INFRARED PHASES OF 2D ABELIAN GAUGE THEORIES:

from the Schwinger model to symmetric mass generation

Bernardo Zan

Università di Genova

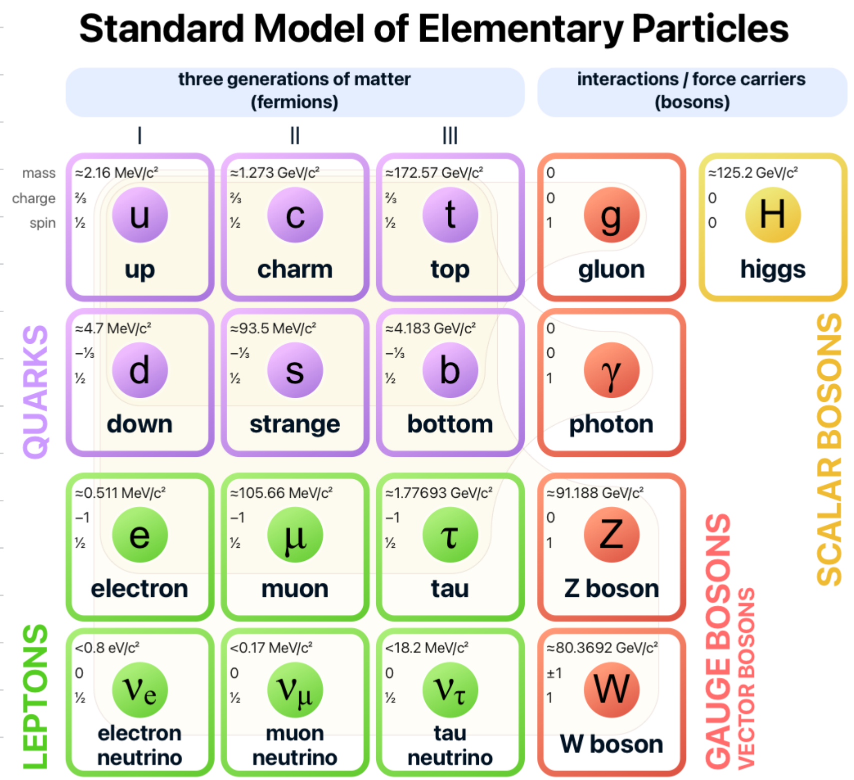
INFN Istituto Nazionale di Fisica Nucleare

based on work with R. Dempsey, I. Klebanov, R. Moulund, S. Pufu, B. Søgaard, D. Tong

QUANTUM FIELD THEORIES

Quantum field theory: at the basis of our understanding of Nature on the most microscopical level.

→ Standard model: a 3+1 dimensional gauge theory, explaining everything we know minus gravity



QUANTUM FIELD THEORIES

QFTs are generically very hard to solve!

* Free theories: easy but boring
* theories close to be free:

  can study them in perturbation theory

  e.g. QCD at high energies
* strongly coupled theories:

  lost of analytic control

  e.g. QCD at low energies

QUANTUM FIELD THEORIES

Our understanding of strongly coupled QFTs relies on numerical approaches, e.g.

* Euclidean lattice QFT

  e.g. Monte-Carlo methods

* Hamiltonian lattice QFT

  e.g. tensor networks, Hamiltonian truncation

* Bootstrap approaches

  e.g. conformal bootstrap, S-matrix bootstrap

$\vdots$

TWO DIMENSIONS

Physics is easier in lower dimensions

Today's topic: gauge theories in 2d

Why (according to a high-energy theorist):

* show some of the same phenomena as in 4d

  confinement, screening, ...

* much easier than 4d
* perfect setting to understand properties of strongly coupled QFTs

TWO DIMENSIONS

2d gauge theories interesting also for

* condensed matter

  effective description of 1+1 d quantum matter

  e.g. spin-$\frac{1}{2}$ antiferromagnetic chains described by CP^1 model ($\approx U(1)$ gauge theory)

  [Haldane '83]

* many body physics

  classical simulation of gauge theories

  e.g. tensor networks (more on this today)

* quantum simulations of gauge theories

Real-time dynamics of lattice gauge theories with a few-qubit quantum computer

Esteban A. Martinez[1]*, Christine A. Muschik[2,3]*, Philipp Schindler[1], Daniel Nigg[1], Alexander Erhard[1], Markus Heyl[2,4], Philipp Hauke[2,3], Marcello Dalmonte[2,3], Thomas Monz[1], Peter Zoller[2,3] & Rainer Blatt[1,2]

2016

N = 4 sites lattice Schwinger model

using ultra cold atoms

String-Breaking Mechanism in a Lattice Schwinger Model Simulator

Ying Liu[1,2,*], Wei-Yong Zhang[1,2,*], Zi-Hang Zhu[1,2], Ming-Gen He[1,2], Zhen-Sheng Yuan[1,2,3], and Jian-Wei Pan[1,2,3]

