$M_\odot = 2\times10^{33}\,g$ $\qquad$ $M_\oplus \approx 6\times10^{27}\,g$

$R_\odot \approx 700{,}000\,km$ $\qquad$ $R_\oplus \approx 6{,}000\,km$

$\bar{\rho}_\odot \approx 2\,g/cm^3$ $\qquad$ $\bar{\rho}_\oplus \approx 5\,g/cm^3$

$\rho_{c\odot} \approx 150\,g/cm^3$

$M \approx 1.4\,M_\odot$ $\qquad$ $\bar{\rho}_{NS} \approx 7\times10^{14}\,g/cm^3$

$R \approx 10\,km$

$$A = Z + N$$

$\bar{\rho}_{nucl} \approx \frac{A}{R_{Nu}^3}$ $\qquad$ $R_{Nu} \propto A^{1/3}$

$\bar{\rho}_{NS} > \rho_{nucl}$

$\rho_{nucl} \approx const \approx 2.3\times10^{14}\,g/cm^3$

E_{Grav} $\frac{GM^2}{R}$ $E_G^{NS} \simeq 5\times10^{53}\,\text{erg} \approx 0.2\,Mc^2$

$E_G^{WD} \approx 2\times10^{-4}\,Mc^2$

$R \simeq 10\,\text{km}$ $R_{SCH} \approx 4\,\text{km}$

GR to model structure

Matter interior

$n = \frac{\rho}{m_n}$ $(m_n \simeq m_p)$ N_{baryon} conserved

$\bar{n} \approx 0.5\,\text{fm}^{-3}$ $\bar{l} \approx \frac{1}{\bar{n}^{1/3}} \approx 1\,\text{fm}$

$\bar{l}_{nucl} \approx 2\,\text{fm}$ $\bar{l}_{WD} \approx 10^3\,\text{fm}$ $\bar{l}_{\odot} \approx 10^5\,\text{fm}$

Compare

$\lambda^{DB} = \frac{h}{p}$ $\bar{l} \gg \lambda^{DB}$ no overlap WF

$$\lambda^{\text{Compton}} = \frac{h}{mc}$$

$\Delta x \Delta p \geq \frac{\hbar}{2}$ take $\Delta x = \lambda^{\text{com}}$ $\Delta p \geq \frac{\hbar}{2\lambda_c} \approx mc$

$\bar{l} << \lambda^{\text{com}}$ $\Delta pc >> mc^2$ $p = \frac{E}{c}$ Ultrarelativistic

$\bar{l} >> \lambda^{\text{compton}}$ $\Delta pc << mc^2$ $p = \sqrt{2mE_K}$ non Rel.

Start λ^c

$\lambda^c_{e^-} \approx 2400$ fm $\bar{l}_{NS} << \lambda^c_{e^-}$ e^- UR

$\lambda^c_n \approx 1.3$ fm $\lambda^c \approx \bar{l}_{NS}$

λ^{DB}

e^- $p = \frac{E}{c}$ $E_{th} \approx \frac{3}{2} k_B T$ $T \approx 10^8$ K

$\lambda^{DB} \approx 10^5$ fm $\bar{l}_{e^-} << \lambda^{DB}_{e^-}$ degenerate

n $p = \sqrt{2mE_{th}}$ $\lambda^{DB} \approx 200$ fm

$\bar{l} << \lambda^{DB}_n$ degenerate

	Gravity	SR	QM
WD	Newt. Gravity	Rel e^-	Deg e^-
NS	GR corrections	NR e^- / Rel n	Deg
BH	GR	✓	✓
Sun	Newt. Gravity	Non Rel	non Deg.

