d}T}{\mathrm{d}\tau} = 1 + \frac{\tau}{4e}\frac{\mathrm{d}e}{\mathrm{d}\tau} = 1 - \frac{1}{4e}(e + P_L) = 1 - \frac{1}{4e}\left(\frac{4}{3}e + \pi_\eta^\eta\right) = \frac{2}{3} - \frac{\pi_\eta^\eta}{4e}\,.$$

ii) Show that f_π obeys the following evolution equation in terms of $\tilde{w}$.

$$\tilde{w}\left(\frac{2}{3} - \frac{f_\pi}{4}\right)\frac{\mathrm{d}f_\pi}{\mathrm{d}\tilde{w}} = -\frac{4\pi C_\eta}{C_{\tau\pi}}\tilde{w} f_\pi - \frac{16 C_\eta}{9 C_{\tau\pi}} + \frac{4}{3}f_\pi + f_\pi^2$$

Thus, in terms of $\tilde{w}$, the evolution equation for f_π decouples from the energy density.
We plug in the evolution equations for e and π.

$$\tau\frac{\mathrm{d}f_\pi}{\mathrm{d}\tau} = \frac{\tau}{e}\frac{\mathrm{d}\pi_\eta^\eta}{\mathrm{d}\tau} - \frac{\tau\pi_\eta^\eta}{e^2}\frac{\mathrm{d}e}{\mathrm{d}\tau} = -\frac{\tau}{\tau_\pi}\frac{\pi_\eta^\eta}{e} - \frac{4\eta}{3\tau_\pi e} - \frac{\pi_\eta^\eta}{e}\left(-\frac{4}{3} - \frac{\pi_\eta^\eta}{e}\right) = -\frac{4\pi C_\eta}{C_{\tau\pi}}\tilde{w} f_\pi - \frac{16 C_\eta}{9 C_{\tau\pi}} + \frac{4}{3}f_\pi + f_\pi^2$$

From i) we already know

$$\frac{\tau}{\tilde{w}}\frac{\mathrm{d}\tilde{w}}{\mathrm{d}\tau} = \frac{2}{3} - \frac{f_\pi}{4} \quad \text{and thus} \quad \tau\frac{\mathrm{d}f_\pi}{\mathrm{d}\tau} = \tau\frac{\mathrm{d}\tilde{w}}{\mathrm{d}\tau}\frac{\mathrm{d}f_\pi}{\mathrm{d}\tilde{w}} = \tilde{w}\left(\frac{2}{3} - \frac{f_\pi}{4}\right)\frac{\mathrm{d}f_\pi}{\mathrm{d}\tilde{w}}\,.$$

2 Asymptotic behaviour

Use the evolution equation for $f_\pi(\tilde{w})$ to find

i) the late time asymptotics for $f_\pi(\tilde{w})$, i.e. expand f_π up to linear order in $1/\tilde{w}$.
Plugging the ansatz $f_\pi = f_0 + f_1\tilde{w}^{-1} + \mathcal{O}(\tilde{w}^{-2})$ into the evolution equation, we find at $\mathcal{O}(\tilde{w})$:

$$0 = -\frac{4\pi C_\eta}{C_{\tau\pi}}\tilde{w}f_0 \quad \Rightarrow \quad f_0 = 0$$

and at $\mathcal{O}(1)$:

$$0 = -\frac{4\pi C_\eta}{C_{\tau\pi}}f_1 - \frac{16C_\eta}{9C_{\tau\pi}} \quad \Rightarrow \quad f_1 = -\frac{4}{9\pi}\,.$$

ii) possible values for f_π at $\tilde{w} \to 0$ and their stability at small finite $\tilde{w}$ by using them to reformulate the evolution equation.
In the limit $\tilde{w} \to 0$, we get the algebraic equation

$$0 = -\frac{16C_\eta}{9C_{\tau\pi}} + \frac{4}{3}f_\pi + f_\pi^2$$

with the solutions

$$f_\pm = -\frac{2}{3} \pm \sqrt{\frac{4}{9} + \frac{16}{9}\frac{C_\eta}{C_{\tau\pi}}}\,.$$

f_+ is positive and f_- is negative. Now we can rewrite the evolution equation as

$$\left(\frac{2}{3} - \frac{f_\pi}{4}\right)\frac{\mathrm{d}f_\pi}{\mathrm{d}\tilde{w}} = -\frac{4\pi C_\eta}{C_{\tau\pi}}f_\pi + \tilde{w}^{-1}(f_\pi - f_-)(f_\pi - f_+)\,.$$

At small $\tilde{w}$, we can neglect the first term on the right hand side. For the factor $2/3 - f_\pi/4$, we just need to know that it is positive as $\mathrm{d}\tilde{w}/\mathrm{d}\tau > 0$. Now if we are close to f_-, $f_\pi - f_+$ will be negative and the evolution equation will make f_π decay to f_-. If we are close to f_+, $f_\pi - f_-$ will be positive and the evolution equation will push f_π away from f_+.