UNIVERSITÀ DI SIENA 1240

DIPARTIMENTO DI SCIENZE FISICHE, DELLA TERRA E DELL'AMBIENTE — DSFTA

# Entanglement Properties of Quantum Complex Networks


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February 27, 2026

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# Defining a Complex Network

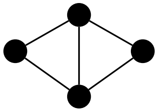
A simple graph is defined as $G = (V, L)$, where

$$V \text{ is the set of vertices,} \qquad |V| = M,$$

$$L \subseteq \{(a, b) \mid a, b \in V,\ a \neq b\} \text{ is the set of edges.}$$

The graph is completely specified by its adjacency matrix $A$, whose entries are:

$$A_{ij} = \begin{cases} 1 & \text{if } (i, j) \in L, \\ 0 & \text{otherwise.} \end{cases} \quad \text{E.g.,} \quad A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \implies$$



The local influence of a vertex $i$ is measured by its degree $d_i = \sum_j A_{ij}$.

More generally, centrality measures can be constructed from powers of $A$, e.g. Katz centrality.

# Statistical–Mechanical Description of Graphs

Once the adjacency matrix $A$ is specified, a graph is completely determined. For fixed $M$, the number of possible simple graphs is:

$$|\mathcal{G}_M| = 2^{\frac{M(M-1)}{2}},$$

since each pair of vertices can either be connected or not.

Therefore, it is natural to **describe graphs statistically**, by introducing a probability measure $P(G)$ over the configuration space $\mathcal{G}_M$.

Following the maximum entropy principle, we maximise Park and Newman (2004):

$$S = -\sum_G P(G) \log P(G),$$

subject to constraints on the observables of the graph $x_i(G)$.

This yields the canonical form:

$$P(G) = \frac{e^{-H(G)}}{Z}, \qquad H(G) = \sum_i \theta_i x_i(G).$$

# Erdős–Rényi Ensemble

Consider the constraint on the expected number of edges:
$x(G) = |L(G)| \equiv \sum_{i<j} A_{ij}(G)$.

Choosing the Hamiltonian $H(G) = \lambda\, x(G)$, the canonical ensemble becomes

$$P(G) = p^{x(G)}(1-p)^{\binom{M}{2}-x(G)}, \quad p \equiv \frac{1}{e^{\lambda}+1},$$

that is, a Bernoulli distribution with $p$ interpreted as the probability of linking.

Basic properties:

$$\langle x \rangle = p\binom{M}{2}, \qquad \langle k \rangle = p(M-1).$$

Let $S_k$ denote the size of the $k$-th largest connected component of the graph. In the sparse scaling $p = c/M$, we have

$$\begin{cases} c < 1: & S_1 \sim O(\log M), \\ c = 1: & S_1 \sim O(M^{2/3}), \\ c > 1: & S_1 \sim O(M). \end{cases}$$

# From Classical to Quantum Graphs

We now associate a Hilbert space $\mathcal{H}_i \simeq \mathbb{C}^2$ to each vertex $i \in V$.

We associate to each node a generic state $|\phi\rangle$ and to each edge $(i,j) \in L$ a two–body unitary operator $U_{ij}$.
We assume commuting pairwise interactions: Hein et al. (2006)

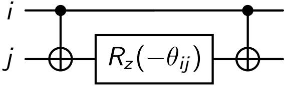
$$[U_{ij}, U_{jk}] = 0, \quad \forall i,j,k \in V. \tag{1}$$

This natural condition alone implies also that the graph is undirected, that is $U_{ij} \equiv U_{ji}$. Indeed, the condition (1) implies an Ising hamiltonian of the kind:

$$H = -\frac{1}{2}\sum_{i<j} \theta_{ij}\sigma_z^{(i)}\sigma_z^{(j)}$$

up to local $\sigma_z^{(i)}$ unitary operation. Viewed as a quantum circuit, the pairwise operator takes the form

$$U_{ij} = \Pi_0^{(i)} e^{i\frac{\theta_{ij}}{2}\sigma_z^{(j)}} + \Pi_1^{(i)} e^{-i\frac{\theta_{ij}}{2}\sigma_z^{(j)}} \quad \equiv$$



