$$L_{GW} = \frac{1}{32\pi} \sum_{l=2}^{\infty} \sum_{m=-l}^{l} \left\langle \left| {}^{l+1} \underset{\substack{\uparrow \\ \text{mass}}}{I^{lm}} \right|^2 + \left| {}^{l+1} \underset{\substack{\uparrow \\ \text{current}}}{S^{lm}} \right|^2 \right\rangle$$

Thorne 1980

$$I^{lm} = f(l,m) \int T_{00} Y^*_{lm} r^l d^3x$$

$$S^{lm} = g(l,m) \int -T_{0j} \, {}^{B}Y^{j*}_{lm} r^l d^3x$$

$${}^{B}Y^{lm} = \frac{r \times \nabla Y^{lm}}{\sqrt{l(l+1)}}$$

lowest order

$T_{00} = \rho \qquad T_{0j} = -\rho v_j$

GW emission from modes of oscillation

$\delta\rho(r,\theta,\varphi)\, e^{i\omega t} \qquad v = \alpha A(r,\theta,\varphi)\, e^{i\omega t}$

$$
\begin{cases}
\partial_t v^j + v^i \nabla_i v^j + \dfrac{\nabla^j P}{\rho} = -\nabla^j \phi \\
\partial_t \rho + \nabla^i (\rho v_i) = 0 \\
\nabla^2 \phi = 4\pi G \rho
\end{cases}
$$

$\downarrow$ perturb (assume a small perturbation)

$$\rho = \rho_0 + \delta\rho$$

$$v^i = 0 + \frac{\partial \xi^i}{\partial t}$$

$\downarrow$ do it

$$
\begin{cases}
\dfrac{\partial^2 \xi^j}{\partial t^2} = -\dfrac{\nabla^j \delta P}{\rho} + \dfrac{\delta\rho}{\rho}\dfrac{\nabla^j P}{\rho} - \nabla^j \delta\phi \\
\delta\rho + \nabla^i(\rho \xi_i) = 0 \\
\nabla^2 \delta\phi = 4\pi G \delta\rho
\end{cases}
$$

regular at the center

At the surface $r=R$ $\quad \Delta P = \delta P + \xi^i \nabla_i P(R) = 0$

$$\frac{\partial^2 \xi^i}{\partial t^2} = -\nabla^i \left(\frac{\delta P}{\rho}\right) + \frac{P}{\rho} \Gamma_1 A^i \left(\nabla^j \xi_j\right)$$ ← g modes

↑ sound waves $\quad c_s^2 = \left.\frac{\partial P}{\partial \rho}\right|_{ad}$ $\quad x_i = \frac{n_i}{n}$

$\Gamma_1 \Rightarrow \frac{\delta P}{P} = \Gamma_1 \frac{\delta \rho}{\rho}$ $\quad A^i = \frac{\nabla^i \rho}{\rho} \left(1 - \frac{\Gamma_e}{\Gamma_1}\right)$

$$\Gamma_e = \overbrace{\frac{\rho}{P} \frac{\partial P}{\partial \rho}}^{\Gamma_1}\Big|_{x_i, T} + \underbrace{\frac{\rho}{P} \frac{\partial P}{\partial x_i}\Big|_{\rho, T} \frac{\partial x^i}{\partial \rho}}_{< 0} + \frac{\rho}{P} \frac{\partial P}{\partial T}\Big|_{x_i, \rho} \frac{\partial T}{\partial \rho}$$

Generally in a NS $\Gamma_e \approx \Gamma_1$

in reality $\Gamma_e < \Gamma_1$

f-mode mode no nodes

Approximate $\delta\rho = \delta\Phi = 0$ $\quad \Gamma_e = \Gamma_1$ $\quad \rho = \text{const}$

$$\nabla \times \left(\frac{\partial^2 \xi^i}{\partial t^2} = -\frac{\nabla^i \delta P}{\rho}\right)$$

$$\nabla^i \xi_i = 0$$

$$\nabla \times \xi = 0 \qquad \xi^i = \nabla^i \psi$$

$$\psi = r^l Y_{lm} e^{i\omega t}$$ solution if $\omega^2 = l \frac{GM}{R^2}$

$$\omega \propto \sqrt{\rho}$$

full calculation $\omega^2 = \frac{l(l-1)}{2l+1} \frac{8\pi G\rho}{3}$ Kelvin

For CW searches r-mode interesting

Coriolis force restoring force

$$\delta v = \alpha \left(\frac{r}{R}\right)^l R \Omega Y^B_{lm} e^{-i\omega_r t}$$

