Analysis of a quantum Digital-Analog algorithm for the Heisenberg model

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Outline

- Modeling a system of neutrinos
- Simulate neutrino dynamics on a quantum computer
  - Digital-Analog approach: first order Trotter
  - Digital-Analog approach: second order Trotter
  - Results
- Further steps

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Modeling a system of neutrinos

Neutrinos are particles equipped with a *flavor* degree of freedom.

Investigate the collective neutrino dynamics of a system of neutrinos on a quantum computer.

Two-flavor SU(2) model: direct mapping between flavor and qubit states

$$\nu_e \mapsto |0\rangle, \quad \nu_x \mapsto |1\rangle$$

$$H = H_{\text{vac}} + H_{\nu\nu} = \sum_{i=1}^{N} \vec{b} \cdot \vec{\sigma}_i + \sum_{\substack{p,q=1 \\ p<q}}^{N} g_{ij}\ \vec{\sigma}_i \cdot \vec{\sigma}_j$$

Heisenberg Hamiltonian

The geometry of the system depends on the structure of the couplings constants.

Peaked neutrino beam

Consider a dense gas of neutrinos forming a peaked beam.

- (self-refracting) dense neutrino gas

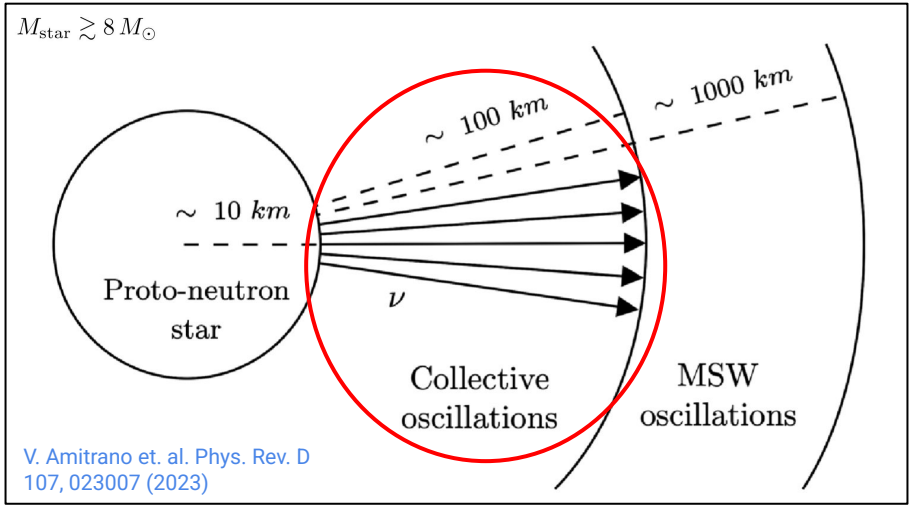
$$g_{ij} \coloneqq \frac{\mu}{N}\left(1-\cos\theta_{ij}\right), \quad \mu = \sqrt{2}G_F n_\nu$$

- peaked beam: fix grid of angles

$$\theta_{ij} = \arccos(\upsilon)\frac{|i-j|}{N-1}, \quad \upsilon \in [-1,1)$$

The parameter υ is related to the "peakness" of the beam.

Also: $\Theta \coloneqq \max_{ij}\left[1-\cos\theta_{ij}\right]$



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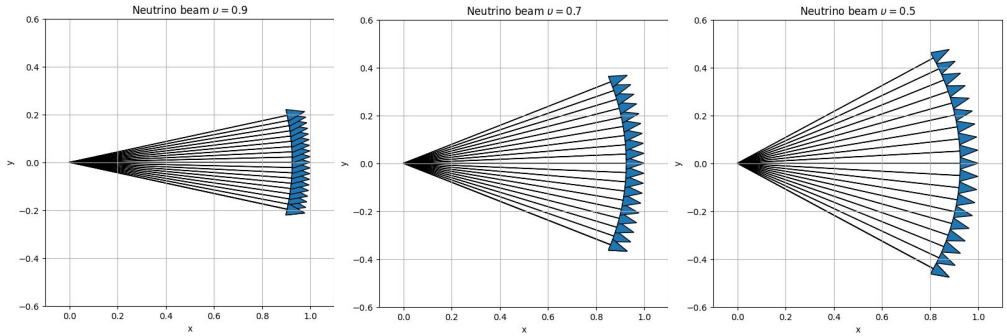