# Entanglement Distance

As an Entanglement Quantifier, we use the Entanglement Distance (ED) given by Cocchiarella et al. (2020); Vesperini et al. (2023a,b, 2024):

$$E(|G\rangle) = M - \sum_i \left[\langle G|\boldsymbol{\sigma}^{(i)}|G\rangle\right]^2, \quad \boldsymbol{\sigma}^{(i)} = (\sigma_1^{(i)}, \sigma_2^{(i)}, \sigma_3^{(i)}) \tag{2}$$

where the graph state $|G\rangle$ is given by:

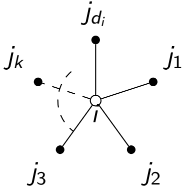
$$|G\rangle = \left[\prod_{(i,j)\in L} U_{ij}\right] |\phi\rangle^{\otimes M} .$$

From (2), we see that the contribution of the single node $i$ to the ED is $E^{(i)} = 1 - \left[\langle G|\boldsymbol{\sigma}^{(i)}|G\rangle\right]^2$. However, denoting with $\mathcal{N}(i)$ the set of vertices adjacent to the vertex $i$, the particular form of $U_{ij}$ allows us to write:

$$U^{(i)} = \prod_{j\in\mathcal{N}(i)} U_{ij} = \Pi_0^{(i)} \prod_{k=1}^{d_i} R_z^{j_k}(-\theta_{ij_k}) + \Pi_1^{(i)} \prod_{k=1}^{d_i} R_z^{j_k}(\theta_{ij_k}) \tag{3}$$

# Computing the Entanglement Distance

The operator $U^{(i)}$ in (3), hence, denotes the subgraph associated to the neighbourhood of node $i$, as a star graph:




$$A = \begin{bmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

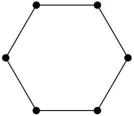
Then, denoting by $m = \langle\phi|\sigma_z|\phi\rangle$, the computation of $E^{(i)}$ with the operator in (3) yields De Simone and Franzosi (2025a,b):

$$E^{(i)} = (1 - m^2)(1 - \prod_{k=1}^{d_i} r_{ij_k}^2), \quad r_{ij_k}^2 = 1 - (1 - m^2)\sin^2\theta_{ij_k} \tag{4}$$

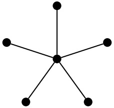
In the framework of direct linking via $A_{ij}$, the network structure enters via $\theta_{ij} \to \theta_{ij} A_{ij}$

# Topology dependence: Ring vs Star

**Ring graph** $C_M$

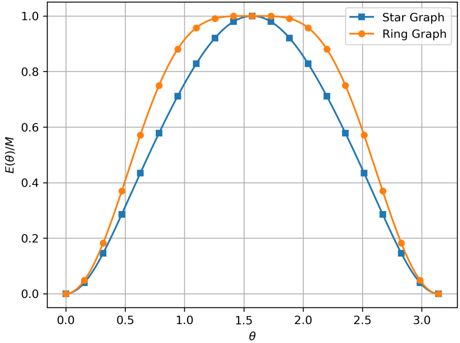


**Star Graph** $S_M$



$$E_{C_M}(\theta)/M = (1 - m^2)(1 - r^4)$$

$$E_{S_M}(\theta)/M = (1 - m^2)\left(1 - \frac{1}{6}r^2(5 + r^8)\right)$$

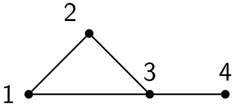


# Katz-weighted interactions

So far, interactions were purely local, $\theta_{ij} \propto A_{ij}$. Recall that:

$$(A^k)_{ij} = \text{number of walks of length } k \text{ between } i \text{ and } j.$$

*Example:* Consider the following graph




$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

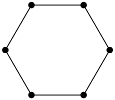
Nodes 1 and 4 are not adjacent, yet they are connected by a path. Should their interaction vanish?