$\omega_r = -\frac{2}{3}\Omega$ ($l=m=2$ leading order)

$\omega_i = \omega_r + m\Omega$ $\qquad \omega_i = \frac{4}{3}\Omega$ → Newtonian

in GR $\omega = f\left(\frac{M}{R}\right)$

$E_r = E_i - \Omega J \qquad E = -\partial_t \qquad J = \partial\varphi \qquad \frac{J}{E} = \frac{m}{\omega_i}$

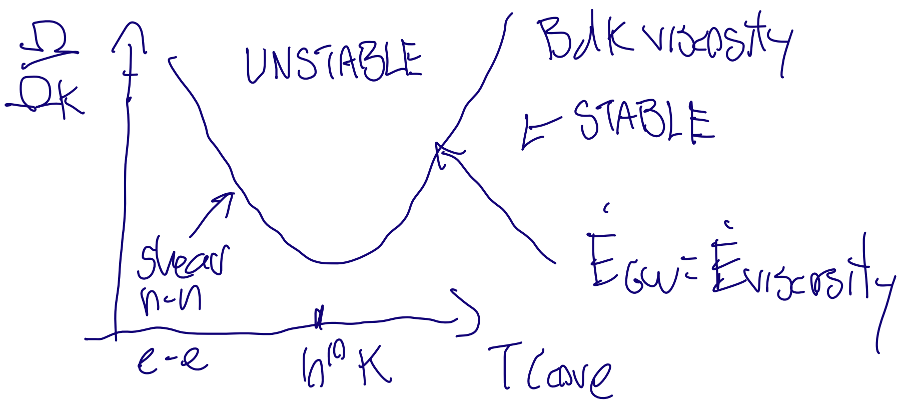
$$\dot{J} = \frac{m}{\omega_i}\dot{E}_i$$

$$\dot{E}_r = \dot{E}_i - \Omega\dot{J} = \dot{E}_i\left[1 - \frac{m\Omega}{\omega_i}\right] = \dot{E}_i\left[\frac{\omega_i - m\Omega}{\omega_i}\right] = \dot{E}_i \frac{\omega_r}{\omega_i}$$

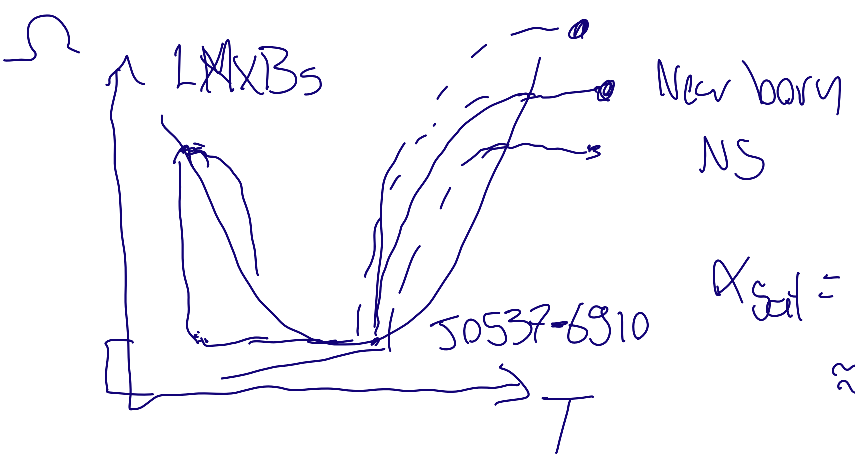
$\dot{E}_i = -L_{GW} \qquad \omega_i > 0 \qquad \omega_r < 0 \Rightarrow \dot{E}_r > 0$

