Non-perturbative thermal QCD up to the electroweak scale

MICHELE PEPE

I N F N Sez. Milano – Bicocca

Milan (I t a l y)

PLAN OF THE TALK

- QCD on the lattice: a brief introduction
- Why the QCD Equation of State at high T matters
- Numerical challenges and strategy to high T regime
- The numerical study and results
- Mesonic and baryonic screening masses
- Conclusions and perspectives

QCD on the lattice

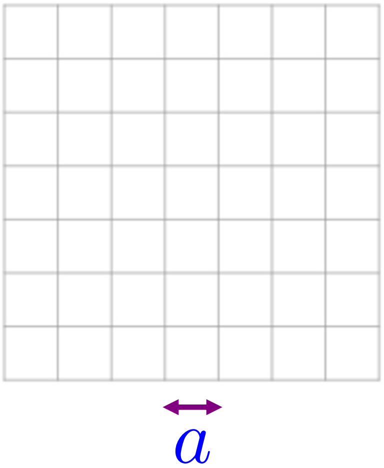
- The lattice regularization is the only currently known framework where QCD can be defined and investigated non-perturbatively.

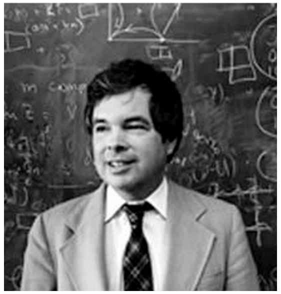
- In 1974 Wilson proposed to regularize QCD by making continuous space-time discrete

  $g_{\mu\nu} = \delta_{\mu\nu}$: Euclidean metric

  a: lattice spacing

  a^{-1}: cutoff in momenta





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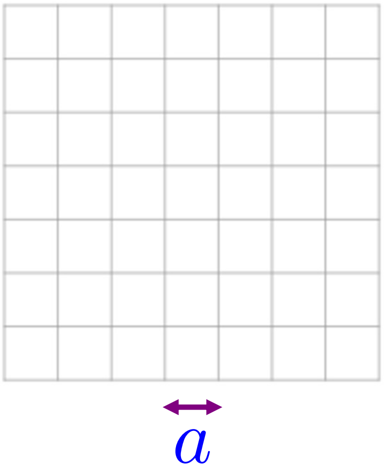

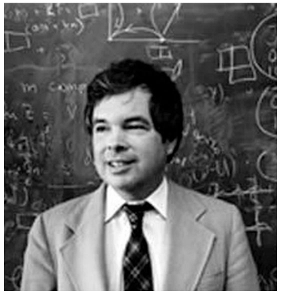

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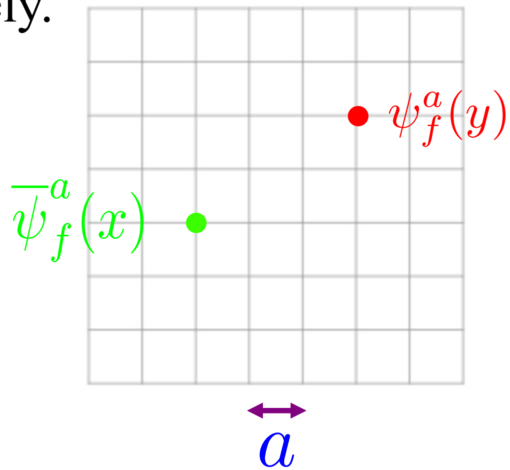

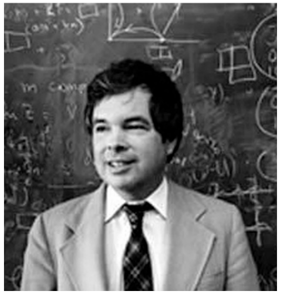

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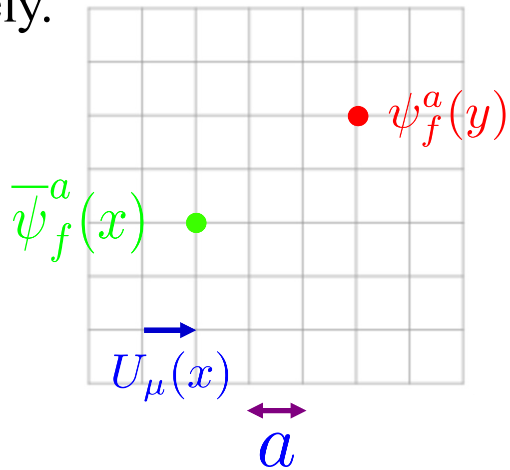

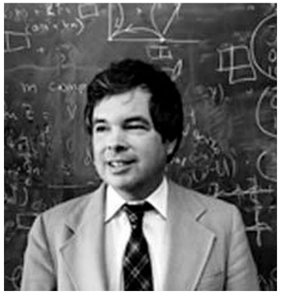

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$A_\mu(x)$ connection between different points in space-time $G(x,y) = \exp\left[i \int_{\mathcal{C}_{xy}} A \cdot ds\right] \in$ group SU(3)

the link field: $U_\mu(x) = \exp\left[i \int_x^{x+a\hat{\mu}} A_\mu(y) dy_\mu\right] \in SU(3)$

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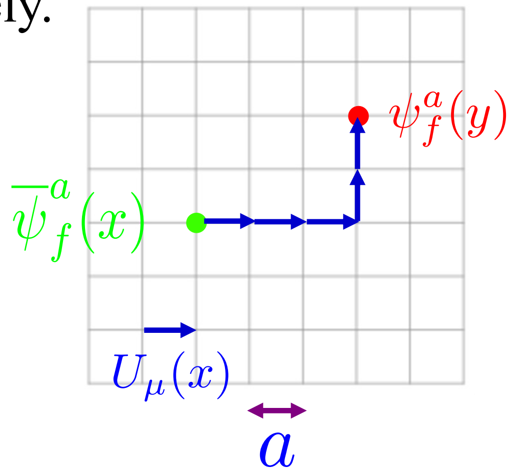

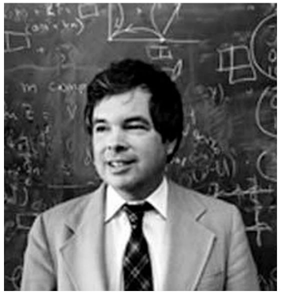

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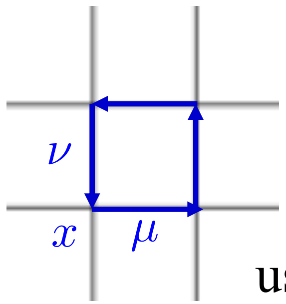
- Matter fields at different lattice points are connected by a path-ordered products of links

The action: the gauge sector

- The plaquette



$$U_{\mu\nu}(x) = U_\mu(x) U_\nu(x + a\hat{\mu}) U_\nu(x + a\hat{\nu})^\dagger U_\nu(x)^\dagger = \exp\left[i a^2 F_{\mu\nu}(x) + \mathcal{O}(a^3) \right]$$

$$S_G = \frac{2}{g_0^2} \sum_x \sum_{\mu<\nu} \mathrm{Re\ Tr}\left[1 - U_{\mu\nu}(x)\right] \xrightarrow[a\to 0]{} \frac{a^4}{2g_0^2} \sum_x \sum_{\mu\nu} \mathrm{Tr}\left[F_{\mu\nu}(x) F_{\mu\nu}(x)\right]$$

usually rewritten as $S_G = \beta \sum_x \sum_{\mu<\nu} \mathrm{Re\ Tr}\left\{\frac{1}{3}\left[1 - U_{\mu\nu}(x)\right]\right\}; \qquad \beta = \frac{6}{g_0^2}$ Wilson action

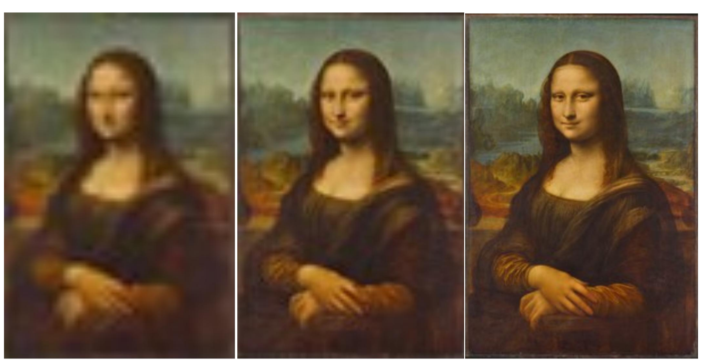
- The Yang-Mills theory is defined and SU(3)-gauge-invariant: $U'_\mu(x) = \Omega(x) U_\mu(x) \Omega(x + a\hat{\mu})$

- From asymptotic freedom $a(g_0) = \frac{1}{\Lambda}\left[b_0 g_0^2\right]^{-b_1/(2b_0^2)}\, e^{-\frac{1}{2b_0 g_0^2}}$

  the gauge coupling is the handle to change the lattice spacing

We finally have to remove the regulator: the continuum limit

$a \to 0$ ⟶ vanishing gauge coupling



- The physics must stay the same: dimensionful quantities $\xi = 1/m$ diverge in lattice units ξ/a in the continuum limit

The action: the quark sector

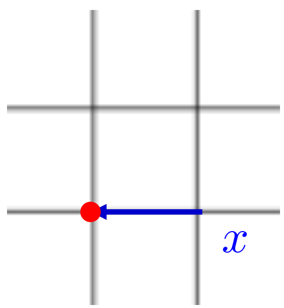
- Putting fermions on the lattice is not straightforward: Wilson fermions, staggered fermions, variants, …
- Wilson fermions: fully theoretically solid

$$S_F = a^4 \sum_x \overline{\psi}(x)(D+m_0)\psi(x), \qquad D = D_W$$

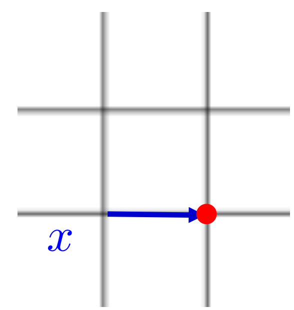
$$D_W = \frac{1}{2}\left\{\gamma_\mu(\nabla^*_\mu + \nabla_\mu) - a\nabla^*_\mu\nabla_\mu\right\}$$

Dirac-Wilson operator

$$a\nabla^*_\mu\psi(x) = \psi(x) - U^\dagger_\mu(x-a\hat{\mu})\psi(x-a\hat{\mu})$$



$$a\nabla_\mu\psi(x) = U_\mu(x)\psi(x+a\hat{\mu}) - \psi(x)$$



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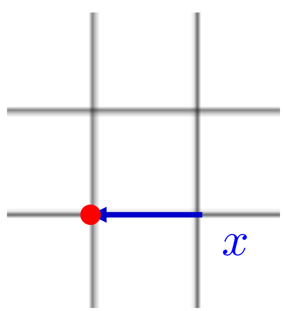
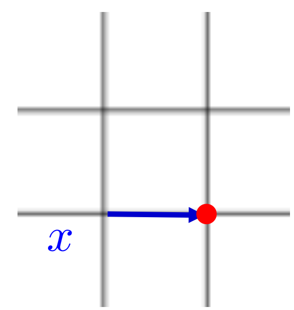
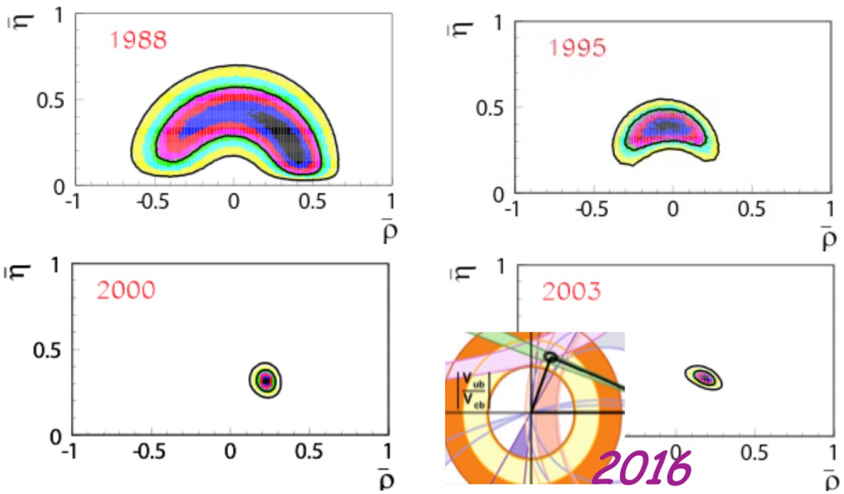




- Lattice discretization artifacts are $\mathcal{O}(a)$ and they can be reduced to $\mathcal{O}(a^2)$
- The lattice action is gauge-invariant: $\psi'(x) \to \Omega(x)\psi(x) \qquad \overline{\psi}'(x) \to \overline{\psi}(x)\Omega^\dagger(x)$
- $\beta = \dfrac{6}{g_0^2}$ handle on a, m_0 modifies the quark masses: lines of constant physics