$$\frac{dP}{dr} = -\frac{G\rho M}{r^2}$$

$$\frac{dM}{dr} = 4\pi r^2 \rho$$

$$P = P(\rho, T)$$ ideal EoS $P = k\rho T$

$$\frac{dT}{dr} = -\frac{3\hat{\kappa}\rho L}{16\pi a c T^3 r^2}$$ $a = \frac{4\sigma_{SB}}{c}$

$$\frac{dL}{dr} = 4\pi r^2 \rho \, \varepsilon(T)$$

No reactions → collapse

$M_p \lesssim 8 M_\odot \rightarrow$ WD

$8 \leq M_p \lesssim 25 M_\odot \rightarrow$ NS

$p + e^- \rightarrow n + \nu_e$

$M_p \gg 25 M_\odot \rightarrow$ BH

$P \rightarrow P(\rho)$

ρ high enough $kT \ll E_F$

probability that a state of energy E is occupied (Fermions)

$$f(E) = \frac{1}{e^{\frac{E-\mu}{kT}} + 1}$$

μ chemical potential

$$\lim_{T \to 0} f(E) = \begin{cases} 1 & E < E_F \\ 0 & E > E_F \end{cases}$$

$$\frac{dN}{d^3x} = \frac{dN_{states}}{d^3x} f(p) \leftarrow \text{occupation}$$

$$\frac{dN_{st}}{d^3x} = \frac{g_s}{h^3} d^3p \qquad g_s = 2S+1$$

neutrons $S=\frac{1}{2}$ $g_s=2$

$$n = \int_0^{p_F} \frac{dN}{d^3x} d^3p = \int_0^{p_F} \frac{g_s}{h} 4\pi p^2 dp = \frac{8\pi}{3h^3} p_F^3$$

$$\Rightarrow p_F = \hbar (3\pi^2 n)^{1/3}$$

$$\mathcal{E} = \frac{g_s}{h^3} \int E f(p) d^3p$$

$$P = \frac{1}{3} \frac{g_s}{h^3} \int p v f(p) d^3p$$

$\Rightarrow P(\rho)$ Non interacting Fermi gas

$$P = K\rho^\Gamma \qquad \Gamma = 5/3 \text{ non Rel.} \qquad \Gamma = 4/3 \text{ UR}$$

e^- $E_F = mc^2$ $\rho \simeq 10^6$ g/cm³ $\qquad$ $n \to 6\times10^{15}$ g/cm³

Solve Newt. eq. (Lane-Emden)

$M \propto R^{\frac{3-n}{1-n}}$ $\qquad \Gamma = 1 + \frac{1}{n}$

$\Gamma = \frac{5}{3}$ NR $\qquad M \sim \frac{1}{R^3}$

$\Gamma = 4/3$ $\qquad M = M_{ax}$

(WD $M_{Ch} \approx 1.4 M_\odot$)

$n = 1$ $\quad \Gamma = 2$ $\qquad \rho = \rho_c \frac{\sin \xi}{\xi}$ $\quad \xi = \frac{r\pi}{R}$

Core NS $\Gamma \approx 2$

$M_{max}^{[Ch]}$ (Newton) $\approx 6 M_\odot$ WRONG

Missing GR

How to add it?

$$d\tau^2 = -e^{-2\phi} dt^2 + e^{2\Lambda} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta \, d\varphi^2 \right)$$

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} \qquad T^{\alpha\beta} = (\rho + p) U^\alpha U^\beta + p g^{\alpha\beta}$$

TOV equations

$P \propto \rho \frac{\langle v \rangle^2}{3}$ $\qquad \frac{P}{c^2} \ll O\left(\frac{v}{c}\right)^2$

$$\frac{dP}{dr} = -\frac{G\rho M}{r^2} \frac{\left[1 + \frac{P}{\rho c^2}\right]\left[1 + \frac{4\pi r^3 P}{M c^2}\right]}{\underbrace{1 - \frac{2GM}{c^2 r}}_{1 - \frac{R_{sch}}{r}}}$$

