

Elliptical Polarization in Partial Wave Analysis of Two Spinless Meson Photoproduction

Based on D. I. Glazier, V. Mathieu, Phys. Rev. D 112, 116015

Derek Glazier

University of Glasgow

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Mathematical Ambiguities in PWA

- PWA is not always unique: multiple different sets of amplitudes can perfectly describe the same experimental data.
- This makes it difficult to draw firm physics conclusions.
- **Source of Ambiguity I:** We only measure intensity $\propto |A|^2$
- **Source of Ambiguity II:** Not enough experimental information

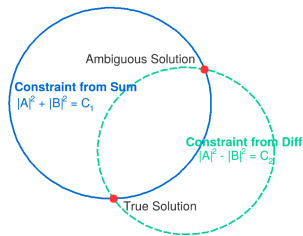
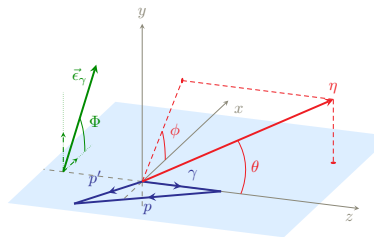


Figure: Ambiguities come from elliptical constraint equations

The PWA Formalism

Mathieu et al. , Phys. Rev. D 100, 054017

- The intensity $\mathcal{I}(\Omega, \Phi)$ depends on decay angles (Ω) and the photon polarization angle (Φ).
- For real photoproduction it decomposes into four components stemming from the photon polarization density matrix.



Two spin 0 Photoproduction

Intensity Decomposition

$$\mathcal{I}(\Omega, \Phi) = \mathcal{I}^0 - P_{\gamma L} \mathcal{I}^1 \cos(2\Phi) - P_{\gamma L} \mathcal{I}^2 \sin(2\Phi) - P_{\gamma C} \mathcal{I}^3$$

From Intensity to Moments

To analyze the data, we decompose each intensity component (\mathcal{I}^α) into a series of spherical harmonic moments, $H^\alpha(LM)$. These moments are the experimentally measurable quantities that contain all the information about the decay dynamics.

Spherical Harmonic Decomposition

The total intensity expands into four independent moment sums:

$$\mathcal{I}^0(\Omega) = \kappa \sum_{L,M} H_{LM}^0 Y_L^M(\Omega)$$

$$\mathcal{I}^1(\Omega) = -\kappa \sum_{L,M} H_{LM}^1 Y_L^M(\Omega) P_{\gamma L} \cos(2\Phi)$$

$$\mathcal{I}^2(\Omega) = -\kappa \sum_{L,M} H_{LM}^2 Y_L^M(\Omega) P_{\gamma L} \sin(2\Phi)$$

$$\mathcal{I}^3(\Omega) = -\kappa \sum_{L,M} H_{LM}^3 Y_L^M(\Omega) P_{\gamma C}$$

Key Idea

By fitting the data's angular distribution, we extract the moments $H^\alpha(LM)$. These moments are the bridge between experimental data and the underlying partial-wave amplitudes.

From Moments to SDMEs

- The moments, $H^\alpha(LM)$, are linear combinations of the Spin-Density Matrix Elements (SDMEs), $\rho_{mm'}^{\alpha, \ell \ell'}$.

The Moment-SDME Relationship

The connection is made via a sum over all contributing waves, weighted by Clebsch-Gordan coefficients that ensure angular momentum conservation

$$H^\alpha(LM) = \sum_{\ell \ell', mm'} C_{\ell' m' LM}^{\ell m} \cdot \rho_{mm'}^{\alpha, \ell \ell'}$$

Key Idea

The SDMEs are the fundamental quantities describing the spin state of the produced meson resonance. By measuring the moments, we can solve for these SDMEs.

From SDMEs to Partial-Wave Amplitudes

- The SDMEs are themselves bilinear products of the underlying partial-wave amplitudes, $[\ell]_m^{(\epsilon)}$.
- This is where the sensitivity to the amplitude magnitudes and phases comes from.

