Beyond the Factorization Scheme in Weak Nuclear Transitions

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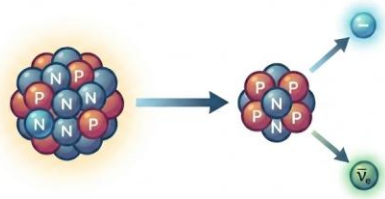
Theory of beta decay



$$A(Z,N) \rightarrow A(Z+1,N-1) + e^- + \bar{\nu}_e$$

- Fermi's Golden Rule:

$$\lambda = \frac{2\pi}{\hbar} \left|\langle f|H_{int}|i\rangle\right|^2 \frac{dn}{dE}(e,\bar{\nu}_e)$$

- Matrix element:

$$|M_{fi}|^2 \equiv |\langle f|H_{\text{int}}|i\rangle|^2 = \left|\int \psi_D^*(\mathbf{r})\psi_{\bar{\nu}_e}^*(\mathbf{r})\psi_e^*(\mathbf{r})H_{\text{int}}\psi_P(\mathbf{r})d\mathbf{r}\right|^2$$ [1]

Anti-neutrino

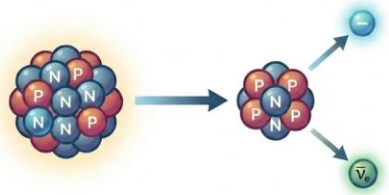
Electron

Daughter nucleus

Parent nucleus

[1] *K. Heyde,* Basic Ideas and Concepts in Nuclear Physics: An Introductory Approach, Second Edition

Theory of beta decay: Factorization



$$|M_{fi}|^2 = \left| \int \psi_D^*(\mathbf{r}) \psi_{\bar{\nu}_e}^*(\mathbf{r}) \psi_e^*(\mathbf{r}) H_{\text{int}} \psi_P(\mathbf{r}) d\mathbf{r} \right|^2$$

What about the leptons? We don't have the exact wave functions.

Electron: Only a plane wave would be too crude. We use the Fermi's function to account for the Coulomb's distortion.

$$|\psi_e(\mathbf{r_0})|^2 \simeq \frac{1}{V} F(Z, p_e)$$

$$F(Z, p_e) = |\psi_e(\mathbf{r_0})|^2_{\text{dist}} / |\psi_e(\mathbf{r_0})|^2_{\text{free}}$$

Anti-neutrino: Plane Wave

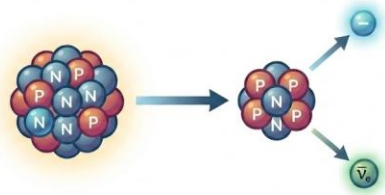
$$\psi_{\bar{\nu}_e}(\mathbf{r}) \simeq \frac{1}{\sqrt{V}} e^{i\mathbf{p}_{\bar{\nu}_e} \cdot \mathbf{r}}$$

Under the assumption they are almost constant over the volume of the nucleus

$$|M_{fi}|^2 = |\psi_{\bar{\nu}_e}^* \psi_e^*|^2 \left| \int \psi_D^*(\mathbf{r}) H_{\text{int}} \psi_P(\mathbf{r}) d\mathbf{r} \right|^2$$

Factorized!

Non-factorized scheme



What if we had the exact wave functions for the electron? No need to factorize anymore!

Let's introduce an effective Hamiltonian for beta decays:

$$H_{\text{eff}} = \frac{G_F V_{ud}}{\sqrt{2}} \int d\mathbf{x} \left[\bar{\ell}(\mathbf{x}) \gamma^\mu (1-\gamma^5) \nu_\ell(\mathbf{x}) J_\mu(\mathbf{x}) + \bar{\nu}_\ell(\mathbf{x}) \gamma^\mu (1-\gamma^5) \ell(\mathbf{x}) J_\mu^\dagger(\mathbf{x}) \right]$$ [2]

$$J_\mu(\mathbf{x}) = V_\mu(\mathbf{x}) - A_\mu(\mathbf{x})$$ Hadron current

where the transition element, for the beta – decay, becomes:

$$\left\langle e^-(p_e) \bar{\nu}_e(p_\nu) f \right| H_{\text{eff}} \left| i \right\rangle = \frac{G_F V_{ud}}{\sqrt{2}} \int d\mathbf{x} \bar{\psi}^{(-)}_{e^-, p_e, s_e}(\mathbf{x}) \gamma^\mu (1-\gamma_5) v_{s_\nu}(\mathbf{p}_\nu) e^{-i\mathbf{p}_\nu \cdot \mathbf{x}} \left\langle f \right| J_\mu(\mathbf{x}) \left| i \right\rangle$$ [2]