Leading and driving role of the Rome groups

Many fundamental contributions
Flavour Physics to mention one…



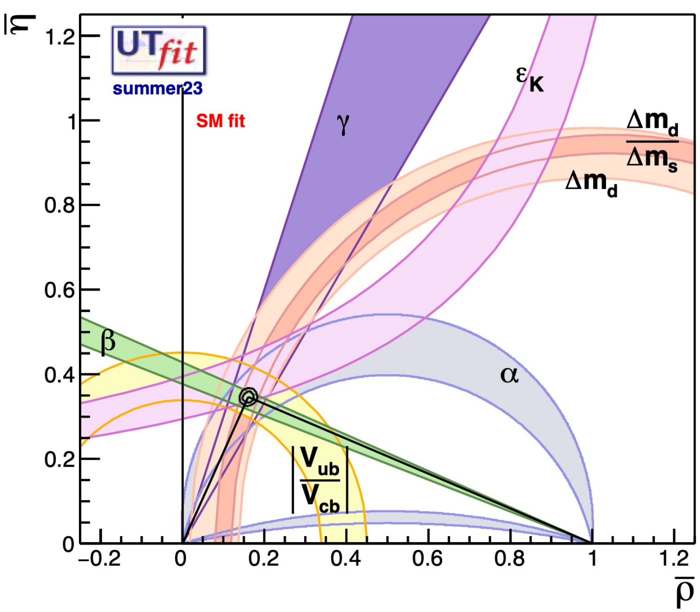
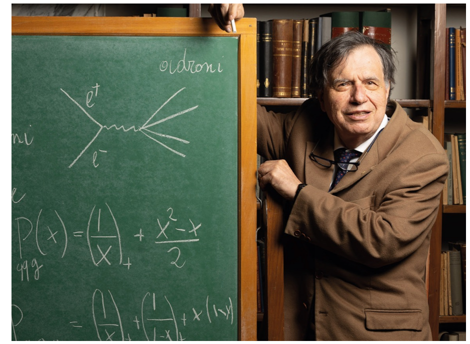
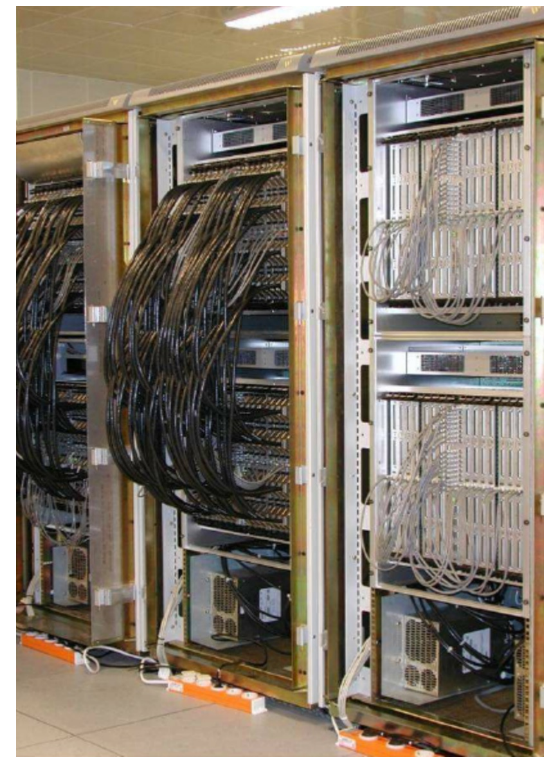


Image courtesy from
G. Martinelli's talk

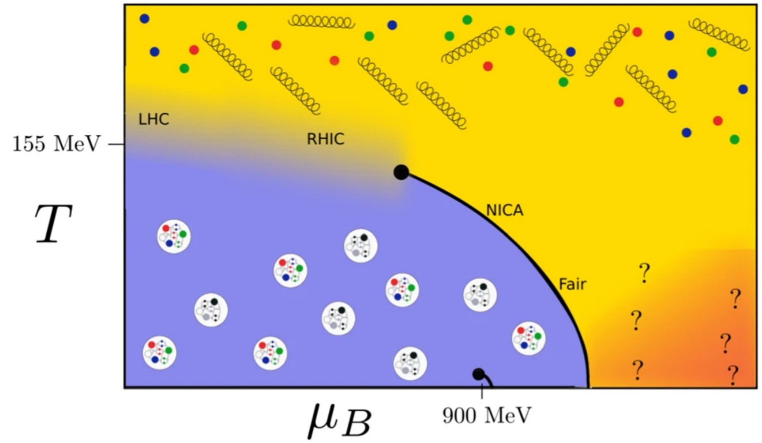




APE project: top performing supercomputers

Why the QCD EoS at high T matters

- At the hadronic scale, T~ 100 MeV, EoS is relevant in understanding the results on the QGP formation at the heavy ion colliders



QCD phase diagram

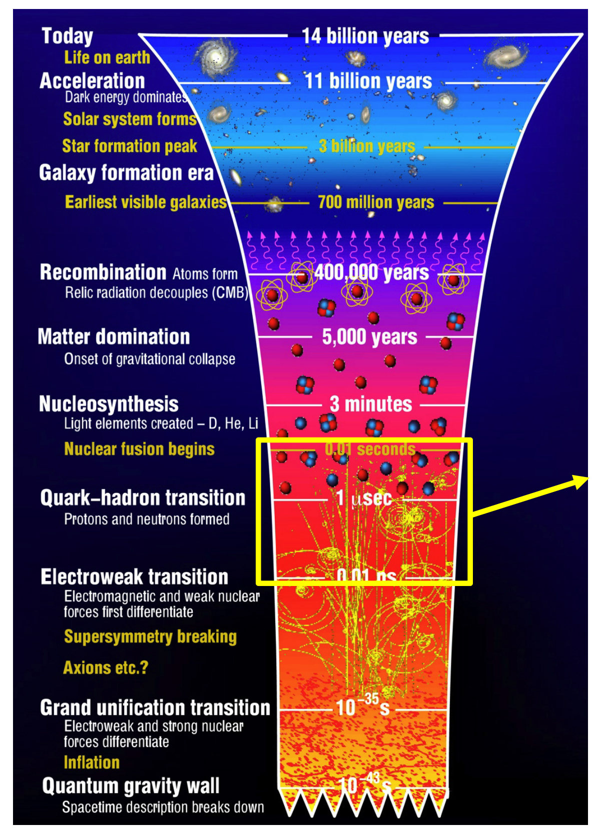
J. N. Guenther, EPJA 57 (2021) 136

- At high T, up to the electroweak scale T~ 100 GeV, EoS controlled the evolution of the early Universe

It determined the cooling rate of the Universe with an impact on primordial Gravitational Waves

$$\frac{1}{T}\frac{dT}{dt} = -\frac{\sqrt{24\pi}}{m_{Pl}}\frac{\sqrt{e(T)s(T)}}{e'(T)}$$

- Understand the QCD dynamics as the temperature changes



Chronology of the Universe

Particle Era

Source: S. Hawking Centre for Theoretical Cosmology University of Cambridge

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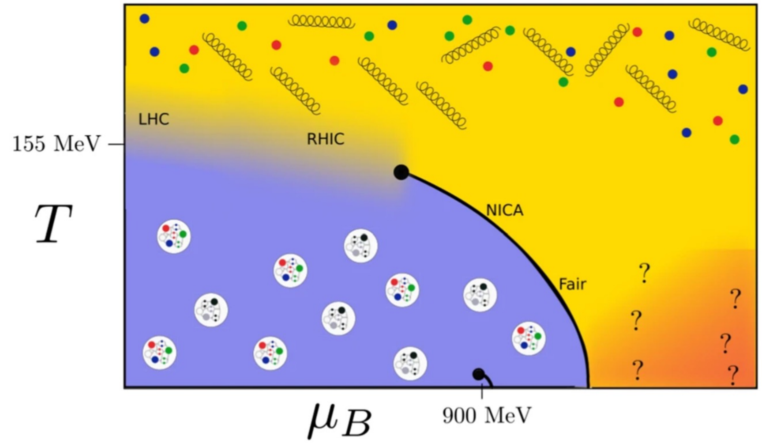

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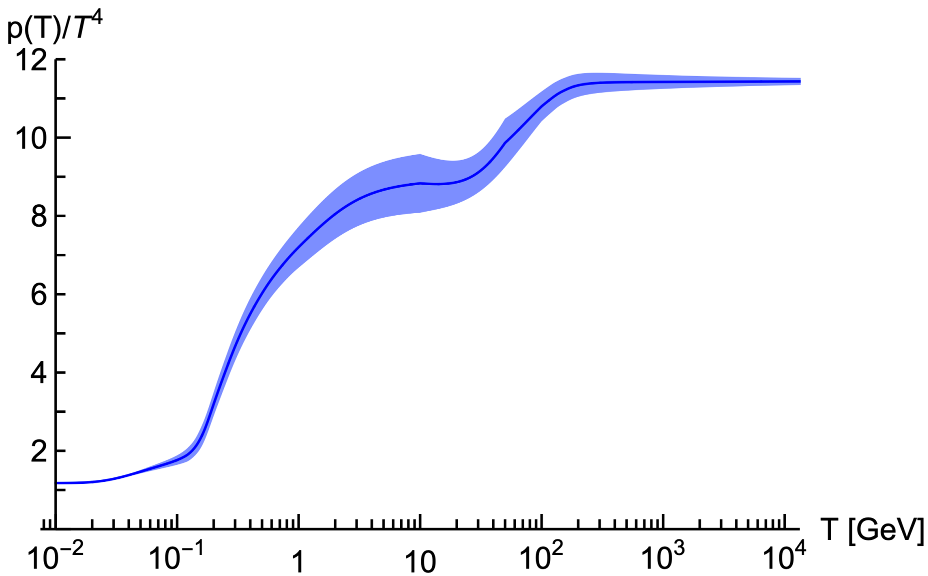

K. Saikawa and S. Shirai, JCAP 05 (2018) 035

- Understand the QCD dynamics as the temperature changes

A glimpse of the large range of energy we are spanning …

- At the hadronic scale, T ~ 100 MeV

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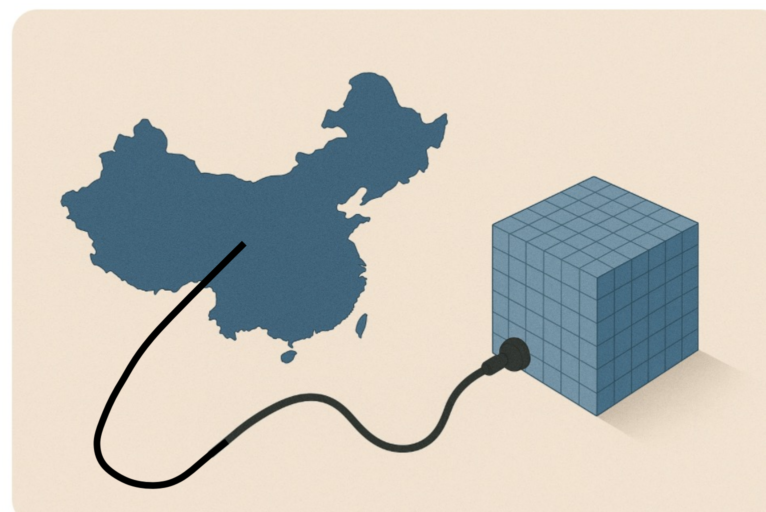
total annual electric consumption of
China stored in the same nano-cube



… illustrating the extreme scale separation involved.

The QCD EoS at $\mu_\mathbf{B} = 0$

- The EoS describes the thermal properties of the QCD plasma at equilibrium

$$p(T) = -f(T) = -\frac{T}{V} \log Z_{QCD}$$

pressure

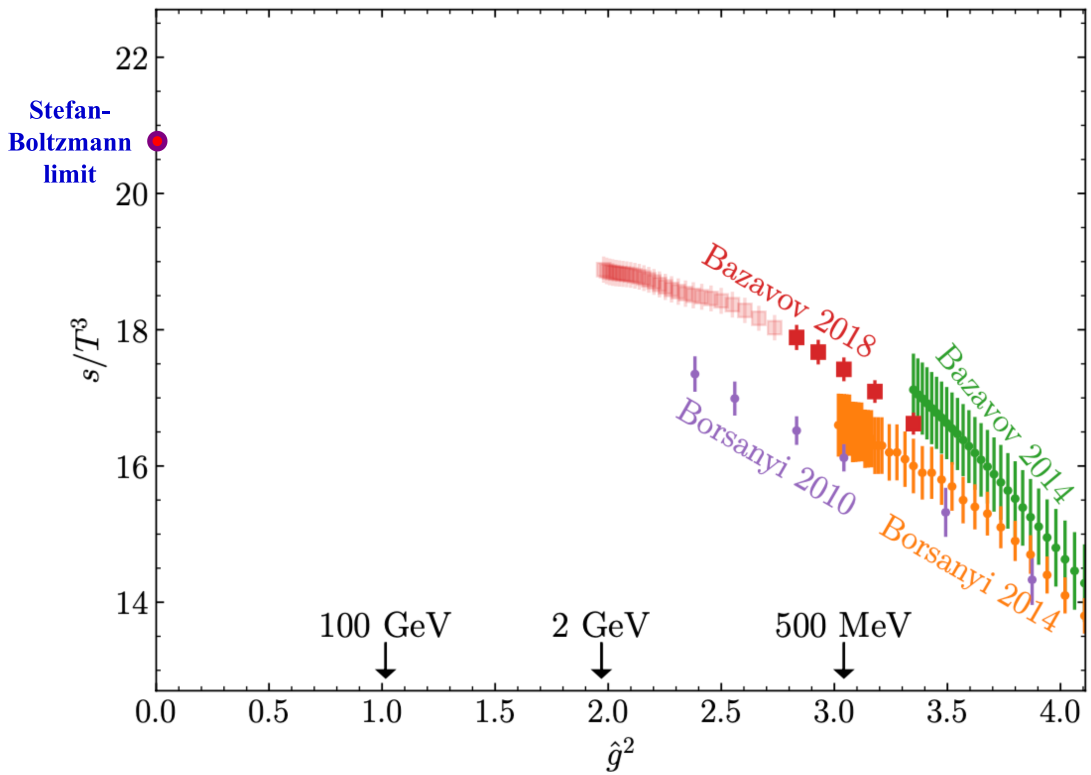
$$s(T) = \frac{\partial\, p(T)}{\partial\, T}$$

entropy density

$$e(T) = T s(T) - p(T)$$

energy density

- For QCD with 2+1 flavours we have results for $T \lesssim 1$ GeV



S. Borsanyi et al., JHEP 11 (2010) 077
S. Borsanyi et al., PLB 730 (2014) 99
A. Bazavov et al., PRD 90 (2014) 094503
A. Bazavov et al., PRD 97 (2018) 014510

