Exploring the internal structure of the nucleon

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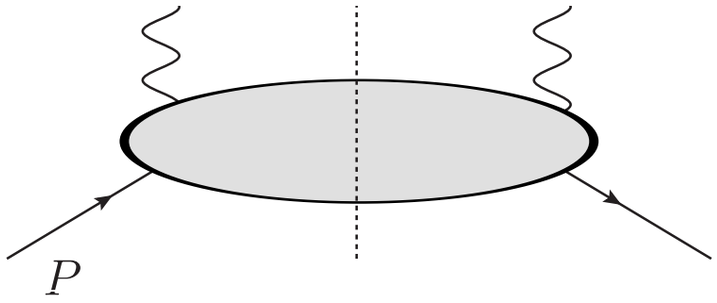
From structure functions to quark distributions

DIS cross section in terms of structure functions

$$\frac{d^3\sigma}{dx_B dy d\phi_S} = \frac{\alpha^2\, y}{2\, Q^4}\, L_{\mu\nu}(l, l', \lambda_e)\; 2MW^{\mu\nu}(q, P, S)$$

$$2MW^{\mu\nu} = \frac{1}{x}\left[-g_\perp^{\mu\nu} F_{UU,T} + \hat{t}^\mu \hat{t}^\nu F_{UU,L}\right]$$

$F_{UU,T}(x, Q^2)$



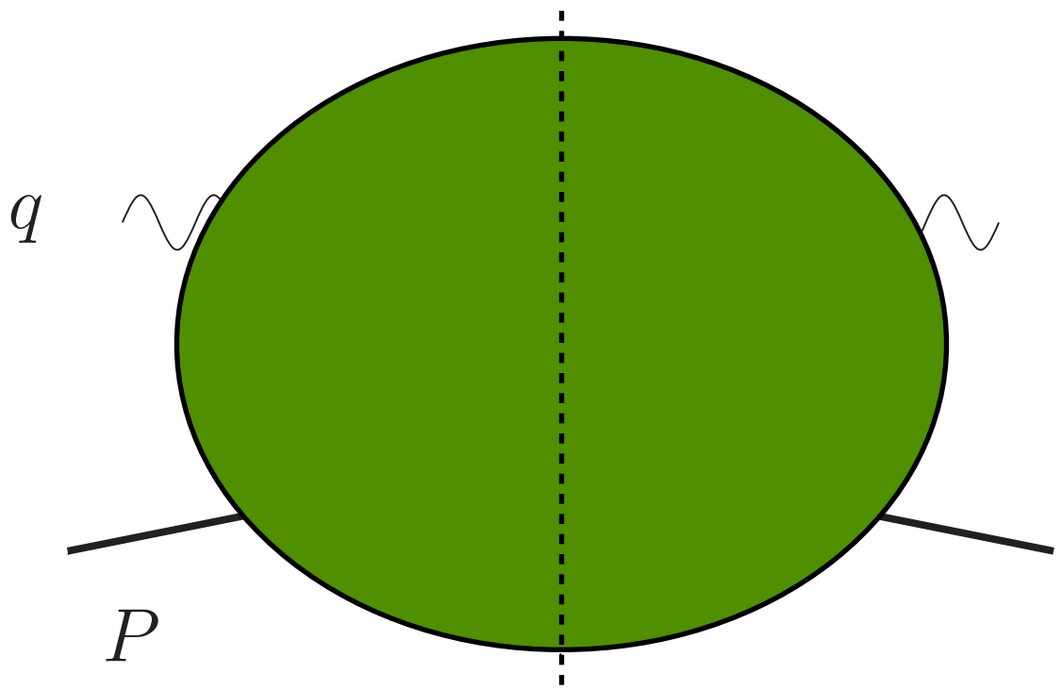
$$\frac{d\sigma}{dx_B\, dy\, d\phi_S} = \frac{2\alpha^2}{x_B y Q^2}\left\{\left(1 - y + \frac{y^2}{2}\right) F_{UU,T} + (1 - y)\, F_{UU,L}\right\}$$

see, e.g., A.B., Diehl, Goeke, Metz, Mulders, Schlegel, JHEP093 (07)

Hadronic tensor in terms of correlation function

This is the result, then we are going to motivate it

$$2MW^{\mu\nu}(q,P,S) \approx \sum_q e_q^2 \, \frac{1}{2} \mathrm{Tr}\left[\Phi(x_B,S)\,\gamma^\mu \gamma^+ \gamma^\nu\right].$$



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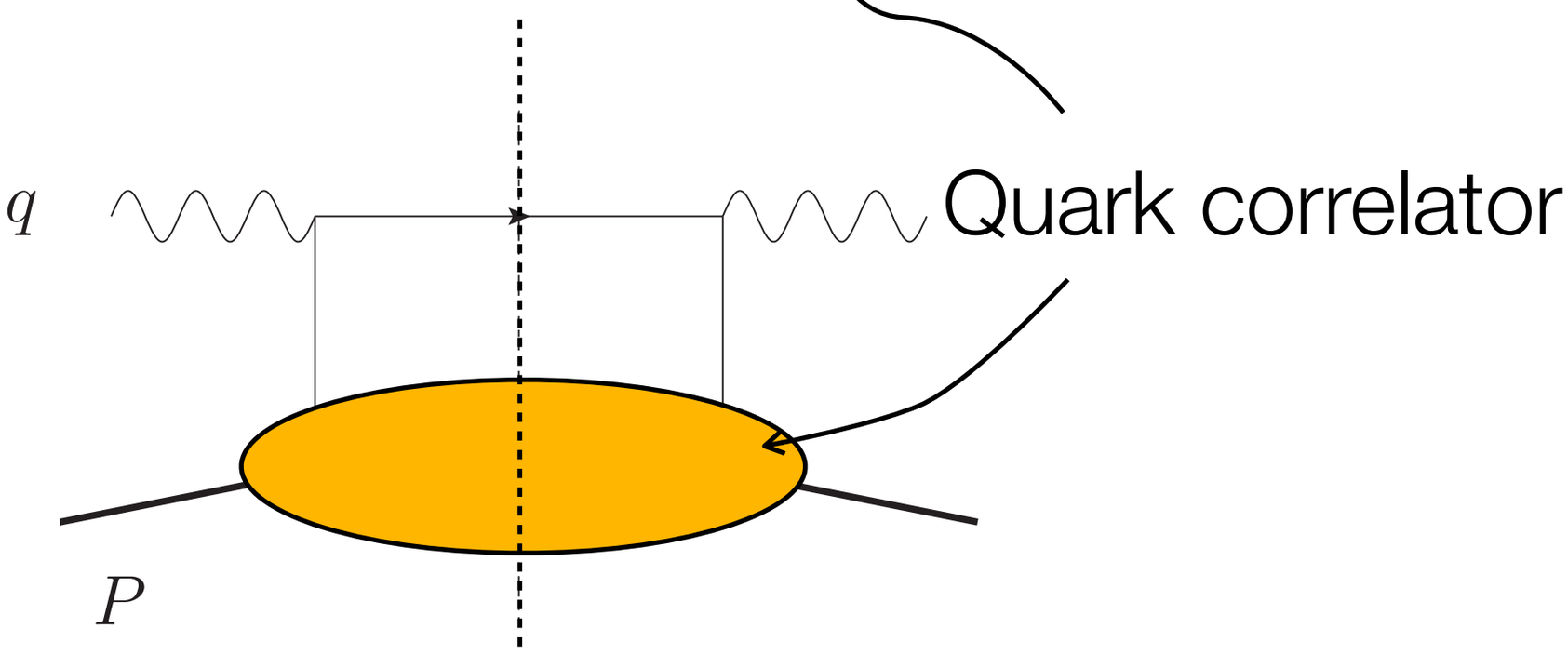
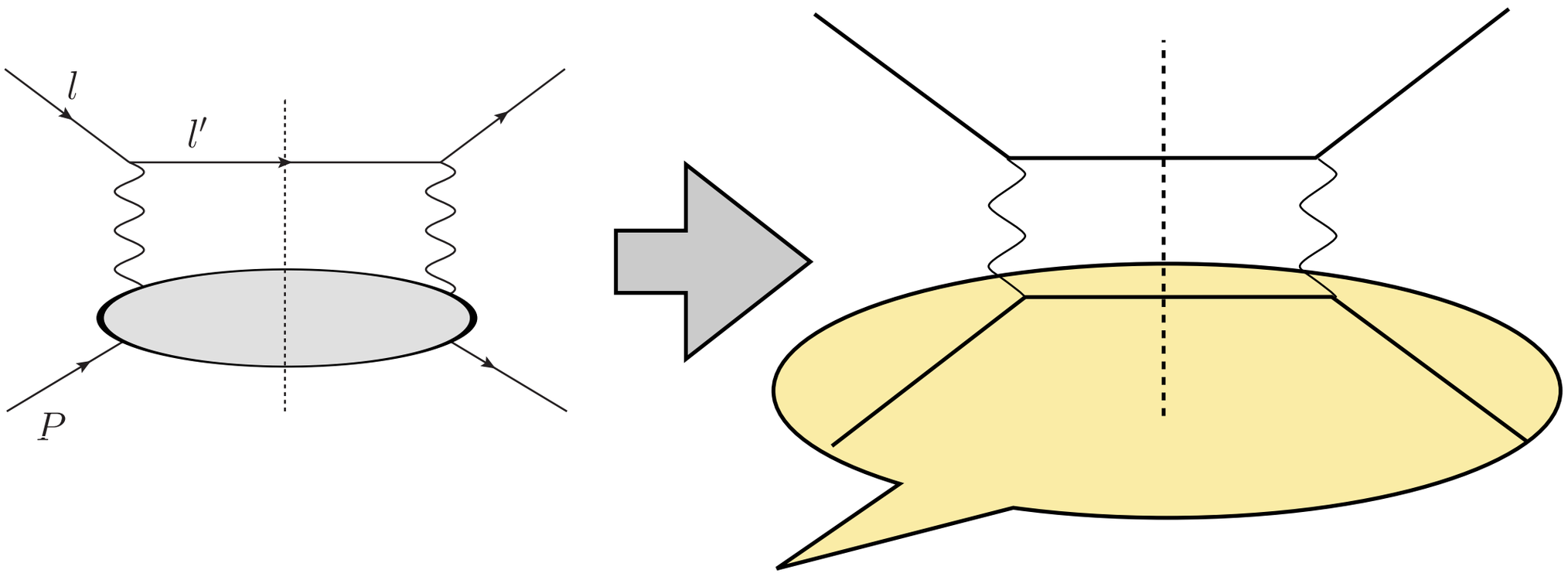