Anti-neutrino

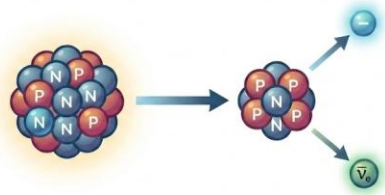
Parent nucleus

Electron

Daughter nucleus

[2] Wataru Horiuchi, Toru Sato, Yuichi Uesaka, Kenichi Yoshida, Electron wave functions in beta-decay formulas revisited (I): Gamow–Teller and spin-dipole contributions to allowed and first-forbidden transitions, *Progress of Theoretical and Experimental Physics*, Volume 2021, Issue 10, October 2021, 103D03, https://doi.org/10.1093/ptep/ptab069

Multipole expansions for leptons



$$\left\langle e^-(p_e)\bar{\nu}_e(p_\nu)f\right|H_{\text{eff}}\left|i\right\rangle = \frac{G_F V_{ud}}{\sqrt{2}}\int d\mathbf{x}\bar{\psi}^{(-)}_{e^-,p_e,s_e}(\mathbf{x})\gamma^\mu(1-\gamma_5)v_{s_\nu}(\mathbf{p}_\nu)e^{-i\mathbf{p}_\nu\cdot\mathbf{x}}\left\langle f\right|J_\mu(\mathbf{x})\left|i\right\rangle_{[2]}$$

Anti-neutrino: Plane wave

$$v_{s_\nu}(\mathbf{p}_\nu)e^{-i\mathbf{p}_\nu\cdot\mathbf{r}} = \sum_{\kappa_\nu,m_\nu,\mu_\nu}\frac{4\pi}{\sqrt{2}}i^{-\ell_{\kappa_\nu}}Y^*_{\ell_{\kappa_\nu}m_\nu}(\hat{p}_\nu)(\ell_{\kappa_\nu},m_\nu,1/2,-s_\nu|j_{\kappa_\nu},\mu_\nu)(-1)^{1/2-s_\nu}\begin{pmatrix}-if_{\kappa_\nu}(r)\chi^{\mu_\nu}_{-\kappa_\nu}\\ g_{\kappa_\nu}(r)\chi^{\mu_\nu}_{\kappa_\nu}\end{pmatrix}_{[2]}$$

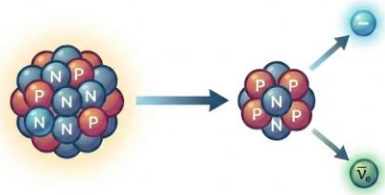
Spherical Bessel functions

Electron: Scattering function

$$\psi^{(-)}_{e,p_e,s_e}(\mathbf{x}) = \sum_{\kappa_e,m_e,\mu_e}(4\pi)i^{\ell_{\kappa_e}}(\ell_{\kappa_e},m_e,1/2,s_e|j_{\kappa_e},\mu_e)Y^*_{\ell_{\kappa_e},m_e}(\hat{p}_e)e^{-i\Delta_{\kappa_e}}\begin{pmatrix}G_{\kappa_e}(r)\chi^{\mu_e}_{\kappa_e}\\ iF_{\kappa_e}(r)\chi^{\mu_e}_{-\kappa_e}\end{pmatrix}_{[2]}$$

Exact solutions of the Dirac equation

Transition density



Putting everything together we obtain:

$$H_{\text{eff}} = \frac{G_F V_{ud}}{\sqrt{2}} \sqrt{\frac{(4\pi)^3}{2}} \sum_{el} \sum_{neu} \sum_{L,J} (j_e, -\mu_e, j_\nu, \mu_\nu | J, M)(-1)^{1/2-\mu_e} \Xi_{JLM}(\kappa_e, \kappa_\nu) \quad [2]$$

$$S_{KLJ}(\kappa', \kappa) = \sqrt{2(2j_\kappa + 1)(2j_{\kappa'} + 1)(2l_\kappa + 1)(2l_{\kappa'} + 1)(2K + 1)} \times (l_\kappa, 0, l_{\kappa'}, 0 | L, 0) \begin{Bmatrix} l_{\kappa'} & 1/2 & j_{\kappa'} \\ l_\kappa & 1/2 & j_\kappa \\ L & K & J \end{Bmatrix} \quad [2]$$