$\frac{2GM}{c^2 r} \rightarrow R_{sch}$

$$\frac{dM}{dr} = 4\pi r^2 \rho$$

Newt $\Gamma = 4/3$

m 3

GR unst ρ

($\Gamma = 5/3$ non int neutrons)

$M_{max} \simeq 0.8\, M_\odot$

OBSERVED (Binary Pulsars)

$M_{max} \approx 2\, M_\odot$

Need interactions

^{56}Fe

Pressure ionized $\bar{l} \approx a_0 = \frac{\hbar}{Z\alpha m c}$ For $^{56}Fe \approx 10^6 g/cm^3$ (really $10^4 g/cm^3$)

$E^{e^-}_{Fermi} = kT$?

$\frac{p_F^2}{2m_e} = kT$ $\rho \approx 10 \, g/cm^3 \left(\frac{T}{10^6 K}\right)^{3/2}$ degenerate

UR $\rho \gtrsim 10^6 \, g/cm^3$

p, n ? Ions

$kT = \frac{Z^2 e^2}{\bar{l}}$ inter ion

Polytropic Γ

$\Gamma = \frac{Z^2 e^2}{\bar{l} kT}$

$\Gamma > 1$ liquid

$\Gamma \gtrsim 175$ solid

^{56}Fe $\Gamma > 175$ $\rho \gtrsim 100 \left(\frac{T}{10^6 K}\right)^3 g/cm^3$

Crystalline crust: ions + sea e^-

$P \approx K \rho(n_e)^{5/3}$ + Electrostatic corrections

$n \to p + e^- + \bar{\nu}$

$e^- + p \to n + \nu$

Beta eq. Cooling

$\mu_e + \mu_p = \mu_n$

happens when

$Q = m_n - m_p = 1.29 \ \frac{\text{MeV}}{c^2}$

$E_e^F > Q$

$\rho \gtrsim 10^7 \ g/cm^3$

$\frac{n_p}{n_n} < 1$ non. int. Fermi gas

$\frac{n_p}{n_n} \to \frac{1}{8}$ $\rho \to \infty$

How to describe the NS crust $\rho \gtrsim 10^7 \ g/cm^3$

?

$$\mathcal{E} = \mathcal{E}_e(n_e) + \mathcal{E}_n(n_n) + \mathcal{E}_{lattice} + n_{nuclei} M(A,Z)$$

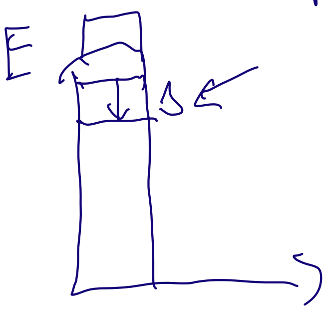
$$M(A,Z) = Z(m_p + m_e)c^2 + (A-Z)m_n c^2 - BE = \mathcal{E}(n_b, A, Z, n_n)$$

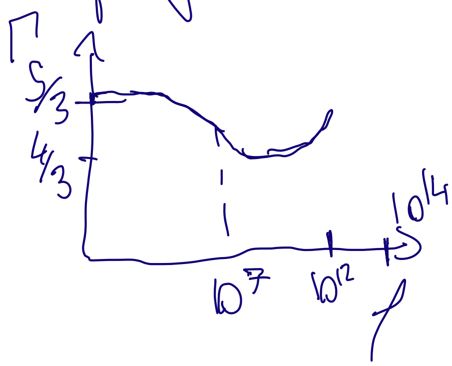
On Earth just $M(A,Z)$ $\left.\frac{\partial E}{\partial Z}\right|_A = 0$ $Z \sim \frac{A}{2}$