Katz's idea: a node influences another if it can be reached through walks of any length. Therefore, we define an effective coupling:

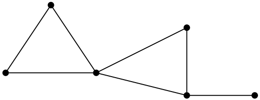
$$\theta_{ij} = \theta(A_{ij} + \sum_{k=2}^{K} \beta^{k-1} (A^k)_{ij}).$$

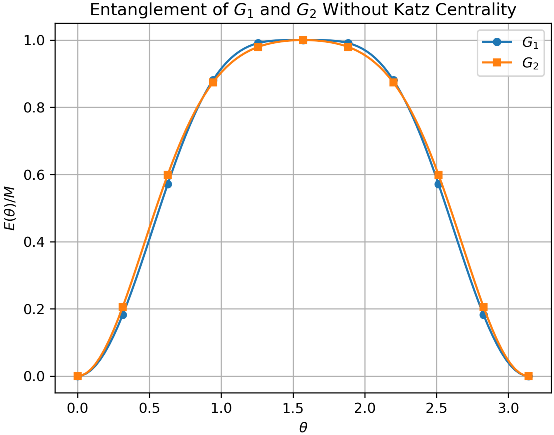
# Example **without** Katz Centrality

**Graph** $G_1$



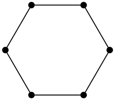
**Graph** $G_2$



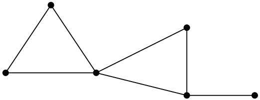


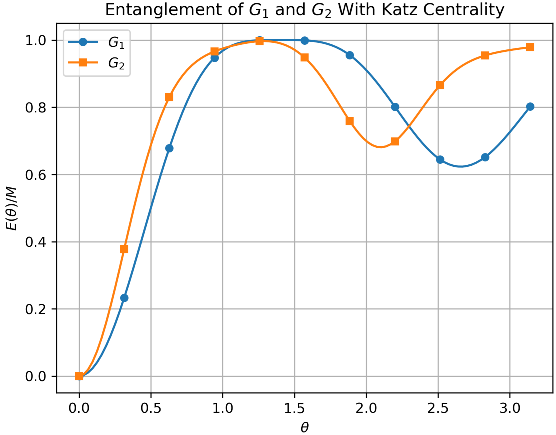
# Example **with** Katz Centrality

**Graph** $G_1$



**Graph** $G_2$





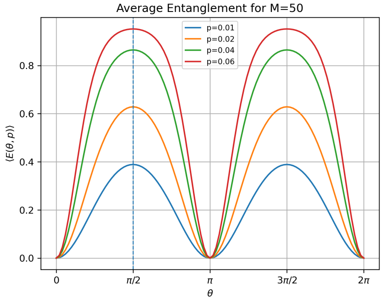
# What about Erdős–Rényi Ensemble?

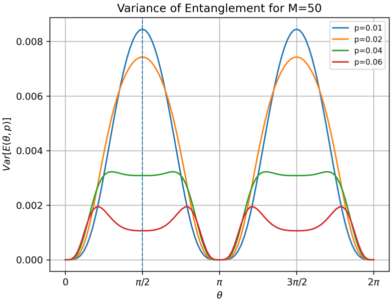
In a graph ensemble $\mathcal{G}_M$, each realization $G$ occurs with probability $\mathbb{P}(G)$. In this framework, we can calculate averaged quantities. In the simplest case $\theta_{ij} = \theta A_{ij}$ and $m = 0$, for the Erdős–Rényi ensemble mean and fluctuations of $E(G)$ read:

$$\langle E(\theta, p) \rangle = 1 - \beta^{M-1}.$$

$$\mathrm{Var}[E(\theta, p)] = \frac{1}{M}\alpha^{M-1} + \frac{M-1}{M}\alpha\beta^{2(M-2)} - \beta^{2(M-1)}$$

with $\alpha = 1 - p + pt^2$, $\beta = 1 - p + pt$ and $t = \cos^2\theta$.