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pressure **entropy density** **energy density**

- For QCD with 2+1 flavours we have results for $T \lesssim 1$ GeV

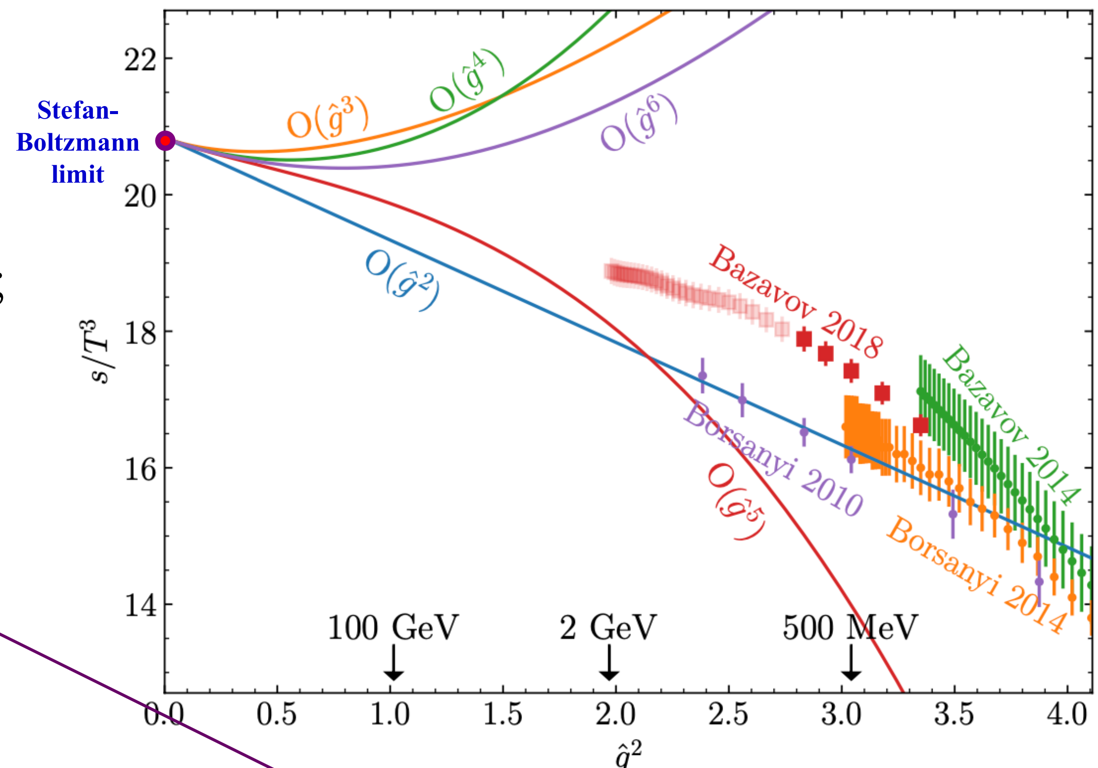
- For high T: lack of NP results; thanks to asymptotic freedom

↓

perturbative approach

$p(T)$ computed up to $\mathcal{O}(\hat{g}^6)$

E. Braaten, L. Yaffe, L. McLerran, R. Pisarski, E. Shutyak, J.P. Blaziot, E. Iancu, M. Laine, K. Kajantie, …



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- Even at T~100 GeV, perturbation theory converges poorly and non-perturbative contributions appear at

Unexplored high-T regime: numerical challenges

- Theoretical issue: we lack knowledge of a non-pert. renormalization strategy for covering the broad range of T from the hadron scale to the electroweak scale

- Algorithmic challenge: the integral method is based on computing p(T) from the trace anomaly $\Theta(T) = e(T) - 3\,p(T)$ of the energy-momentum tensor

$$\frac{\Theta(T)}{T^4} = T\frac{d}{dT}\left(\frac{p(T)}{T^4}\right) \quad \rightarrow \quad \frac{p(T)}{T^4} = \frac{p(T_0)}{T_0^4} + \int_{T_0}^{T} \frac{\Theta(T')}{T'^5} dT'$$

G. Boyd et al. NPB 469 (1996) 419

successful but two scales, T_0 and T, are involved and it can be numerically demanding if they are much separated

strategy to overcome these difficulties

- Use the non-pert. running of a renormalized coupling $g_R^2(\mu)$ to determine $a \leftrightarrow g_0$

L. Giusti and M. Pepe, PRL 2014, PRD 2015, PLB 2017

M. Dalla Brida et al., JHEP 04 (2022) 034

- Formulate thermal QCD in a moving reference frame: $s(T)$ is the primary observable

L. Giusti and H. Meyer, PRL 2011, JHEP 2011 and 2013

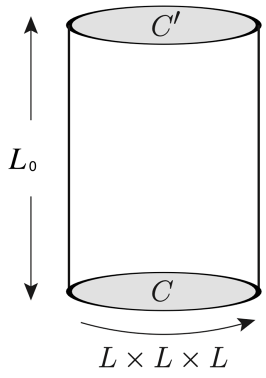
Non-perturbative renormalization strategy

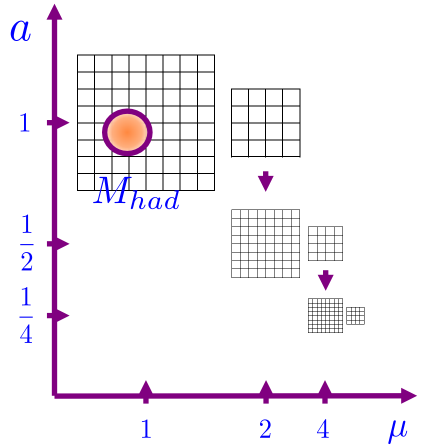
- The problem: renormalize the theory at very high T with a hadronic scheme is challenging

$$a \ll \frac{1}{T} \ll \frac{1}{M_{had}} \ll L$$

a very fine lattice and a very large system

- The solution: use finite volume schemes, like the Schrödinger Functional, to define a non-perturbative ren. coupling $g^2_{SF}(\mu)$ at $\mu = \frac{1}{L_0}$ and exploit step-scaling techniques to move to high energies non-perturbatively.

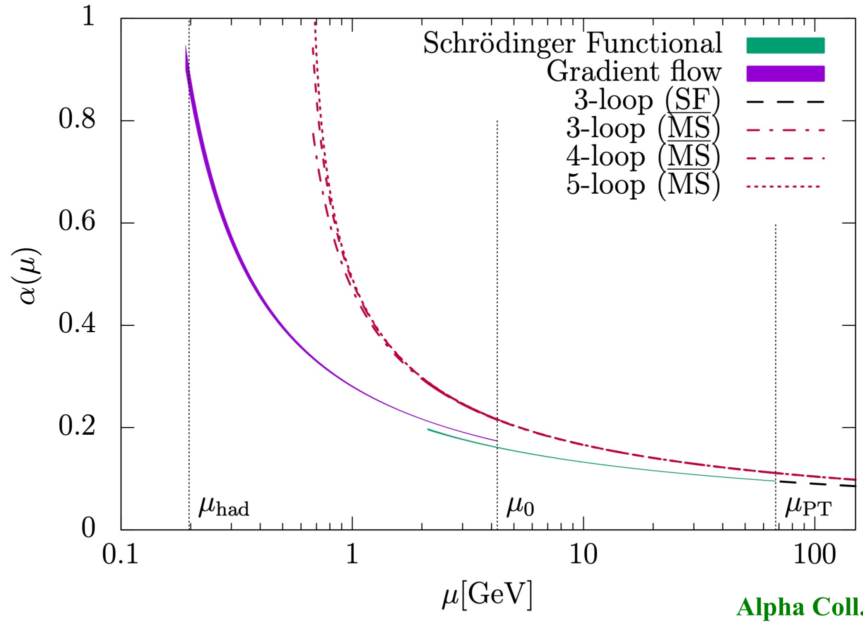




M Lüscher et al.
NPB 359 (1991) 221

Non-perturbative renormalization strategy

The non-perturbative running of the ren. coupling $g^2_{SF}(\mu)$ has been computed from M_{had} up to very high energy



Alpha Coll. 2016, 2017, 2018

M. Bruno et al.
PRL 119 (2017) 10, 102001

Once the ren. coupling is fixed at some temperature $\mu_0 \sim T_0$ then

$$\ln\left(\frac{\mu}{\mu_0}\right) = \int_{g_{SF}(\mu_0)}^{g_{SF}(\mu)} \frac{dg}{\beta(g)}$$

We choose $\mu \sim T$ and renormalize the theory by fixing the value of the ren. coupling at finite lattice spacing requiring that

$$g^2_{SF}(g^2_0, a\mu) = g^2_{SF}(\mu)$$

L. Giusti and M. Pepe, PLB 769 (2017) 385

M. Dalla Brida, L. Giusti, T. Harris, D. Laudicina, M. Pepe
JHEP 04 (2022) 034

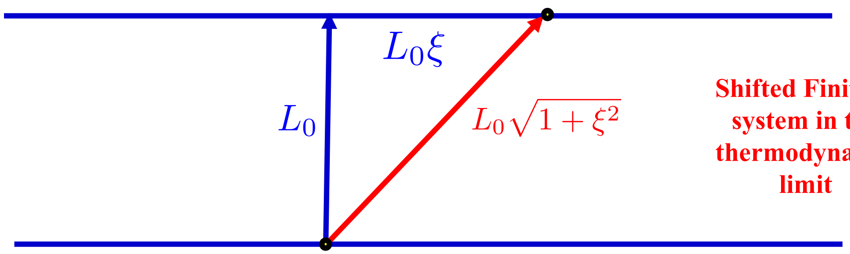
This way we determined the bare couplings to consider in the Monte Carlo simulations so to reach very high temperatures.

QCD at finite T: the moving frame

- Thermal theory in a moving reference frame $Z(L_0, \xi) = \mathrm{Tr}\left[e^{-L_0(H - \xi_k \hat{T}_{0k})}\right]$

Energy-Momentum Tensor

That corresponds to introducing a spatial shift $L_0\boldsymbol{\xi}$ for the fields when closing the b.c. along the compact direction of extension L_0



$$U_\mu(x_0 + L_0, \mathbf{x}) = U_\mu(x_0, \mathbf{x} - L_0\xi)$$

$$\psi(x_0 + L_0, \mathbf{x}) = -\psi(x_0, \mathbf{x} - L_0\xi)$$

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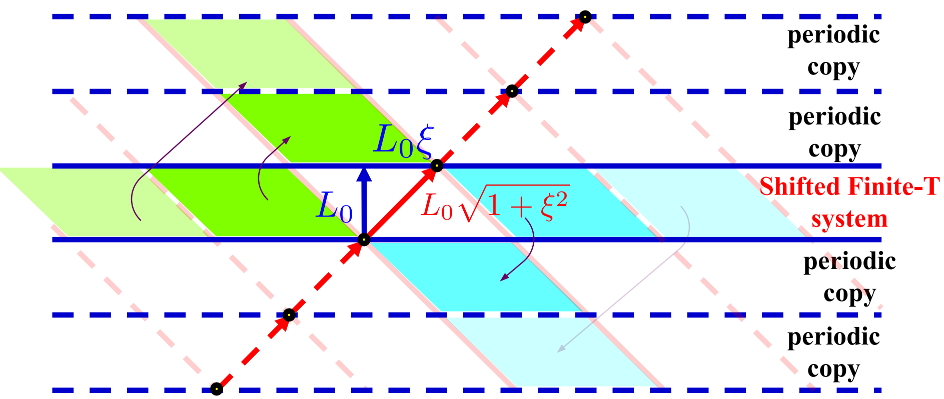