In a NS crust $\frac{Z}{A} \sim \frac{1}{A^{1/3}}$

at $\rho_{ND} \approx 4\times10^{11}\ g/cm^3$ $\quad n_n > 0$

neutrons pair → superfluid $\quad kT \ll \Delta$ pairing





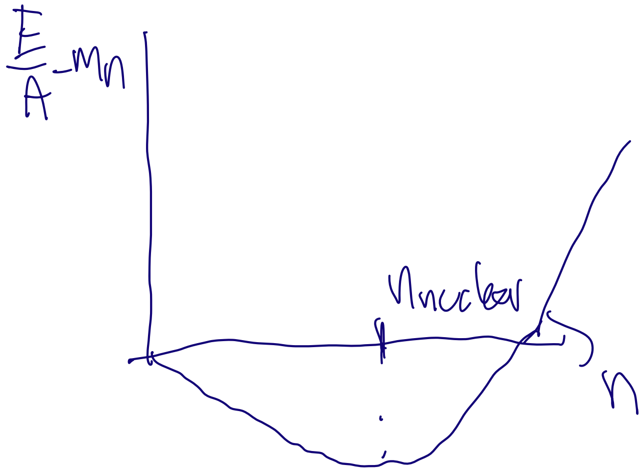
Above n_{sat} → fluid core

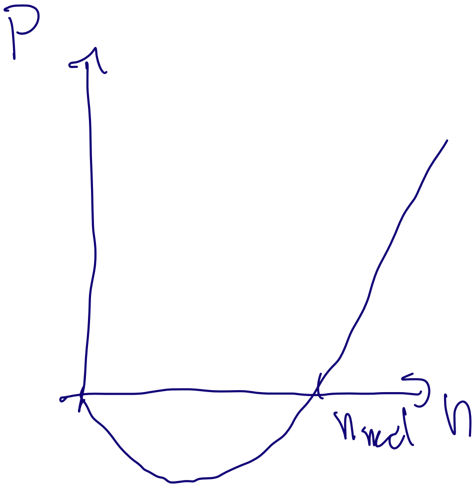
$$\mathcal{E} = \mathcal{E}(n_e) + \mathcal{E}(n_n) + \mathcal{E}(n_p) + V(n_b)$$

What is $V(n_b)$?
What are the constraints?

$$BE = \frac{\mathcal{E}}{A} - m_n c^2 = -16\ \mathrm{MeV}$$

$$\frac{d}{dn}\left(\frac{\mathcal{E}}{n}\right) = 0 \quad n = n_{nuclear}$$





true $n \lesssim n_{nucl}$ but also $\frac{Z}{A} \simeq \frac{1}{2}$

$$\alpha = \frac{n_n - n_p}{n} = \frac{N-Z}{A}$$

correct $n = n_{nucl}$

$$V(n,\alpha) \simeq V(n,0) + \alpha^2 S(n)$$ Symmetry

$$S(\rho) = S_0 + \frac{1}{3} L \left(\frac{\rho - \rho_{nucl}}{\rho}\right) + \frac{1}{8} K_{sym} \left(\frac{\rho-\rho_0}{\rho}\right)^2$$

$$L = 3\rho_0 \left(\frac{\partial S}{\partial \rho}\right)_{\rho=\rho_0} \qquad K_{sym} = 9\rho_0 \left(\frac{\partial^2 S}{\partial \rho^2}\right)_{\rho=\rho_0}$$

Construct models

$M_{max} \gtrsim 2 M_\odot$ $\quad \Gamma \approx 2$

Haensel, Potekhin, Yakovlev → Neutron Stars 1

Chamel & Haensel 2008 LRR 11, 10

High density

$n + n \to n + \Lambda$

$n + n \to p + \Sigma^-$

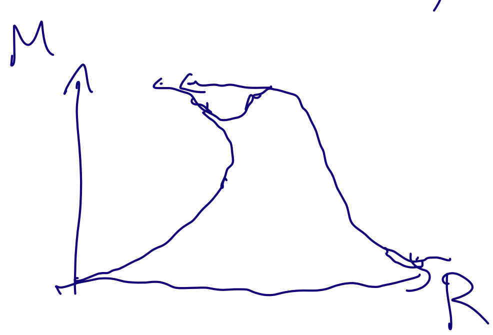
MIT bag model

m = 0 bag
m = ∞ outside

$\varepsilon = \alpha n^{4/3} + B$ $\qquad$ $P = \frac{\rho c^2}{3} - \frac{4B}{3}$ self bound

$\rho < \rho_0 = \frac{B}{c^2} \approx 5 \times 10^{14}\ g/cm^3 \Rightarrow \rho$ const.