SDME Structure

Each SDME is a sum of bilinear terms:

$$\rho_{mm'}^{0,\ell\ell'} \propto [\ell]_m^{(+)} [\ell']_{m'}^{(+)*} + (-1)^{m-m'} [\ell]_{-m}^{(+)} [\ell']_{-m'}^{(+)*} + \dots$$

The Crucial Difference

The different types of polarization give us access to different parts of the phase information:

- Unpolarized & Linear ($H^{0,1,2}$) $\implies \cos(\Delta\phi) \rightarrow$ Real parts.

This is why a complete set of polarization observables is needed.

From SDMEs to Partial-Wave Amplitudes

- The SDMEs are themselves bilinear products of the underlying partial-wave amplitudes, $[\ell]_m^{(\epsilon)}$.
- This is where the sensitivity to the amplitude magnitudes and phases

Critical Assumption: Vertex Factorization

We assume the scattering amplitude factorizes into independent top (beam) and bottom (target) vertices **at high (JLab12) energies**.

This implies the meson production amplitude is **independent** of the nucleon spin transition.

Consequently, summing over nucleon spins yields a scalar factor, preserving the **coherence (Rank-1)** of the meson state. **If factorization fails, the meson state becomes mixed, requiring a polarized target.**

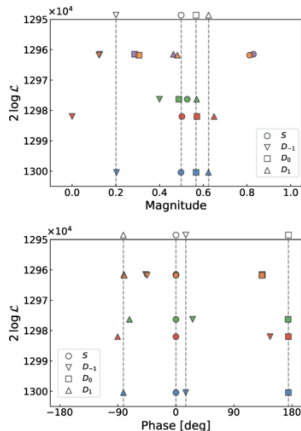
- Unpolarized & Linear ($H^{0,1,2}$) $\implies \cos(\Delta\phi) \rightarrow$ Real parts.

This is why a complete set of polarization observables is needed.

The Work of Smith et al. (JPAC)

Phys. Rev. D 108, 076001

- Showed that for the photoproduction of two spinless mesons with a **linearly polarized beam**, the mathematical ambiguities from Barrelet zeros do not exist.
- Different ambiguous solutions, which produce the same unpolarized intensity, will produce **different** polarized intensity distributions (H^1, H^2).
- The existence of many local minima in the likelihood.



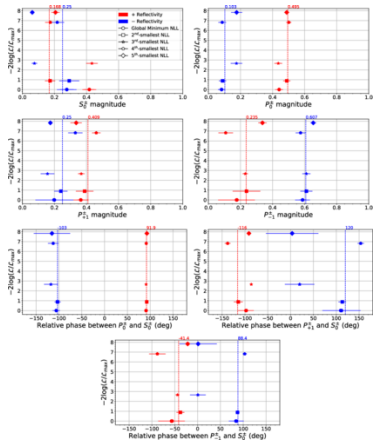
Conclusion

mathematically linearly polarized photoproduction is largely free from ambiguities, leaving only a complex conjugate ambiguity to be solved.

The Work of Guo et al.

Phys. Rev. D 112, 016025

- Confirmed that discrete ambiguities are absent, but revealed that **continuous ambiguities** appear for certain simple wave sets
- This occurs when the system of equations is **under-constrained**—for example, the $\{P_{\pm 1}^{\pm}\}$ wave set has 6 unknown parameters but only 5 constraining moment equations
- Highlighted that noise can lead to true minima no longer having lowest likelihood.



Conclusion

Even in linearly polarized photoproduction, the choice of wave set is critical. Some physically simple models can still be mathematically ambiguous.

The Addition of H^3 : Accessing Imaginary Parts

To solve the final ambiguity, we need to constrain the *sine* of the phases.

1. The Starting Point

The H^3 moment is proportional to a difference of two terms:

$$H^3(L, M) \propto -\sum C... \left(\underbrace{[l]_m^{(\epsilon)} [l']_{m'}^{(\epsilon)*}}_Z - (-1)^{m-m'} \underbrace{[l]_{-m}^{(\epsilon)} [l']_{-m'}^*}_{\text{Term 2}} \right)$$

2. The Key Insight

Using symmetries of the Clebsch-Gordan coefficients, one can prove that the second term is the exact complex conjugate of the first term (Z):

3. The Result

The expression therefore simplifies to the difference between a complex number and its conjugate, which is purely imaginary:

$$H^3(L, M) \propto -(Z - Z^*) = -2i \cdot \text{Im}(Z)$$

The circular polarization (H^3) directly probes the imaginary part of the PW.

