# Experimental methods in Hadron Spectroscopy

## The dynamical part of the amplitude

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# Overview

- Dynamical Functions
  - Resonance formation: S-T-K matrices
  - Breit-Wigner, Flatté functions & co.
  - Centrifugal barrier
  - Resonance production: P & Q vectors
  - N/D method
- Fitting methods and practical issues

# The dynamic part of the amplitude

- Breit-Wigner functions widely used in the first analyses AND by High Energy Experiments
  - Useful only for one resonance at a time without any interference
    - Single resonance
    - Single decay channel

- Much more complicated situation
  - Resonances have several decay channels
  - Several resonances sharing the same decay channel (interference)
  - Line shapes distortions due to threshold openings and sudden change of available phase space

- Thorough approach needed
- Simplifications to be applied in a second step

# T vs S matrix

- Recall: scattering amplitude via partial wave expansion

$\Rightarrow$ T: transition amplitude

$$f_{fi}(\Omega) = \frac{1}{q_i}\sum_J (2J+1) T^J_{fi}(s)\ D^{J*}_{\lambda\mu}(\theta,\phi,0)$$

$\Rightarrow$ Differential cross section

$$\frac{d\sigma_{fi}}{d\Omega} = \frac{1}{(8\pi)^2 s}\frac{q_f}{q_i}\left|M_{fi}\right|^2 = \left|f_{fi}(\Omega)\right|^2$$

- Scattering matrix (unitarity): $S_{if} = \langle i|S|f\rangle$ $\quad \boldsymbol{S = 1 + 2i\, T}$
    - Its elements express the probability to find a $|f\rangle = |cd, JM\rangle$ final state starting from a $|i\rangle = |ab, JM\ \lambda_a\lambda_b\rangle$ initial state

# *K* vs *S* vs *T* matrix

- ***T*** is a n×n matrix representing n incoming and n outgoing channels
  - from its unitarity properties follows:
    - $(\boldsymbol{T}^{\dagger})^{-1} - \boldsymbol{T}^{-1} = 2i\,\mathbf{1}$

- ***K***-matrix definition

$$\boxed{\boldsymbol{K}^{-1} = T^{-1} + i\ \mathbf{1}}$$

  - ***K*** is a hermitian operator
  - Due to time invariance: ***K*** REAL matrix
    - Under threshold it can be continued analytically
    - ***T*** and ***K*** commute

$$\boxed{\boldsymbol{T} = \boldsymbol{K}(\mathbf{1} - i\boldsymbol{K})^{-1} = (\mathbf{1} - i\boldsymbol{K})^{-1}\boldsymbol{K}}$$

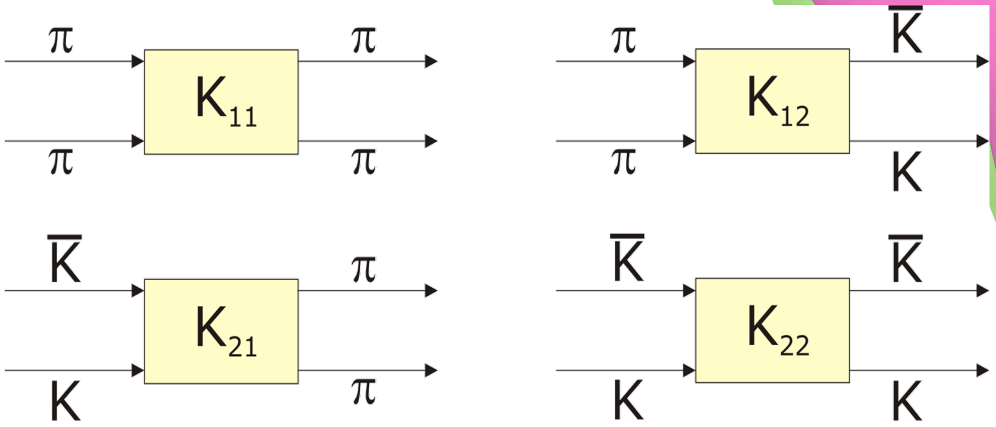
# Single channel problem

- **T matrix**: $T = (e^{2i\delta}-1)/2i = e^{i\delta}\sin\delta$
  - Argand plot: C(0, i/2), r=1/2
- **S matrix**: $S = e^{2i\delta}$
- **K matrix**:
  - $K^{-1} = T^{-1} - i = ctg\delta$
  - $K = tg\delta \quad \Rightarrow$ pole for $\delta = \pi/2$
    - Recall Breit-Wigner function derivation…
- **Cross section** (S wave):

$$\sigma = \left(\frac{4\pi}{q_i^2}\right)\sin^2\delta$$

# 2 channels problem (i.e. $\bar{K}K + \pi\pi$)

$$\boldsymbol{K} = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}$$




determinant

$$D = K_{11}K_{22} - K_{12}^2$$

- **K** is real, $K_{12} = K_{21}$
- **T** matrix:

$$\boldsymbol{T} = \frac{1}{1 - D - i(K_{11} + K_{22})} \begin{pmatrix} K_{11} - D & K_{12} \\ K_{21} & K_{22} - D \end{pmatrix}$$