$M = \frac{4}{3} \pi \rho_0 R^3$



Additional

- Rotation
- B field
- Crustal elasticity

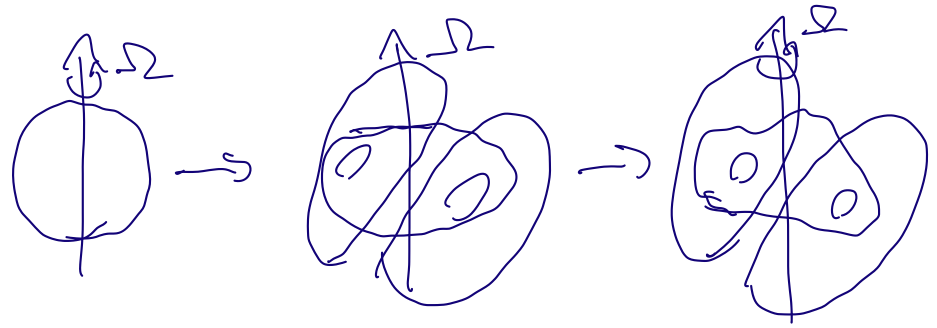
→ EM emission

non-spherical

↓

GW emission

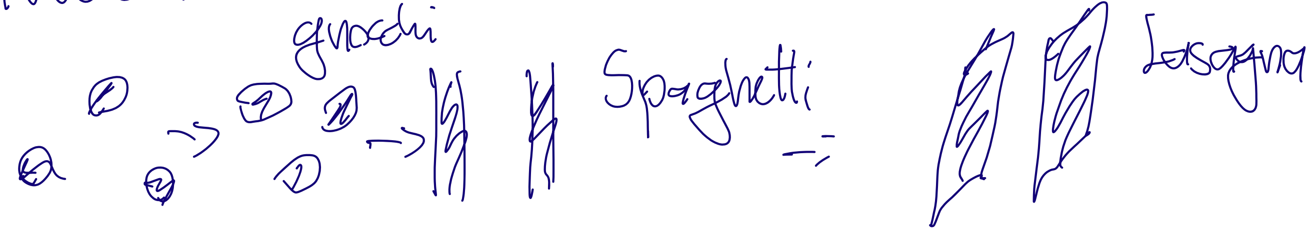
- Thermal effects



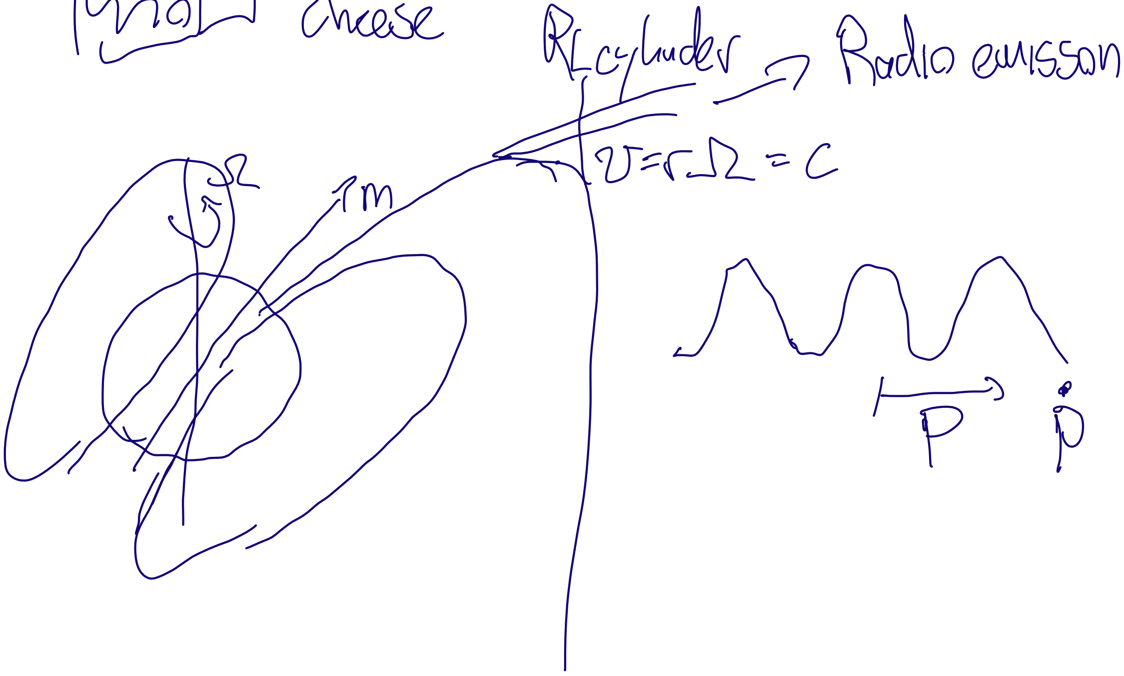
$$I_{jk} = \int_V \rho(r,\theta,\varphi)\,(r^2\delta_{jk} - x_j x_k)\, dV$$

$$\varepsilon_{ROT} = \frac{I_{xx} - I_{zz}}{I_{zz}} \qquad \varepsilon_{GW} \simeq \frac{I_{xx} - I_{yy}}{I_{zz}}$$

Nuclear Pasta $\quad n \simeq n_{nuclear}$



Swiss cheese



$$\frac{\nabla P}{\rho} = -\nabla \phi + F_L$$ Lorentz Force

$\sigma \to \infty$ ideal MHD

long wavelength, low freq. limit

$L \gg \lambda_{screen} \to$ charge neutral $\quad \lambda_{TF} \approx 0.01\,\mathrm{fm}$

$\tau \sim 10^{-15}\,s \quad \Omega \ll \Omega_c = \frac{qB}{m_i c} \approx 10^{15}\,\mathrm{Hz}$

cgs units

$$\nabla E = 4\pi \rho_q$$

$$\nabla B = 0$$

$$\nabla \times E = -\frac{1}{c}\frac{\partial B}{\partial t}$$