$$\Phi^q_{ji}(x,S) = \int \frac{d\xi^-}{2\pi}\, e^{-\mathrm{i}p\cdot\xi}\left\langle P,S\right|\bar\psi^q_i(\xi)\psi^q_j(0)\left|P,S\right\rangle\Big|_{\xi^+=\boldsymbol{\xi}_T=0,\ p^+=xP^+}$$

Are there more “relevant” and less “relevant” parts?

To address this problem, let’s first look at a full QED process like electron—muon scattering, where the cross section is proportional to the contraction of two leptonic tensors.



$$\mathrm{Tr}[(\not{P}' + M)\gamma^\mu(\not{P}' + \not{q} + M)\gamma^\nu] = 4P^\mu(P+q)^\nu + 4P^\nu(P+q)^\mu - 4g^{\mu\nu}P\cdot(P+q) + 4M^2 g^{\mu\nu}$$

Are there more “relevant” and less “relevant” parts?

$$\mathrm{Tr}[(\not{P}+M)\gamma^\mu(\not{P}+\not{q}+M)\gamma^\nu] = 4P^\mu(P+q)^\nu + 4P^\nu(P+q)^\mu - 4g^{\mu\nu}P\cdot(P+q) + 4M^2 g^{\mu\nu}$$

Introduce light-cone vectors n_+ and n_- so that $(n_+)^2=(n_-)^2=0$, $n_+\cdot n_- =1$.

Write P and q in terms of n_+ and n_-

$$P^\mu = P^+(n_+)^\mu + \frac{M^2}{2P^+}(n_-)^\mu$$

$$q^\mu = -x_B P^+(n_+)^\mu + \frac{Q^2}{2x_B P^+}(n_-)^\mu$$

Neglecting terms of order M^4/Q^4

$$(P+q)^\mu = (1-x_B)P^+(n_+)^\mu + \left(\frac{M^2}{2P^+} + \frac{Q^2}{2x_B P^+}\right)(n_-)^\mu$$

$$\Longrightarrow x_B = 1 - \frac{M^2}{Q^2}$$

Are there more “relevant” and less “relevant” parts?

$$\mathrm{Tr}[(\not{P}+M)\gamma^\mu(\not{P}+\not{q}+M)\gamma^\nu] = 4P^\mu(P+q)^\nu + 4P^\nu(P+q)^\mu - 4g^{\mu\nu}P\cdot(P+q) + 4M^2 g^{\mu\nu}$$

Introduce light-cone vectors n_+ and n_- so that $(n_+)^2=(n_-)^2=0$, $n_+\cdot n_- =1$.

Write P and q in terms of n_+ and n_-

$$P^\mu = P^+(n_+)^\mu$$

$$q^\mu = -x_B P^+(n_+)^\mu + \frac{Q^2}{2x_B P^+}(n_-)^\mu$$

$$(P+q)^\mu = \frac{Q^2}{2x_B P^+}(n_-)^\mu$$

$$\Longrightarrow x_B = 1$$

It is sufficient to take into consideration the + component of P and the – component of P+q

Neglecting terms of order M^2/Q^2

Light cone coordinates

Light-cone vectors will be indicated as

$$a^\mu = \left[a^-,\ a^+,\ \boldsymbol{a}_T\right] = \left[\frac{a^0 - a^3}{\sqrt{2}},\ \frac{a^0 + a^3}{\sqrt{2}},\ a^1,\ a^2\right]. \tag{1}$$

The dot-product in light-cone components is

$$a \cdot b = a^+ b^- + a^- b^+ - \boldsymbol{a}_T \cdot \boldsymbol{b}_T \tag{2}$$

The light-cone decomposition of a vector can be written in a Lorentz covariant fashion using two light-like vectors n_+ and n_- satisfying $n_\pm^2 = 0$ and $n_+ \cdot n_- = 1$ and promoting $\boldsymbol{a}_T$ to a four-vector $a_T^\mu = [0, 0, \boldsymbol{a}_T]$ so that

$$a^\mu = a^+ n_+^\mu + a^- n_-^\mu + a_T^\mu, \tag{3}$$

where

$$a^+ = a \cdot n_-, \qquad a^- = a \cdot n_+, \qquad a_T \cdot n_+ = a_T \cdot n_- = 0. \tag{4}$$

Note that

$$a_T \cdot b_T = -\boldsymbol{a}_T \cdot \boldsymbol{b}_T \tag{5}$$

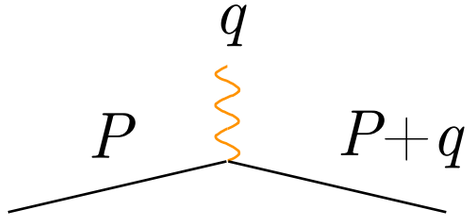
Choice of frame (not necessary, but useful)

The most symmetric choice is (Breit frame)

$$P^+ = \frac{Q}{x_B\sqrt{2}}$$

but by changing P^+ we can select any other collinear frame (e.g. proton rest frame)

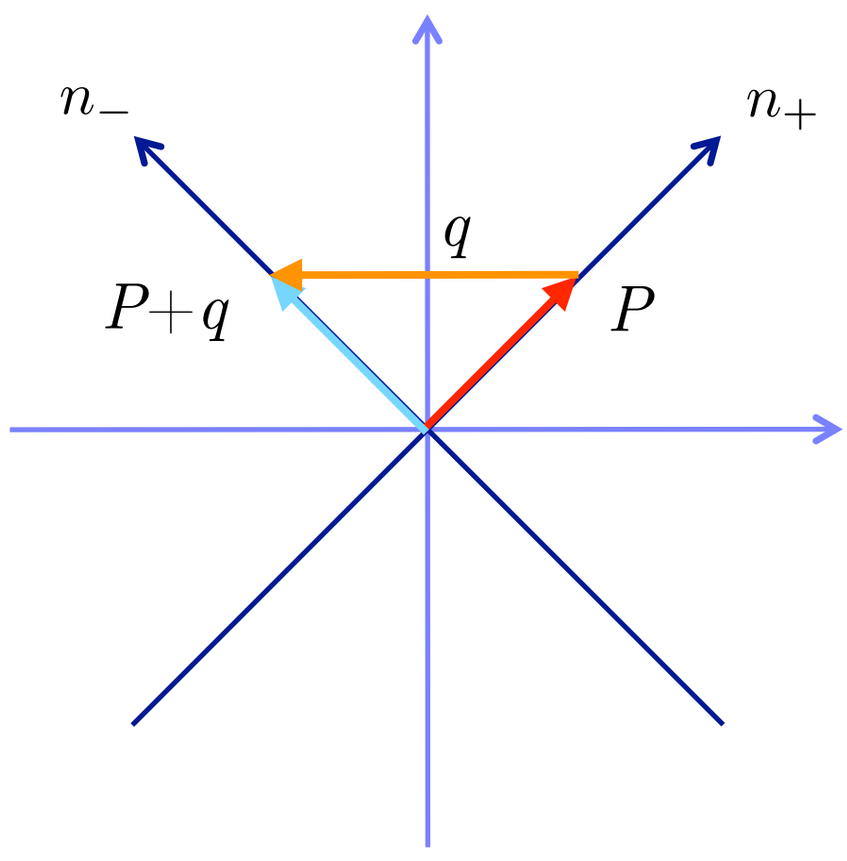
Neglecting masses and setting x_B=1 in this trivial case



$$P^\mu = \left[0,\ \frac{Q}{\sqrt{2}},\ 0,\ 0\right]$$

$$q^\mu = \left[\frac{Q}{\sqrt{2}},\ -\frac{Q}{\sqrt{2}},\ 0,\ 0\right]$$

$$(P+q)^\mu = \left[\frac{Q}{\sqrt{2}},\ 0,\ 0,\ 0\right]$$



Key points

- The relevant components are the light-cone + components before the absorption of the photon and the – components after the absorption.