The Quasi-Real Photon Limit ($Q^2 \rightarrow 0$)

The Virtual Photon Connection:

- In electroproduction ($ep \rightarrow e'X$), the scattered electron defines a polarization axis.
- As $Q^2 \rightarrow 0$, the longitudinal mode vanishes ($\epsilon_L \rightarrow 0$).

Stokes Parameters via Electron Scattering

$$P_{\gamma L} = \epsilon$$

$$P_{\gamma C} = hP_e \sqrt{1 - \epsilon^2}$$

Photon Density Matrix (Eq. B2)

The photon spin density matrix in the helicity basis is given by:

$$\rho_\gamma = \frac{1}{2} \begin{pmatrix} 1 + P_{\gamma C} & -P_{\gamma L} e^{-2i\Phi} \\ -P_{\gamma L} e^{2i\Phi} & 1 - P_{\gamma C} \end{pmatrix}$$

Where Φ is the angle between the scattering plane and the reaction plane.

Implication for CLAS12

CLAS12 data with a polarized beam is inherently **Elliptically Polarized**.
The formalism presented here applies directly to low- Q^2 electroproduction.

Solving Ambiguities with Polarization (1/3)

Unpolarized Data

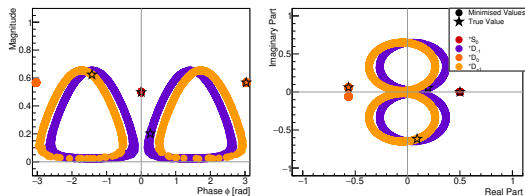
- With no polarization, we can only measure the H^0 moments.
- The system of equations is **under-constrained**.

Solving Ambiguities with Polarization (1/3)

Unpolarized Data

- With no polarization, we can only measure the H^0 moments.
- The system of equations is **under-constrained**.
- This leads to a **continuous ambiguity**, where a family of solutions exists.
- Here only involves $D_{\pm 1}$ waves

Using Smith amplitudes : $S, D_{0,\pm 1}$ +ve refl.



Results of numerical inversion with H^0

Solving Ambiguities with Polarization (2/3)

$\frac{1}{2}$ Linearly Polarized Data

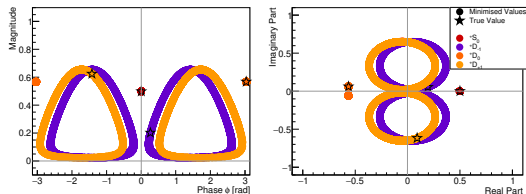
- A linearly polarized beam gives access to the H^1 and H^2 moments.
- These add crucial constraints, resolving the continuous ambiguity.
- First just consider adding only H^0 and H^1

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- These add crucial constraints, resolving the continuous ambiguity.
- First just consider adding only H^0 and H^1
- For this waveset no change from H^0 alone

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Results of numerical inversion with H^0, H^1

Solving Ambiguities with Polarization (2/3)

Linearly Polarized Data

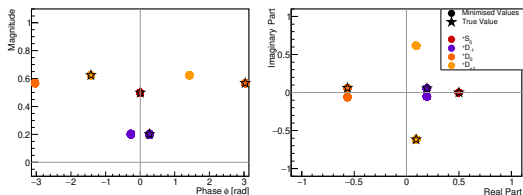
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Solving Ambiguities with Polarization (2/3)

Linearly Polarized Data

- A linearly polarized beam gives access to the H^1 and H^2 moments.
- These add crucial constraints, resolving the continuous ambiguity.
- Now consider H^0 , H^1 and H^2
- Continuous ambiguity resolved. Left with 2 complex conjugate solutions

Using Smith amplitudes : $S, D_{0,\pm 1}$ +ve refl.



Results of numerical inversion with H^0, H^1, H^2

Solving Ambiguities with Polarization (3/3)

Elliptically Polarized Data

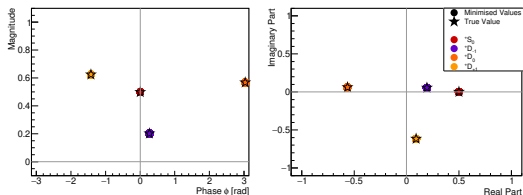
- An elliptically polarized beam also provides the H^3 moments.
- This probes the *imaginary* parts of the amplitude products, which constrains the sine of the relative phases.