$$\nabla \times B = \frac{1}{c}\left(4\pi J + \cancel{\frac{\partial E}{\partial t}}\right) \qquad \frac{\partial E}{\partial t} \approx \frac{E}{t_{dyn}} \ll 1$$

$$\Rightarrow J = \frac{c}{4\pi} \nabla \times B$$

$$F_L = \cancel{\overset{\sim 0}{\rho_c E}} + \frac{J \times B}{c} = -\frac{B \times (\nabla \times B)}{4\pi}$$

$$\frac{\partial B}{\partial t} = \nabla \times (\overset{\text{dynamo}}{v \times B}) - \nabla \times (\eta \nabla \times B) \qquad \eta = \frac{c^2}{4\pi\sigma}$$

$$\frac{\partial B}{\partial t} = 0 \quad \text{MHD} \qquad \begin{matrix} \eta \to 0 \\ \sigma \to \infty \end{matrix}$$

$\Downarrow$

Flux freezing $\quad \Phi = \int_S B \cdot dS \qquad \frac{\partial \Phi}{\partial t} = 0$

$$\nabla \times \left(\frac{\nabla P}{\rho}\right) = \nabla \times \left(-\nabla \phi + \frac{(\nabla \times B) \times B}{4\pi\rho}\right)$$

$$\rho = \rho_{sph} + \delta\rho(r,\theta,\varphi) \qquad \delta\rho \approx O(B^2) \nearrow O\left(\frac{E_{mag}}{E_{grav}}\right)$$