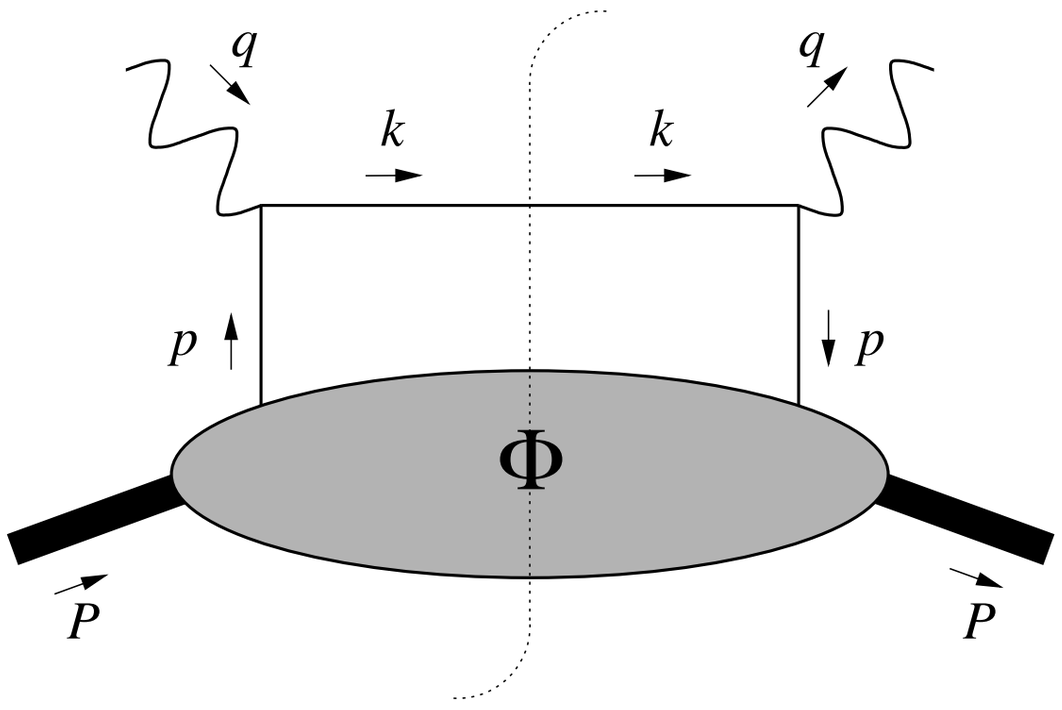
- The identification of the relevant components can be done in a frame-independent way, however it is convenient to work in certain frames to simplify the discussion

Back to the hadron tensor

$$2MW^{\mu\nu}(q,P,S) = \sum_q e_q^2 \int \mathrm{d}^4p\, \delta\left(\left(p+q\right)^2 - m^2\right)\, \theta\left(p^0+q^0-m\right)$$
$$\times\, \mathrm{Tr}\left[\Phi(p,P,S)\gamma^\mu\left(\not{p}+\not{q}+m\right)\gamma^\nu\right]$$

compared to our lepton tensor analogy, there is an extra integration over the phase space of the final quark (assumed to go unobserved, but on shell)

$$\Phi_{ji}(p,P,S) = \frac{1}{(2\pi)^4}\int \mathrm{d}^4\xi\, \mathrm{e}^{-\mathrm{i}p\cdot\xi}\left\langle P,S\right|\bar{\psi}_i(\xi)\,\psi_j(0)\left|P,S\right\rangle$$
$$= \sum_X \int \frac{\mathrm{d}^3\boldsymbol{P}_X}{(2\pi)^3\, 2P_X^0}\,\left\langle P,S\right|\bar{\psi}_i(0)\left|X\right\rangle\left\langle X\right|\psi_j(0)\left|P,S\right\rangle\,\delta^{(4)}\left(P-p-P_X\right)$$



Relevant parts

$$p^\mu = \left[\frac{p^2 + |\boldsymbol{p}_T|^2}{2xP^+},\ xP^+,\ \boldsymbol{p}_T\right]$$

Fraction of light-cone "+" momentum

Transverse momentum

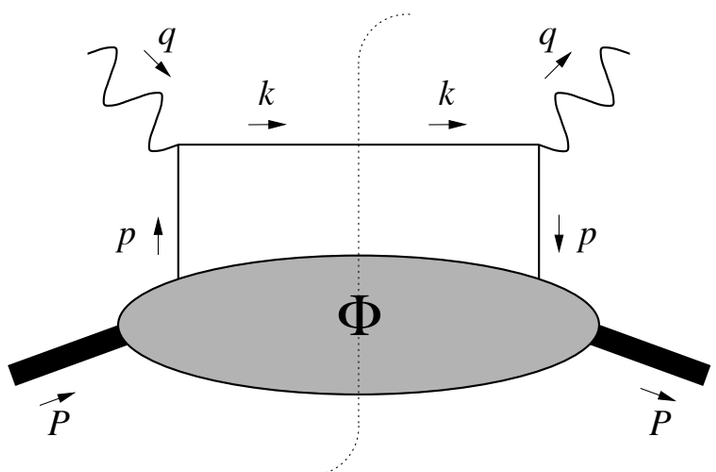
$$d^4p = d^2\boldsymbol{p}_T\, dp^-\, P^+\, dx$$

$$\delta\left(\left(p+q\right)^2 - m^2\right) \approx \delta(p^+ + q^+) \approx P^+\,\delta(x - x_B)$$

We assume that virtuality and transverse momentum of the quark are small compared to Q

$$\begin{aligned}
2MW^{\mu\nu}(q,P,S) &\approx \sum_q e_q^2 \int d^2\boldsymbol{p}_T dp^- dx\ \frac{P^+}{2P\cdot q}\,\delta\left(x - x_B\right) \\
&\quad \times \mathrm{Tr}\left[\Phi^q(p,P,S)\,\gamma^\mu\,(\not{p} + \not{q} + m)\,\gamma^\nu\right] \\
&= \sum_q e_q^2\ \frac{1}{2}\mathrm{Tr}\left[\Phi^q(x_B,S)\,\gamma^\mu \frac{P^+}{P\cdot q}\,(\not{p} + \not{q} + m)\,\gamma^\nu\right] \\
&= \sum_q e_q^2\ \frac{1}{2}\mathrm{Tr}\left[\Phi^q(x_B,S)\,\gamma^\mu\gamma^+\gamma^\nu\right]
\end{aligned}$$

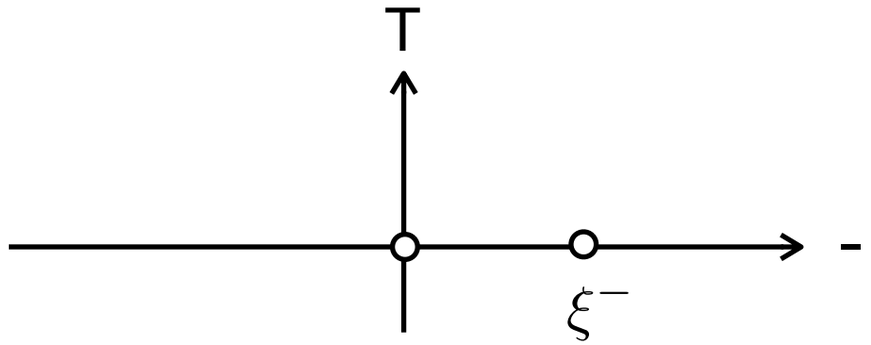
Correlator



$$2MW^{\mu\nu}(q,P,S) \approx \sum_q e_q^2 \, \frac{1}{2}\mathrm{Tr}\left[\Phi(x_B,S)\,\gamma^\mu\gamma^+\gamma^\nu\right].$$

$$\Phi_{ij}(x,S) = \int d^2\boldsymbol{p}_T dp^- \; \Phi_{ij}(p,P,S)\Big|_{p^+=xP^+}$$

$$= \int \frac{d\xi^-}{2\pi} \; e^{ip\cdot\xi}\left\langle P,S\right|\bar{\psi}_j(0)\,\psi_i(\xi)\left|P,S\right\rangle\Big|_{\xi^+=\boldsymbol{\xi}_T=0}$$



$$\Phi_{ij}(p,P,S) = \frac{1}{(2\pi)^4}\int d^4\xi \; e^{ip\cdot\xi}\left\langle P,S\right|\bar{\psi}_j(0)\,\psi_i(\xi)\left|P,S\right\rangle$$