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Unitary implementation of Heisenberg propagator

Target quantum evolution: unitary time evolution operator $U(dt) = e^{-iHdt}$

$$U(dt) = e^{-i(H_{\text{vac}}+H_{\nu\nu})dt} = e^{-iH_{\text{vac}}dt}\ e^{-iH_{\nu\nu}dt} := U_1(dt)U_2(dt)$$

$$[H_{\text{vac}}, H_{\nu\nu}] = 0$$

Free evolution: easy, $U_1(dt)$ is the direct product of single qubit rotations.

Interaction evolution:

- digital approach: combine 1 and 2-qubit gates to approximate $U_2(dt)$.
- digital-analog approach: combine native evolution of the chip (analog gate) and single qubit gates to approximate $U_2(dt)$.

Digital-Analog approach: construction of the analog gate

Decomposition of the interaction hamiltonian:

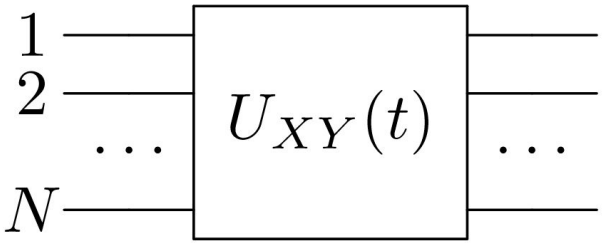
$$H^N_{\nu\nu} = H^N_{XY} + H^N_{XZ} + H^N_{YZ}$$

$$H_{AB} = \sum_{\substack{p,q=1 \\ p<q}}^{N} \frac{g_{pq}}{2} \left(A_p A_q + B_p B_q\right)$$

All-to-all capacitively coupled qubits at resonance in RWA:

$$\tilde{H}|_{\text{res}} = \sum_{\substack{p,q=1 \\ p<q}}^{N} \frac{g_{pq}}{2} \left(X_p X_q + Y_p Y_q\right)$$

Analog gate: $U^N_{XY}(t) = \exp\left[-iH^N_{XY}t\right]$



We use the analog gate as the building block of the simulation.

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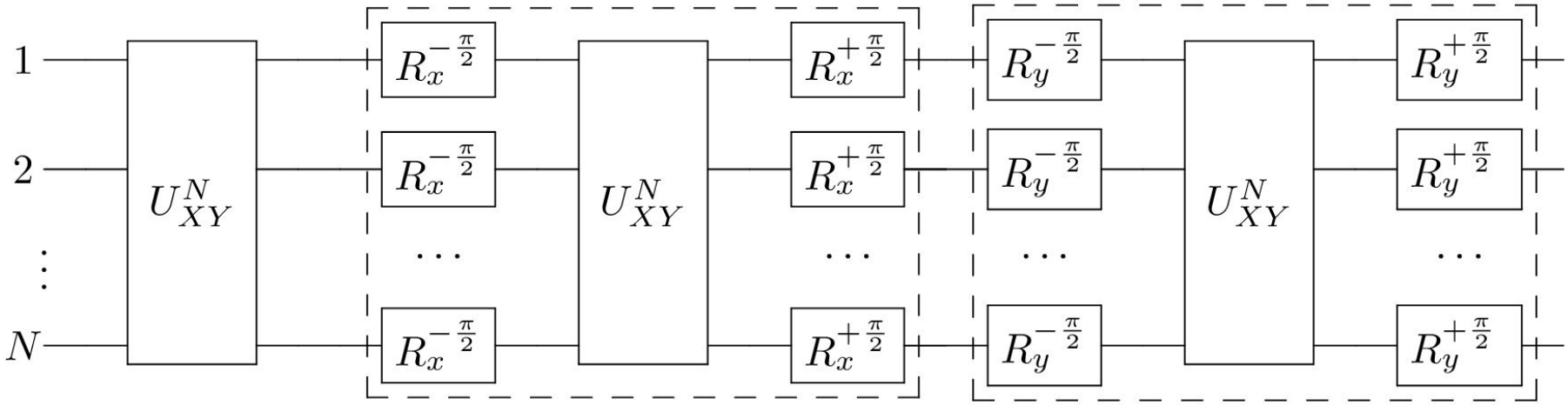
First order digital-analog algorithm