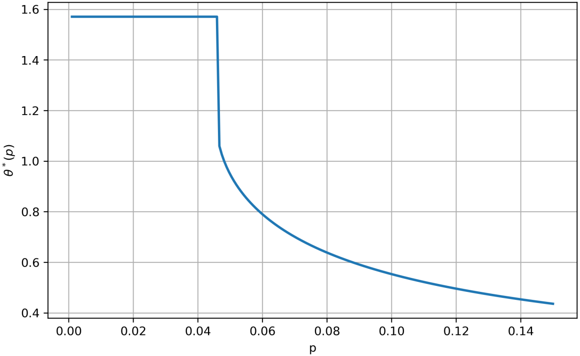


# Entanglement transition?

Curiously, the maximum in the variance, $\theta^*(p) = \arg\max_{\theta\in[0,\pi]} \mathrm{Var}[E(\theta, p)]$, changes as a function of $p$. In particular:

$$\begin{cases} \theta^* = \frac{\pi}{2} & \text{for} \quad p < p_{lim}(M) \\ \theta^* < \frac{\pi}{2} & \text{for} \quad p > p_{lim}(M) \end{cases} \qquad p_{lim} \simeq \frac{1.83}{M}$$



# Toward a full quantum graph?

Above we saw the exact hamiltonian of Erdős–Rényi model, that is $H(G) = \lambda\, m(G)$. What if we add some more term? The hamiltonian

$$H(G) = -J \sum_i d_i^2(G) - B \sum_i d_i(G)$$

is known as 2-star model Park and Newman (2004).

Erdős–Rényi ensemble arises as the mean field description of this model. What if we quantize the graph itself? Namely, by introducing basis states $|s_{ij}\rangle$ such that $A_{ij}|s_{ij}\rangle = \frac{1}{2}(1 + s_{ij})\,|s_{ij}\rangle$, with $s_{ij} = \pm 1$. Equivalently, $A_{ij} = \frac{1}{2}\left(1 + \tau_{ij}^{(z)}\right)$, with $\tau_{ij}^{(z)}$ acting as a $\sigma_z$ matrix for the link-qubit $ij$.

However, this hamiltonian is diagonal on this basis $\longrightarrow$ we introduce a transverse field acting on the link degrees of freedom

# Toward a full quantum graph?

The full hamiltonian takes the form:

$$H = -J\sum_i d_i^2 - B\sum_i d_i - \Gamma\sum_{i<j}\tau_{ij}^{(x)} - \frac{1}{2}\sum_{i<j}\theta_{ij}\sigma_z^{(i)}\sigma_z^{(j)}.$$

Now, many calculations...

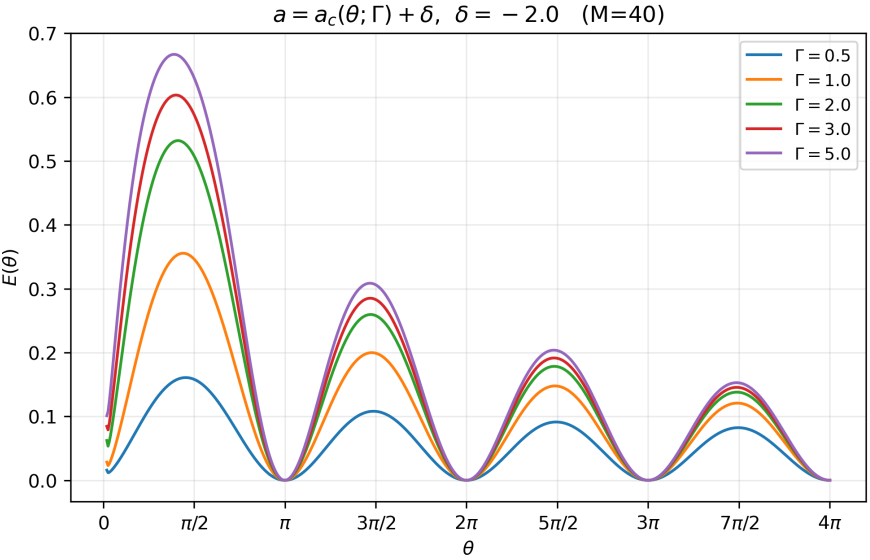
Goal: we want averaged quantities. We need the partition function.
Important steps: 1) Suzuki-Trotter, 2) Hubbard–Stratonovich $\phi$, 3) Mean-field approximation.
Recall that, in a Bernoulli approximation, the mean entanglement is still:

$$\langle E\rangle = (1-m^2)\left[1-\left(1-p+p\,r^2\right)^{M-1}\right]$$

but, now, $m = m(\Gamma,\theta,a)$ and $p = p(\Gamma,\theta,a)$ with $a \equiv J\phi + B$.

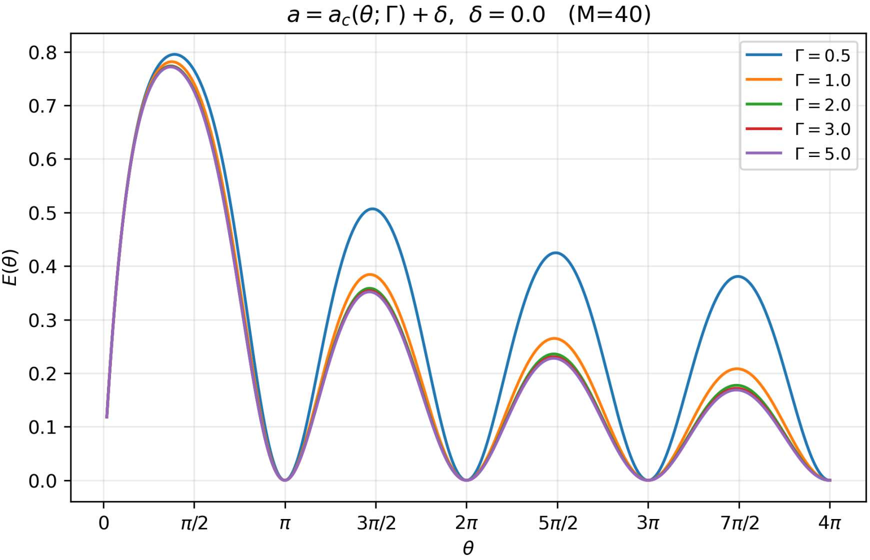
There are 3 different regimes!

# Case $a < 1$



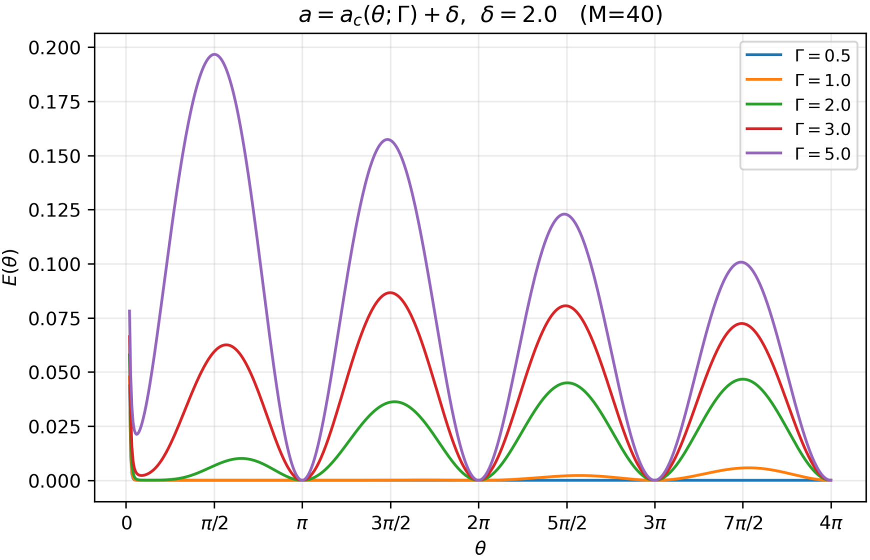
$a = a_c(\theta;\Gamma) + \delta,\ \ \delta = -2.0\quad$ (M=40)
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
$E(\theta)$
0
$\pi/2$
$\pi$
$3\pi/2$
$2\pi$
$5\pi/2$
$3\pi$
$7\pi/2$
$4\pi$
$\theta$
$\Gamma = 0.5$
$\Gamma = 1.0$
$\Gamma = 2.0$
$\Gamma = 3.0$
$\Gamma = 5.0$

