Fermion Soliton Black Holes

Master’s Degree in Physics

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The problem of gravitational singularities

Black holes are at the heart of several open questions in fundamental physics. Foremost among these is the **problem of gravitational singularities**:

- General Relativity predicts singularities within black holes, via the **Hawking-Penrose theorems**;
- the **cosmic censorship conjecture** posits that singularities must be hidden by event horizons;
- singularities may be an **artifact of the classical theory's breakdown** at small energy scales;
- **regular black holes** consist in quantum **effective black hole spacetimes** with no singularity.

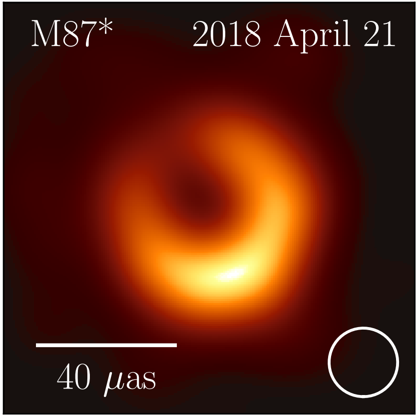
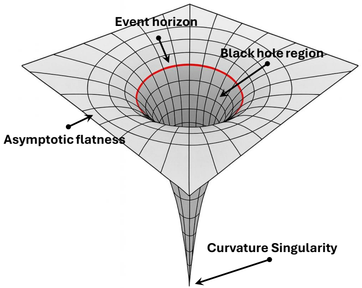


Figure from Akiyama et al. (2024).

Black holes and singularities



What is a black hole?

Asymptotically flat spacetime with **no-escape region** for causal signals. **Event horizon** as region boundary.

Asymptotically flat → similar to Minkowski at infinity.

What is a singularity?

Metric singularity: $g_{\mu\nu}$ diverge or $\det(g_{\mu\nu}) = 0$.
Coordinate singularity: removable via extension.
Curvature singularity: curvature invariants diverge.

Curvature invariant → scalar polynomial in $R_{\mu\nu\alpha\beta}$.

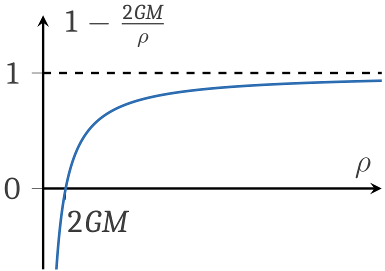
An example: the Schwarzschild black hole

Adopt standard spherical coordinates $\{t, \rho, \theta, \varphi\}$, with $d\Omega^2 \equiv d\theta^2 + \sin^2\theta \, d\varphi^2$. Metric:

$$ds^2 = -\left(1 - \frac{2GM}{\rho}\right) dt^2 + \left(1 - \frac{2GM}{\rho}\right)^{-1} d\rho^2 + \rho^2 d\Omega^2.$$

- Asymptotically flat, as $\lim_{\rho\to+\infty} \frac{2GM}{\rho} = 0$
- $\rho = 2GM$: removable singularity (event horizon);
- $\rho = 0$: curvature singularity, where **Kretschmann curvature invariant** diverges:

$$K \equiv R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} = \frac{48(GM)^2}{\rho^6}.$$



A spacetime is called **regular** at a point if no curvature singularity emerges at that point.

Regular black hole → black hole with no curvature singularities.

Restrict to **non-rotating** spacetimes and consider standard spherical coordinates.

${R_{\mu\nu}}^{\alpha\beta}$ components are:

- **pairwise diagonal**;
- **four are independent**.

Scalar curvature invariants are **scalar polynomials** in ${R_{\mu\nu}}^{\alpha\beta}$.

Spacetime regular at a point iff

$$
\begin{aligned}
K_1 &\equiv -{R_{01}}^{01}, \\
K_2 &\equiv -{R_{02}}^{02} = -{R_{03}}^{03}, \\
K_3 &\equiv -{R_{12}}^{12} = -{R_{13}}^{13}, \\
K_4 &\equiv -{R_{23}}^{23}.
\end{aligned}
$$

are finite.