Direct extension of the same circuit proposed for two qubits by Salathé et al. [2].



The circuit represents the single step of the first order Trotterization of the Heisenberg model in the digital-analog approach.

First order error bounds with small prefactors

The error of the approximation for a small time step is determined by a result of Childs et al. [3]:

$$\varepsilon_1(dt) := \left\| \mathcal{S}_1(dt) - e^{-iHt} \right\| \leq \frac{dt^2}{2} \sum_{\gamma_1=1}^{\Gamma} \left\| \left[\sum_{\gamma_2=\gamma_1+1} H_{\gamma_2}, H_{\gamma_1} \right] \right\|$$

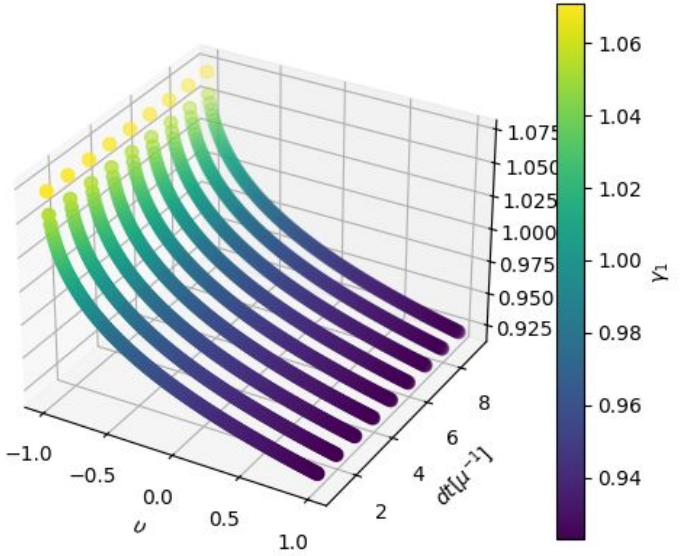
$$\Gamma = 3: \ \varepsilon_1(dt) \leq \frac{dt^2}{2} \left\{ \left\| \left[H_{XZ}^N, H_{XY}^N \right] \right\| + \left\| \left[H_{YZ}^N, H_{XY}^N \right] \right\| + \left\| \left[H_{YZ}^N, H_{XZ}^N \right] \right\| \right\}$$

In order to determine the scaling of the error in terms of the parameter of the system, we developed numerical routines to compute this expression with complete generality.

First order Trotter scalings

- determine the N dependance of the error
- fit data points generated by numerical routines
- first order error model:

$$e_1\left(dt, N, \{g_{ij}\}; K_1, \gamma_1\right) = \frac{1}{8} K_1 \mu^2 \Theta^2 dt^2 N^{\gamma_1}$$



Single-step error	Number of steps	Circuit complexity
$\mathcal{O}(\mu^2 dt^2 N^{\gamma_1})$	$\mathcal{O}\left(\mu^2 \frac{T^2}{\varepsilon} N^{\gamma_1}\right)$	$\mathcal{O}\left(\mu^2 \frac{T^2}{\varepsilon} N^{\gamma_1+1}\right)$