Solving Ambiguities with Polarization (3/3)

Elliptically Polarized Data

- An elliptically polarized beam also provides the H^3 moments.
- This probes the *imaginary* parts of the amplitude products, which constrains the sine of the relative phases.
- This breaks the final ambiguity, yielding a **unique solution**.

Using Smith amplitudes : $S, D_{0,\pm 1}$ +ve refl.



*Results of numerical inversion with
 H^0, H^1, H^2, H^3*

Stability against Noise and Ambiguities

Full Wave Set Inversion

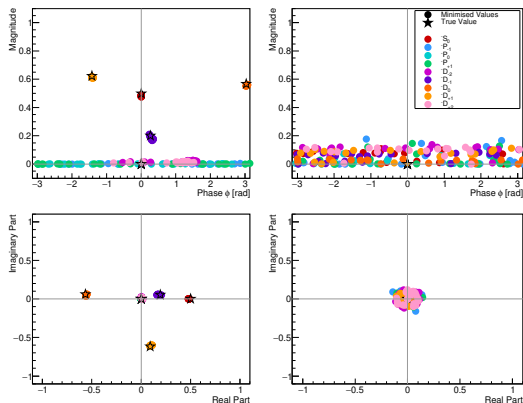
- But what about the P-waves ?
- And the -ve reflectivity ?

Stability against Noise and Ambiguities

Full Wave Set Inversion

- But what about the P-waves ?
- And the -ve reflectivity ?
- Inverting with the full set of waves still recovers the true non-zero waves.
- Non-zero waves noisy. Amplitudes up to 0.1

Now inverting with : $^{+,-}S, ^{+,-}P_{0,\pm 1}, ^{-}D_{0,\pm 1,\pm 2}$
+ve reflectivity **-ve reflectivity**



With H^0, H^1, H^2, H^3

A Special Case: The $\{P_{\pm 1}^{\pm}\}$ System

- The paper by Guo et al. showed that for this simplified wave set, a **continuous ambiguity persists** even with linear polarization
- This occurs because the system is under-constrained, with only 5 independent equations for 6 unknown parameters.

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- This occurs because the system is under-constrained, with only 5 independent equations for 6 unknown parameters.
- The five moment equations that constrain the system are shown on the right.

The 5 Independent Moment Equations

$$H^0(0,0) = 2 \sum_{\epsilon=\pm} (|P_{+1}^{\epsilon}|^2 + |P_{-1}^{\epsilon}|^2)$$

$$H^1(2,2) = \frac{\sqrt{6}}{5} \sum_{\epsilon=\pm} \epsilon (|P_{+1}^{\epsilon}|^2 + |P_{-1}^{\epsilon}|^2)$$

$$H^2(2,2)/i = \frac{\sqrt{6}}{5} \sum_{\epsilon=\pm} \epsilon (|P_{-1}^{\epsilon}|^2 - |P_{+1}^{\epsilon}|^2)$$

$$H^0(2,2) = -\frac{2\sqrt{6}}{5} \sum_{\epsilon=\pm} |P_{+1}^{\epsilon}| |P_{-1}^{\epsilon}| \cos(\phi_{P_{-1}}^{\epsilon})$$

$$H^1(0,0) = 4 \sum_{\epsilon=\pm} \epsilon |P_{+1}^{\epsilon}| |P_{-1}^{\epsilon}| \cos(\phi_{P_{-1}}^{\epsilon})$$

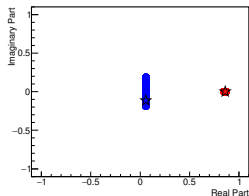
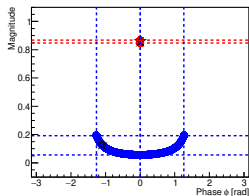
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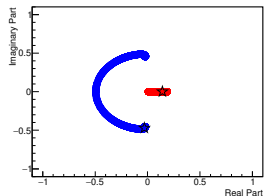
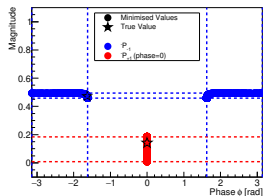
Additional Information

The five moment equations can be used to define the allowed solution ranges - shown as dashed lines.