$$\Xi_{JLM}(\kappa_e, \kappa_\nu) = S_{\kappa_e} \int d\boldsymbol{r} \Bigg\{$$

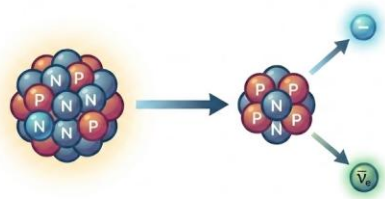
Leptonic part

Transition density (Nuclear part): parametrized or theoretically calculated

$$\Bigg\{ \begin{aligned}
&- i\, Y_{LM}(\hat{r})\, V_0 \delta_{L,J} \left(G_{\kappa_e} f_{\kappa_\nu}\, S_{0JJ}(\kappa_e, -\kappa_\nu) - F_{\kappa_e} g_{\kappa_\nu}\, S_{0JJ}(-\kappa_e, \kappa_\nu) \right) \\
&+ \left[Y_L(\hat{r}) \otimes V_i \right]_{JM} \left(G_{\kappa_e} g_{\kappa_\nu}\, S_{1LJ}(\kappa_e, \kappa_\nu) + F_{\kappa_e} f_{\kappa_\nu}\, S_{1LJ}(-\kappa_e, -\kappa_\nu) \right) \\
&+ i\, Y_{LM}(\hat{r})\, A_0 \delta_{L,J} \left(G_{\kappa_e} f_{\kappa_\nu}\, S_{0JJ}(\kappa_e, -\kappa_\nu) - F_{\kappa_e} g_{\kappa_\nu}\, S_{0JJ}(-\kappa_e, \kappa_\nu) \right) \\
&- \left[Y_L(\hat{r}) \otimes A_i \right]_{JM} \left(G_{\kappa_e} g_{\kappa_\nu}\, S_{1LJ}(\kappa_e, \kappa_\nu) + F_{\kappa_e} f_{\kappa_\nu}\, S_{1LJ}(-\kappa_e, -\kappa_\nu) \right)
\end{aligned} \Bigg\}$$

- Vectorial temporal
- Vectorial spatial
- Axial temporal
- Axial spatial

Transition density and rate



$$\Xi_{JLM}(\kappa_e,\kappa_\nu) = S_{\kappa_e}\int d\boldsymbol{r}\Big\{$$

$$- \; i\,Y_{LM}(\hat{r})\,V_0\delta_{L,J}\left(G_{\kappa_e}f_{\kappa_\nu}\,S_{0JJ}(\kappa_e,-\kappa_\nu) - F_{\kappa_e}g_{\kappa_\nu}\,S_{0JJ}(-\kappa_e,\kappa_\nu)\right) \longrightarrow \text{Vectorial temporal}$$

$$+ \; \left[Y_L(\hat{r})\otimes V_i\right]_{JM}\left(G_{\kappa_e}g_{\kappa_\nu}\,S_{1LJ}(\kappa_e,\kappa_\nu) + F_{\kappa_e}f_{\kappa_\nu}\,S_{1LJ}(-\kappa_e,-\kappa_\nu)\right) \longrightarrow \text{Vectorial spatial}$$

$$+ \; i\,Y_{LM}(\hat{r})\,A_0\delta_{L,J}\left(G_{\kappa_e}f_{\kappa_\nu}\,S_{0JJ}(\kappa_e,-\kappa_\nu) - F_{\kappa_e}g_{\kappa_\nu}\,S_{0JJ}(-\kappa_e,\kappa_\nu)\right) \longrightarrow \text{Axial temporal}$$

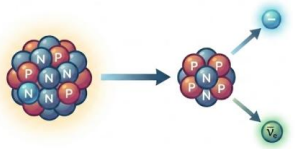
$$- \; \left[Y_L(\hat{r})\otimes A_i\right]_{JM}\left(G_{\kappa_e}g_{\kappa_\nu}\,S_{1LJ}(\kappa_e,\kappa_\nu) + F_{\kappa_e}f_{\kappa_\nu}\,S_{1LJ}(-\kappa_e,-\kappa_\nu)\right)\Big\} \longrightarrow \text{Axial spatial}$$