An example: the Bardeen regular black hole

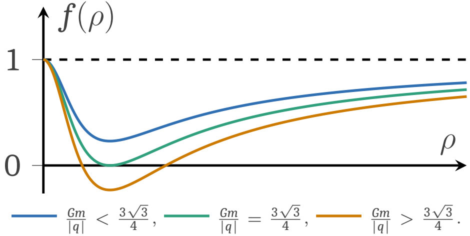
First proposed **regular black hole**, with no specified source, by Bardeen (1968):

$$ds^2 = -f(\rho)\,dt^2 + f(\rho)^{-1}d\rho^2 + \rho^2 d\Omega^2,$$

$$f(\rho) \equiv 1 - \frac{2Gm\,\rho^2}{(\rho^2+q^2)^{3/2}}.$$

in standard coordinates.

- Asymptotically flat, as $\lim_{\rho\to+\infty} f(\rho) = 1$;
- **regular**, as $\{K_i\}$ are finite $\forall \rho \geq 0$.



Outermost $f(\rho)$ zero is the event horizon.

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Action and scalar potential

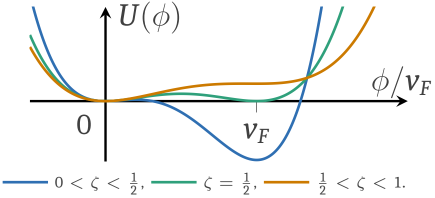
Fermion scalar theory **minimally coupled** to General Relativity.
Dirac fermion ψ and **real scalar** ϕ interacting via a **Yukawa term**:

$$S = \int d^4x \sqrt{-g} \left[\frac{R}{16\pi G} - \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - U(\phi) - \overline{\psi}\left(\gamma^\mu D_\mu + m_f\right)\psi + f\phi\,\overline{\psi}\psi \right].$$

Effective fermion mass: $m_{\text{eff}} = m_f - f\phi$,
with $f = \frac{m_f}{v_F}$. **Scalar potential**:

$$U(\phi) = \frac{\mu^2 v_F^2}{12}\frac{v_F}{v_B}\left(\frac{\phi}{v_F}\right)^2\left[3\left(\frac{\phi}{v_F}\right)^2 - 4\left(\frac{\phi}{v_F}\right)\left(1+\frac{v_B}{v_F}\right) + 6\frac{v_B}{v_F}\right],$$

Restrict to $0 < \zeta < 1$, with $\zeta \equiv \frac{v_B}{v_F}$.
Two **vacua** at $\phi = 0$ (true), $\phi = v_F$ (false).



Semiclassical treatment

We apply a **semiclassical treatment** to the fermion scalar theory.

- Scalar ϕ and metric $g_{\mu\nu}$ are **classical**;
- **static** and **spherically symmetric** spacetime, in standard coordinates:

$$ds^2 = -e^{2\bar{u}(\rho)}dt^2 + e^{2\bar{v}(\rho)}d\rho^2 + \rho^2 d\Omega^2.$$

Scalar ϕ has spacetime symmetries: $\phi(\rho)$.

Dirac fermion ψ **quantized in curved spacetime** and **fermion ground state** constructed at fixed particle number.

Apply:

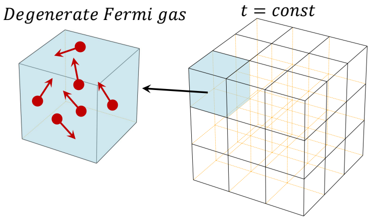
- **canonical quantization** to ψ;
- **Thomas-Fermi approximation**.

Thomas-Fermi approximation

Physical assumption:

$\bar{u}, \bar{v}$ and ϕ **vary slowly** over three-space with respect to fermion dynamics.