2024

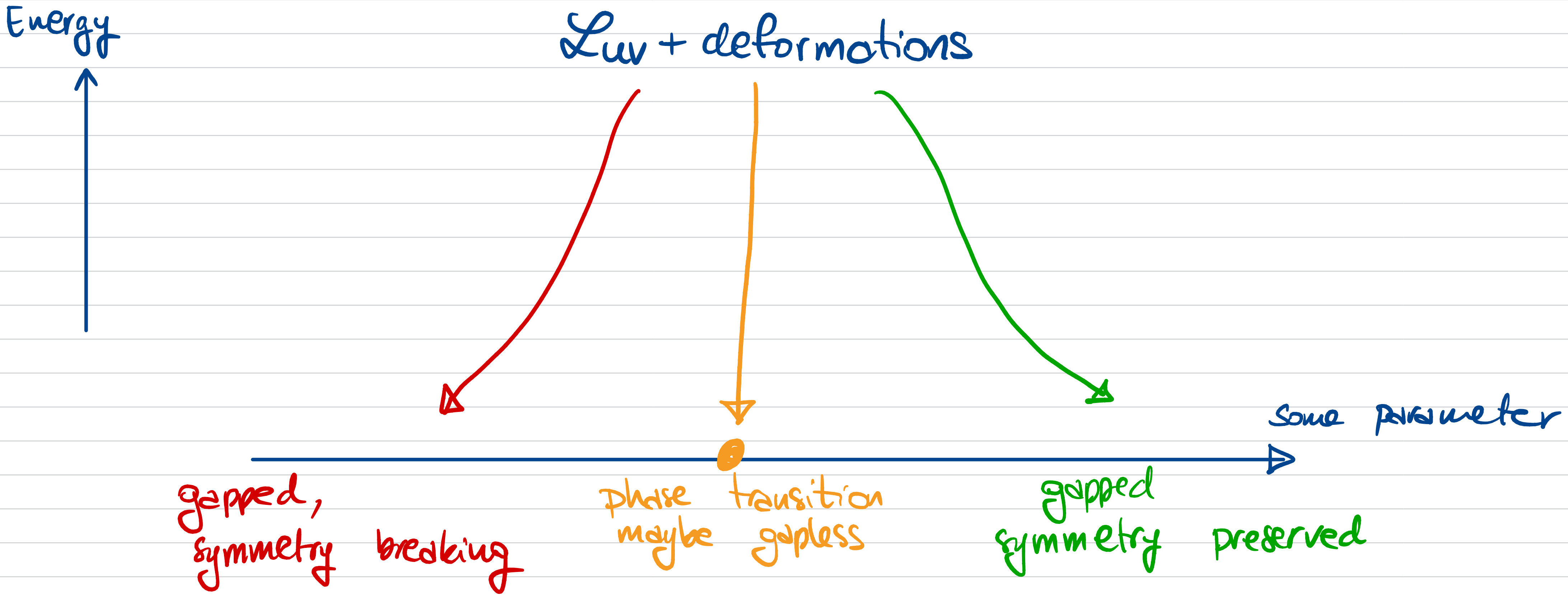
up to N = 10

THE QUESTIONS

Given the UV Lagrangian, what happens at low energies?

→ mass gap or not?

→ symmetries spontaneously broken?



OUR TOOLS

How will we answer these questions?

* semiclassics: control over the theory when the coupling is very large
* bosonization: in 2d fermions $\leftrightarrow$ bosons (with caveats)
* numerics: will use Hamiltonian methods

  DMRG aka MPS aka tensor networks

  density matrix renormalization group

  matrix product states

THE THEORIES WE'LL CONSIDER

U(1) gauge theory with fermions

$$\mathcal{L} = -\frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu} + \frac{\theta}{4\pi}\varepsilon^{\mu\nu}F_{\mu\nu} + \text{matter}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

total derivative: it does not affect the equation of motions, but it plays a role in determining the symmetries and the IR behavior of the theory

$\theta \sim \theta + 2\pi$ identification

WARM-UP: PURE MAXWELL

In 4d photons have 2 degrees of freedom

$\Updownarrow$

2 polarizations for light

What about 2d?

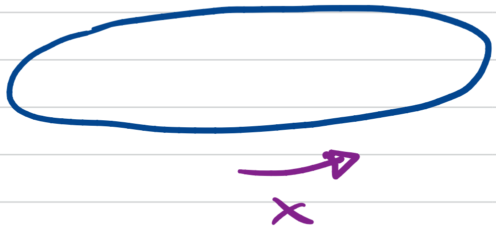
In d-dimensions $\to$ d-2 degrees of freedom

In 2d photons have no propagating degrees of freedom!

Things in 2d are easier.

PURE MAXWELL

work on a spatial circle:



Electric field can change when we meet a fermion.

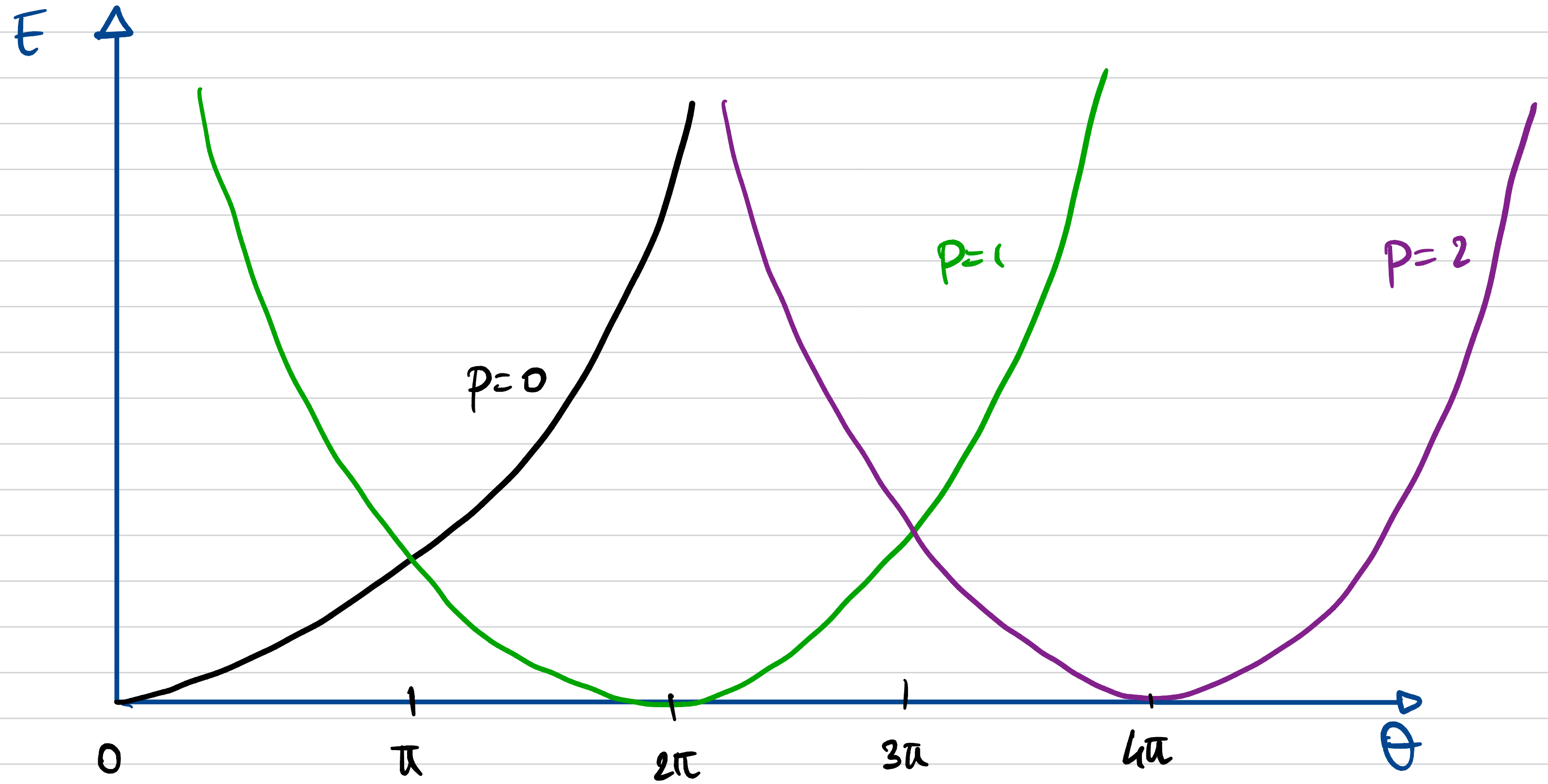
No fermions $\rightarrow$ constant electric field.