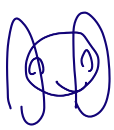
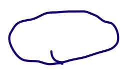
$\nabla \times \left(\frac{F_L}{\rho}\right) = 0$ 1^{st} order $\rho = \rho_{spherical}$

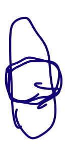
Model for B field, given EoS

$$\text{Spherical} \xrightarrow{1^{st}\text{ order}} B \longrightarrow \delta\left(\frac{\nabla P}{\rho}\right) = -\delta(\nabla\phi) + F_L$$

$\frac{\nabla P}{\rho} = -\nabla\phi$ $\qquad \nabla \times \left(\frac{F_L}{\rho}\right) = 0$ $\qquad \downarrow \delta\rho$

From $\delta\rho$ $\quad \varepsilon = \frac{I_{xx} - I_{yy}}{I_{zz}}$ $\qquad \varepsilon \approx \pm 10^{-12}\left(\frac{B}{10^{12} G}\right)^2$

Poloidal  $\varepsilon = \oplus$ 

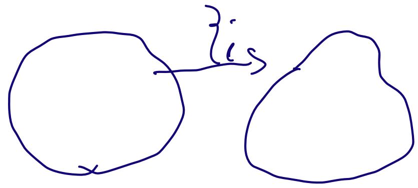
Toroidal  $\varepsilon = \ominus$ 

Purely Poloidal/Toroidal unstable

Mastrano 2011

$$\Lambda = \frac{\int B_p^2 \, dV}{\int B^2 \, dV} \qquad \mathcal{E} = 5\times10^{-7}\left(\frac{B}{10^{14}G}\right)^2\left(1-\frac{0.385}{\Lambda}\right)$$

Elasticity in the crust



Strain tensor $\quad t_{ik} = \frac{1}{2}\left(\frac{\partial \xi_i}{\partial x_k} + \frac{\partial \xi_k}{\partial x_i}\right)$

$$t_{ik} = t_{ik}^c + t_{ik}^s \qquad t_{ik}^c = \frac{1}{3}\delta_{ik}\nabla^k \xi_k \qquad dV' = dV(1+\nabla^i\xi_i)$$

$$t_{ik}^s = t_{ik} - t_{ik}^c$$

Equilibrium

$$\nabla_j T^{ij} - \rho\nabla^i\phi = 0 \qquad T_{ik} = -p\,\delta_{ik} + 2\overset{\text{shear modulus}}{\mu}\, t_{ik}^s$$

$\sigma_{ik} = \sqrt{\frac{3}{2}t_{ik}^s}$ how large before it breaks?

$\Rightarrow \sqrt{\sigma^{ab}\sigma_{ab}} < \sigma_{max}$

Van Mises criterion

From simulations $\sigma_{max} \simeq 0.1$ Horowitz + Kadau 2009

$$\varepsilon_{max} \simeq 10^{-6} \left(\frac{\sigma_{max}}{0.1} \right) \quad \text{Crust}$$

Crystalline phases in the core (quark) $\varepsilon_{max}^{CORE} > \varepsilon_{max}^{crust}$

$\varepsilon_{max}^{CFL} \simeq 10^{-4}$

Gravitational waves from isolated NSs

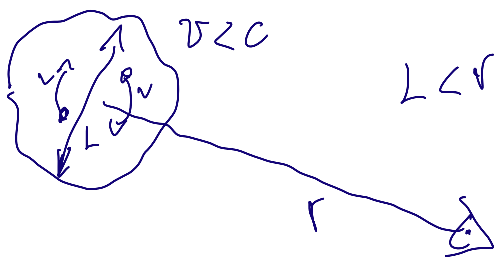
$c = G = 1$

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} \qquad |h| \ll 1 \qquad \Box = -\frac{\partial^2}{\partial t^2} + \nabla^2$$