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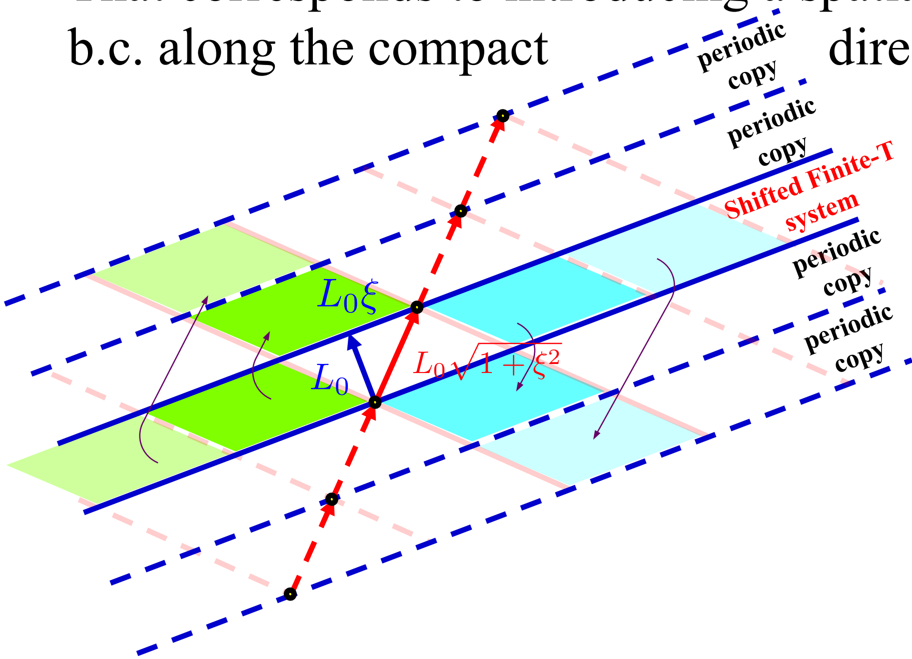

$$U_\mu(x_0+L_0,\mathbf{x}) = U_\mu(x_0,\mathbf{x}-L_0\xi)$$

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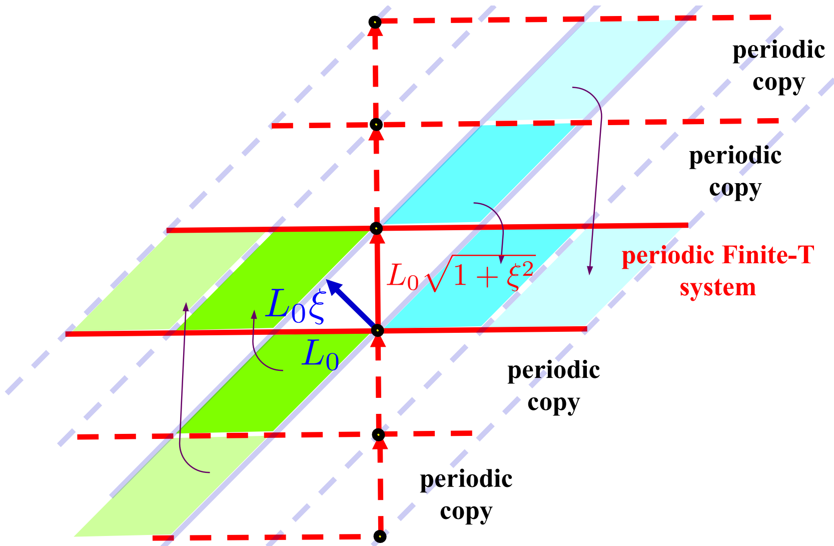
QCD at finite T: the moving frame

- Thermal theory in a moving reference frame

$$Z(L_0,\xi) = \mathrm{Tr}\left[e^{-L_0(H-\xi_k \hat{T}_{0k})}\right]$$

Energy-Momentum Tensor

That corresponds to introducing a spatial shift $L_0\boldsymbol{\xi}$ for the fields when closing the b.c. along the compact direction of extension L_0



$$U_\mu(x_0+L_0,\mathbf{x}) = U_\mu(x_0,\mathbf{x}-L_0\xi)$$

$$\psi(x_0+L_0,\mathbf{x}) = -\psi(x_0,\mathbf{x}-L_0\xi)$$

$$\bar{\psi}(x_0+L_0,\mathbf{x}) = -\bar{\psi}(x_0,\mathbf{x}-L_0\xi)$$

QCD at finite T: the moving frame

- Thermal theory in a moving reference frame $Z(L_0,\xi) = \mathrm{Tr}\left[e^{-L_0(H-\xi_k \hat{T}_{0k})}\right]$

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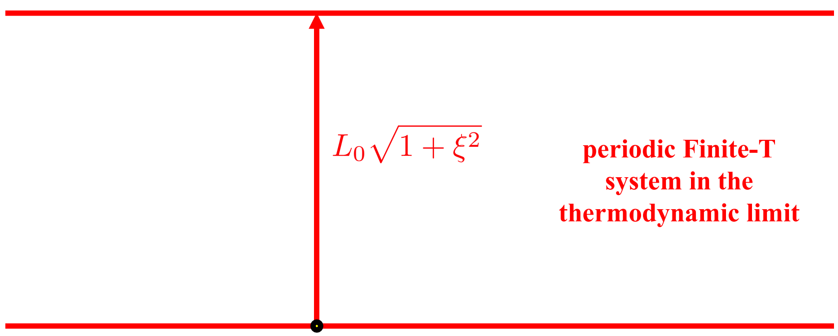


by Lorentz invariance, the free-energy density $f_{\boldsymbol{\xi}}(L_0)$ of the shifted system is

$$f\left(L_0\sqrt{1+\boldsymbol{\xi}^2}\right) = -\lim_{V\to\infty}\frac{1}{L_0 V}\log Z(L_0,V,\boldsymbol{\xi}) = f_{\boldsymbol{\xi}}(L_0)$$

equivalent to a periodic system at the temperature $T = \dfrac{1}{L_0\sqrt{1+\boldsymbol{\xi}^2}}$

- Entropy density is now the primary observable

$$\frac{s}{T^3} = \frac{1+\boldsymbol{\xi}^2}{\xi_k}\frac{1}{T^4}\frac{\partial f_{\boldsymbol{\xi}}}{\partial \xi_k} = -\frac{L_0^4(1+\boldsymbol{\xi}^2)^3}{\xi_k}\langle T_{0k}\rangle_\xi$$

Not vanishing with shifted b.c.

No need to work at two different temperatures but only at the temperature of interest

Systematics of high T regime

- three energy scales are relevant:

E. Braaten, L. Yaffe, L. McLerran, R. Pisarski, S. Huang, M. Lissia, J._P. Blaziot, E. Iancu, M. Laine, …

T: temperature

gT: effective mass of A_0 field

g^2T: energy scale of A_i field: 3d YM confining theory

Finite size effects are related to the lightest energy scale involved → e^{-g^2TL} → $g^2TL \gg 1$

- the fluctuations among topological sectors are strongly suppressed

$$\langle Q^2 \rangle \propto \frac{L^3}{T^b} \qquad b \sim 8-9$$

S. Borsanyi et al. PLB 752 (2016) 175

Y. Taniguchi et al. PRD 95 (2017) 054502

C. Bonati et al. 2016 - 2022

L. Giusti and M. Lüscher EPJC 79 (2019) 3, 207

only some fluctuation at lowest T , effect smaller than statistical accuracy.

The numerical study

M. Bresciani, M. Dalla Brida,
L. Giusti, and M. Pepe
PRL 134 (2025) 20, 201904
PRD 113 (2026) 3, 034506

- QCD on the lattice with $N_f = 3$ quarks in the chiral limit

- 9 values of the temperature in the range $3 - 165$ GeV

- large spatial volumes to have finite volume effects under control (checked explicitly): $g^2 \sim 1;\ LT \sim 10 - 25$

- shifted boundary conditions $\xi = (1, 0, 0)$

- reduced lattice artifacts: $O(a)$ - improved Wilson fermions

- continuum limit extrapolation: $L_0/a = 4, 6, 8, 10; L/a = 144$

T	T (GeV)
T_0	164.6(5.6)
T_1	82.3(2.8)
T_2	51.4(1.7)
T_3	32.8(1.0)
T_4	20.63(63)
T_5	12.77(37)
T_6	8.03(22)
T_7	4.91(13)
T_8	3.040(78)

The numerical study

- The entropy density is the primary observable of our study

$$\frac{s}{T^3} = \frac{1+\boldsymbol{\xi}^2}{\xi_k}\frac{1}{T^4}\frac{\partial f_{\boldsymbol{\xi}}}{\partial \xi_k} = -\frac{L_0^4(1+\boldsymbol{\xi}^2)^3}{\xi_k}\langle T_{0k}\rangle_\xi \qquad \langle T_{0k}\rangle_\xi = Z_G(g_0^2)\,\langle T^G_{0k}\rangle_\xi + Z_F(g_0^2)\,\langle T^F_{0k}\rangle_\xi$$

M. Dalla Brida, L. Giusti and M. Pepe, JHEP 04 (2020) 043

$$\frac{\partial}{\partial \xi_k} f(\xi) = -\frac{1}{L_0 V}\frac{\partial}{\partial \xi_k}\log \mathcal{Z}(L_0,\boldsymbol{\xi}) \longrightarrow -\frac{1}{L_0 V \Delta\xi}\log\frac{\mathcal{Z}(\boldsymbol{\xi}_+)}{\mathcal{Z}(\boldsymbol{\xi}_-)}$$

- it is convenient to rewrite

$$\log\frac{\mathcal{Z}(m_0,g_0^2,\boldsymbol{\xi}_+)}{\mathcal{Z}(m_0,g_0^2,\boldsymbol{\xi}_-)} = \log\frac{\mathcal{Z}(m_0,g_0^2,\boldsymbol{\xi}_+)}{\mathcal{Z}(\infty,g_0^2,\boldsymbol{\xi}_+)} + \log\frac{\mathcal{Z}(\infty,g_0^2,\boldsymbol{\xi}_-)}{\mathcal{Z}(m_0,g_0^2,\boldsymbol{\xi}_-)} + \log\frac{\mathcal{Z}(\infty,g_0^2,\boldsymbol{\xi}_+)}{\mathcal{Z}(\infty,g_0^2,\boldsymbol{\xi}_-)}$$

- poor overlap of relevant phase space: numerically challenging

$$\log\frac{\mathcal{Z}(m_0,g_0^2,\boldsymbol{\xi})}{\mathcal{Z}(\infty,g_0^2,\boldsymbol{\xi})} = -\int_\infty^{m_0}\langle\bar\psi\psi\rangle_\xi^{\bar m_0}\,d\bar m_0$$

Monte Carlo simulations in QCD

$$\log\frac{\mathcal{Z}_{YM}(g_0^2,\boldsymbol{\xi}_+)}{\mathcal{Z}_{YM}(g_0^2,\boldsymbol{\xi}_-)} - \log\frac{\mathcal{Z}_{YM}(0,\boldsymbol{\xi}_+)}{\mathcal{Z}_{YM}(0,\boldsymbol{\xi}_-)} = -\int_0^{g_0^2}\left[\Delta_\xi\langle\frac{\partial S_G}{\partial \bar g_0^2}\rangle_\xi\right]\frac{d\bar g_0^2}{\bar g_0^2}$$

Monte Carlo simulations in Yang-Mills theory

The numerical study

- The entropy density is the primary observable of our study

$$\frac{s}{T^3} = \frac{1+\boldsymbol{\xi}^2}{\xi_k}\frac{1}{T^4}\frac{\partial f_{\boldsymbol{\xi}}}{\partial \xi_k} = -\frac{L_0^4(1+\boldsymbol{\xi}^2)^3}{\xi_k}\langle T_{0k}\rangle_\xi \qquad \langle T_{0k}\rangle_\xi = Z_G(g_0^2)\ \langle T_{0k}^G\rangle_\xi + Z_F(g_0^2)\ \langle T_{0k}^F\rangle_\xi$$

M. Dalla Brida, L. Giusti and M. Pepe, JHEP 04 (2020) 043

$$\frac{\partial}{\partial \xi_k} f(\xi) = -\frac{1}{L_0 V}\frac{\partial}{\partial \xi_k}\log \mathcal{Z}(L_0,\boldsymbol{\xi}) \longrightarrow -\frac{1}{L_0 V \Delta\xi}\log\frac{\mathcal{Z}(\boldsymbol{\xi}_+)}{\mathcal{Z}(\boldsymbol{\xi}_-)}$$

- it is convenient to rewrite

$$\log\frac{\mathcal{Z}(m_0,g_0^2,\boldsymbol{\xi}_+)}{\mathcal{Z}(m_0,g_0^2,\boldsymbol{\xi}_-)} = \underbrace{\log\frac{\mathcal{Z}(m_0,g_0^2,\boldsymbol{\xi}_+)}{\mathcal{Z}(\infty,g_0^2,\boldsymbol{\xi}_+)} + \log\frac{\mathcal{Z}(\infty,g_0^2,\boldsymbol{\xi}_-)}{\mathcal{Z}(m_0,g_0^2,\boldsymbol{\xi}_-)}}_{\mathbf{A}} + \underbrace{\log\frac{\mathcal{Z}(\infty,g_0^2,\boldsymbol{\xi}_+)}{\mathcal{Z}(\infty,g_0^2,\boldsymbol{\xi}_-)}}_{\mathbf{B}}$$

- poor overlap of relevant phase space: numerically challenging

A

$$\log\frac{\mathcal{Z}(m_0,g_0^2,\boldsymbol{\xi})}{\mathcal{Z}(\infty,g_0^2,\boldsymbol{\xi})} = -\int_\infty^{m_0}\langle\bar\psi\psi\rangle_\xi^{\bar m_0}\, d\bar m_0$$