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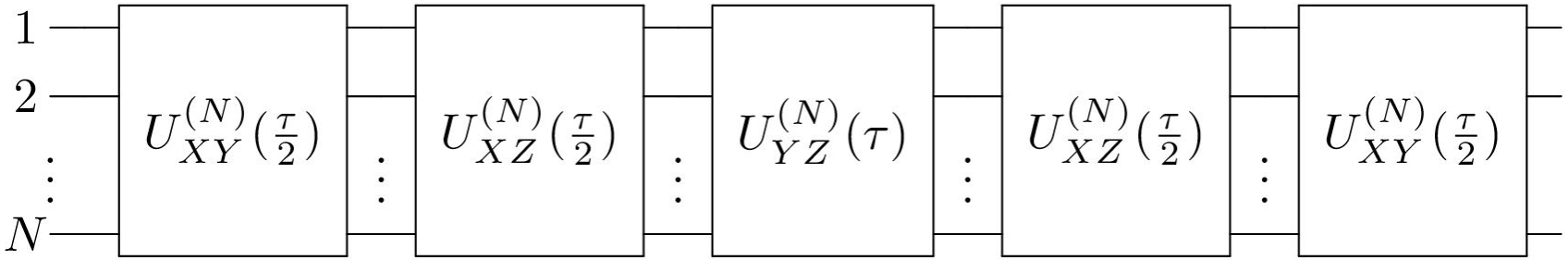
Second order digital-analog algorithm



The circuit represents the single step of the second order Trotterization of the Heisenberg model in the digital-analog approach.

First order error bounds with small prefactors

The error of the approximation for a small time step is determined by a result of Childs et al. [3]:

$$\varepsilon_2(dt) := \left\| \mathcal{S}_2(dt) - e^{-idtH} \right\| \leq \frac{dt^3}{12} \sum_{\gamma_1=1}^{\Gamma} \left\| \left[\sum_{\gamma_3=\gamma_1+1}^{\Gamma} H_{\gamma_3}, \sum_{\gamma_2=\gamma_1+1}^{\Gamma} [H_{\gamma_2}, H_{\gamma_1}] \right] \right\| + \frac{dt^3}{24} \sum_{\gamma_1=1}^{\Gamma} \left\| \left[H_{\gamma_1}, \left[H_{\gamma_1}, \sum_{\gamma_2=\gamma_1+1}^{\Gamma} H_{\gamma_2} \right] \right] \right\|$$

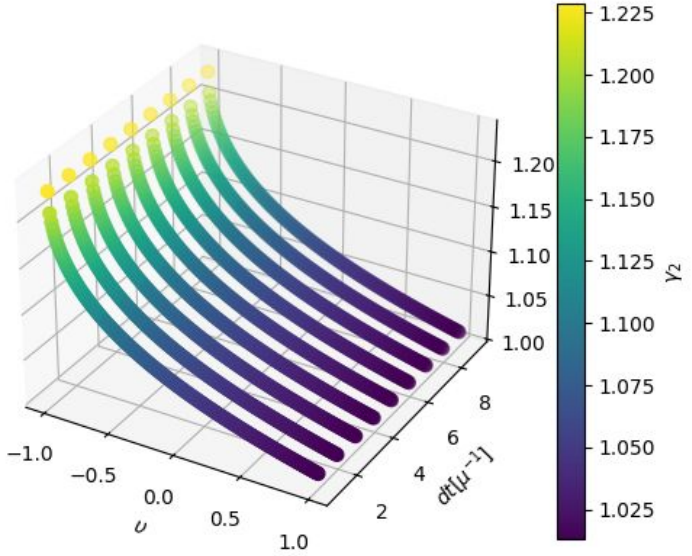
$$\begin{aligned} \Gamma = 3: \ \varepsilon_2(dt) \leq \frac{dt^3}{12} \Big\{ & \|[H_{XZ}^N, [H_{XZ}^N, H_{XY}^N]]\| + \|[H_{XZ}^N, [H_{YZ}^N, H_{XY}^N]]\| + \|[H_{YZ}^N, [H_{XZ}^N, H_{XY}^N]]\| + \\ & + \|[H_{YZ}^N, [H_{YZ}^N, H_{XY}^N]]\| + \|[H_{YZ}^N, [H_{YZ}^N, H_{XZ}^N]]\| \Big\} + \\ & + \frac{dt^3}{24} \Big\{ \|[H_{XY}^N, [H_{XY}^N, H_{XZ}^N]]\| + \|[H_{XY}^N, [H_{XY}^N, H_{YZ}^N]]\| + \|[H_{XZ}^N, [H_{XZ}^N, H_{YZ}^N]]\| \Big\} \end{aligned}$$