General form of correlation function

Available vectors $p,\ P$

Available Dirac matrices $\mathbf{1},\ \gamma_5,\ \gamma^\mu,\ \gamma^\mu\gamma_5,\ i\sigma^{\mu\nu}\gamma_5$ $\qquad \sigma^{\mu\nu} \equiv \frac{i}{2}\left[\gamma^\mu, \gamma^\nu\right].$

Constraints

$$\text{Hermiticity:} \qquad \Phi(p,P,S) = \gamma^0\,\Phi^\dagger(p,P,S)\,\gamma^0, \tag{1a}$$

$$\text{parity:} \qquad \Phi(p,P,S) = \gamma^0\,\Phi(\tilde{p},\tilde{P},-\tilde{S})\,\gamma^0 \tag{1b}$$

see, e.g., Mulders, Tangerman, NPB 461 (96)
Goeke, Metz, Schlegel, PLB 618 (05)

Parton correlation function for unpolarized hadron

Question: guess at least a term in the decomposition of the correlator

$$\Phi(p,P) = M A_1 + \not{P} A_2 + \not{p} A_3 + \frac{i}{2M}[\not{P}, \not{p}]\, A_4$$

A_i are real scalar functions with dimensions $1/[m]^4$.
They are sometimes called “parton correlation functions” (PCF) or
“fully unintegrated parton distribution functions”

If we keep only the leading terms in $1/P^+$ (leading twist)

$$\Phi(p,P) \approx P^+ (A_2 + x A_3)\, \not{n}_+ + P^+ \frac{i}{2M}\big[\not{n}_+, \not{p}_T\big] A_4,$$

DIS: Parton Distribution Functions

$$\Phi(p,P) \approx P^+ \left(A_2 + xA_3\right) \not{n}_+ + P^+ \frac{i}{2M} \left[\not{n}_+, \not{p}_T\right] A_4,$$

$$\Phi(x) \equiv \int d^2\boldsymbol{p}_T dp^- \Phi(p,P) = f_1(x) \frac{\not{n}_+}{2}$$

$$f_1(x) = 2P^+ \int d^2\boldsymbol{p}_T dp^- \left(A_2 + xA_3\right)$$

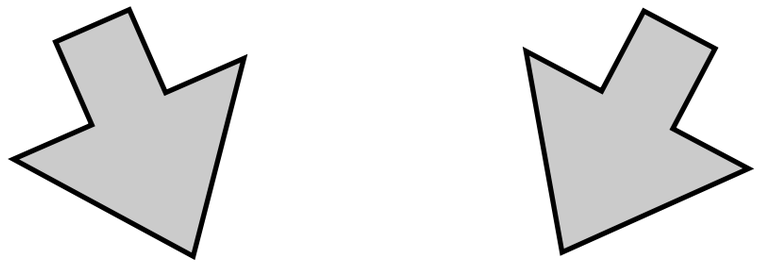
Parton Distribution Function, or standard PDF, or collinear PDF…
It comes from the decomposition of the correlation function integrated over – and transverse components

Results for structure functions in inclusive DIS

$$2MW^{\mu\nu} = \frac{1}{x}\left[-g_\perp^{\mu\nu} F_{UU,T} + \hat{t}^\mu \hat{t}^\nu F_{UU,L}\right]$$

$$2MW^{\mu\nu}(q,P,S) \approx \sum_q e_q^2 \, \frac{1}{2}\mathrm{Tr}\left[\Phi(x_B,S)\,\gamma^\mu\gamma^+\gamma^\nu\right]$$

$$\Phi(x) = f_1(x)\frac{\gamma^-}{2}$$

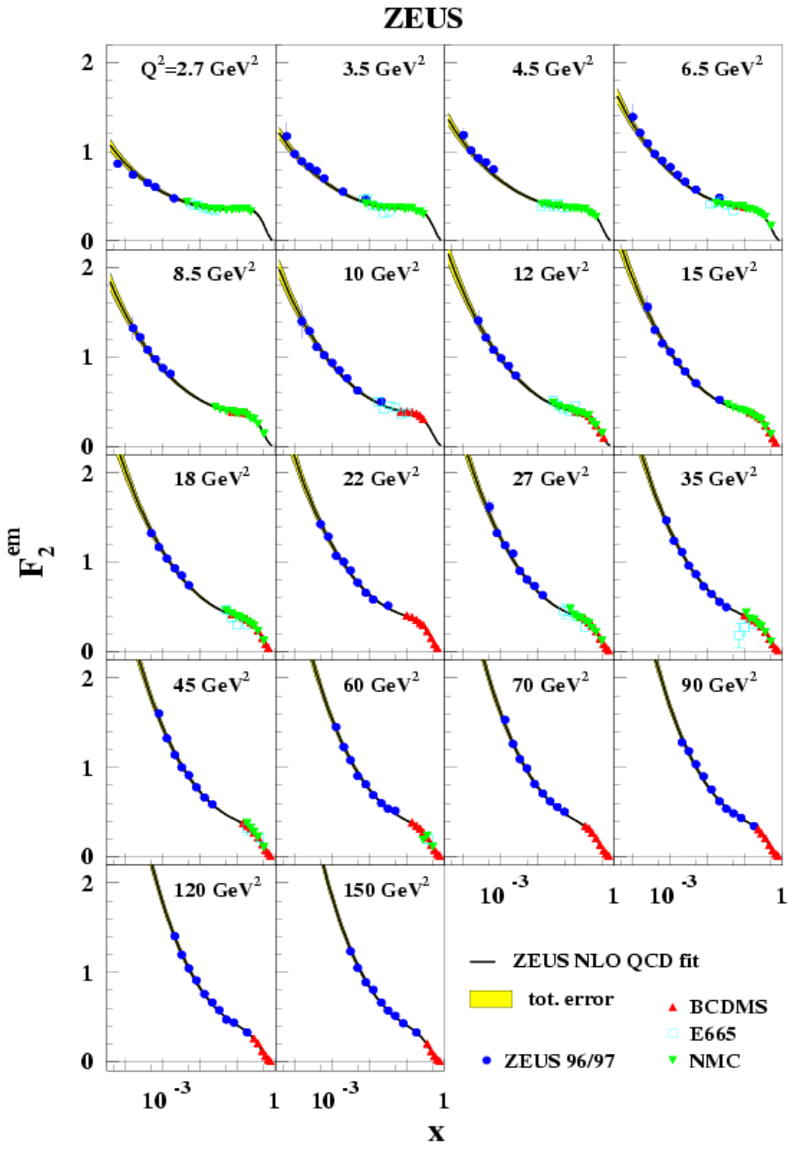


$$F_{UU,T} = x_B \sum_a e_a^2 \, f_1^a(x_B) \qquad\qquad F_{UU,L} = 0$$

$$F_2 = x_B \sum_a e_a^2 \, f_1^a(x_B)$$

Competition

- Form groups
- Compute the structure function F_2 using Parton Distribution Functions from the LHAPDF library (lhapdf.hepforge.org)
- LHAPDF can be used in the Colab environment (see, e.g., colab.research.google.com/drive/1EVIuYQ4HTgvRk2sCBA6N_Bi28BY9ibwz) or in Mathematica (using the ManeParse package ncteq.hepforge.org/mma/)
- Compare the predictions with some data (see, for instance, https://www.hepdata.net/search/?q=f2&page=1&observables=F2 or https://github.com/JeffersonLab/CJ-JAM-database/tree/main/data)



What about polarization?

Results for polarized inclusive DIS

$$2MW^{\mu\nu} = \frac{1}{x}\left[-g_\perp^{\mu\nu} F_{UU,T} + \hat{t}^\mu \hat{t}^\nu F_{UU,L} + iS_L \epsilon_\perp^{\mu\nu} F_{LL} - i\hat{t}^{[\mu}\epsilon_\perp^{\nu]\rho} S_\rho F_{LT}^{\cos\phi_S}\right]$$

$$2MW^{\mu\nu}(q,P,S) \approx \sum_q e_q^2 \, \frac{1}{2}\mathrm{Tr}\left[\Phi(x_B,S)\,\gamma^\mu\gamma^+\gamma^\nu\right].$$

$$\Phi(x,S) = \frac{1}{2}\left\{f_1\gamma^- + S_L\, g_1\gamma_5\gamma^- + h_1\,\gamma_5\; \not{S}_T\gamma^-\right\}$$

Results for polarized inclusive DIS

$$F_{UU,T} = x_B \sum_a e_a^2 \; f_1^a(x_B)$$

$$F_{UU,L} = 0$$

$$F_{LL} = x_B \sum_a e_a^2 \; g_1^a(x_B)$$

$$F_{LT}^{\cos\phi_S} = 0$$

Polarized twist-3 (just for your information)

$$\Phi(x,S) = \frac{1}{2}\left\{ f_1 \gamma^- + S_L\, g_1 \gamma_5 \gamma^- + h_1\, \gamma_5\, \not{S}_T \gamma^- \right\}$$

Twist 2

$$+ \frac{M}{2P^+}\left\{ e + g_T\, \gamma_5\, \not{S}_T + h_L\, S_L\, \frac{[\gamma^-,\gamma^+]\gamma_5}{2} \right\}$$