Monte Carlo simulations in QCD

B

$$\log\frac{\mathcal{Z}_{YM}(g_0^2,\boldsymbol{\xi}_+)}{\mathcal{Z}_{YM}(g_0^2,\boldsymbol{\xi}_-)} - \log\frac{\mathcal{Z}_{YM}(0,\boldsymbol{\xi}_+)}{\mathcal{Z}_{YM}(0,\boldsymbol{\xi}_-)} = -\int_0^{g_0^2}\left[\Delta_\xi\langle\frac{\partial S_G}{\partial \bar g_0^2}\rangle_\xi\right]\frac{d\bar g_0^2}{\bar g_0^2}$$

Monte Carlo simulations in Yang-Mills theory

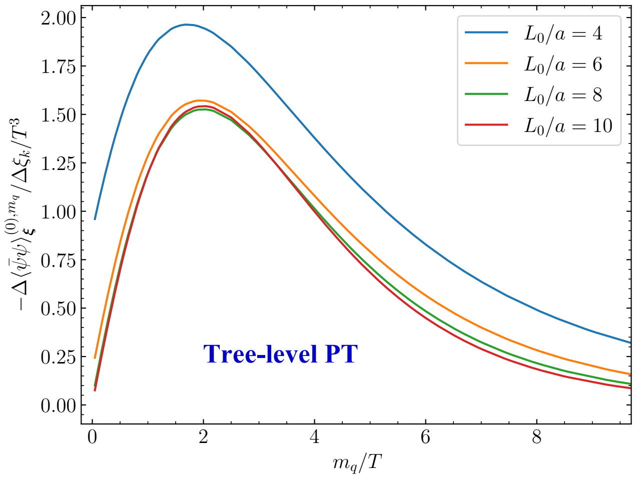
- the computation can be performed with high precision with affordable effort

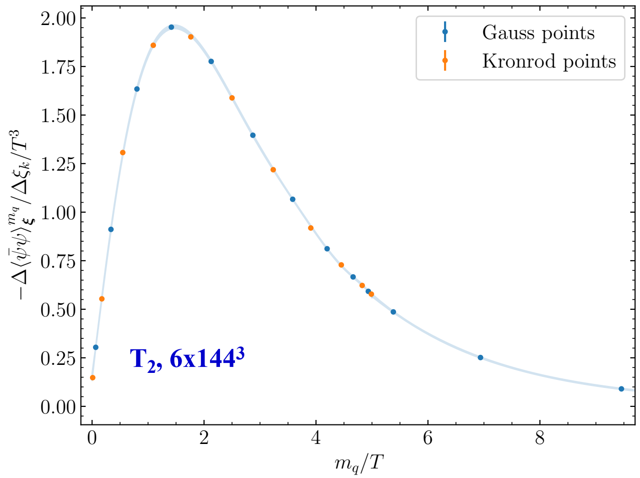
The computational strategy

- Part **A**: Gaussian quadrature

$$\log\frac{\mathcal{Z}(m_{cr},g_0^2,\boldsymbol{\xi}_+)}{\mathcal{Z}(\infty,g_0^2,\boldsymbol{\xi}_+)}-\log\frac{\mathcal{Z}(m_{cr},g_0^2,\boldsymbol{\xi}_-)}{\mathcal{Z}(\infty,g_0^2,\boldsymbol{\xi}_-)}=\int_0^\infty\frac{\Delta\langle\bar\psi\psi\rangle_\xi^{m_q}}{\Delta\xi_k}\,d\bar m_q$$

$$m_q = m_0 - m_{cr}$$





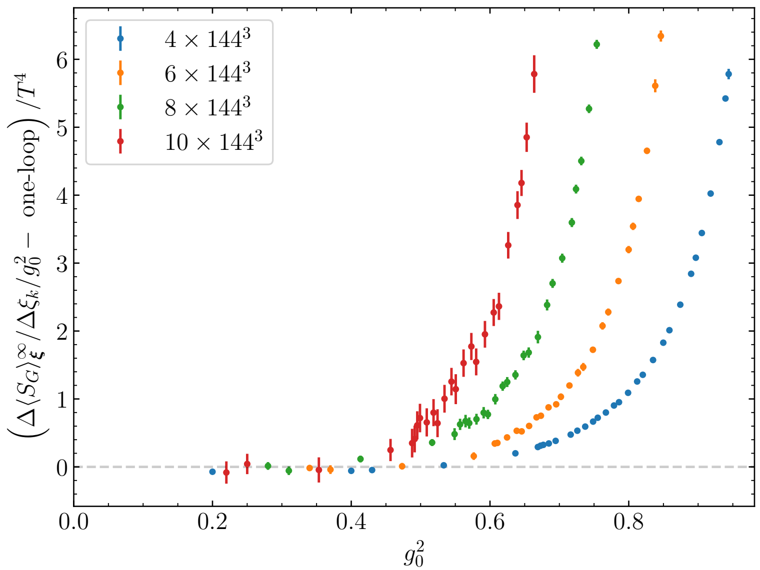
- combination 3 ranges:
  [0, 5] : 10-point Gaussian quadrature
  [5, 20]: 7-pont Gaussian quadrature
  [20,∞]: 3-point Gaussian quadrature in κ

- Check of systematics with Gauss-Kronrod method: difference several times smaller than the statistical accuracy

- Part **B**: Gaussian quadrature

$$\log\frac{\mathcal{Z}_{YM}(g_0^2,\boldsymbol{\xi}_+)}{\mathcal{Z}_{YM}(g_0^2,\boldsymbol{\xi}_-)}-\log\frac{\mathcal{Z}_{YM}(0,\boldsymbol{\xi}_+)}{\mathcal{Z}_{YM}(0,\boldsymbol{\xi}_-)}=-\int_0^{g_0^2}\left[\frac{\Delta}{\Delta\xi_k}\langle\frac{\partial S_G}{\partial\bar g_0^2}\rangle_\xi\right]\frac{d\bar g_0^2}{\bar g_0^2}$$

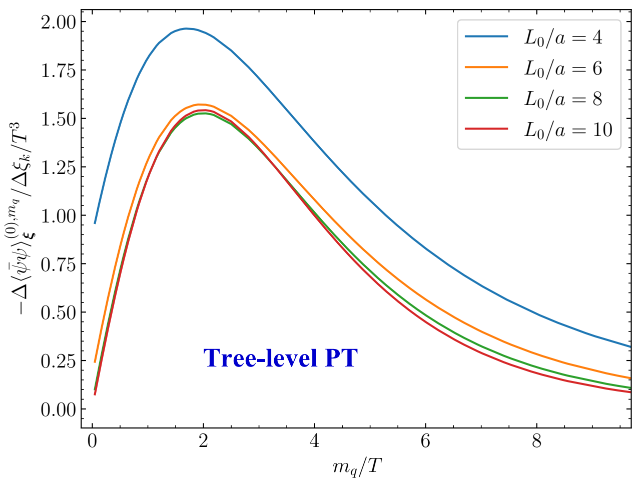
interval	quadrature	
$0\le g_0^2\le 6/15$	3 (Simpson) 2 (trapezoid)	$L_0/a=4$ $L_0/a=6,8,10$
$6/15\le g_0^2\le 6/9$	3 (Gauss-Legendre)	
$6/9\le g_0^2\le g_0^2\vert_{T_0}$	3 (Gauss-Legendre) 1 (midpoint)	$L_0/a=4$ $L_0/a=6$
$6/9\le g_0^2\le g_0^2\vert_{T_1}$	3 (Gauss-Legendre)	
$g_0^2\vert_{T_{i-1}}\le g_0^2\le g_0^2\vert_{T_i}$	3 (Gauss-Legendre) 5 (Gauss-Legendre)	$1<i<7$ $i=7,8$

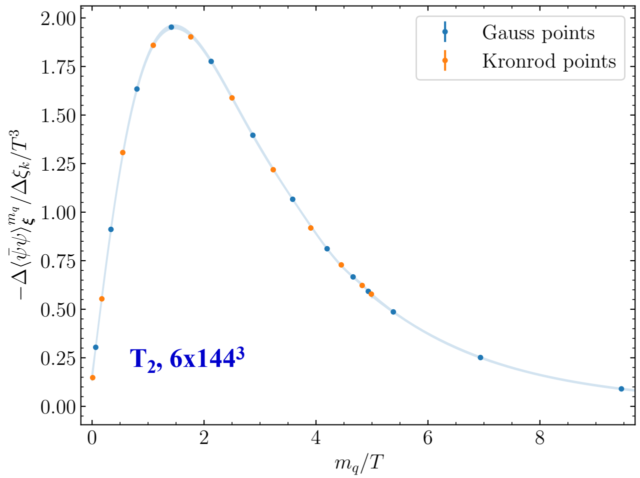


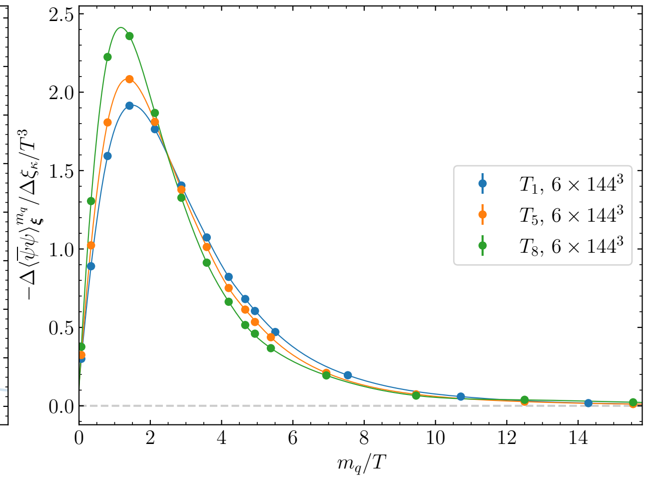
The computational strategy

- Part **A**: Gaussian quadrature

$$\log \frac{\mathcal{Z}(m_{cr}, g_0^2, \boldsymbol{\xi}_+)}{\mathcal{Z}(\infty, g_0^2, \boldsymbol{\xi}_+)} - \log \frac{\mathcal{Z}(m_{cr}, g_0^2, \boldsymbol{\xi}_-)}{\mathcal{Z}(\infty, g_0^2, \boldsymbol{\xi}_-)} = \int_0^\infty \frac{\Delta \langle \bar\psi \psi \rangle_\xi^{m_q}}{\Delta \xi_k} \, d\bar m_q \qquad m_q = m_0 - m_{cr}$$







- Part **B**: Gaussian quadrature

$$\log \frac{\mathcal{Z}_{YM}(g_0^2, \boldsymbol{\xi}_+)}{\mathcal{Z}_{YM}(g_0^2, \boldsymbol{\xi}_-)} - \log \frac{\mathcal{Z}_{YM}(0, \boldsymbol{\xi}_+)}{\mathcal{Z}_{YM}(0, \boldsymbol{\xi}_-)} = -\int_0^{g_0^2} \left[\frac{\Delta}{\Delta \xi_k} \langle \frac{\partial S_G}{\partial \bar g_0^2} \rangle_\xi \right] \frac{d\bar g_0^2}{\bar g_0^2}$$

interval	quadrature	
$0 \le g_0^2 \le 6/15$	3 (Simpson) 2 (trapezoid)	$L_0/a = 4$ $L_0/a = 6, 8, 10$
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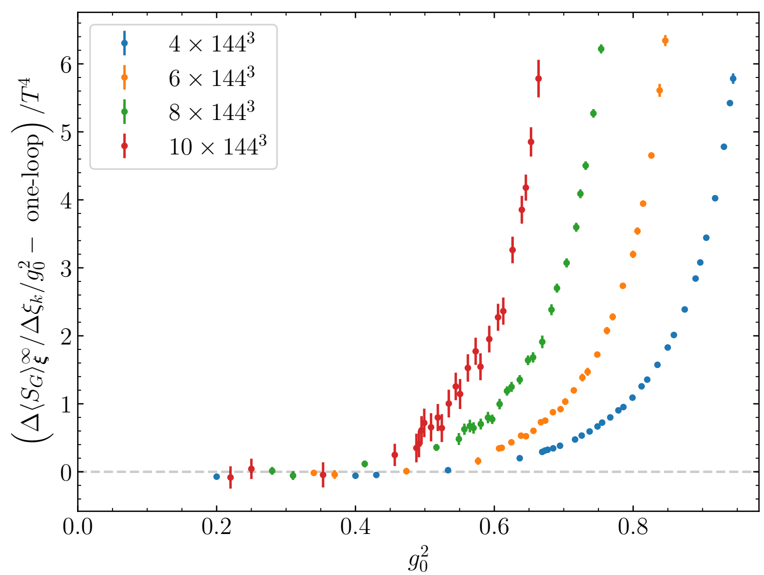
$6/9 \le g_0^2 \le g_0^2\vert_{T_0}$	3 (Gauss-Legendre) 1 (midpoint)	$L_0/a = 4$ $L_0/a = 6$
$6/9 \le g_0^2 \le g_0^2\vert_{T_1}$	3 (Gauss-Legendre)	
$g_0^2\vert_{T_{i-1}} \le g_0^2 \le g_0^2\vert_{T_i}$	3 (Gauss-Legendre) 5 (Gauss-Legendre)	$1 < i < 7$ $i = 7, 8$