+ve reflectivity



-ve reflectivity



With H^0, H^1, H^2

A Special Case: The $\{P_{\pm 1}^{\pm}\}$ System

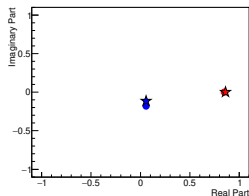
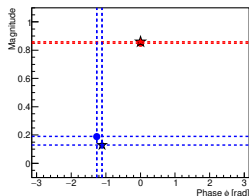
- Additional constraints come from circular polarisation

Additional Information

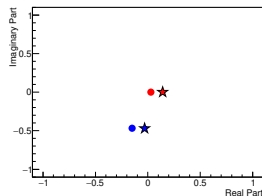
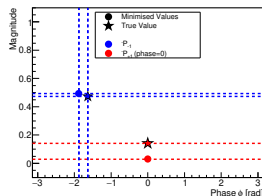
$$H^3(2, 2) = \frac{2\sqrt{6}}{5} [|P_{+1}^+| |P_{-1}^+| \sin(\phi_{P_{+1}^+} - \phi_{P_{-1}^+}) + |P_{+1}^-| |P_{-1}^-| \sin(\phi_{P_{+1}^-} - \phi_{P_{-1}^-})]$$

- Replace continuous ambiguity with discrete
- Both solutions calculable from moments equations - dashed lines

+ve reflectivity



-ve reflectivity

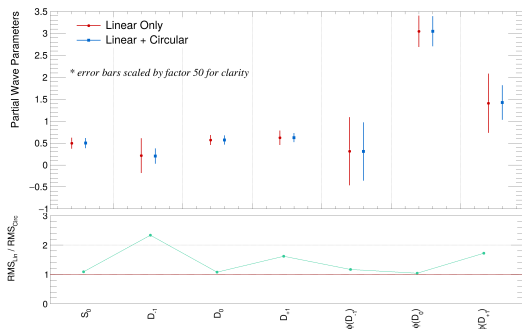


With H^0, H^1, H^2, H^3

Quantifying the Impact: Toy Monte Carlo Study (1/2)

Methodology:

- Generated 100,000 toy events based on amplitudes from Smith et al.
- Extracted partial waves using MCMC sampling.
- Compared statistical uncertainties (RMS of posterior) for two scenarios.



Quantifying the Impact: Toy Monte Carlo Study (2/2)

Scenarios Compared:

- **Linear Only:** $P_{\gamma L} = 0.38$
- **Elliptical:** $P_{\gamma L} = 0.38, P_{\gamma C} = 0.65$

Uncertainty Reduction (RMS Ratio):

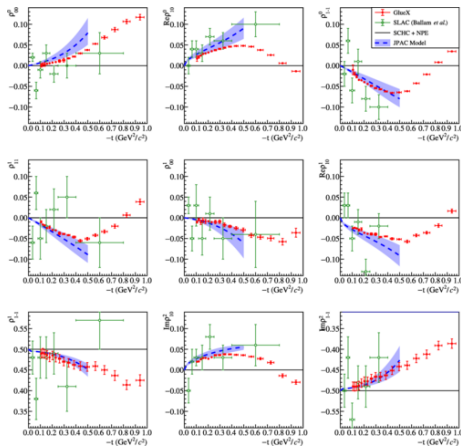
- $m = 0$ waves: $\sim 10\%$ reduction
- $m = +1$ waves: $\sim 40\%$ reduction
- $m = -1$ **waves:** $\sim 50\%$ **reduction**

The "Beamtime" Equivalent

Since statistical uncertainty scales as $\sigma \propto 1/\sqrt{N}$, a **50%** reduction in uncertainty is mathematically equivalent to increasing the experimental beamtime by a factor of **4**.