Twist 3

Polarized twist-3 (just for your information)

$$F_{UU,T} = x_B \sum_a e_a^2 \; f_1^a(x_B)$$

$$F_{UU,L} = 0$$

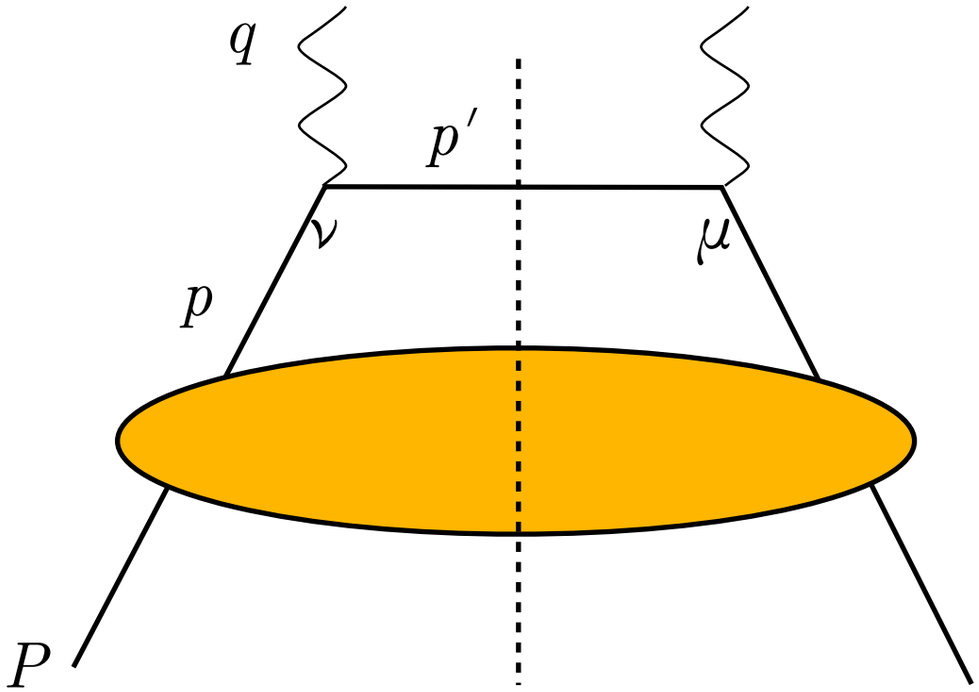
$$F_{LL} = x_B \sum_a e_a^2 \; g_1^a(x_B)$$

$$F_{LT}^{\cos\phi_S} = -\gamma x_B \sum_a e_a^2 \; g_T^a(x_B)$$

$$\gamma = \frac{2x_B M}{Q}$$

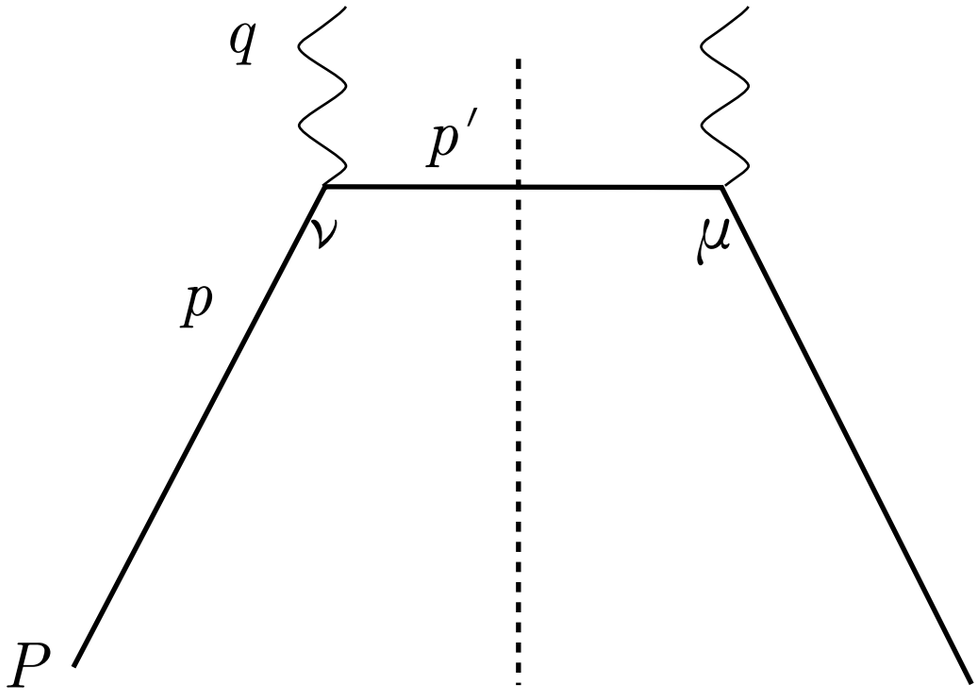
Beyond the parton model

Quark-in-quark case



At order α^0, the calculation should be simple and give us a result very similar to the leptonic tensor

Quark-in-quark case



$$\Phi^q_{ji}(x,S) = \int \frac{d\xi^-}{2\pi}\, e^{-\mathrm{i}p\cdot\xi} \Big\langle P,S\Big|\, \overline{\psi}^q_i(\xi)\psi^q_j(0)\,\Big|P,S\Big\rangle\Big|_{\xi^+=\boldsymbol{\xi}_T=0,\ p^+=xP^+}$$

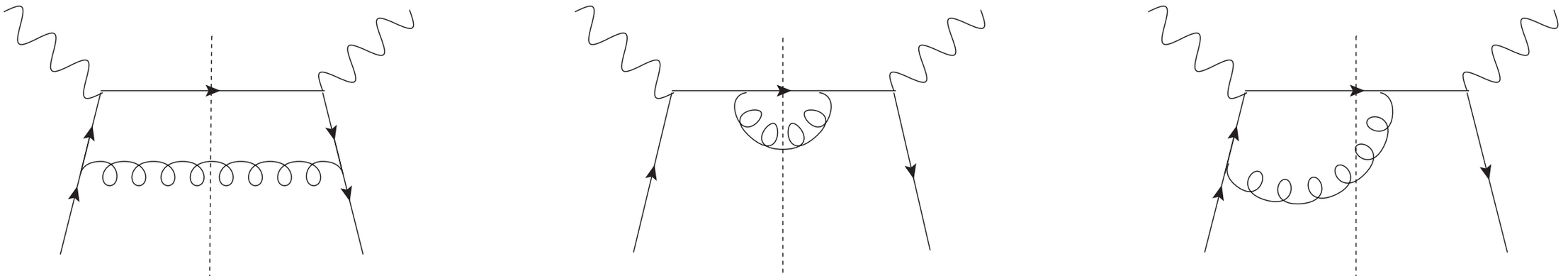
$$\Big|P,S\Big\rangle = c^\dagger(P)\,\Big|0\Big\rangle$$

unpolarized

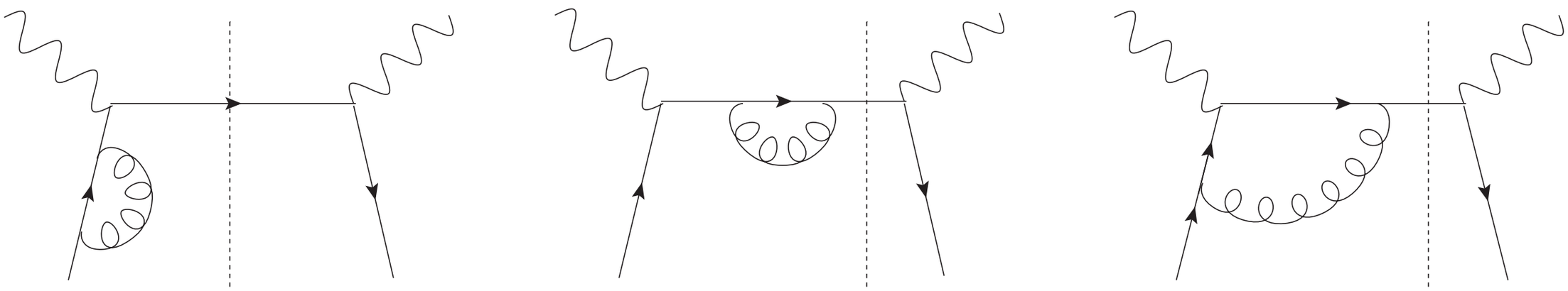
$$\Phi^q_{ji}(x) = \int \frac{d\xi^-}{2\pi}\, e^{-\mathrm{i}(xP^+-P^+)\xi^-} \frac{1}{2}\sum_s u_j(P)\overline{u}_i(P) = \frac{1}{2}\ \not{P}_{ji}\ \delta(xP^+-P^+)$$