- Three-space **partitioned into domains**;
- domains filled with degenerate **Fermi gas**: step Fermi distribution, **Fermi momentum** $k_F(\rho)$;
- in domains, compute number N and fermion stress-energy tensor.



Fermion ground state → **isotropic perfect fluid**.

Field equations and initial conditions

To find $k_F(\rho)$, minimize fermion energy at fixed number, with **Lagrange multiplier** ω_F:

$$\sqrt{k_F^2 + m_{\text{eff}}^2} = \omega_F \, e^{-\bar{u}}.$$

To find $\bar{u}, \bar{v}$ and ϕ, **Einstein's** and **scalar field eqs.**:

$$\begin{aligned}
&e^{-2\bar{v}} - 1 - 2e^{-2\bar{v}}\rho\,\frac{\partial \bar{v}}{\partial \rho} = -8\pi G\rho^2(W + V + U),\\
&e^{-2\bar{v}} - 1 + 2e^{-2\bar{v}}\rho\,\frac{\partial \bar{u}}{\partial \rho} = 8\pi G\rho^2(P + V - U),\\
&e^{-2\bar{v}}\left(\frac{\partial^2 \phi}{\partial \rho^2} + \left(\frac{2}{\rho} + \frac{\partial \bar{u}}{\partial \rho} - \frac{\partial \bar{v}}{\partial \rho}\right)\frac{\partial \phi}{\partial \rho}\right) + fS - \frac{dU}{d\phi} = 0.
\end{aligned}$$

Stress-energy tensor:

- $W, P, S \rightarrow$ fermion quantities;
- $U, V \rightarrow$ scalar quantities.

Initial conditions at $\rho = 0$:

$$\begin{aligned}
&\bar{u}(0) = 0, && \bar{v}(0) = 0,\\
&\phi(0) = v_F(1 - \varepsilon), && \partial_\rho \phi(0) = 0,
\end{aligned}$$

with $\varepsilon \gg 1$.

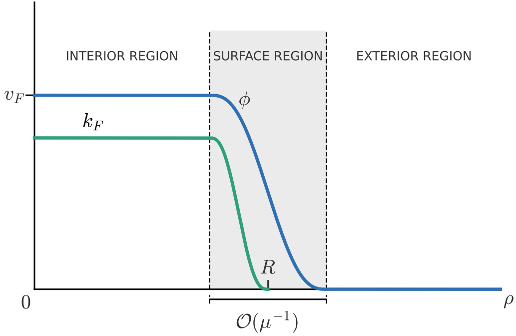
Solitonic configurations

We are interested in **solitonic** solutions.

- **asymptotically flat**: null ϕ at $\rho \to +\infty$;
- **confined fermion core**.

We set $\phi(0) \approx v_F$ ➞ three regions.

Fermion soliton star: regular and static solitonic solution.



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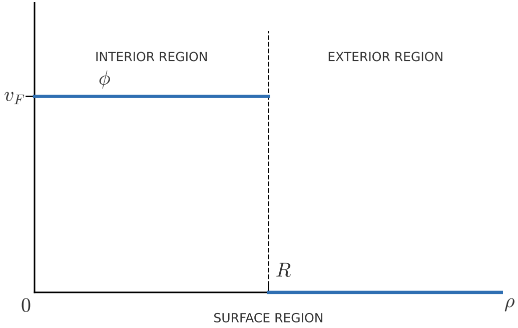
Lee, Pang (1987) found fermion soliton stars in **thin-shell approximation**: surface region reduced to **zero-thickness hypersurface**.

Applicability condition:

$$R\mu \gg 1.$$

Non-restrictive: implied by ϕ classicality.