In order to determine the scaling of the error in terms of the parameter of the system, we developed numerical routines to compute this expression with complete generality.

Second order Trotter scalings

- determine the N dependance of the error
- fit data points generated by numerical routines
- second order error model:

$$e_2\left(dt, N, \{g_{ij}\}; K_2, \gamma_2\right) = \frac{1}{96} K_2 \mu^3 \Theta^3 dt^3 N^{\gamma_2}$$



Single-step error	Number of steps	Circuit complexity
$\mathcal{O}(\mu^3 dt^3 N^{\gamma_2})$	$\mathcal{O}\left(\mu^{3/2}\frac{T^{3/2}}{\sqrt{\varepsilon}} N^{\gamma_2/2}\right)$	$\mathcal{O}\left(\mu^{3/2}\frac{T^{3/2}}{\sqrt{\varepsilon}} N^{(\gamma_2+2)/2}\right)$

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Digital vs. Digital-Analog: scalings

Decomposition type	Single-step error	Number of steps	Circuit complexity
Digital 1st order Trotter	$\mathcal{O}(\mu^2 dt^2 N^2)$	$\mathcal{O}\left(\mu^2 \frac{T^2}{\sqrt{\varepsilon}} N\right)$	$\mathcal{O}\left(\mu^2 \frac{T^2}{\sqrt{\varepsilon}} N^3\right)$
Digital 2nd order Trotter	$\mathcal{O}(\mu^3 dt^3 N)$	$\mathcal{O}\left(\mu^{3/2} \frac{T^{3/2}}{\sqrt{\varepsilon}} N^{1/2}\right)$	$\mathcal{O}\left(\mu^{3/2} \frac{T^{3/2}}{\sqrt{\varepsilon}} N^{5/2}\right)$
D-A 1st order Trotter	$\mathcal{O}(\mu^2 dt^2 N^{\gamma_1})$	$\mathcal{O}\left(\mu^2 \frac{T^2}{\varepsilon} N^{\gamma_1}\right)$	$\mathcal{O}\left(\mu^2 \frac{T^2}{\varepsilon} N^{\gamma_1+1}\right)$
D-A 2nd order Trotter	$\mathcal{O}(\mu^3 dt^3 N^{\gamma_2})$	$\mathcal{O}\left(\mu^{3/2} \frac{T^{3/2}}{\sqrt{\varepsilon}} N^{\gamma_2/2}\right)$	$\mathcal{O}\left(\mu^{3/2} \frac{T^{3/2}}{\sqrt{\varepsilon}} N^{(\gamma_2+2)/2}\right)$

For $\upsilon = 0.9$, both γ_1 and γ_2 are approximately 1.

$\Downarrow$

The digital-analog approach achieves an improved scaling of the circuit complexity.