$s(T)/T^3$: the extrapolation to the continuum limit

- 1-loop improved definition of the entropy density

$$s(\frac{L_0}{a}, g_0^2) \to s(\frac{L_0}{a}, g_0^2)\,\frac{\left[s_0^{(c)} + g^2\, s_1^{(c)}\right]}{\frac{\Delta}{\Delta\xi_k}\left[f_0^{(lat)} + g^2\, f_1^{(lat)}\right]} \qquad g^2 = g^2_{SF}(1/L_0)$$

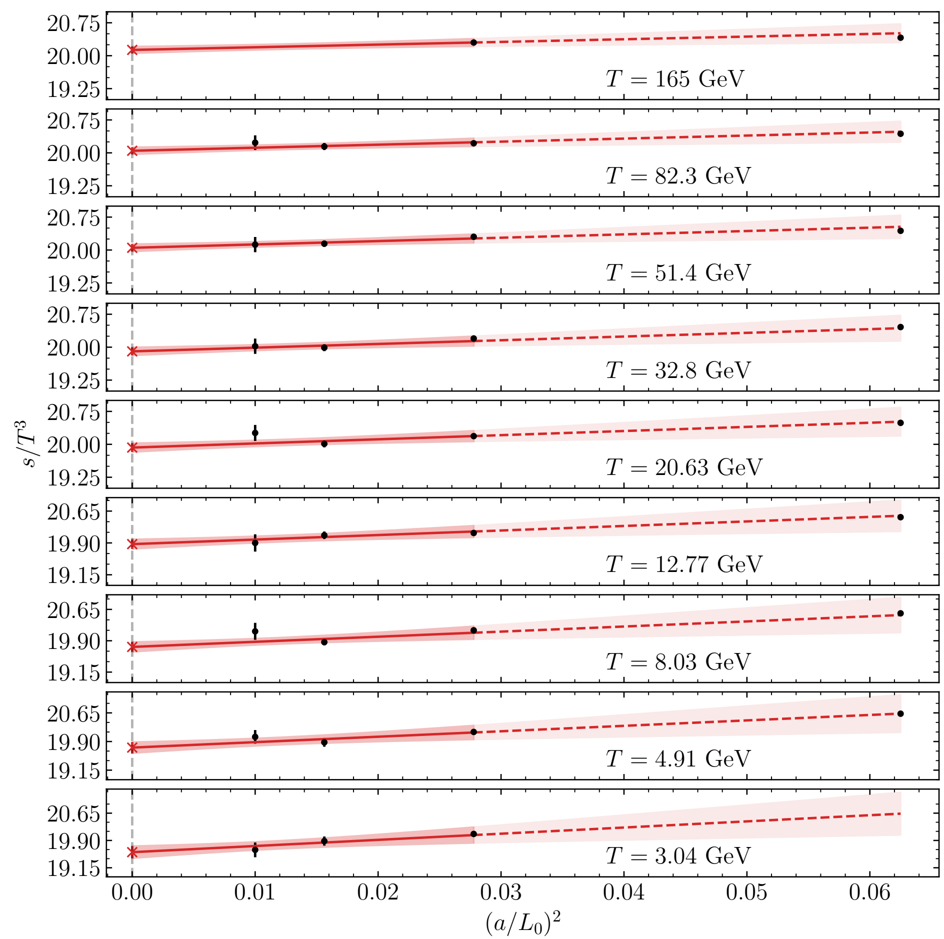
- extrapolation to the continuum limit using a global fit using $L_0/a > 4$

$$\frac{s(T,a)}{T^3} = \frac{s(T)}{T^3} + c\left(\frac{a}{L_0}\right)^2 g^3$$

with $\chi^2/\mathrm{dof} = 0.82$

- final accuracy of 0.5%-1% on $s(T)/T^3$

- Performed various fits for consistency checks, including higher orders in $a,\ g$



The QCD Equation of State

- $\hat{g}^2(T)$: 5-loop $\overline{MS}$ gauge coupling at $\mu = 2\pi T$ — **M. Baikov et al. PRL 118, (2017) 082002**

$$\frac{1}{\hat{g}^2(T)} = \frac{9}{8\pi^2} \ln \frac{2\pi T}{\Lambda_{\overline{MS}}} + \dots \qquad \Lambda_{\overline{MS}} = 341\text{MeV}$$

M. Bruno et al. PRL 119 (2017) 10, 102001

it is a function of T; simpler to compare with Perturbation Theory

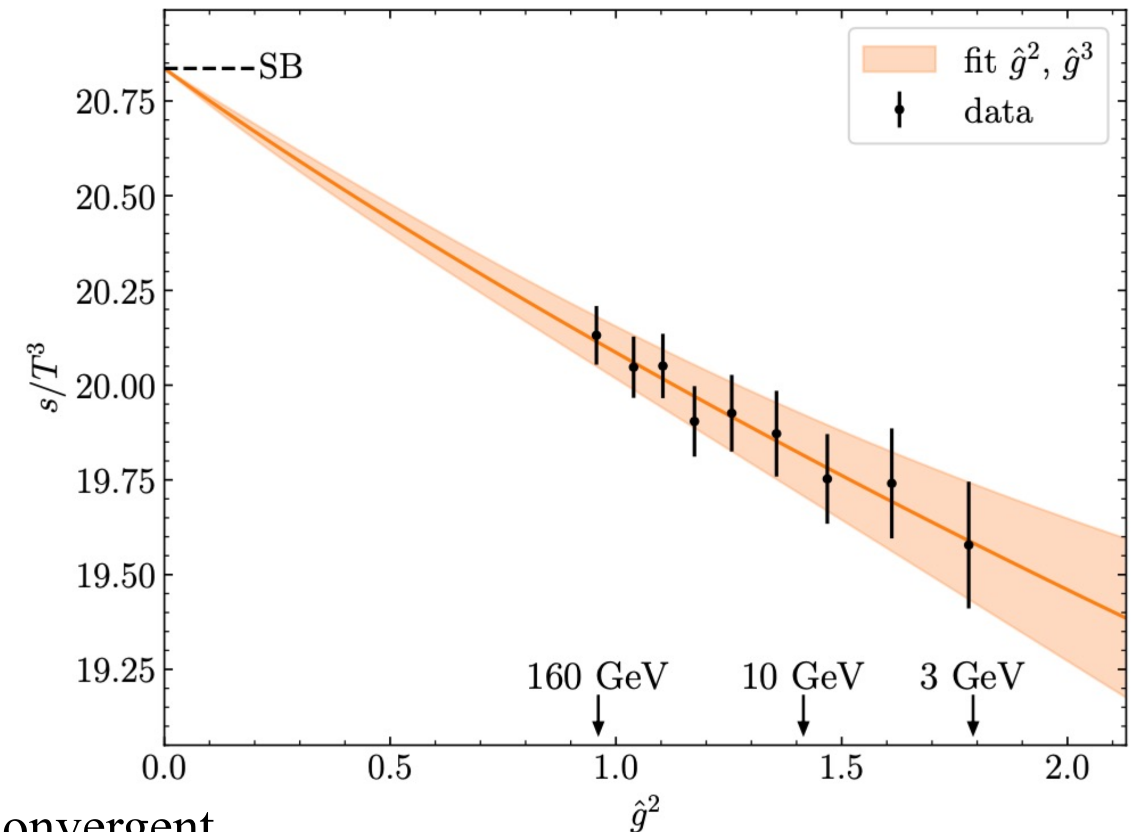
- phenomenological approach: linear fit works well and compatible with SB limit

- enforcing exact SB limit

$$\frac{s(T)}{T^3} = \frac{32\pi^2}{45}\left[s_0 + s_2\left(\frac{\hat{g}}{2\pi}\right)^2 + s_3\left(\frac{\hat{g}}{2\pi}\right)^3\right]$$

$s_2 = -5.1(9); \qquad s_3 = 5(5)$

in PT: $s_2 = -8.438$



but PT is very poorly/slowly convergent…

The QCD Equation of State: PT or not PT?

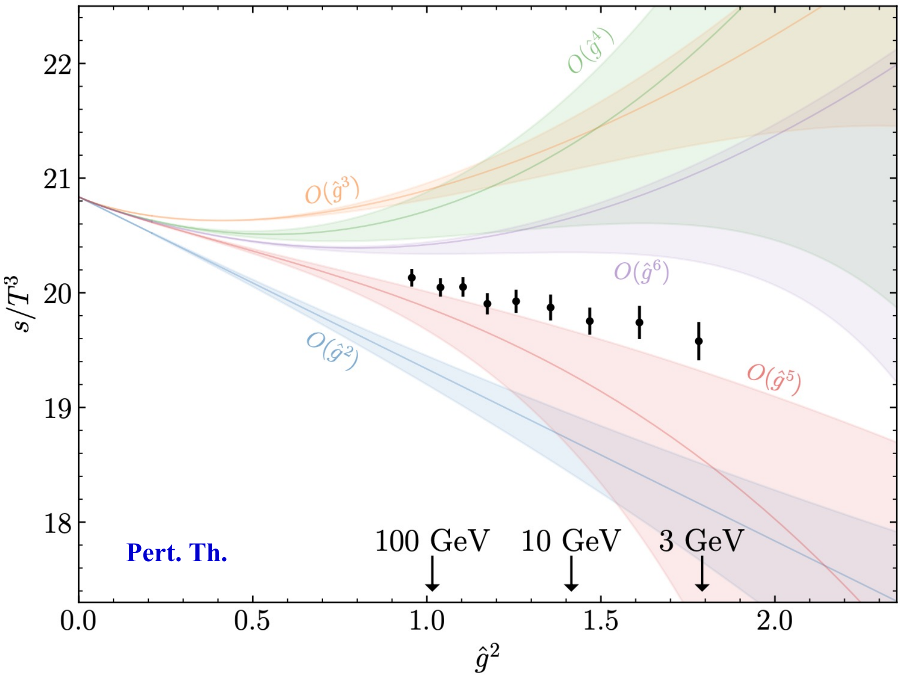
- $p(T)$: computed up to g⁶ in PT when non-perturbative contributions appear

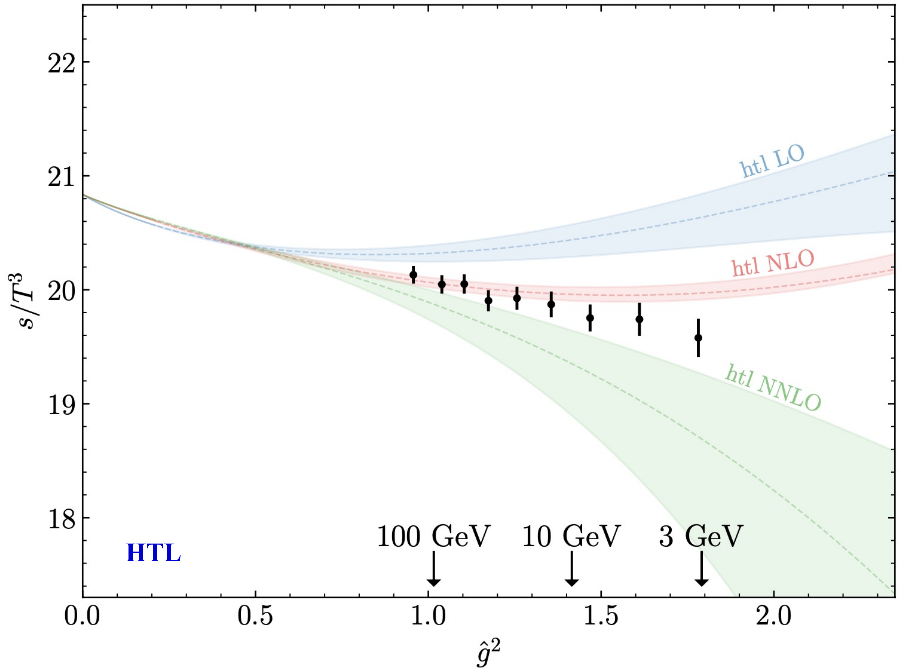
K. Kajantie et al. Phys.Rev.D 67 (2003) 105008

P. Navarrete and Y. Schröder, JHEP 11 (2024) 037

$$\frac{p(T)}{T^4} = \frac{8\pi^2}{45}\left[\sum_{k=0}^{6} p_k \left(\frac{\hat{g}(T)}{2\pi}\right)^k + q_c \left(\frac{\hat{g}(T)}{2\pi}\right)^6\right]$$

$$\frac{s(T)}{T^3} = \frac{1}{T^3}\frac{d\,p(T)}{dT} = \frac{32\pi^2}{45}\left[\sum_{k=0}^{6} s_k \left(\frac{\hat{g}(T)}{2\pi}\right)^k + q_c \left(\frac{\hat{g}(T)}{2\pi}\right)^6\right]$$





J. Andersen et al. JHEP 08 (2011) 053

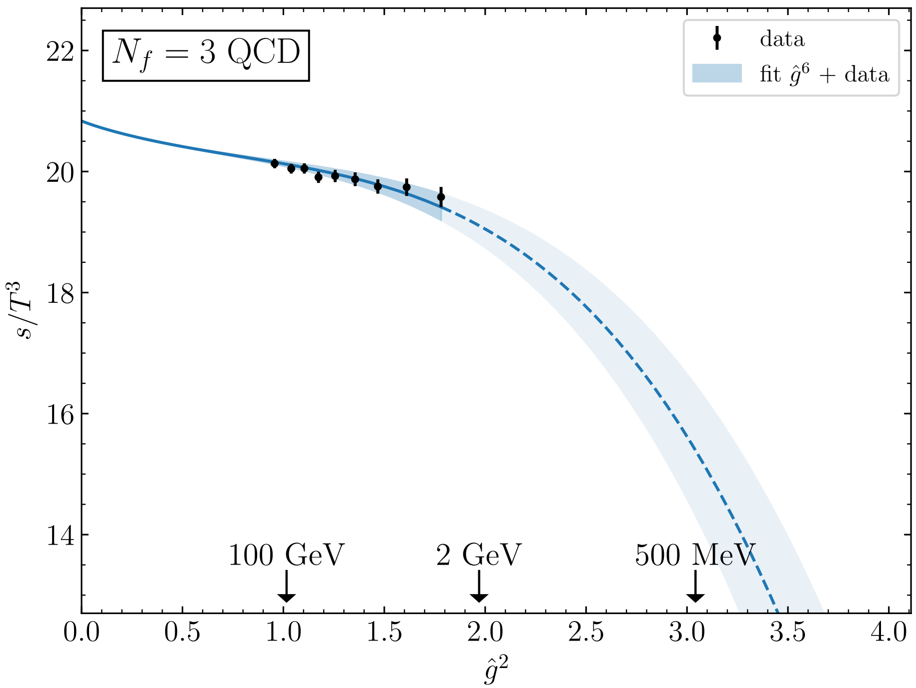
bands represent the effect of changing the scale $\mu = 2\pi T$ by a factor 2 or ½.