mode grows on timescale $\frac{1}{\tau_{GW}} = \frac{\dot{E}_{GW}}{2E}$

Grows only if $|\dot{E}_{GW}| > |\dot{E}_{viscosity}|$



Highly EoS dependent

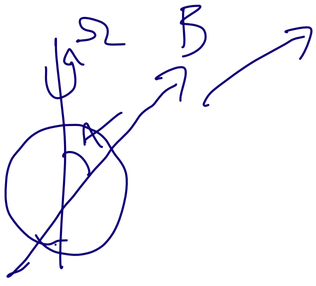


α_{sat} = depends on EoS

≈ 0.001

$\dot{\nu} = K\nu^{n}$ for r-modes $n=7$

Pulsar



Radio

P, $\dot{P}$

$$m = \frac{BR^3}{2}\hat{m}\, e^{i\Omega t} \qquad |m| = \frac{BR^3}{2}$$

$$\dot{E}_{rad} = -\frac{2}{3c^3}|\ddot{m}|^2 = -\frac{B^2R^6\Omega^4\sin^2\alpha}{6c^3}$$

$$\dot{E}_{rot} = -\dot{E}_{rad} \qquad \dot{E}_{rot} = \frac{d}{dt}\left(\frac{1}{2}I\Omega^2\right)$$

$\dot{\Omega} = \beta\,\Omega^3$ $n=3$ dipole EM radiation

$$B^2 = \frac{3c^3 I P\dot{P}}{8\pi^2R^6\sin^2\alpha}$$

$$B \gtrsim 3.2 \times 10^{19}\,G \left(\frac{P\dot{P}}{S}\right)^{1/2}$$

$M = 1.4\, M_\odot$
$R = 10$ Km

Age estimate

$\dot{P} = \frac{B}{P}$ integrate from $t=0 \longrightarrow \tau_c$

assume $P(0) \ll P(\tau_c)$

$\tau_c = \frac{P}{2\dot{P}}$ Assumes dipole rad. constant B

$\dot{\nu} = \alpha \nu^n$ $\qquad n = \frac{\nu \ddot{\nu}}{\dot{\nu}^2}$

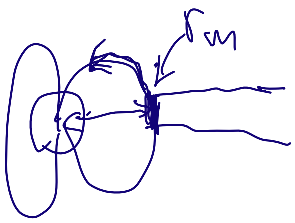
$n = 3 + \frac{\dot{\alpha}}{\alpha} \frac{\Omega}{\dot{\Omega}}$ if B changes decay $n > 3$