Einstein Equations $\Box \bar{h}_{\alpha\beta} = -16\pi T_{\alpha\beta}$

$$\bar{h}_{\alpha\beta} = h_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} h^{\gamma}{}_{\gamma} \qquad \bar{h}^{\alpha}{}_{\beta,\alpha} = 0$$

$$\bar{h}_{\alpha\beta}(t,x) = 4 \int d^3x' \, \frac{T_{ab}(t - |x - x'|, x')}{|x - x'|}$$



$L < r$

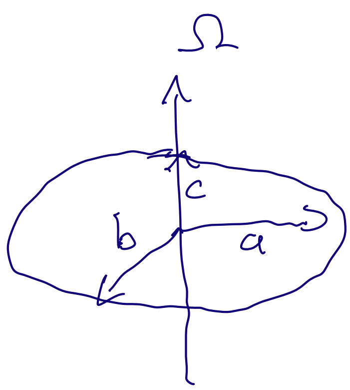
$$\frac{1}{|x-x'|} \simeq \frac{1}{r} + O\left(\frac{L}{r}\right)$$

$$q^{jk} = \int T^{00}(t, x^m)\, x^j x^k\, d^3x$$

$$Q_{ij} = q_{ij} - \frac{1}{3}\delta_{ij}\, q^k{}_k$$

$$\bar{h}^{ik} = \frac{2}{r}\frac{d^2}{dt^2} Q^{ik} + O\left(\frac{v}{c}\right) + \ldots$$

$$L_{GW} = \frac{dE_{GW}}{dt} = \frac{1}{5}\left[\dddot{Q}^{ij}\dddot{Q}_{ij}\right]$$ lowest order



$a \neq b \neq c$

$I_x \neq I_y \neq I_z$

$$I_r = \begin{pmatrix} I_x & & \\ & I_y & \\ & & I_z \end{pmatrix}$$

$$I^j_{\ j} = I_x + I_y + I_z = \text{constant}$$

$$Q_{ij} = I_{ij} + \text{constant}$$

$$I = R^T I_r R \qquad R = \begin{pmatrix} \cos\Omega t & \sin\Omega t & 0 \\ -\sin\Omega t & \cos\Omega t & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$I_{xx} = -I_{yy} = \frac{1}{2}(I_x - I_y)\cos 2\Omega t + \text{cost.}$$

$$I_{xy} = I_{yx} = \frac{1}{2}(I_x - I_y)\sin 2\Omega t + \text{const.}$$

$$\dot{E} = \frac{G}{5c^5}\left\langle \dddot{I}_{xx}^{\,2} + 2\dddot{I}_{xy}^{\,2} + \dddot{I}_{yy}^{\,2} \right\rangle = \frac{32G}{5c^5}(I_x - I_y)^2 \Omega^6$$

$$= -\frac{32G}{5c^5}\varepsilon^2 I_z^2 \Omega^6 \qquad \varepsilon = \frac{I_x - I_y}{I_z}$$

$$E_{rot} = \frac{1}{2} I \Omega^2 \qquad \dot{E}_{rot} = -L_{GW}$$

$$\dot{\Omega} = -\frac{32G}{5c^5} I_z \varepsilon^2 \Omega^5 \qquad \dot{\Omega} = \alpha \Omega^n$$

braking index

for GW "mountains" $n=5$

dipole EM $n=3$

Spindown Limit

Crab pulsar $\qquad \nu \approx 30\,\mathrm{Hz}$ (from radio)

$\dot{\nu} = -3.77 \times 10^{-10}\ \frac{\mathrm{Hz}}{\mathrm{s}}$

entirely GW driven $\Longrightarrow \varepsilon_{SD} \approx 7 \times 10^{-4}$