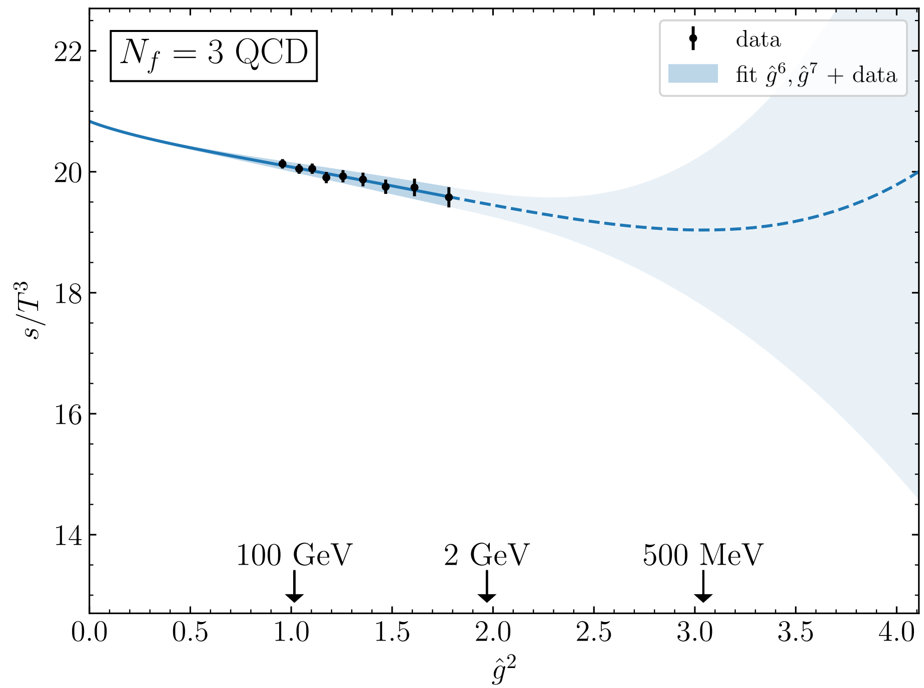
The QCD Equation of State: PT or not PT?

- $s(T)/T^3$: fit unknown contributions with the data

$$\frac{s(T)}{T^3} = \frac{1}{T^3}\frac{d\,p(T)}{dT} = \frac{32\pi^2}{45}\left[\sum_{k=0}^{6} s_k \left(\frac{\hat{g}(T)}{2\pi}\right)^k + q_c \left(\frac{\hat{g}(T)}{2\pi}\right)^6\right]$$

$$\frac{s(T)}{T^3} = \frac{1}{T^3}\frac{d\,p(T)}{dT} = \frac{32\pi^2}{45}\left[\sum_{k=0}^{7} s_k \left(\frac{\hat{g}(T)}{2\pi}\right)^k + q_c \left(\frac{\hat{g}(T)}{2\pi}\right)^6\right]$$





- higher order effects, including non-perturbative contributions, are necessary

The QCD Equation of State: PT or not PT?

- $s(T)/T^3$: fit unknown contributions with the data

$$\frac{s(T)}{T^3} = \frac{1}{T^3}\frac{d\,p(T)}{d\,T} = \frac{32\pi^2}{45}\left[\sum_{k=0}^{6} s_k \left(\frac{\hat{g}(T)}{2\pi}\right)^k + q_c \left(\frac{\hat{g}(T)}{2\pi}\right)^6\right]$$

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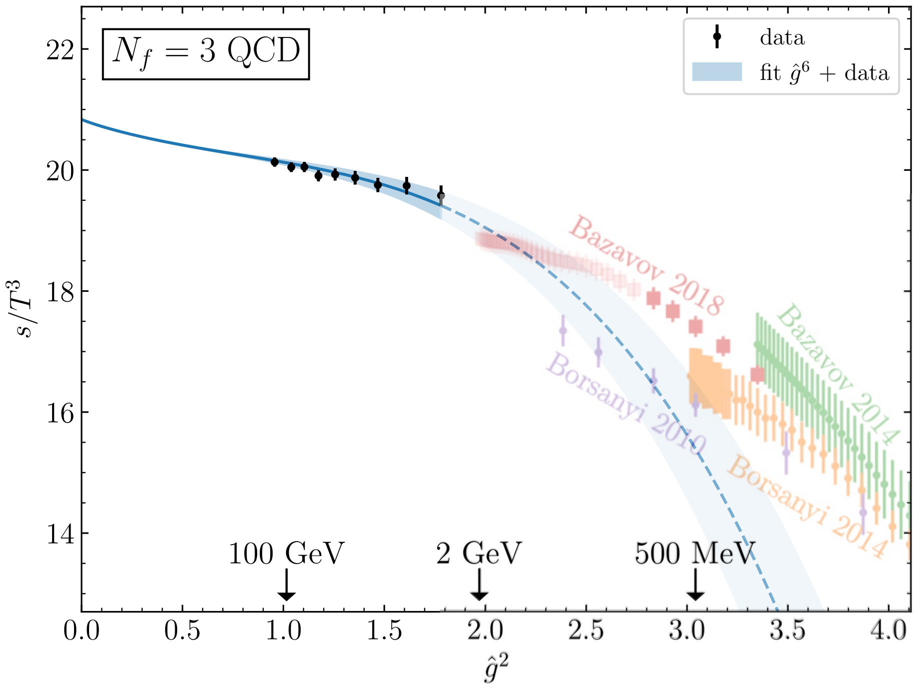
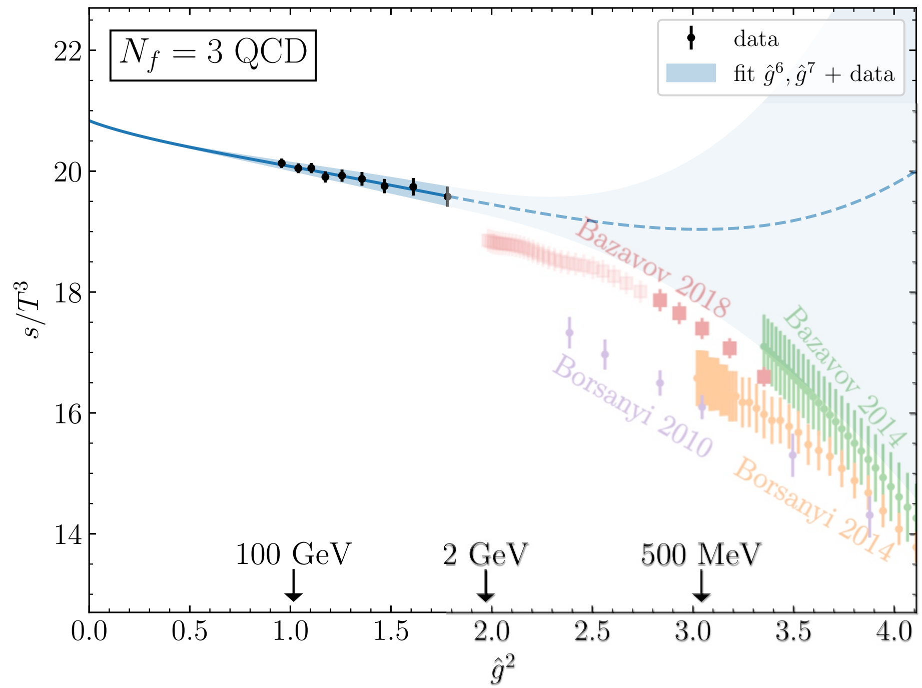
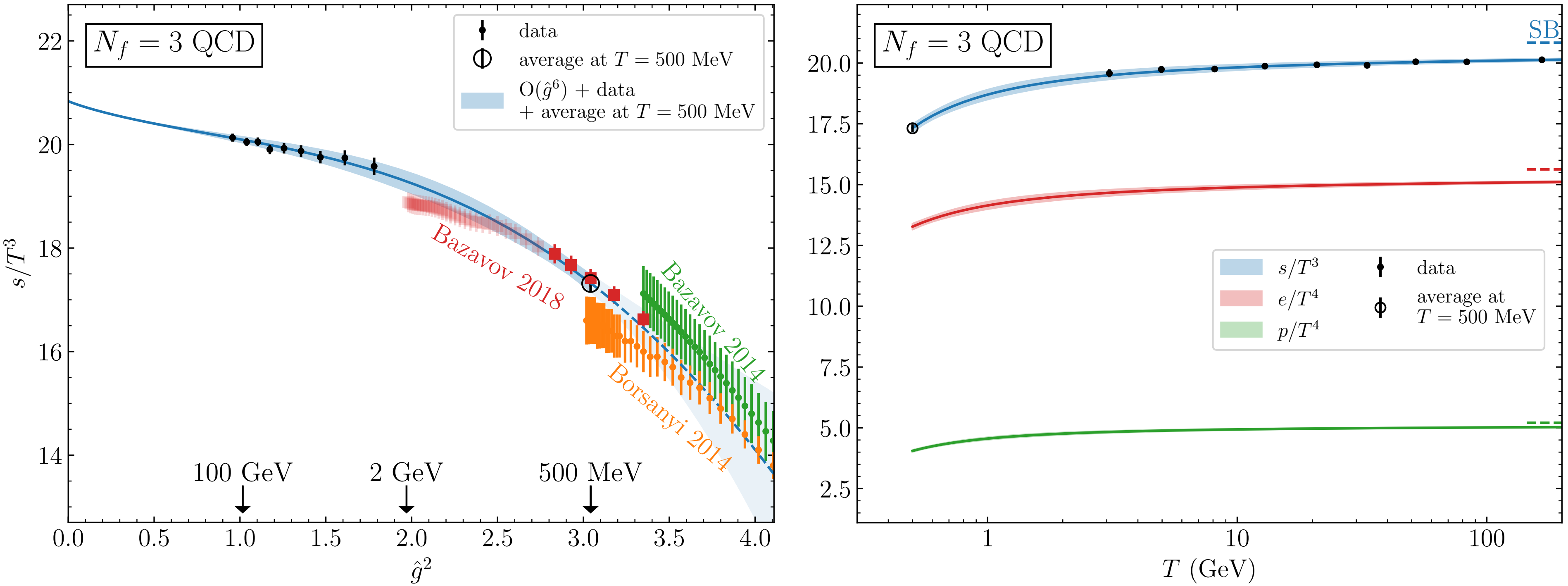




- higher order effects, including non-perturbative contributions, are necessary

The QCD Equation of State: connecting to low T

- $s(T)/T^3$: fit using known $\mathcal{O}(g^6)$ PT $+q_c + s_7\,(\hat{g}(T))^7$ terms



Hadronic screening masses

- They give information on how two hadrons influence each other when inserted into a thermal medium of quarks and gluons
- They are defined from the exponential fall-off of spatial two-point functions
- Mesons: $C_{\mathcal{O}}(x_3) = \int dx_0 dx_1 dx_2 \, \langle \mathcal{O}^a(x) \mathcal{O}^a(0) \rangle \sim \mathrm{e}^{-m_{\mathcal{O}} x_3}$

with $\mathcal{O}^a(x) = \overline{\psi}(x) \Gamma_{\mathcal{O}} \, T^a \, \psi(x)$ where $\Gamma_{\mathcal{O}} = \{\mathbb{1}, \gamma_5, \gamma_\mu, \gamma_\mu \gamma_5\}$ T^a generators of $SU(N_f)$

- Nucleons: $C_{N^\pm}(x_3) = \int dx_0 dx_1 dx_2 \, \mathrm{e}^{i \frac{x_0}{L_0} \pi} \langle P_\pm N(x) \overline{N}(0) \rangle \sim \mathrm{e}^{-m_{N^\pm} x_3}$