$n = 5$ GW mountains

$n = 7$ r-modes

$n = 1$ winds

$\dot{M}$ highly variable

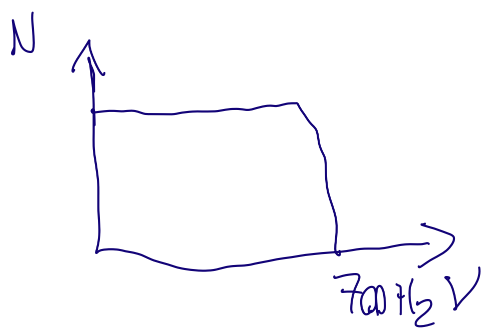


$l = \sqrt{GMr_m}$ $\quad N_{acc} = \dot{M}\sqrt{GMr_m}$

Simplest $r_m = R_{NS}$

$\nu_{max}^{T} \approx 1500\,Hz - 2000\,Hz$

$\nu_{max}^{OBS} \approx 700\,Hz$



Propeller $\quad N = \dot{M}\left(\sqrt{GMr_m} - R^2\Omega\right)$

What is r_m for a magnetised NS

B field truncate disk $\quad \frac{B^2}{8\pi} \sim \rho\frac{v^2}{2}$

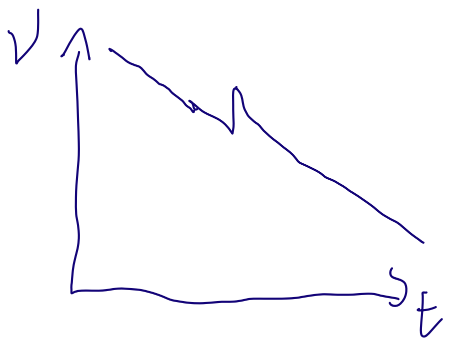
$$B = B_*\left(\frac{R}{r}\right)^3 \qquad r_m = \left(\frac{B_*^4 R^{12}}{GM\dot{M}^2}\right)^{1/7}$$

$$\approx 20\,\text{km}\left(\frac{B}{10^8\,G}\right)^{4/7}\left(\frac{\dot{M}}{10^{-9}M_\odot/\text{yr}}\right)^{-2/7}\left(\frac{M}{1.4M_\odot}\right)^{-1/7}$$

$N_{acc} = N_{GW}$ spin equilibrium $\nu \approx 700$ Hz

if $\varepsilon \approx 10^{-7}$

Glitches



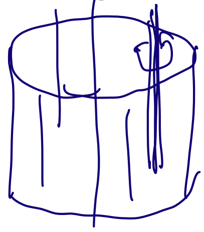
$$\frac{\Delta\nu}{\nu} \sim 10^{-10} - 10^{-5}$$

Superfluid neutrons (in crust?)

$$\psi = \sqrt{n}\, e^{i\varphi} \qquad |\psi^2| = n$$

$$J = \frac{\hbar}{2im}\left(\psi^* \nabla\psi - \psi\nabla\psi^*\right) = nv$$

$$J = \frac{\hbar}{m} n \nabla\phi \Rightarrow v = \frac{\hbar}{m}\nabla\phi \qquad \nabla\times v = 0$$

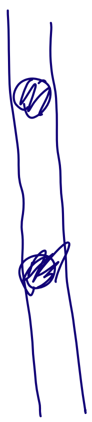


Ω

vortices

$$\oint v \cdot ds = \frac{h}{m_{pair}} \qquad \kappa = \frac{h}{2m_n} \qquad v_\varphi = \frac{\kappa l}{2\pi r}$$

$$2\Omega_n = \kappa n_v$$



$\rho < \rho_{nucl}$ pinning

ρ_{nucl} $\Delta E_{cond}(\rho)$

$$\dot{\Omega}_n = 0$$

$$\dot{\Omega}_p = -\frac{\beta \Omega^3}{I_p} \quad \text{pinned}$$

$$\dot{\Omega}_n = -2\Omega \tilde{B} \Delta\Omega^{np} \qquad \Delta\Omega^{np} = \Omega^n - \Omega^p$$

$$\dot{\Omega}_p = 2\Omega\tilde{B}\Delta\Omega^{np} - \frac{\beta\Omega^3}{I_p}$$

$$\tau = \frac{x_p}{2\Omega \tilde{B}} \qquad \tau \approx 1\,s \quad \text{core}$$

microphysics

neglect $\frac{\beta \Omega^3}{I_p}$ $\quad$ $\Delta\Omega \ll \Omega$

$$\Delta\Omega = \Delta_0 \exp\left(\frac{-t}{\tau}\right)$$

$$J_i = I_p \Omega_p + I_n(\Omega_p + \Delta\Omega) = (I_p + I_n)\Omega_p + I_n \Delta\Omega$$

$$J_F = (I_p + I_n)(\Omega_p + \Delta\Omega^{OBS})$$

$$J_i = J_F$$

pinning

pinning region

$$\Delta\Omega^{OBS} = \Delta\Omega \frac{I_n}{I_p + I_n}$$

Probably $\quad \frac{p_n}{m_n} = v_n + \varepsilon(v_n - v_p)$

Chamel 2013 $\quad \varepsilon_n$ large negative

CW bursts from glitch

$|\dot{v}|_{GL} > |\dot{v}|_{BEFORE}$