Energy levels are very simple to compute

$\rightsquigarrow$ spectrum $\quad E_p = \pi e^2 L \left(p - \dfrac{\theta}{2\pi} \right)^2 \qquad p \in \mathbb{Z}$

SPECTRUM

Energy levels
$$E_p = \pi e^2 L \left(p - \frac{\theta}{2\pi}\right)^2$$



Two degenerate ground states at $\theta = \pi, 3\pi, \ldots$

SYMMETRY

Why 2 degenerate ground states at $\theta = \pi$?

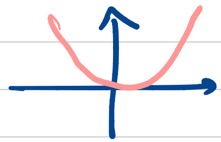
The theory has a $\mathbb{Z}_2$ charge conjugation symmetry

$$C: \quad A_\mu \to -A_\mu$$

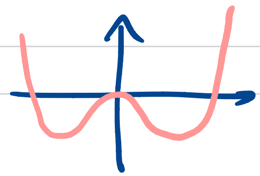
Only a symmetry at $\theta = 0, \pi$ $\qquad \frac{\theta}{4\pi} \varepsilon^{\mu\nu} F_{\mu\nu} \to -\frac{\theta}{4\pi} \varepsilon^{\mu\nu} F_{\mu\nu}$

When is it preserved/broken?

$\mathbb{Z}_2$ not spontaneously broken at $\theta = 0$ $\to$ 1 ground state



$\mathbb{Z}_2$ spontaneously broken at $\theta = \pi$ $\to$ 2 ground states



SCHWINGER MODEL

Now add matter : 1 fermion $\qquad \psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}$

$$\mathcal{L} = -\frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} + \frac{\theta}{4\pi} \varepsilon^{\mu\nu} F_{\mu\nu} + i \bar{\psi} \underbrace{\not{D}}_{D_\mu = \partial_\mu - i A_\mu} \psi + m \bar{\psi} \psi$$

[Schwinger '62]

Gauge transformation $\qquad \psi \to e^{i\alpha(x)} \psi$

$\qquad A_\mu \to A_\mu - \partial_\mu \alpha$

Again, $\mathbb{Z}_2$ symmetry at $\theta = 0, \pi$

* when is $\mathbb{Z}_2$ spontaneously broken? When is it not?

* gapped vs gapless? phase transition between the $\mathbb{Z}_2$ broken and preserving phase?

TOOL #1 : BOSONIZATION

Set $m=0$.

Gauge current $J^\mu = \bar\psi \gamma^\mu \psi$, conserved $\partial_\mu J^\mu = 0$

$[\psi] = 1/2$

$\mathcal{L} \supset J^\mu A_\mu$

Introduce a scalar field ϕ $\rightarrow$ $[\phi] = 0$

$$\frac{1}{2\pi} \varepsilon^{\mu\nu} \partial_\nu \phi$$

* dimension 1, same as J^μ
* conserved, $\varepsilon^{\mu\nu} \partial_\mu \partial_\nu \phi = 0$

→ Write $J^\mu = \frac{1}{2\pi} \varepsilon^{\mu\nu} \partial_\nu \phi$

TOOL #1 : BOSONIZATION

Get rid of fermion, write Lagrangian in terms of boson only

$$\mathcal{L} = \frac{1}{2e^2} F_{01}^2 + \frac{1}{8\pi} (\partial_\mu \phi)^2 + \frac{1}{2\pi} (\phi + \theta) F_{01}$$

only independent component of $F_{\mu\nu} = \begin{pmatrix} 0 & F_{01} \\ -F_{01} & 0 \end{pmatrix}$

Integrate out F_{01}

$$\mathcal{L} = \frac{1}{8\pi} (\partial_\mu \phi)^2 + \frac{e^2}{8\pi^2} (\phi + \theta)^2$$

Shift $\phi \to \phi - \theta$

$$\mathcal{L} = \frac{1}{8\pi} (\partial_\mu \phi)^2 + \frac{e^2}{8\pi^2} \phi^2$$

TOOL #1 : BOSONIZATION

Massless Schwinger model is exactly solvable!

Free massive scalar, mass $M_s = \frac{e}{\sqrt{\pi}}$

First example of mass gap in gauge theories.

Only one ground state: $\mathbb{Z}_2$ is preserved.

For $m \neq 0$, the model is not exactly solvable!

Need different tools!

TOOL #2: SEMICLASSICS

Choose $\theta = 0$ ($\theta = \pi$ follows)

What happens for very large mass, $|m| \gg e$?

* In the limit $m \to \infty$, can integrate out fermion.
  Left w/ $\theta = 0$ pure Maxwell theory!

  $\to$ 1 ground state, $\mathbb{Z}_2$ preserved

* When $m \to -\infty$?
  Need chiral / Schwinger / ABJ anomaly

SCHWINGER ANOMALY

Like in 4d, $U(1)_A$ $\quad \psi \to e^{i\alpha\gamma_5}\psi$ is anomalous in the massless theory.

$\gamma_5 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$

$$U(1)_A: \quad \begin{aligned} \psi &\to e^{i\alpha\gamma_5}\psi \\ \theta &\to \theta - 2\alpha \end{aligned}$$

$$\mathcal{L} = i\bar{\psi}\not{D}\psi - \frac{1}{4e^2}F_{\mu\nu}^2 + \frac{\theta}{4\pi}\varepsilon^{\mu\nu}F_{\mu\nu}$$

Doesn't leave the theory invariant! Not a symmetry

Under $U(1)_A$, mass term transform as

$$m\bar{\psi}\psi \to m\bar{\psi}e^{2i\alpha\gamma_5}\psi$$

Choose $\alpha = \pi/2$:

$$\begin{aligned} \theta &\to \theta - \pi \\ m\bar{\psi}\psi &\to -m\bar{\psi}\psi \end{aligned}$$

SCHWINGER ANOMALY

Not a symmetry but

$$(m, \theta) \text{ theory } \xleftrightarrow[\text{same as}]{} (-m, \theta+\pi) \text{ theory}$$

* theory with $-m \gg e$, $\theta = 0$ is the same as $m \gg e$, $\theta = \pi$.