Polarised ρ Photoproduction

- Our polarimetry study begins with real experimental data: the high-statistics measurement of $\rho(770)$ Spin-Density Matrix Elements (SDMEs) from GlueX
- We assume the reaction is dominated by P-waves, consistent with the GlueX analysis.
- We convert these measured SDMEs into the equivalent spherical harmonic moments, which are then used in the numerical inversion to determine partial-wave amplitudes.



A Novel Application: The Reaction Polarimeter

Via ρ Photoproduction

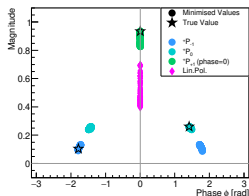
- With elliptical polarization, the PWA system is **overconstrained**.
- This allows us to treat the degrees of polarization ($P_{\gamma L}$ and $P_{\gamma C}$) as **free parameters** in the inversion.

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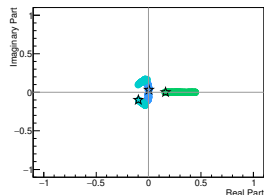
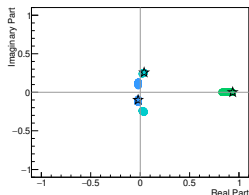
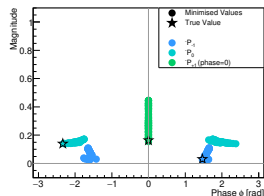
Via ρ Photoproduction

- With elliptical polarization, the PWA system is **overconstrained**.
- This allows us to treat the degrees of polarization ($P_{\gamma L}$ and $P_{\gamma C}$) as **free parameters** in the inversion.
- The reaction itself can be used to measure the beam polarization!
- Linear polarisation alone not sufficient - 10 free parameters, 10 constraint equations

+ve reflectivity



-ve reflectivity



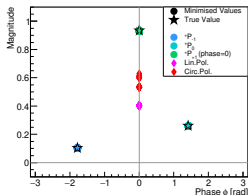
With H^0, H^1, H^2

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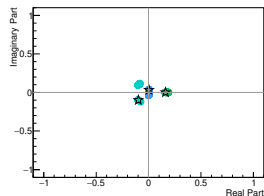
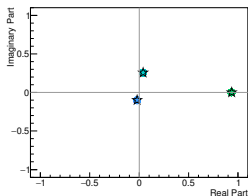
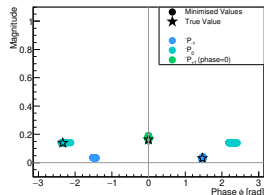
Via ρ Photoproduction

- With elliptical polarization, the PWA system is **overconstrained**.
- This allows us to treat the degrees of polarization ($P_{\gamma L}$ and $P_{\gamma C}$) as **free parameters** in the inversion.
- The reaction itself can be used to measure the beam polarization!
- H^3 provides additional 2 constraints \implies Can solve for polarisations.
- Another discrete ambiguity in solution !

+ve reflectivity



-ve reflectivity



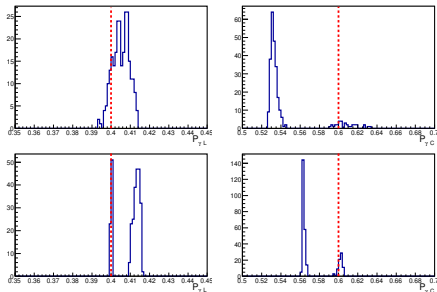
With H^0, H^1, H^2, H^3

Resolving the Polarization Ambiguity

- The polarimetry inversion reveals a discrete ambiguity in the extracted polarization values.

Resolving the Polarization Ambiguity

- The polarimetry inversion reveals a discrete ambiguity in the extracted polarization values.
- We repeated the analysis for two distinct kinematic bins from the GlueX data, centered at
 - $t = 0.230$ and
 - $t = 0.728 \text{ GeV}^2/c^2$.
- \implies Similar performance for partial waves.



Resolving Ambiguity

Analyzing multiple kinematic bins simultaneously gives a consistent true solution, while the ambiguous, unphysical solutions will vary.