Using this we get the decay rate:

$$\Gamma = \frac{(G_F V_{ud})^2}{\pi^2}\int_{m_e}^{E_0} dE_e p_e E_e (E_0 - E_e)^2 \sum_{J,L,\kappa_e,\kappa_\nu}\frac{1}{2J_i+1}\left|\left\langle f\right\|\Xi_{JL}(\kappa_e,\kappa_\nu)\left\|i\right\rangle\right|^2$$

[2]

Neutrino Absorption and Electron Capture



We can repeat the same process with the Neutrino Absorption (NA) and Electron Capture (EC) .

$$\nu_e + A(Z,N) \rightarrow A(Z+1,N-1) + e^-$$

$$e_b^- + A(Z,N) \rightarrow A(Z-1,N+1) + \nu_e$$

$$
\begin{aligned}
&-i\,Y_{LM}(\hat{r})\,V_0\delta_{L,J}\left(G_{\kappa_e} f_{\kappa_\nu} S_{0JJ}(\kappa_e,-\kappa_\nu) + F_{\kappa_e} g_{\kappa_\nu} S_{0JJ}(-\kappa_e,\kappa_\nu)\right)\\
&-\left[Y_L(\hat{r})\otimes V_i\right]_{JM}\left(G_{\kappa_e} g_{\kappa_\nu} S_{1LJ}(\kappa_e,\kappa_\nu) - F_{\kappa_e} f_{\kappa_\nu} S_{1LJ}(-\kappa_e,-\kappa_\nu)\right)\\
&+i\,Y_{LM}(\hat{r})\,A_0\delta_{L,J}\left(G_{\kappa_e} f_{\kappa_\nu} S_{0JJ}(\kappa_e,-\kappa_\nu) + F_{\kappa_e} g_{\kappa_\nu} S_{0JJ}(-\kappa_e,\kappa_\nu)\right)\\
&+\left[Y_L(\hat{r})\otimes A_i\right]_{JM}\left(G_{\kappa_e} g_{\kappa_\nu} S_{1LJ}(\kappa_e,\kappa_\nu) - F_{\kappa_e} f_{\kappa_\nu} S_{1LJ}(-\kappa_e,-\kappa_\nu)\right)
\end{aligned}
$$

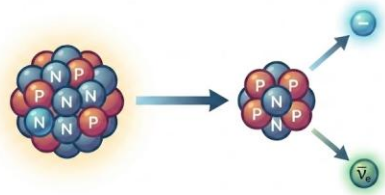
$$
\begin{aligned}
&-i\,Y_{LM}(\hat{r})\,V_0\delta_{L,J}\left(G_{\kappa_x} f_{\kappa_\nu} S_{0JJ}(-\kappa_\nu,\kappa_x) + F_{\kappa_x} g_{\kappa_\nu} S_{0JJ}(\kappa_\nu,-\kappa_x)\right)\\
&-\left[Y_L(\hat{r})\otimes V_i\right]_{JM}\left(G_{\kappa_x} g_{\kappa_\nu} S_{1LJ}(\kappa_\nu,\kappa_x) - F_{\kappa_x} f_{\kappa_\nu} S_{1LJ}(-\kappa_\nu,-\kappa_x)\right)\\
&+i\,Y_{LM}(\hat{r})\,A_0\delta_{L,J}\left(G_{\kappa_x} f_{\kappa_\nu} S_{0JJ}(-\kappa_\nu,\kappa_x) + F_{\kappa_x} g_{\kappa_\nu} S_{0JJ}(\kappa_\nu,-\kappa_x)\right)\\
&+\left[Y_L(\hat{r})\otimes A_i\right]_{JM}\left(G_{\kappa_x} g_{\kappa_\nu} S_{1LJ}(\kappa_\nu,\kappa_x) - F_{\kappa_x} f_{\kappa_\nu} S_{1LJ}(-\kappa_\nu,-\kappa_x)\right)
\end{aligned}
$$

- The EC and NA have the same amplitude structure, since they are almost the same process under time inversion (unlike the standard beta decay).