Order α_S corrections

The complications start at order α_S



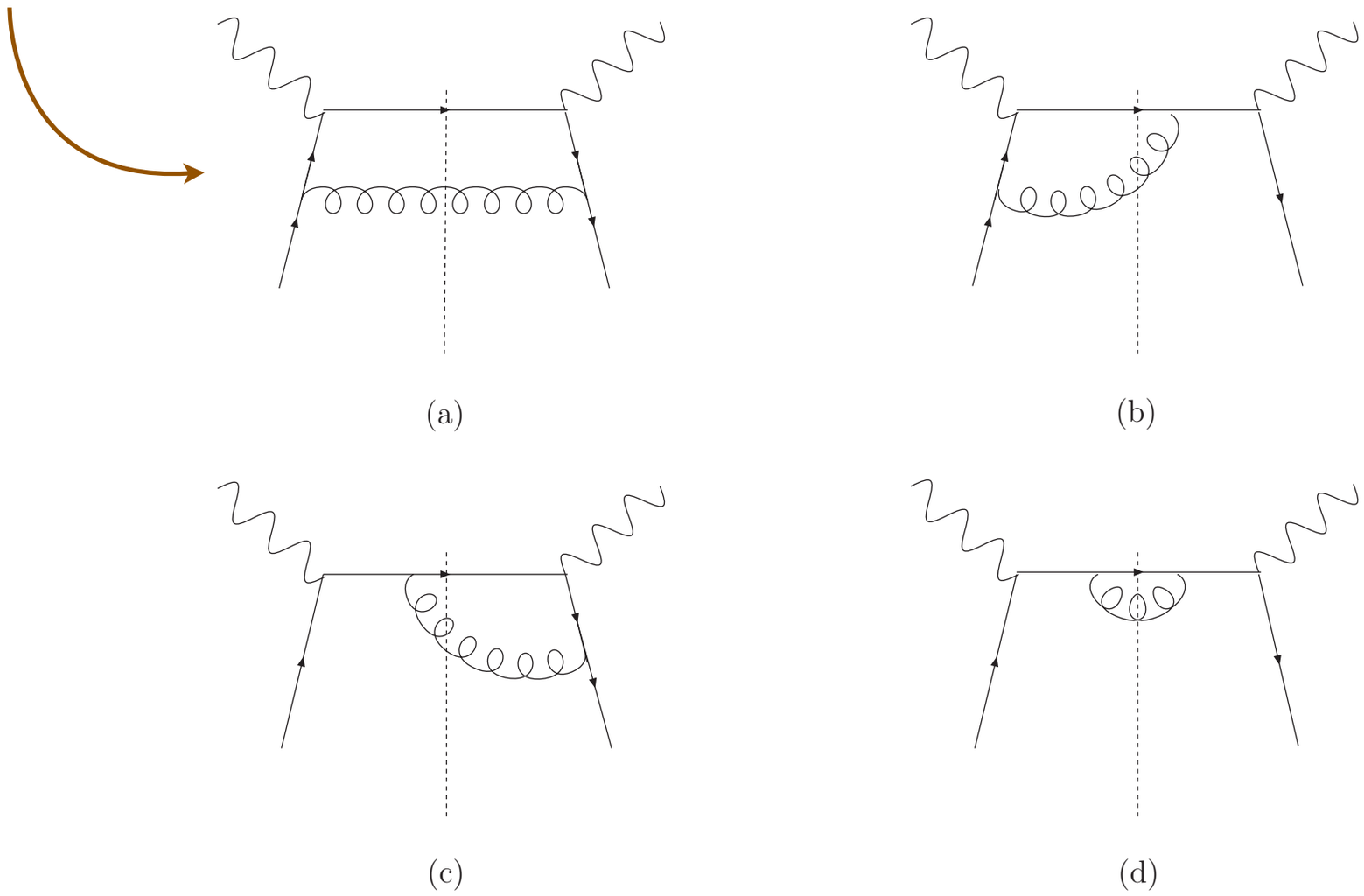
(r1) (r2) (r3)



(v1) (v2) (v3)

Real diagrams

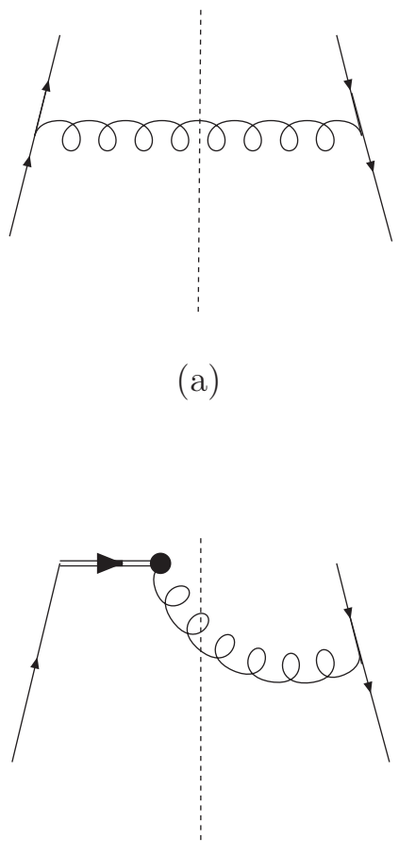
Light-cone gauge



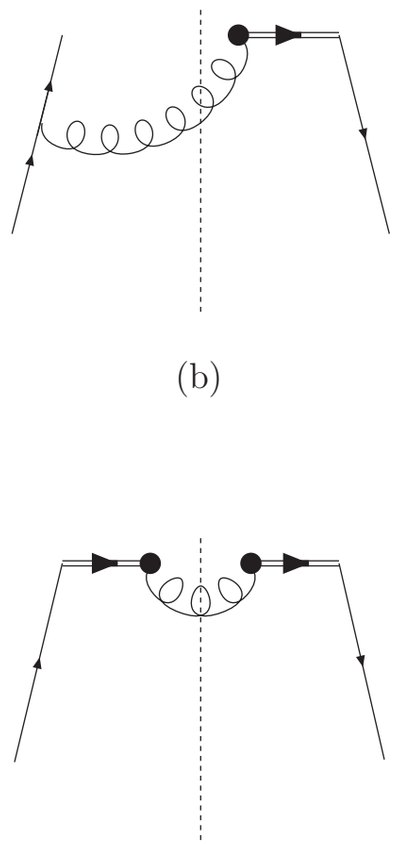
Diagrams (b), (c), (d) do not give leading contributions in the light-cone gauge, but they do in other gauges (e.g., Feynman gauge).

Gauge link and quark-in-quark case

Light-cone gauge



(a)

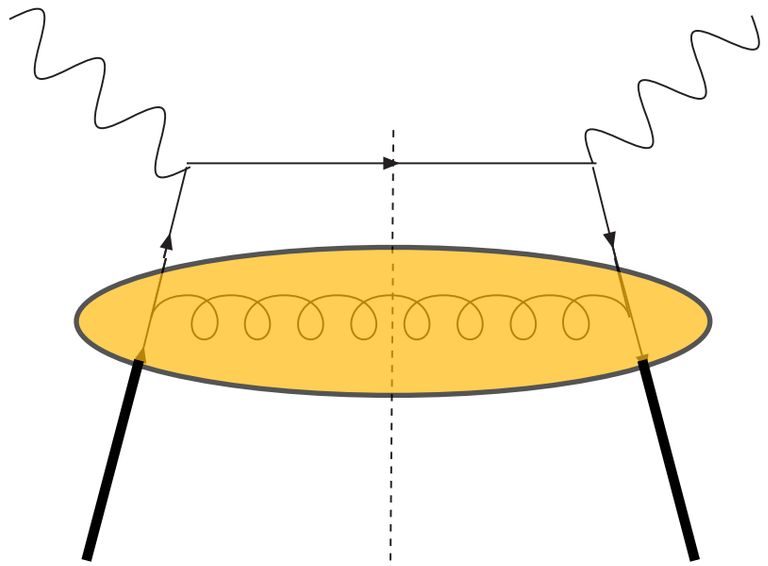


(b)

(c)

(d)

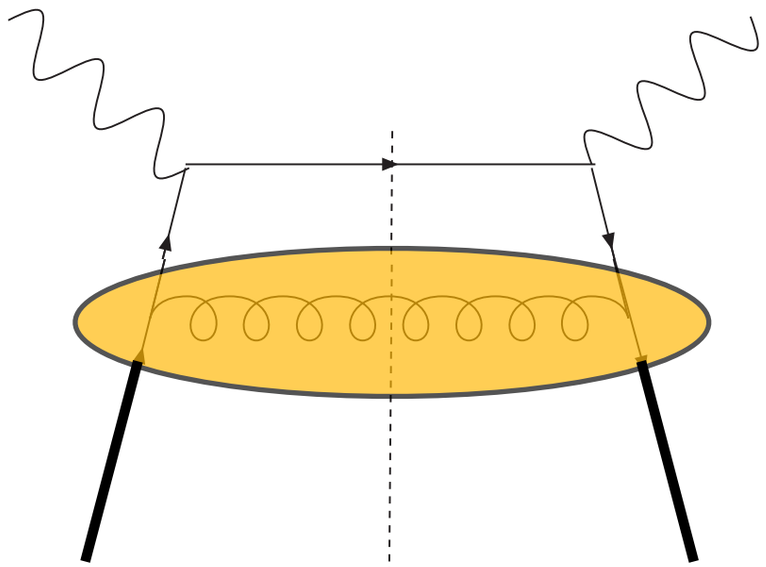
Divergences



We have to solve two problems:

1. The diagram gives divergent results if we integrate over the final state gluon. This is not so surprising because we are not performing a scattering off a free, pointlike quark, but rather off the proton.