For $|m| \to \infty$, pure Maxwell w/ $\theta = \pi$

2 ground states $\to$ $\mathbb{Z}_2$ broken

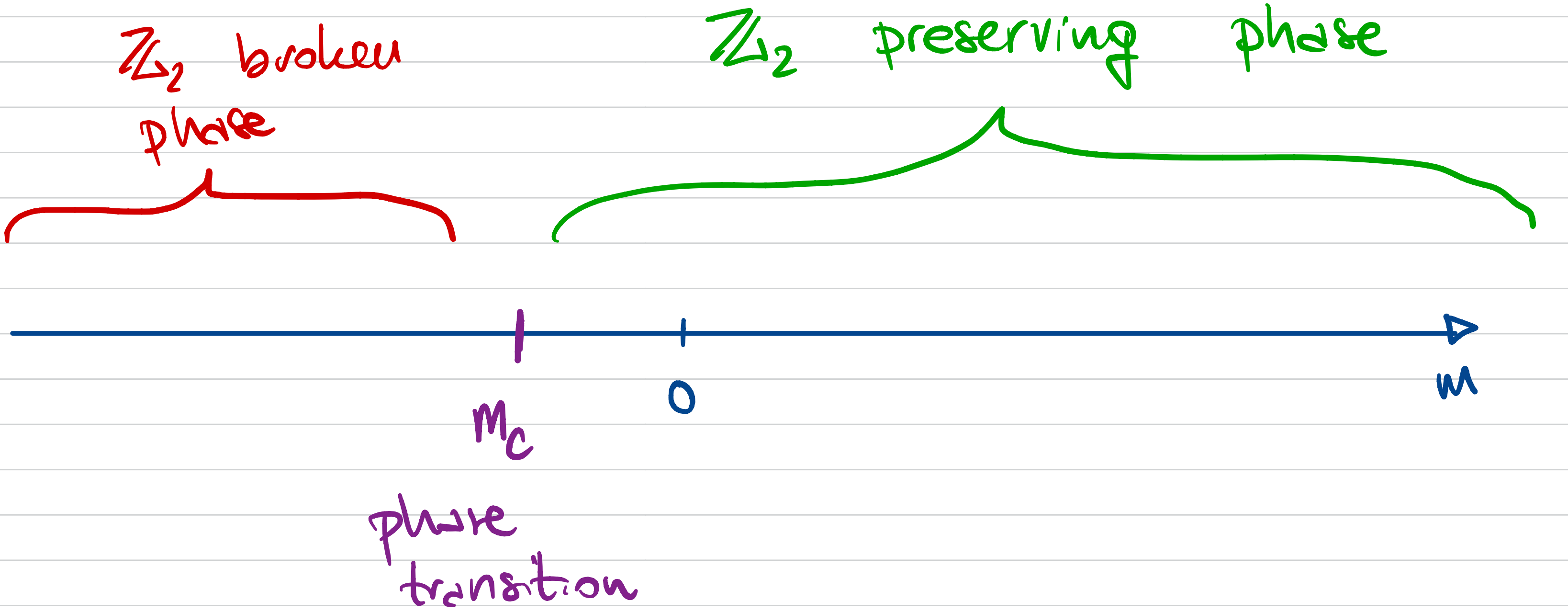
PHASE DIAGRAM

$m \gg e$: $\mathbb{Z}_2$ preserved

$m = 0$: $\mathbb{Z}_2$ preserved

$-m \gg e$: $\mathbb{Z}_2$ broken

Simplest scenario



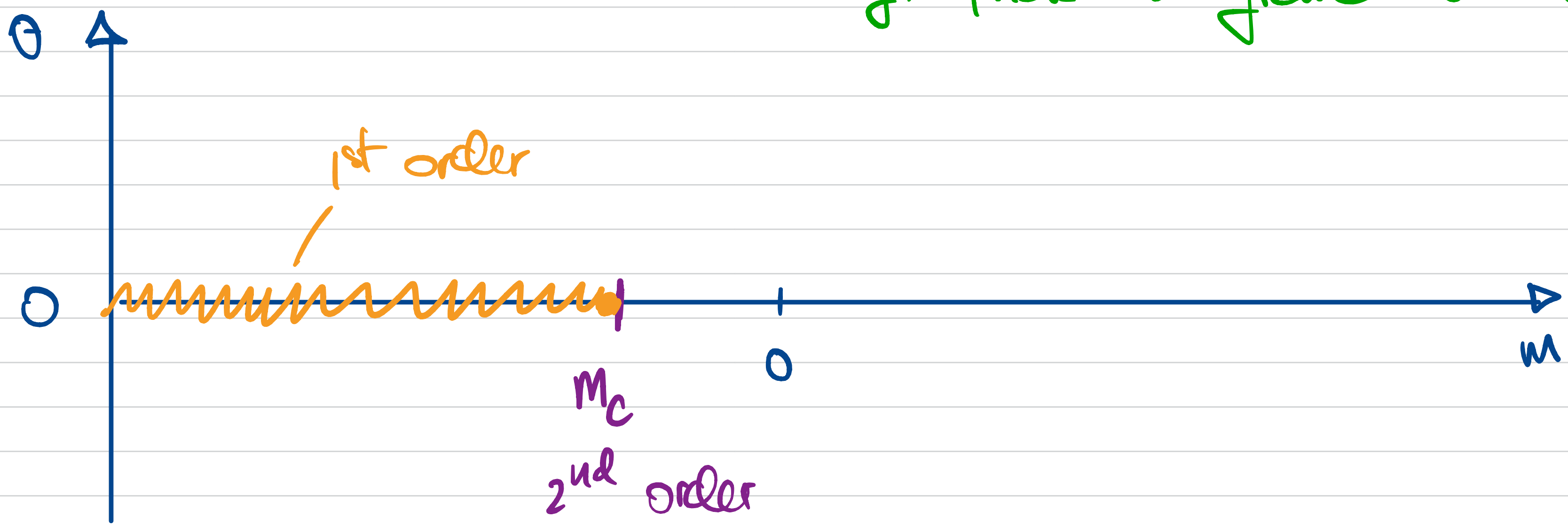
PHASE TRANSITION

Phase transition is 2nd order

* at $-m \gg e$, 1st order phase transition $\theta = 0^- \to \theta = 0^+$

Line of 1st order ph. tr. ends on 2nd order ph. tr.