$N(x) = \epsilon_{abc} \left[u_a^T(x) C \gamma_5 d_b(x) \right] d_c(x)$ $\overline{N}(x) = \epsilon_{abc} \, \overline{d}_c(x) \left[\overline{d}_b(x) C \gamma_5 \overline{u}_a^T(x) \right]$

$P_\pm = \frac{1 \pm \gamma_3}{2}$: projectors on the positive/negative x_3 – parity states

- restoration of chiral symmetry
- numerically simple to compute non-perturbatively on the lattice
- comparison with EFT: computed at 1-loop order in high-T perturbation theory

$$m_{\mathcal{O}}^{PT} = 2\pi T \, (1 + 0.032739961 \, g^2)$$

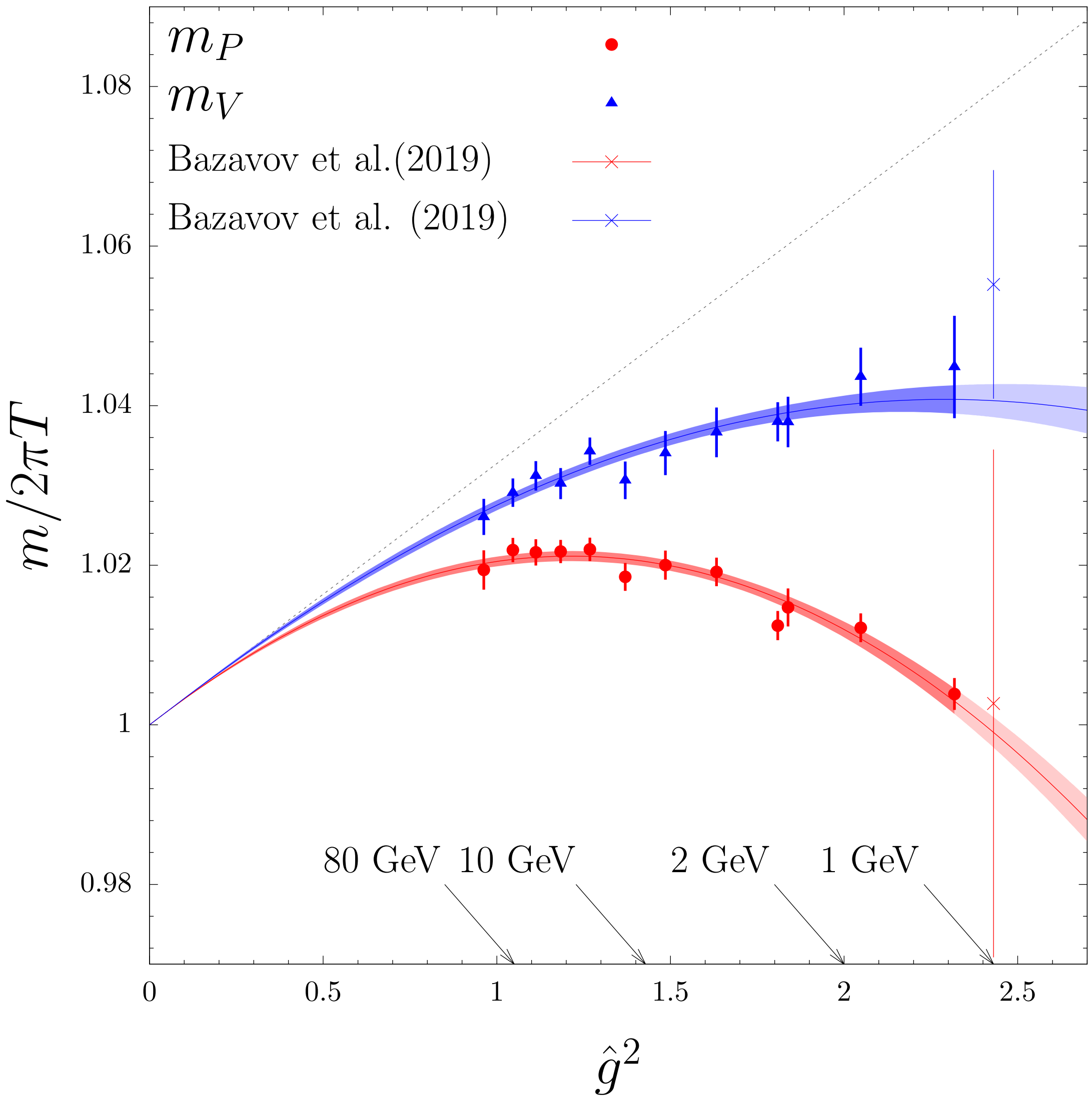
$$m_{N^\pm}^{PT} = 3\pi T \, (1 + 0.046 \, g^2)$$

M. Laine and M. Vepsäläinen, JHEP 02 (2004) 004

T.H. Hansson and I. Zahed, NPB 374 (1992) 277

L. Giusti, M. Laine, D. Laudicina, M. Pepe, P. Rescigno JHEP 06 (2024) 205

The temperature dependence of mesonic screening masses



- 12 values of T: range 1-165 GeV
- continuum limit: $L_0/a = 4, 6, 8, 10$
- Degeneracy: chiral sym. restored

$$V_\mu \leftrightarrow A_\mu \,; PS \leftrightarrow S$$

- A few % away from the $T \to \infty$ limit
- Mass-splitting visible up to T ~165 GeV
- Masses not compatible with 1-loop PT up to 165 GeV

$$\frac{1}{\hat{g}^2(T)} \equiv \frac{9}{8\pi^2} \ln \frac{2\pi T}{\Lambda_{\overline{\rm MS}}} + \frac{4}{9\pi^2} \ln \left(2 \ln \frac{2\pi T}{\Lambda_{\overline{\rm MS}}} \right) , \qquad \Lambda_{\overline{\rm MS}} = 341 \text{ MeV}$$

The mass splitting

- No prediction for such a term in 1-loop PT, computed at next order

$$\frac{\Delta m_{\rm VP}^{\rm lo}}{2\pi T} = g^4 \left(\frac{N_c}{3} + \frac{N_f}{6}\right) \frac{C_F}{8\pi^4} |\hat{\psi}_0(0)|^2 \longrightarrow 0.002376 \cdot g^4$$

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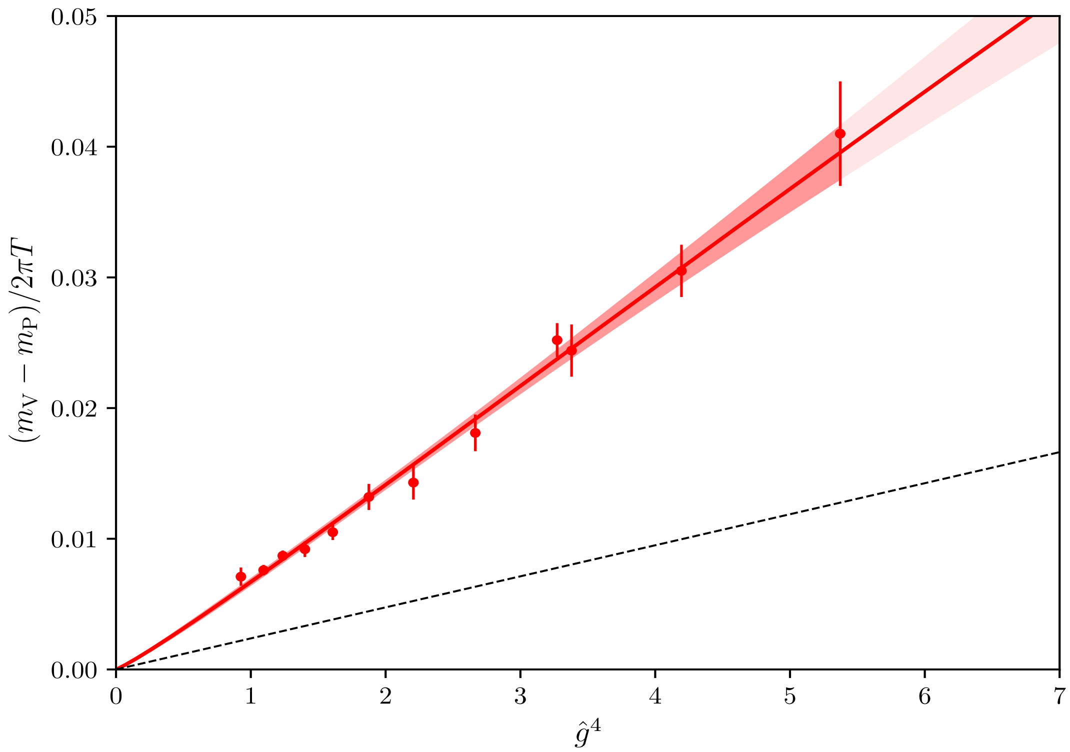
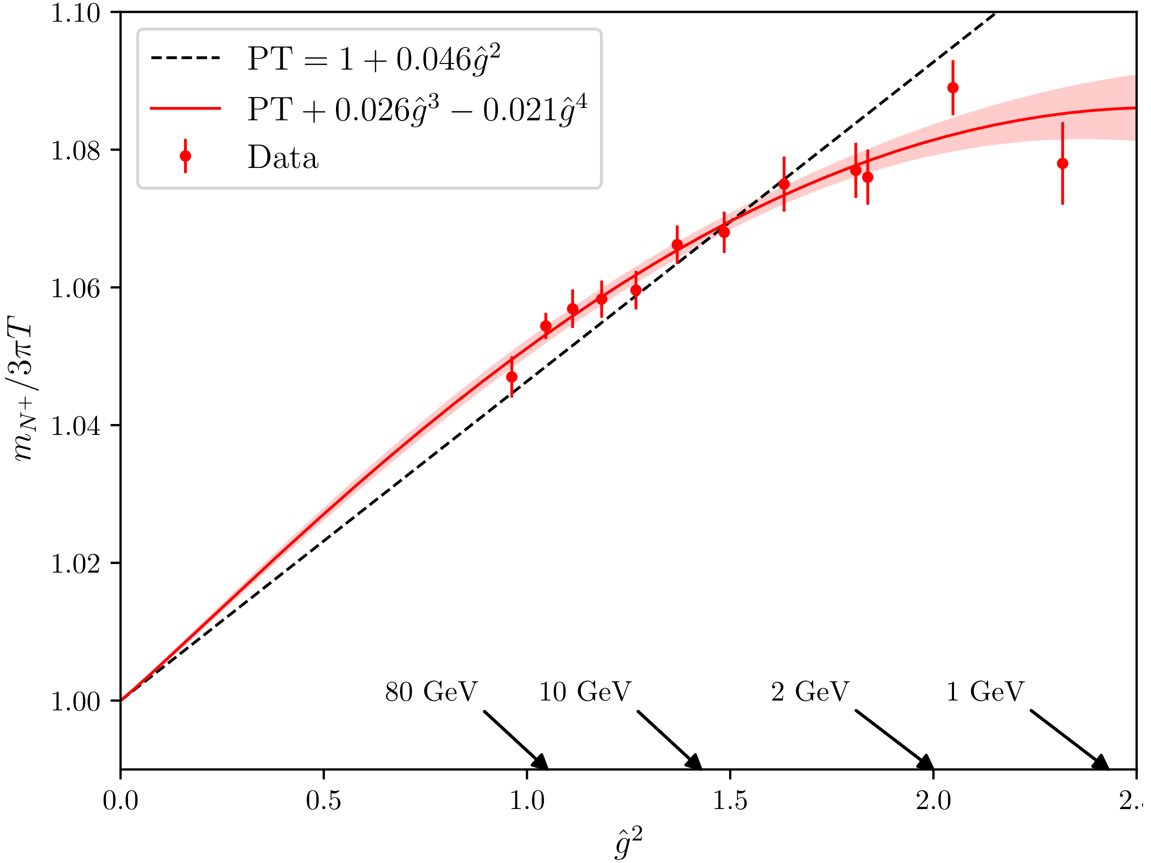
Non-perturbative data show an effective $\hat{g}^4(T)$ over two order of magnitude in T but with a different slope w.r.t. the PT result

$$\frac{\Delta m_{\rm VP}^{\rm lo}}{2\pi T} = 0.002376\, \hat{g}^4 + s_5\, \hat{g}^5 + s_6\, \hat{g}^6$$



mass-splitting clearly visible up to the Electro-Weak scale: high orders, including non-perturbative effects, are relevant

The temperature dependence of baryonic screening mass



- Bulk of the screening mass comes from the free theory but…
- A 4-8 % away from the $T \to \infty$ limit up to very high T
- Red line is a fit to a quartic polynomial

- The 1-loop correction to the baryonic screening mass computed recently

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Conclusions and perspectives

- Current theoretical and computational advances in lattice QCD allow a fully non-perturbative study of thermal QCD up to the electroweak scale.

- QCD EoS (N_f=3): computed with 0.5-1% accuracy in the unexplored range of temperatures between 3 and 165 GeV.

- The method is applicable to N_f = 4, 5 massive flavours: relevant for early-Universe cosmology.

- The thermal properties of the plasma of quark and gluons studied also computing hadronic screening masses with a few permille final accuracy.

- PT insufficient: high-order terms that include non-perturbative contributions, are necessary to describe the data up to the electroweak scale.

- Non-equilibrium physics: in progress determination of the renormalization constants of the energy-momentum tensor, necessary for computing transport coefficients