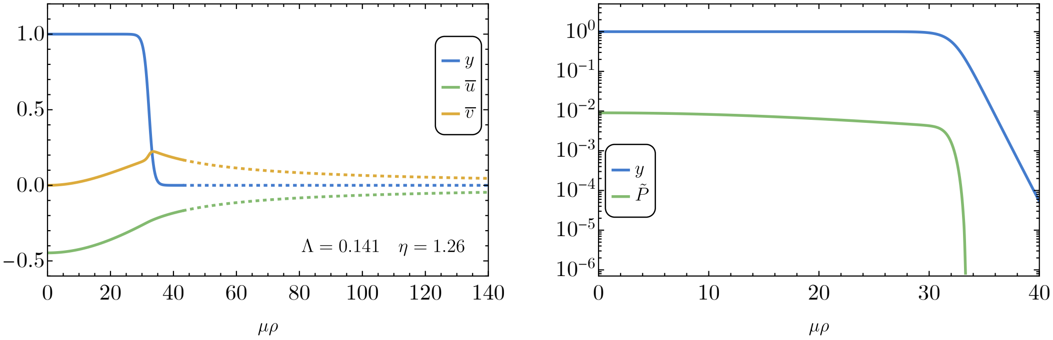
- R → fermion core radius;
- μ^{-1} → scalar Compton wavelegth.



Exact solutions

3 Fermion soliton stars

Fermion soliton stars also found by Del Grosso et al. (2023) as **exact solutions**, via numerical integration. Dimensionless quantities adopted.



Fermion soliton black holes: non-rotating regular black hole solitonic solutions.
Select a coordinate chart:

Isotropic spherical coordinates

$$ds^2 = -A(r)\,dt^2 + B(r)\left(dr^2 + r^2 d\Omega^2\right).$$

$r \longrightarrow$ **isotropic radius.**

Standard spherical coordinates

$$\begin{aligned} ds^2 &= -A(\uprho)\,dt^2 + A(\rho)^{-1}\,d\uprho^2 + \rho(\uprho)^2 d\Omega^2 \\ &= -\bar{A}(\rho)\,dt^2 + \bar{B}(\rho)\,d\rho^2 + \rho^2 d\Omega^2 . \end{aligned}$$

$\rho \longrightarrow$ **standard** radius;
$\uprho \longrightarrow$ **rescaled standard** radius.

Coordinate **transformation** not always possible:
necessary and sufficient condition is $\rho(r) \equiv r\sqrt{B}$ bijective.

Isotropic coordinates: isomorphic regions

Assumptions in isotropic coordinates:

- globally **regular spacetime**, for $r \geq 0$;
- $\boldsymbol{C^\infty}$**—smooth** and **non-diverging** metric functions A, B.

Mutually isomorphic regions if:

1. $\rho(r)$ not globally bijective;
2. $\rho(r)$ **separately bijective** in $r \in I_1, I_2$, with and $\rho(I_1) = \rho(I_2) = J$;
3. **standard reparametrization** in $r \in I_1, I_2$ yields same standard metric functions $\bar{A}, \bar{B}$ in $\rho \in J$.

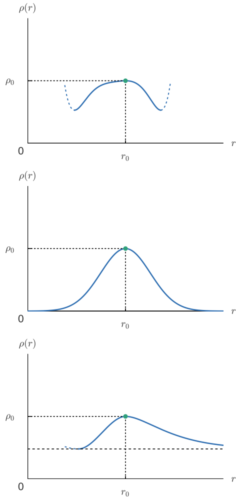
Example: **Schwarzschild wormhole**

$$ds^2 = -\left(\frac{r-a}{r+a}\right)^2 dt^2 + \left(\frac{r+a}{r}\right)^4 (dr^2 + r^2 d\Omega^2).$$

Regions $\{0 < r < a\}$ and $\{r > a\}$ mutually isomorphic, both parametrizing external Schwarzschild black hole with $M = 2a/G$.

Consider matter Lagrangian density **not explicitly position dependent**, and solution with $\rho(r)$ **local extremum** at $r = r_0$.