e.g. phase diagram of water

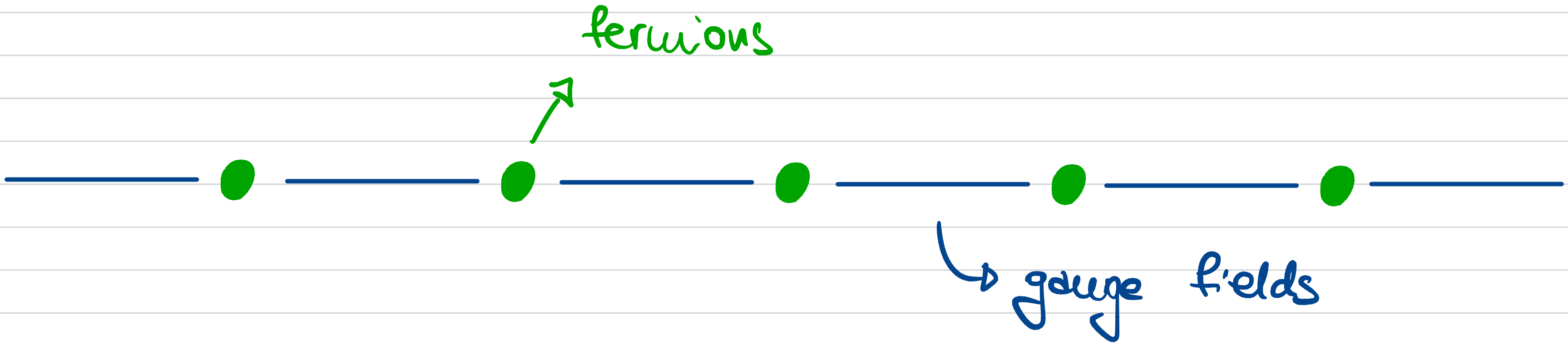


Z_2 breaking : by universality, Ising phase transition.

TOOL #3: NUMERICS

Only thing left: value of critical mass m_c!

Do numerics! Discretize theory (Lorentzian)



Fermions on the lattice are tricky: use staggered fermions

$$\begin{pmatrix} \psi_e \\ \psi_o \end{pmatrix} \qquad \psi_e \to \frac{1}{\sqrt{a}} \chi_{2n} \qquad \psi_o \to \frac{1}{\sqrt{a}} \chi_{2n+1}$$

lattice spacing

[Kogut Susskind '74]

Avoids fermion doubling!

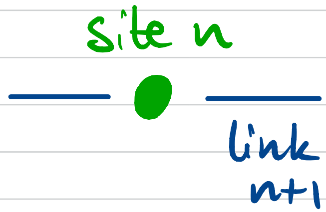
Continuum Hamiltonian ($A_0 = 0$ gauge)

$$H = \int_0^L dx \left[\frac{e^2}{2} \left(E(x) - \frac{\theta}{2\pi} \right)^2 - i \psi^\dagger \gamma^5 (i\partial_1 + i A_1(x)) \psi + m \psi^\dagger \gamma^0 \psi \right]$$

Discrete Hamiltonian

$$H = \frac{e^2 a}{2} \sum_n \left(L_n - \frac{\theta}{2\pi} \right)^2 - \frac{i}{2a} {\sum_n}' \left(\chi_n^\dagger U_{n+1} \chi_{n+1} - h.c. \right)$$

$$+ m_{lat} \sum (-1)^n \chi_n^\dagger \chi_n$$



$$\{\chi_n, \chi_m^\dagger\} = \delta_{nm} \qquad [L_n, U_m] = \delta_{nm} U_m$$

CONTINUUM LIMIT

What to do

* Diagonalize Hamiltonian
* compute properties of low lying states

  (energy gap, correlation function, entanglement entropy...)

* take thermodinamic limit ($N \to \infty$) and

  continuum limit ($a \to 0$)

  ↳ technical improvement: mass shift improves convergence for $a \to 0$ [Dempsey, Klebanov, Pufu, BZ '22]

Usual issue: Hilbert space grows exponentially in N.

TENSOR NETWORKS

Use a matrix product state (MPS) ansatz.

Idea: (ignoring gauge field)

$$|\psi\rangle = \sum_{\{n_i\}} \psi^{n_1 n_2 \dots n_N} |n_1 n_2 \dots n_N\rangle$$

$n_i = 0,1 \to$ occupation number of a site

$\psi^{n_1 \dots n_N} \to 2^N$ numbers

Approximate to

$$|\psi\rangle = \sum_{\{n_i\}} A^{n_1} \cdot A^{n_2} \cdot A^{n_3} \cdot \dots A^{n_4} |n_1 n_2 \dots n_N\rangle$$

A^{n_i}: matrix of size $< \chi \times \chi$

χ: bond dimension

Need to keep only $< 2N \cdot \chi^2$ numbers

TENSOR NETWORKS

* MPS work very well in gapped phase

  need $\chi \sim O(1)$ as $N \to \infty$

* work not as nicely in gapless phases, but still polynomially

  $\chi \sim N^{\alpha}$ as $N \to \infty$

Growth is only polynomial, not exponential!

For Schwinger model can get to $N \sim 10^3$.

Many packages, e.g. ITensor, MPSKit, ...

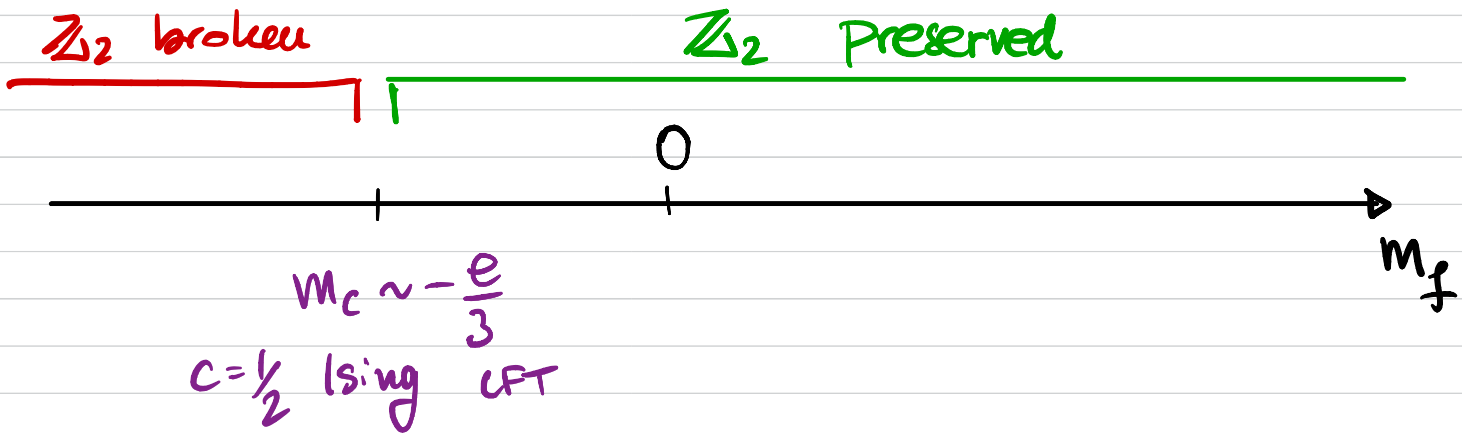
NUMERICS RESULT

Compute order parameter $\mathcal{E} = \frac{1}{N} \sum_i L_i$, $\mathbb{Z}_2 : \mathcal{E} \to -\mathcal{E}$