# Case $a = 1$



$a = a_c(\theta; \Gamma) + \delta, \ \delta = 0.0 \quad$ (M=40)
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
$E(\theta)$
0
$\pi/2$
$\pi$
$3\pi/2$
$2\pi$
$5\pi/2$
$3\pi$
$7\pi/2$
$4\pi$
$\theta$
$\Gamma = 0.5$
$\Gamma = 1.0$
$\Gamma = 2.0$
$\Gamma = 3.0$
$\Gamma = 5.0$

# Case $a > 1$



$a = a_c(\theta;\Gamma) + \delta, \ \delta = 2.0 \quad$ (M=40)
0.200
0.175
0.150
0.125
0.100
0.075
0.050
0.025
0.000
$E(\theta)$
0
$\pi/2$
$\pi$
$3\pi/2$
$2\pi$
$5\pi/2$
$3\pi$
$7\pi/2$
$4\pi$
$\theta$
$\Gamma = 0.5$
$\Gamma = 1.0$
$\Gamma = 2.0$
$\Gamma = 3.0$
$\Gamma = 5.0$

"Always more questions than answers, there are." - Yoda

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# Thank you for your attention!

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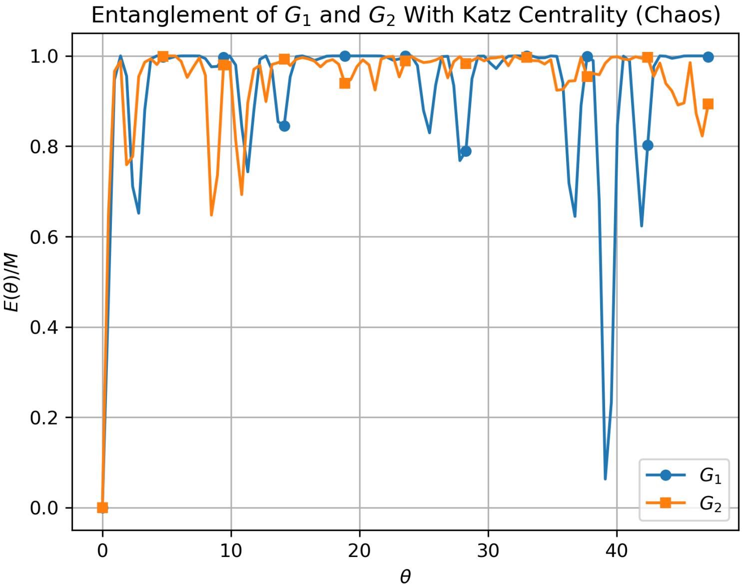
# Chaos



Entanglement of $G_1$ and $G_2$ With Katz Centrality (Chaos)
1.0
0.8
0.6
0.4
0.2
0.0
$E(\theta)/M$
0
10
20
30
40
$\theta$
$G_1$
$G_2$