Summary and Outlook

The Case for Elliptical Polarization

- ✓ **Mathematical Uniqueness:** We have demonstrated that combining Linear and Circular polarization resolves further ambiguities inherent to 2-meson PWA.
 - ✓ **Statistical Power:** Toy Monte Carlo studies show a significant reduction in uncertainty some waves.
Possibly $m < 0$ projections most improved.
- ⇒ Equivalent to a $4\times$ increase in experimental beamtime in some cases

Constraint: Success relies on **High-Energy Factorization**.

We trade the need for a *polarized target* for the validity of the *Regge limit*.

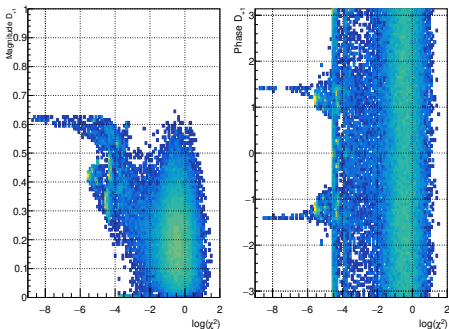
Next Steps :

- 1 **Validation:** Use CLAS12 $\pi\pi$ and KK events as a reaction polarimeter to cross-check $P_{\gamma L}$ and $P_{\gamma C}$ extraction. CLAS12 polarisations are well known.
- 2 **Extension:** Generalize the formalism to **Electroproduction** ($Q^2 > 0$) to support future EIC and CLAS12 physics programs.

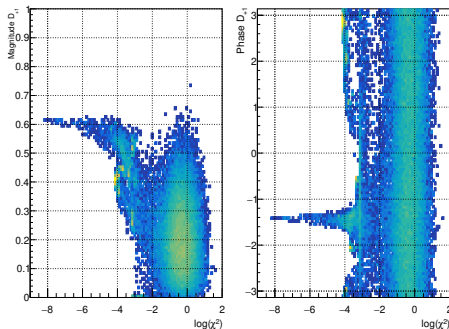
Backup: Investigating the χ^2 Landscape

- 50,000 numerical inversions for S, P, and D waves.
- \implies Final amplitude against the minimized χ^2 .

Linearly Polarized ($H^{0,1,2}$)



Elliptically Polarized ($H^{0,1,2,3}$)



The true solution ($\chi^2 \rightarrow 0$) is found, but many local minima exist nearby.

The true solution is found more cleanly, with fewer and more separated local minima.

Backup: The Photon Spin-Density Matrix (SDM)

Pure State (Linear Basis)

The pure-state density matrix for an elliptically polarized photon is given by [?]:

$$\rho_{\text{linear}} = \frac{1}{a^2 + b^2} \begin{pmatrix} a^2 & -iabe^{i2\Phi} \\ iabe^{-i2\Phi} & b^2 \end{pmatrix}$$

Transformation to Helicity Basis

Using the unitary transformation U , the SDM is rotated into the helicity basis via $\rho_{\lambda\lambda'} = U\rho_{\text{linear}}U^\dagger$. This yields the final form used in the analysis (see Appendix ??):

$$\rho_{\lambda\lambda'} = \frac{1}{2} \begin{pmatrix} 1 + P_{\gamma C} & P_{\gamma L}e^{-i2\Phi} \\ P_{\gamma L}e^{i2\Phi} & 1 - P_{\gamma C} \end{pmatrix}$$

where $P_{\gamma L}$ and $P_{\gamma C}$ are the degrees of linear and circular polarization.

Backup: Derivation of the H^3 Moment

- **Goal:** Prove that $H^3(L, M)$ is purely imaginary and thus sensitive to the *sine* of phase differences.

1. Starting Point

The moment is proportional to a sum over partial-wave products (see Appendix ??):

$$H^3(L, M) \propto - \sum C... \left(\underbrace{[\ell]_m^{(\epsilon)} [\ell']_{m'}^{(\epsilon)*}}_Z - (-1)^{m-m'} \underbrace{[\ell]_{-m}^{(\epsilon)} [\ell']_{-m'}^*}_{\text{Term 2}} \right)$$

2. Key Step

Using symmetries of the Clebsch-Gordan coefficients, Term 2 is the exact complex conjugate of the first term (Z):

$$\sum C... (-1)^{m-m'} [\ell]_{-m}^{(\epsilon)} [\ell']_{-m'}^* = Z^*$$

Backup: GlueX $\rho(770)$ SDME Data Used

Amplitudes for the polarimetry study in Section ?? were derived from the following GlueX SDME data at $t = -0.728 \text{ GeV}^2/c^2$ [?].