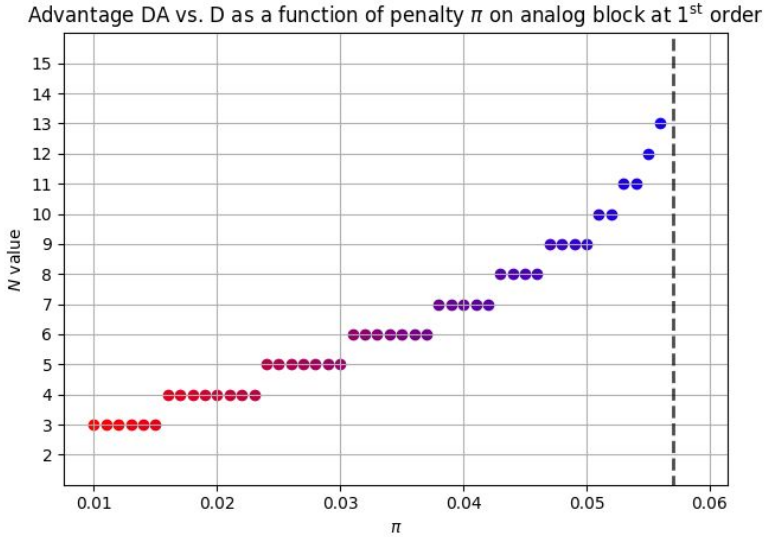
Digital vs. Digital-Analog: fidelity of single first order Trotter step

Compute the fidelity of a single first order step for the two approaches.

- **Digital:** $\mathcal{F}_D^{T1}(N) = (\mathcal{F}_{sqr})^{\Upsilon_{sqr}^{T1D}(N)} (\mathcal{F}_{uq})^{\Upsilon_{uq}^{T1}(N)} (\mathcal{F}_{zz})^{\Upsilon_{zz}^{T1}(N)}$
- **D-A:** $\mathcal{F}_{DA}^{T1}(N;\pi) = (\mathcal{F}_{sqr})^{\Upsilon_{sqr}^{T1DA}(N)} [\mathcal{F}_{xy}(N,\pi)]^{\Upsilon_{xy}}$

Linear model of analog gate fidelity:
$\mathcal{F}_{xy}(N;\pi) = 0.99 - \pi(N-2)$

Gate cost for single first order step	
Digital gates	**Gate count**
$\Upsilon_{\text{sqr}}^{T1D}$	$N + 3N(N-1)/2$
$\Upsilon_{\text{uq}}^{T1}$	$6N(N-1)/2$
$\Upsilon_{\text{zz}}^{T1}$	$3N(N-1)/2$
DA gates	**Gate count**
$\Upsilon_{\text{sqr}}^{T1DA}(N)$	$N + 4N$
$\Upsilon_{\text{xy}}^{T1}$	3



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Further steps

- further investigate validity of fitting method (role of energy scale)
- develop an analytical understanding of the fitting functions
- simulate digital-analog Heisenberg dynamics with QuTiP leveraging on the numerical routines
- develop an understanding of quantum gates and fidelities to improve the (very basic) linear model

Thank you for your attention!

Milano:

Edoardo Emil Riva (Università di Milano-Bicocca)
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Danilo Labranca (Università di Milano-Bicocca & INFN Milano-Bicocca)
Andrea Giachero (Università di Milano-Bicocca & INFN Milano-Bicocca)

Trento:

Alessandro Roggero (Università di Trento & INFN TIFPA)