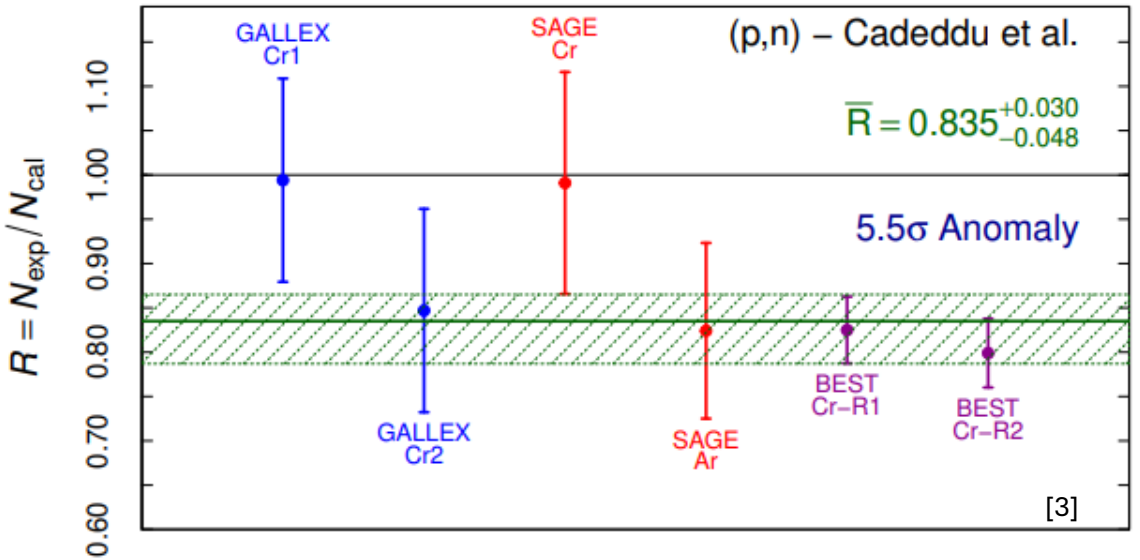
$$\nu_e + A(Z,N) \rightarrow A(Z+1,N-1) + e^- + \cancel{\bar{\nu}_e}$$

- We still need to impose the bound condition for the initial electron in the x orbital into the functions G_{κ_x} and F_{κ_x}, by solving the Dirac Equation.

Applications



- *Gallium Anomaly:* 30 years long 5σ discrepancy between the expected and observed rate for the process $\nu_e + {}^{71}Ge \rightarrow {}^{71}Ga + e^-$ [3]
- Using the appropriate transition density it's possible to resolve the anomaly without BSM physics. [4]
- Other nuclear decays processes may be revisited using this formalism.

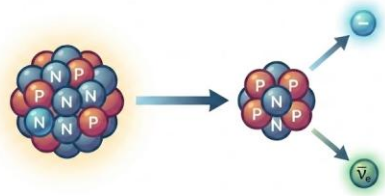


[3] Reassessing the gallium anomaly using self-consistent electron wave functions
M. Cadeddu (INFN, Cagliari), N. Cargioli (INFN, Cagliari), G. Carotenuto (Cagliari U.), F. Dordei (INFN, Cagliari), L. Ferro (INFN, Cagliari and Cagliari U.) et al.
e-Print: 2507.13103 [hep-ph]
DOI: 10.1103/pz8g-zz1b (publication)
Published in: Phys.Rev.D 113 (2026) 3, 033006

[4] **A possible solution to the gallium anomaly moving beyond the leptonic wave function factorization**
M. Cadeddu (INFN, Cagliari), N. Cargioli (INFN, Cagliari), F. Dordei (INFN, Cagliari), L. Ferro (INFN, Cagliari and Cagliari U.), C. Giunti (INFN, Turin) et al.
e-Print: 2512.20560 [hep-ph]

Thank you for your attention.

Back up



$$\sum_{el} = \sum_{\kappa_e,\mu_e} i^{-l_{\kappa e}} e^{i\Delta_{\kappa e}} (l_{\kappa_e}, m_e, 1/2, s_e | j_{\kappa_e}, \mu_e) Y_{l_{\kappa e} m_e}(\hat{p}_e)$$

$$\sum_{neu} = \sum_{\kappa_\nu,\mu_\nu} i^{-l_{\kappa\nu}} (-1)^{1/2-s_\nu}$$

$$\langle f | \Xi_{JLM}(\kappa_e, \kappa_\nu) | i \rangle = \frac{(J_i, s_i, J, M | J_f, s_f)}{\sqrt{2J_f + 1}} \langle f || \Xi_{JL}(\kappa_e, \kappa_\nu) || i \rangle$$ [2]