Source	ρ_{00}^0	$\Re\rho_{10}^0$	ρ_{1-1}^0	ρ_{00}^1	ρ_{11}^1
Original Data	0.0873	0.0257	-0.0211	-0.0496	-0.0203
Recalculated	0.0871	0.0259	-0.0198	-0.0481	-0.0210

Source	ρ_{10}^1	ρ_{1-1}^1	$\Im\rho_{10}^2$	$\Im\rho_{1-1}^2$	
Original Data	-0.0360	0.4365	0.0179	-0.4119	
Recalculated	-0.0379	0.4282	0.0146	-0.4185	

- The "Recalculated" values are derived from the partial-wave solution found by inverting the "Original Data".
- Their close agreement demonstrates the self-consistency of the formalism.

Future Directions

1. Analysis of GlueX Data

- **Elliptical Polarization Analysis:** Apply formalism to GlueX data with linear + circular polarization.
- **Reaction Polarimetry:** Use high-stats $\rho(770)$ events to measure $P_{\gamma L}, P_{\gamma C}$ from reaction dynamics ("Self-Calibrating" detector).

2. Extending the Formalism

Generalizing the GLM extraction to include:

- **Electroproduction ($Q^2 > 0$):**
 - Longitudinally polarized e^- (CLAS12, EIC).
 - Access to longitudinal modes ($[I]_L$).
- **Spin Degrees of Freedom:**
 - Polarized Targets (\vec{N}).
 - Recoil Polarization ($\gamma p \rightarrow K \vec{\Lambda}$).
 - Target spin index k in amplitudes.

Electroproduction + Polarized Target:

$$\rho_{mm'}^0 = i\kappa \sum_k [I]_{T-m;k}^\epsilon [I']_{T-m';k}^{*\epsilon} + (-1)^\delta [I]_{Tm;k}^\epsilon [I']_{Tm';k}^{*\epsilon}$$

$$\rho_{mm'}^1 = -i\epsilon\kappa \sum_k (-1)^m [I]_{T-m;k}^\epsilon [I']_{Tm';k}^{*\epsilon} + \dots$$

$$\rho_{mm'}^2 = \epsilon\kappa \sum_k (-1)^m [I]_{T-m;k}^\epsilon [I']_{Tm';k}^{*\epsilon} - \dots$$

$$\rho_{mm'}^3 = i\kappa \sum_k [I]_{T-m;k}^\epsilon [I']_{T-m';k}^{*\epsilon} - (-1)^{m'-m} \dots$$

$$\rho_{mm'}^4 = 2i\kappa \sum_k [I]_{Lm;k}^\epsilon [I']_{Lm';k}^{*\epsilon} \quad \text{(Longitudinal)}$$

$$\rho_{mm'}^5 = -\frac{i\kappa}{\sqrt{2}} \epsilon \sum_k (-1)^m \left([I]_{Lm;k}^\epsilon [I']_{Tm';k}^{*\epsilon} - [I]_{Tm;k}^\epsilon [I']_{Lm';k}^{*\epsilon} \right) +$$

$$\rho_{mm'}^6 = -\frac{\kappa}{\sqrt{2}} \sum_k [I]_{Lm;k}^\epsilon [I']_{Tm';k}^{*\epsilon} - [I]_{Tm;k}^\epsilon [I']_{Lm';k}^{*\epsilon} - \dots$$

$$\rho_{mm'}^7 = -\frac{i\kappa}{\sqrt{2}} \epsilon \sum_k (-1)^m \left([I]_{Lm;k}^\epsilon [I']_{Tm';k}^{*\epsilon} + [I]_{Tm;k}^\epsilon [I']_{Lm';k}^{*\epsilon} \right) +$$

$$\rho_{mm'}^8 = -\frac{\kappa}{\sqrt{2}} \sum_k [I]_{Lm;k}^\epsilon [I']_{Tm';k}^{*\epsilon} - [I]_{Tm;k}^\epsilon [I']_{Lm';k}^{*\epsilon} + \dots$$