$\hookrightarrow \langle \mathcal{E} \rangle = 0$ $\quad \mathbb{Z}_2$ preserved

$\hookrightarrow \langle \mathcal{E} \rangle \neq 0$ $\quad \mathbb{Z}_2$ spontaneously broken

Finite size scaling of entanglement entropy: find critical mass and central charge of CFT



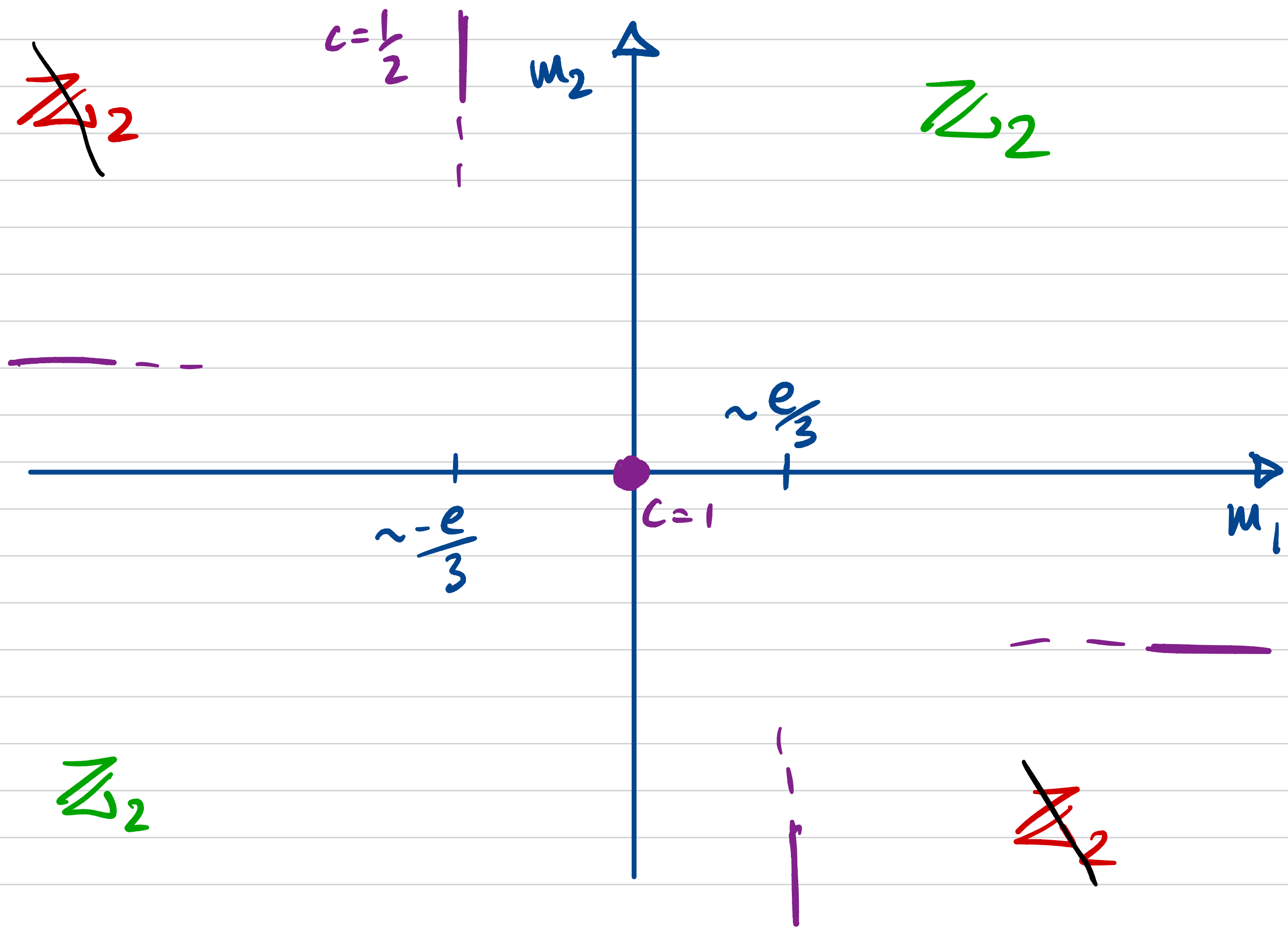
Best result yet $m_c/e = -0.333561(4)$ [Cruz, Tarnopolsky, Xin '24]

2-FLAVOR SCHWINGER MODEL

Two fermions with mass m_1, m_2. $\theta = 0$.