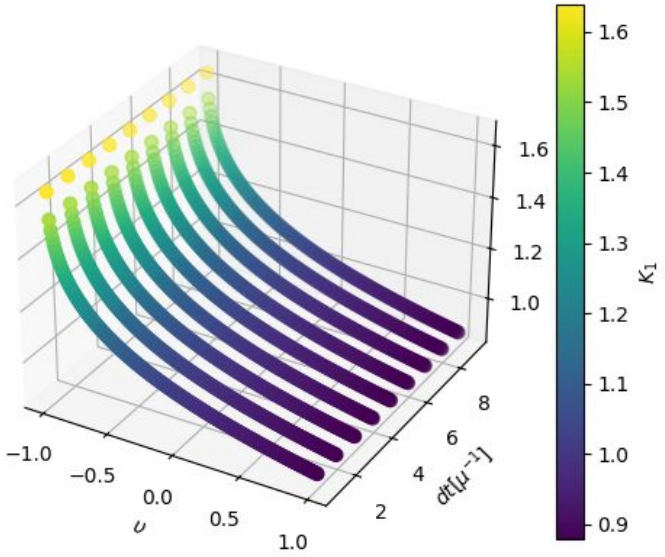
References

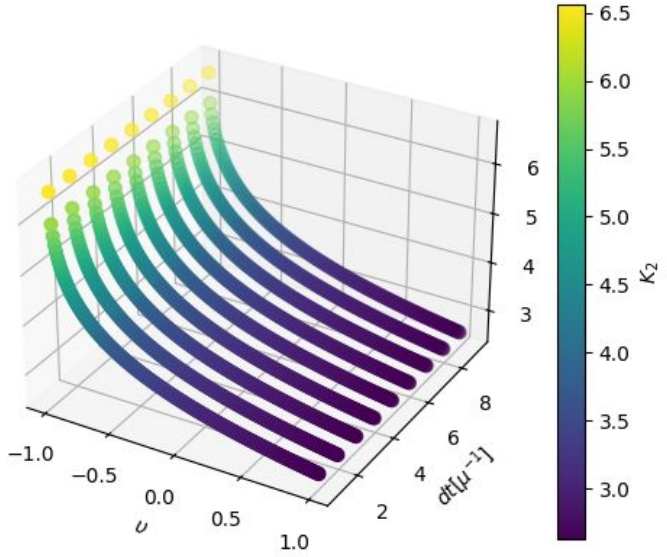
[1] Valentina Amitrano et al. “Trapped-ion quantum simulation of collective neutrino oscillations”. In: Phys. Rev. D 107 (2 Jan. 2023), p. 023007. DOI: 10.1103/PhysRevD.107.023007.

[2] Yves Salathé et al “Digital Quantum Simulation of Spin Models with Circuit Quantum Electrodynamics”. In: Phys. Rev. X 5 (17 Jun. 2015), p. 021027. DOI: https://doi.org/10.1103/PhysRevX.5.021027

[3] Andrew M. Childs et al. “Theory of Trotter Error with Commutator Scaling”. In: Phys. Rev. X 11 (1 Feb. 2021), p. 011020. DOI: https://doi.org/10.1103/PhysRevX.11.011020

Fitting procedure: plots for K_1, K_2





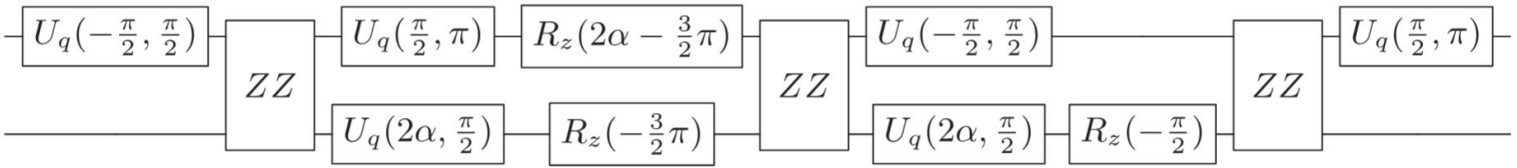
$$e_1\left(dt, N, \{g_{ij}\}; K_1, \gamma_1\right) = \frac{1}{8} K_1 \mu^2 \Theta^2 dt^2 N^{\gamma_1}$$

$$e_2\left(dt, N, \{g_{ij}\}; K_2, \gamma_2\right) = \frac{1}{96} K_2 \mu^3 \Theta^3 dt^3 N^{\gamma_2}$$

Structure of digital algorithm

Digital approach: $U_2(dt) \simeq \prod_{i<j}^{N} u_{ij}(dt)$

$u_{ij}(dt)$:



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