- There exist **left** and **right neighborhoods** of $r = r_0$ related to mutually isomorphic regions;
- $\rho(r)$ has opposite **monotonicity** on each neighborhood;
- neighborhoods extend until **new critical points** for $\rho(r)$ are reached on both sides, where $\rho(r)$ takes same value;
- **exception** when right extension reaches $r = +\infty$: ρ' always vanishes at $r = +\infty$, irrespectively of other side.



Isotropic coordinates: non-existence argument

Consider fermion scalar solution with:

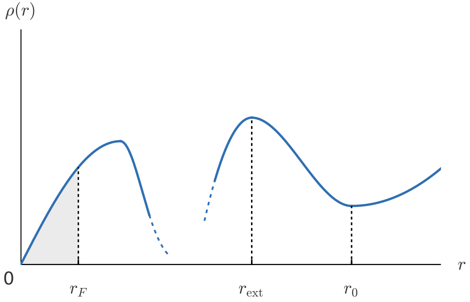
- **confined fermion core** of radius r_F, no local extrema of $\rho(r)$ within;
- **one local extremum** for $\rho(r)$, in $r > r_F$.

Black hole if

- asymptotically flat: $\rho(r) \approx r$ as $r \to +\infty$;
- local extremum as ∂_t-Killing horizon.

Fermion scalar Lagrangian density **explicitly position independent** for $r > r_F$.

Theorem applies to outermost extremum r_0, but thesis denied: **absurd**.



Standard coordinates: no-go theorem

Region $\{a < \rho < b\}$ with $A(a) = 0$ (or a center) and $A(b) = 0$ (or b infinity) is:

- **static** (R) if $A > 0$;
- **non-static** (T) if $A < 0$.

Consider stress-energy tensor $T_{\mu\nu}$. If

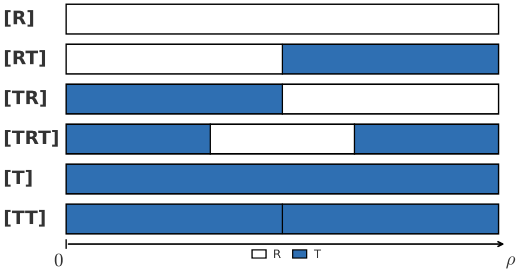
$$-T^t{}_t + T^\theta{}_\theta \leq 0 \text{ where } A \leq 0,$$

and $\{a < \rho < b\}$ static, then:

i. $A'(a), A'(b) \neq 0$;

ii. $A \neq 0$ for $\rho < a$, $\rho > b$.

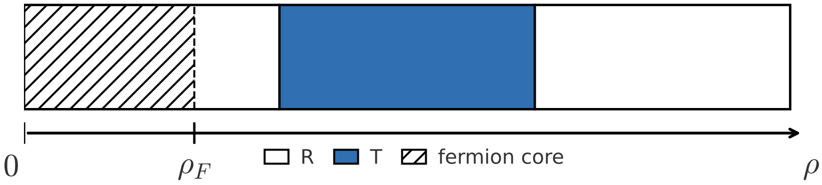
Spacetimes are **sequences of R,T regions.** Theorem **restricts admissible sequences.**



Standard coordinates: non-existence argument

Consider hypothetical fermion soliton black hole: **more than one causal region**, innermost and outermost regions **static**, fermion core of radius ρ_F **confined in innermost region**.

Call h innermost radius where $A(h) = 0$. Theorem applies in $\rho_F < \rho < h$, since theory **purely scalar** for $\rho > h$: **no admissible causal sequence** allows outermost static region.



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Conclusions

We proved the **non-existence** of fermion soliton black holes:

- discussing mutually isomorphic regions in **isotropic coordinates**.
- via a **no-go theorem** in **standard coordinates**.

The non-existence arguments are **complementary** for **maximally extended spacetimes**.

Future developments:

- verify **numerically** the existence of exact fermion soliton black holes in isotropic coordinates, allowing **derivative discontinuities** of the functions;
- study **rotating fermion soliton stars**: regular solitonic solutions to the fermion scalar theory, in stationary and axially symmetric spacetimes.