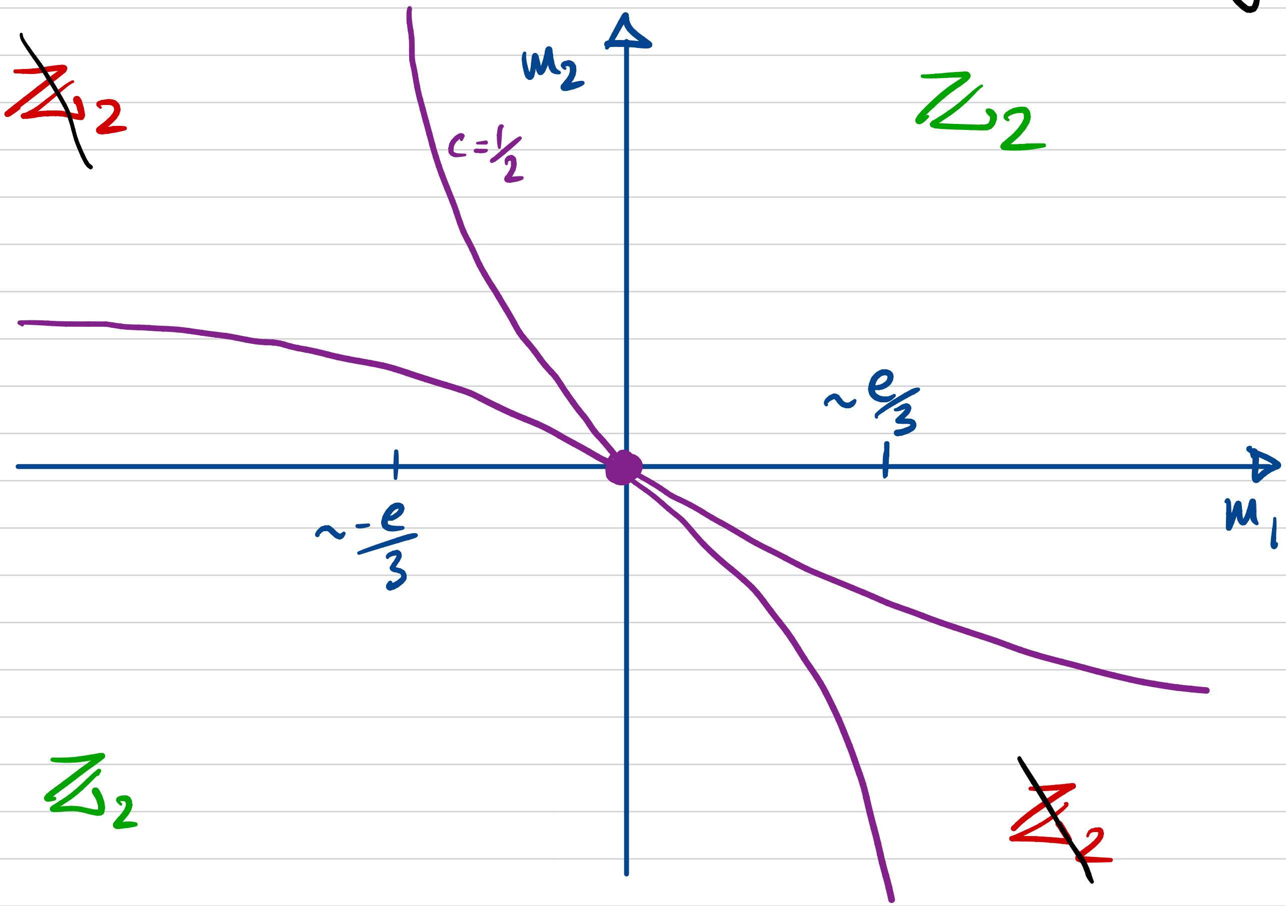
Bosonization: $m_1 = m_2 = 0 \to c = 1$ CFT $(SU(2)_1$ WZW model$)$

Semiclassics:



2-FLAVOR SCHWINGER MODEL

Numerics: understand the $m_1, m_2 \sim O(e)$ region



[Dempsey, Klebanov, Pufu, Søgaard, BZ '23]

Similar to the Dashen phase in QCD: CP broken for $m_u - m_d$ large enough.

[Dashen '71, Creutz '13]

CHIRAL THEORIES

&

SYMMETRIC MASS GENERATION

CHIRAL SYMMETRIES

Chiral symmetry: acts differently on left/right handed fermions.

E.g. $\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}$ $\quad U(1)_V : \psi \to e^{i\alpha}\psi$ $\quad \psi_\pm \to e^{i\alpha}\psi_\pm$

$U(1)_A : \psi \to e^{i\alpha\gamma_5}\psi$ $\quad \psi_\pm \to e^{\pm i\alpha}\psi_\pm$

$\gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Write charges

e.g.

$\psi_+ \to e^{iq_+\alpha}\psi_+$

	ψ_+	ψ_-
$U(1)_V$	1	1
$U(1)_A$	1	-1

CHIRAL SYMMETRIES

Chiral symmetries don't allow mass terms

$$U(1)_A : \quad m\bar{\Psi}\Psi \to m\bar{\Psi} e^{2i\alpha\gamma_5} \Psi$$

Q: can I gap out the fermions without breaking chiral symmetries?

Mass terms are not allowed, but it's possible through non-perturbative effects.

$\hookrightarrow$ symmetric mass generation

SYMMETRIC MASS GENERATION

Inherently a non-perturbative phenomenon

In 2d discussed in chiral theories

[Wang, You '22 review], ...

tried on the lattice as well

[Zeng, Zhu, Wang, You '22]

In 4d, argued to happen in some gauge theories

[Razamat, Tong '20]

observed in some lattice model

[Butt, Catterall, Toga '21]

Related to placing chiral fermions on the lattice.

Nielsen – Ninomiya theorem: on the lattice, cannot have

* locality
* unitarity
* no fermion doublers
* chiral symmetry.

SMG could be a way of gapping out the unwanted chirality of fermions, w/o breaking chiral symmetries.

CHIRAL THEORIES

Can also give different charges to fermions

e.g. $\psi_+ \to e^{iq\alpha}\psi_+$ $\quad q \in \mathbb{Z}$

Consider 2 Dirac fermions, 2 global $U(1)$'s.

	ψ_+^1	ψ_+^2	ψ_-^1	ψ_-^2
$U(1)_A$	q_L^1	q_L^2	q_R^1	q_R^2
$U(1)_B$	p_L^1	p_L^2	p_R^1	p_R^2

Global symmetries can have 't Hooft anomalies that prevent us from gauging them.

ANOMALIES

	ψ_+^1	ψ_+^2	ψ_-^1	ψ_-^2
$U(1)_A$	q_L^1	q_L^2	q_R^1	q_R^2
$U(1)_B$	p_L^1	p_L^2	p_R^1	p_R^2

* No $U(1)_A$ anomaly if

$$(q_L^1)^2 + (q_L^2)^2 = (q_R^1)^2 + (q_R^2)^2$$

* No mixed anomaly if

$$q_L^1 p_L^1 + q_L^2 p_L^2 = q_R^1 p_R^1 + q_R^2 p_R^2$$

→ No $U(1)_A$, $U(1)_B$ anomaly, but mixed anomaly:

Can gauge $U(1)_A \to U(1)_B$ breaks (ABJ anomaly)

Cannot gauge both

→ No $U(1)_{A,B}$ anomaly, no mixed anomaly

Can gauge both!

3450 MODEL

	ψ_+^1	ψ_+^2	ψ_-^1	ψ_-^2
$U(1)_A$	3	4	5	0
$U(1)_B$	2	1	2	1

All anomalies vanish.

Cannot add mass term! No relevant singlet operator. Singlets are marginal or irrelevant.

Can I gap out the theory? Yes

(chiral theory: very hard to do numerics).