Divergences



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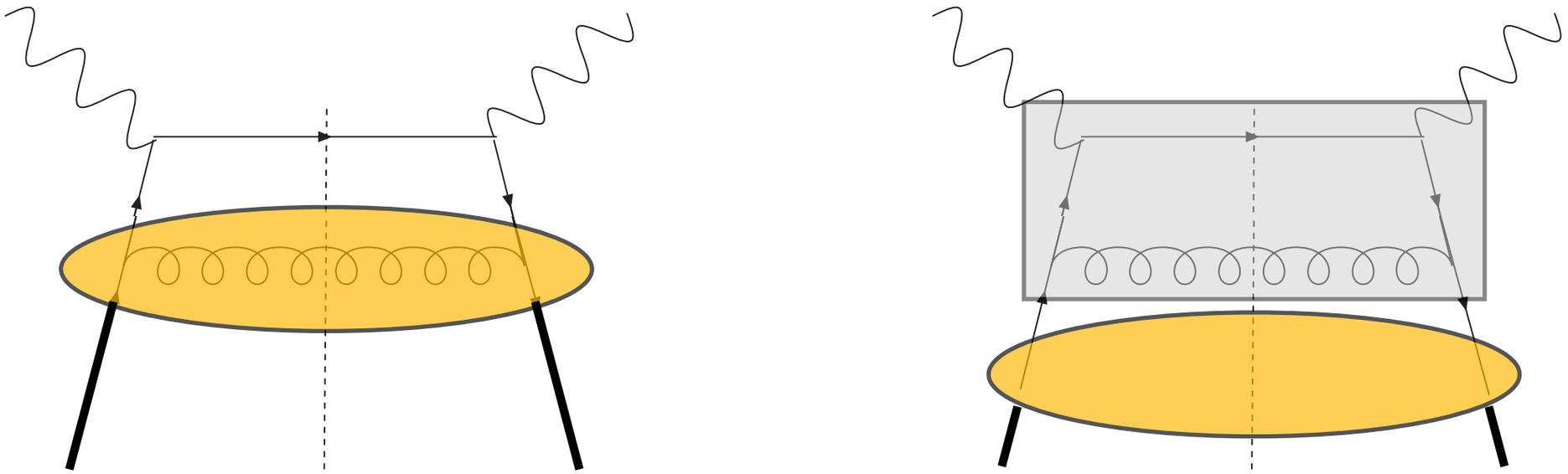
1. The diagram gives divergent results if we integrate over the final state gluon. This is not so surprising because we are not performing a scattering off a free, pointlike quark, but rather off the proton.

2. We have to separate what we include in the partonic distribution function and what goes into the hard factor

Divergences



We have to solve two problems:

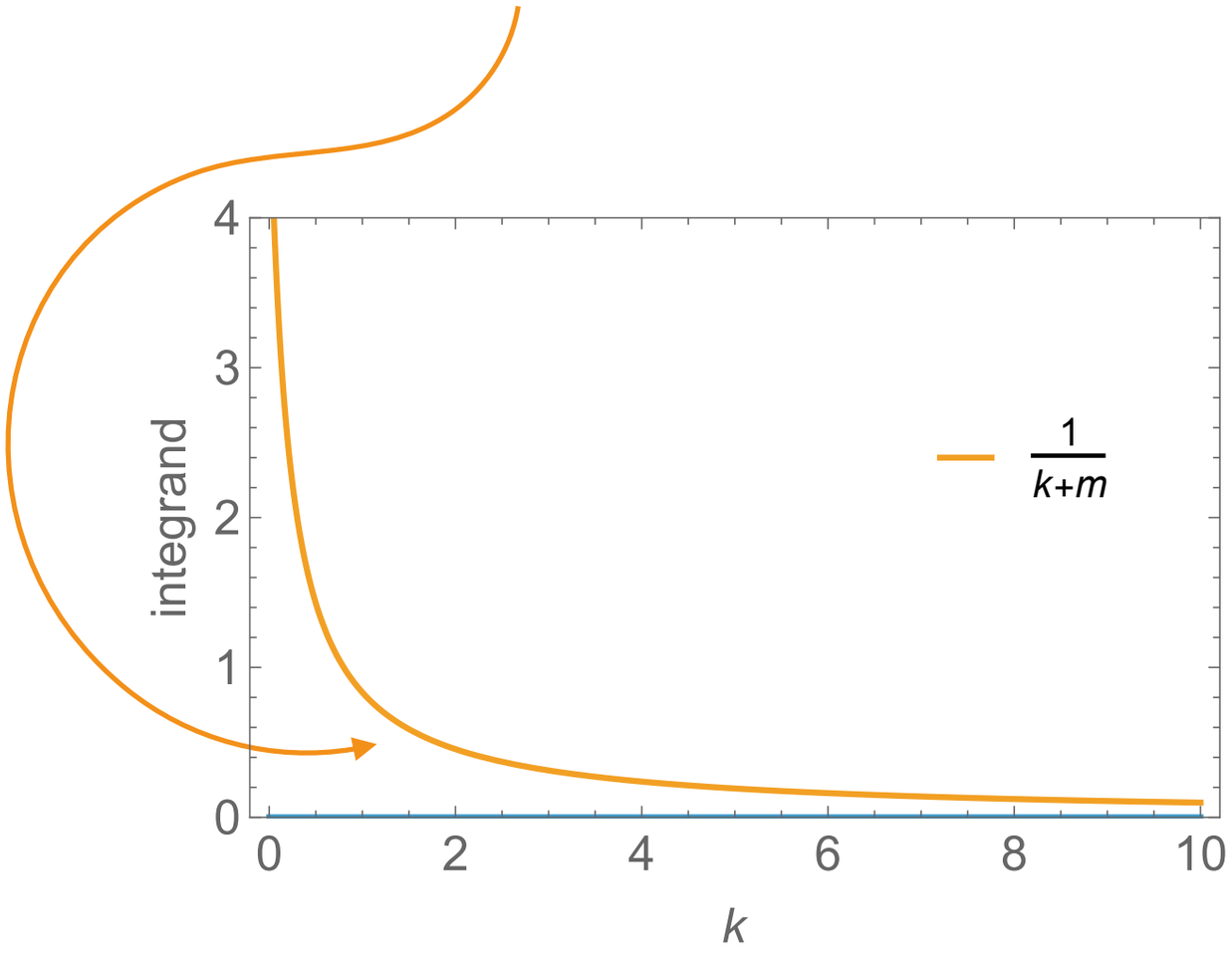
1. The diagram gives divergent results if we integrate over the final state gluon. This is not so surprising because we are not performing a scattering off a free, pointlike quark, but rather off the proton.