Consider a larger theory [Moulaud, Tong, Zou '25]

	ψ_+^1	ψ_+^2	ψ_-^1	ψ_-^2	ϕ_1	ϕ_2
gauged G_1	3	4	5	0	1	0
gauged G_2	2	1	2	1	0	1
global $\hat{G}_1$	0	0	0	0	−1	0
global $\hat{G}_2$	0	0	0	0	0	−1

scalars

$$\mathcal{L} \supset \sum_i |\partial_\mu \phi_i|^2 + m_i^2 |\phi_i|^2 + \lambda |\phi_i|^4$$

Scalar mass term is a singlet, can tune it.

$m_i^2 \to \pm\infty$

$m_i^2 \to \infty$: integrate out ϕ_i, use bosonization

Gapped theory with one ground state

$m_i^2 \to -\infty$: Higgs phase, use bosonization again.

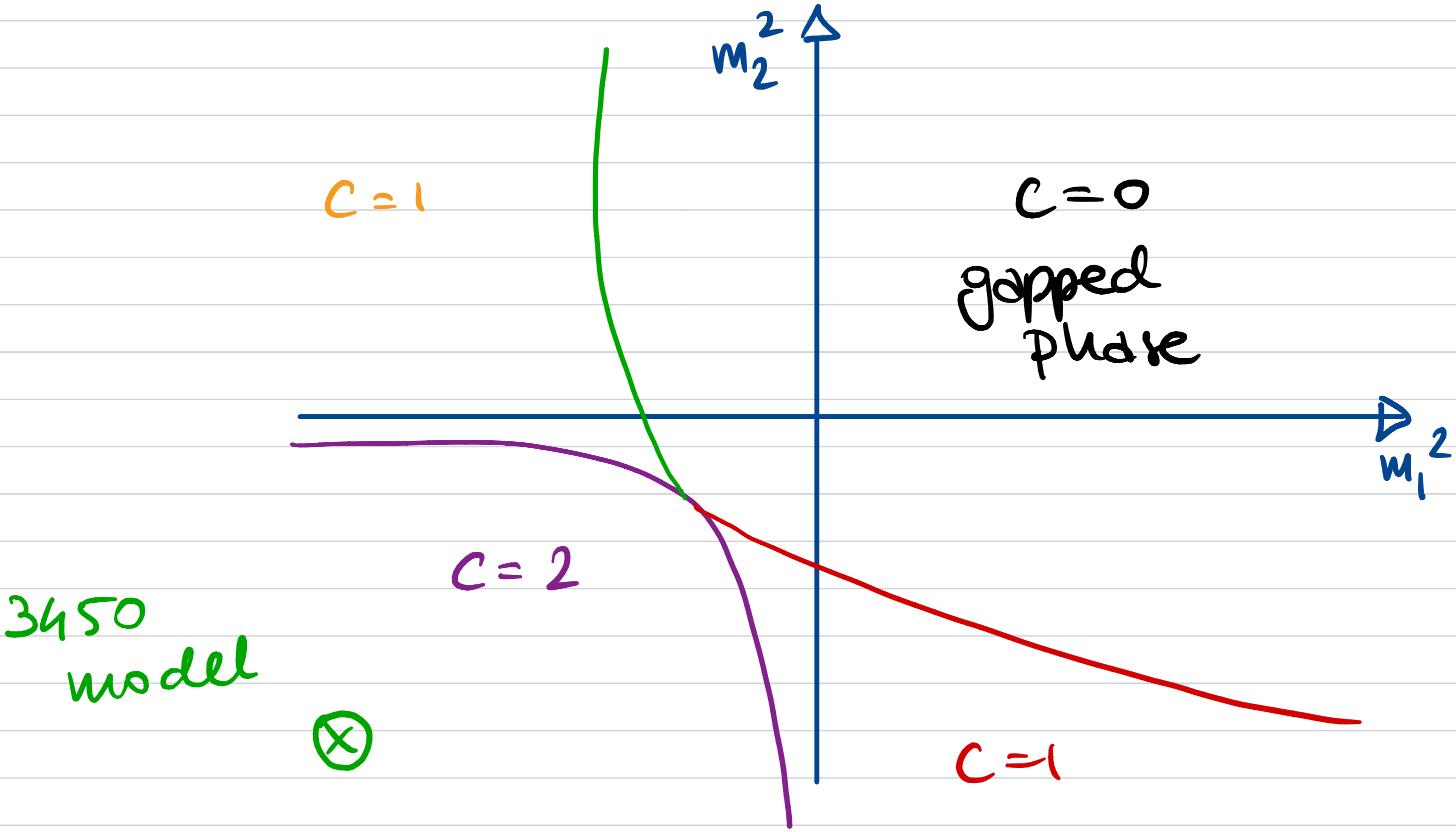
Get a theory of free fermions,

Twisted global symmetries survive in the IR

		$\tilde{\Psi}_+^1$	$\tilde{\Psi}_+^2$	$\tilde{\Psi}_-^1$	$\tilde{\Psi}_-^2$
global	$\tilde{G}_1$	3	4	5	0
	$\tilde{G}_2$	2	1	2	1

PHASE DIAGRAM

Obtain $|m_i| \to \infty$ behavior, fill in the lines



3450 model + marginal deformation: 6-fermion term eventually becomes relevant, gaps the theory out

SMG!

TO SUM UP

- 2d gauge theories important for high energy, condensed matter and many-body physics

- through semiclassics, bosonization and numerics
  → phase diagram of abelian gauge theories

- 3450 model: chiral symmetries prevent mass terms, but can gap the theory out through symmetric mass generation

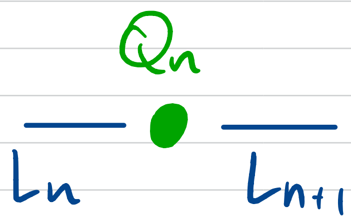
BACKUP SLIDES

MASS SHIFT

Gauss law: ensures gauge invariance

$$L_n - L_{n-1} = Q_n$$

↳ charge of site n



Subtleties when defining Q_n

→ identification between continuum and lattice mass is non-trivial. With common choice for Gauss law

$$m_{lat} = m - \underbrace{\frac{e^2 a}{8}}_{\text{mass shift}}$$

[Dempsey, Klebanov, Pufu, BZ '22]

MASS SHIFT

Conceptually, mass shift preserves a discrete remnant of the chiral anomaly at the massless point

$\pi/2$ chiral rotation : $H(m=0, \theta) \to H(m=0, \theta+\pi)$
(1-site lattice translation)

Practically: mass shift improves convergence to the continuum limit $(a \to 0)$. Better numerics.