2. We have to separate what we include in the partonic distribution function and what goes into the hard factor

Toy-model discussion of factorization

Suppose the true observable corresponds to this integral

$$I_{\text{phys}}(Q,m) = \int_0^Q \frac{dk}{k+m} + C = \ln\frac{Q+m}{m} + C$$



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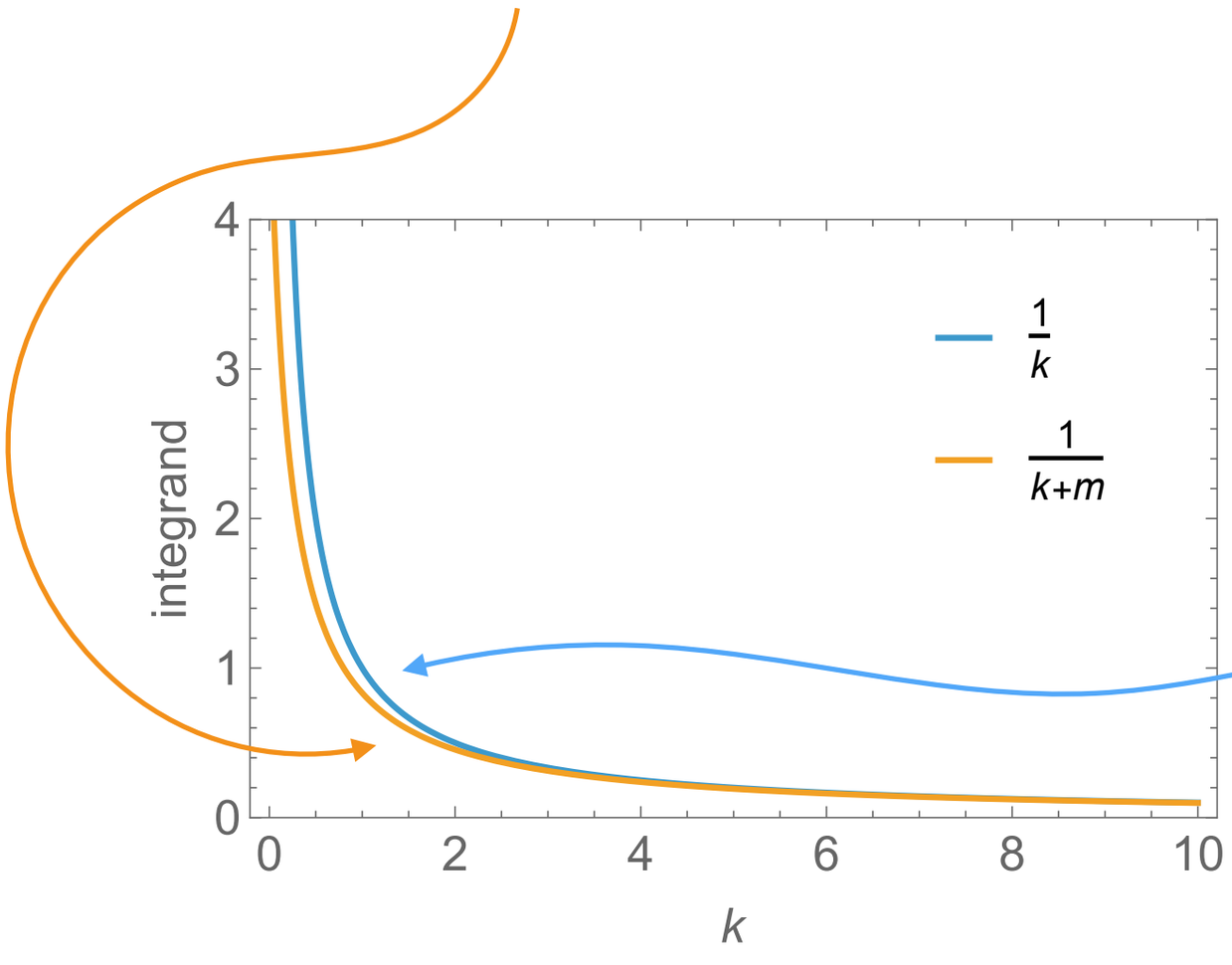


However, we are able to compute the integrand only in the limit for $m = 0$

$$I'(Q) = \int_0^Q \frac{dk}{k} + C$$

The integral is infrared divergent

Separating different regions

Step 1: introduce a regulator (M)

$$I'_{\mathrm{reg}}(Q,M) = \int_M^Q \frac{dk}{k} + C = \ln\frac{Q}{M} + C$$

Step 2: separate two complementary regions

$$I'_{\mathrm{reg}}(Q,M) = I_{\mathrm{low}}(M,\mu) + I_{\mathrm{high}}(\mu,Q) = \int_M^\mu \frac{dk}{k} + \int_\mu^Q \frac{dk}{k} + C$$

$$I_{\mathrm{low}}(M,\mu) = \int_M^\mu \frac{dk}{k} = \ln\frac{\mu}{M}$$

The lower part depends on the infrared regulator and is unreliable

$$I_{\mathrm{high}}(Q,\mu) = \int_\mu^Q \frac{dk}{k} + C = \ln\frac{Q}{\mu} + C$$

The upper part does not depend on the infrared regulator and is reliable

Dependence on arbitrary separation scale

$$I_{\text{low}}(M,\mu) = \ln\frac{\mu}{M} \quad \Rightarrow \quad \frac{dI_{\text{low}}}{d\ln\mu} = \mu\frac{dI_{\text{low}}}{d\mu} = +1$$

$$I_{\text{high}}(Q,\mu) = \ln\frac{Q}{\mu} \quad \Rightarrow \quad \frac{dI_{\text{high}}}{d\ln\mu} = \mu\frac{dI_{\text{high}}}{d\mu} = -1$$

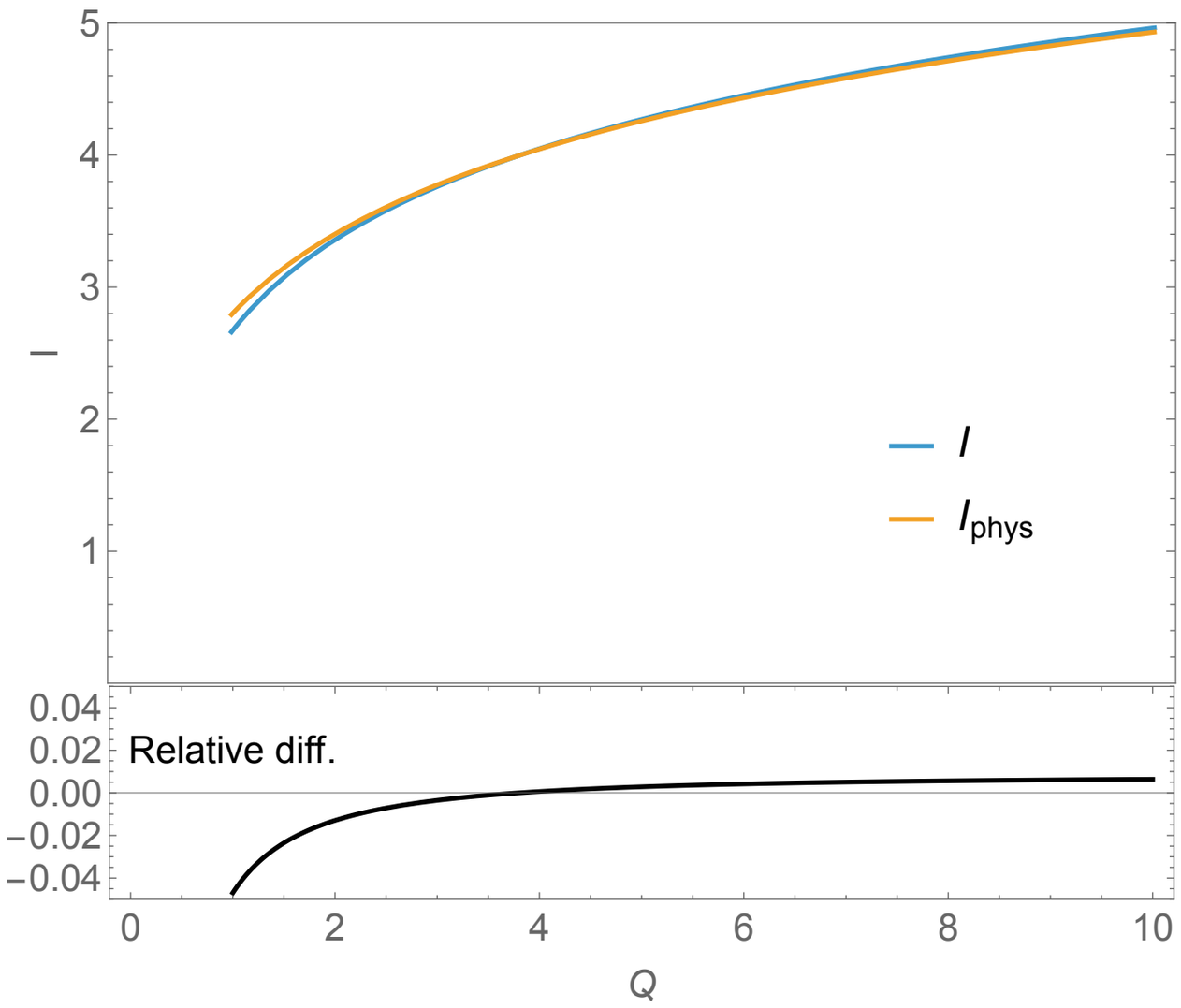
The dependence on the separation scale must cancel in the observable, i.e., knowing the dependence of I_{high}, it is possible to predict the dependence of I_{low}

Eventually, I_{low} can be replaced by a different function, which is fixed by comparison with the data, but with a predicted dependence on μ

$$I_{\text{low}}(M,\mu) = \ln\frac{\mu}{M} \xrightarrow{\text{replaced}} I_{\text{low}}(m_{\text{eff}},\mu) = \ln\frac{\mu}{m_{\text{eff}}}$$

$$I(Q,m) = \ln\frac{\mu}{m_{\text{eff}}} + \ln\frac{Q}{\mu} + C = \ln\frac{Q}{m_{\text{eff}}} + C \qquad I(Q,m) = I_{\text{phys}}(Q,m) + \mathcal{O}(m/Q)$$

Numerical example



$$m = 0.2, \quad m_{\text{eff}} = 0.19$$

Connection with real factorization procedure

Example	QCD
Separation scale μ	Factorization scale μ
I_{high}	Hard part
I_{low}	Parton distribution function
Dependence on μ	Evolution equations
m/Q corrections	Power corrections

Outcome of factorization procedure

- Structure functions in the parton model

$$F_{UU,T}(x, Q^2) = x \sum_a e_a^2 \, f_1^a(x)$$

- Structure functions based on QCD factorization

$$F_{UU,T}(x, Q^2) = x \sum_a C_a\left(\alpha_S, \frac{Q^2}{\mu^2}, \frac{x}{\xi}\right) \otimes f_1^a(\xi, \mu^2)$$

Coefficient functions
(calculable to a certain order in α_S)

longitudinal convolution

scale dependence