# Relativistic extension

- ***T*** is not relativistically invariant
  - One must insert the phase space-matrix elements of the initial and final state

- ***ρ***: phase-space matrix, diagonal
  - Can become **complex under threshold**

$$\rho = \begin{pmatrix} \frac{2q_1}{m} & 0 \\ 0 & \frac{2q_2}{m} \end{pmatrix}$$

- Covariant description of ***T, S*** and ***K***:

$$\boldsymbol{T} = \{\boldsymbol{\rho}\}^{\frac{1}{2}}\, \widehat{\boldsymbol{T}}\, \{\boldsymbol{\rho}\}^{\frac{1}{2}}$$

$$\boldsymbol{S} = \mathbf{1} + 2i\{\boldsymbol{\rho}\}^{\frac{1}{2}}\, \widehat{\boldsymbol{T}}\, \{\boldsymbol{\rho}\}^{\frac{1}{2}}$$

$$\boldsymbol{K} = \{\boldsymbol{\rho}\}^{\frac{1}{2}}\, \widehat{\boldsymbol{K}}\, \{\boldsymbol{\rho}\}^{\frac{1}{2}} \qquad \widehat{\boldsymbol{K}}^{-1} = \widehat{\boldsymbol{T}}^{-1} + i\boldsymbol{\rho}$$

# Let's go back to Breit-Wigner functions…

- Single resonance with single decay channel
  - simplest dynamical function

- Several derivations possible

1. From phase variation
   - Resonance: $\delta = \pi/2$
2. From the decay of unstable states
3. From field theory
4. From ***K***-matrix

# Breit Wigner pdf from wave functions and particle decays

- The wave function for a non-stationary state of frequency $\omega_R = E_R/\hbar$ and lifetime $\tau=\hbar/\Gamma$ can be written as

$$\psi(t) = \Psi_0 e^{-i\omega_R t} e^{-i\frac{t}{2\tau}}$$

- Its Fourier transform gives, as a function of $\omega$:

$$\Psi(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \psi(t) e^{-i\omega t} dt = \frac{\kappa}{(E_R - E) - i(\Gamma/2)}$$

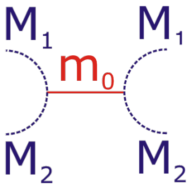
- Since $\sigma_{el} \sim \Psi^*\Psi$:

$$\boxed{\Psi(E) = \frac{\Gamma/2}{(E_R - E) - i(\Gamma/2)}}$$

$$\boxed{\sigma^2{}_{el} = \frac{4\pi}{k}(2l+1)\frac{\Gamma^2/4}{(E_R - E)^2 - i(\Gamma^2/4)}}$$

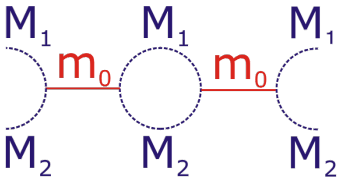
# T-matrix & field theory: resonances

- In field theory a **resonance is described by a propagator**




$$T = V_{12} \frac{1}{E_0 - E} V_{12}$$

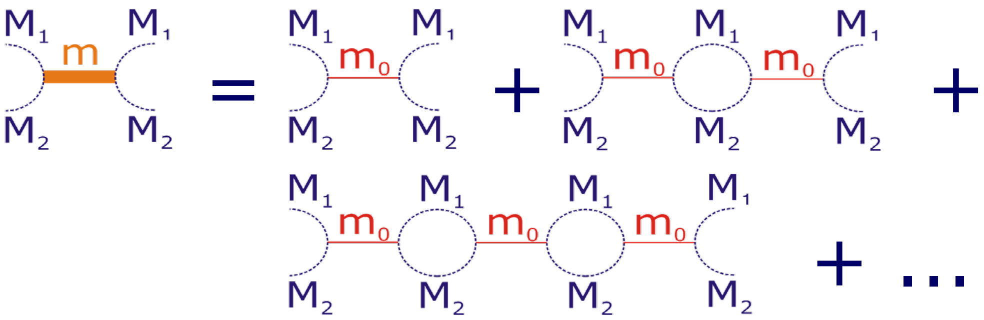
- If a self-energy term is present




$$T = V_{12} \frac{1}{E_0 - E} c \frac{1}{E_0 - E} V_{12} = c \frac{V_{12} c V_{12}}{(E_0 - E)^2}$$

- Every loop involves a complex coupling c
- If the coupling is small, the expansion converges like a geometric series

# T-matrix perturbative treatment




$$T = \frac{V_{12}V_{12}}{E_0 - E}\left(1 + \frac{c}{E_0 - E} + \frac{c^2}{(E_0 - E)^2} + \cdots\right) = \frac{V_{12}V_{12}}{E_0 - E}\left(\frac{1}{1 - \frac{c}{E_0 - E}}\right)$$

$$= \frac{V_{12}V_{12}}{E_0 - E - c}$$

defining the “dressed” energy

$E_R = E_0 - \Re e(c)$

$$\boxed{T = \frac{V_{12}V_{12}}{E_R - E - i\Im m(c)}}$$

# Relativistic Breit-Wigner function

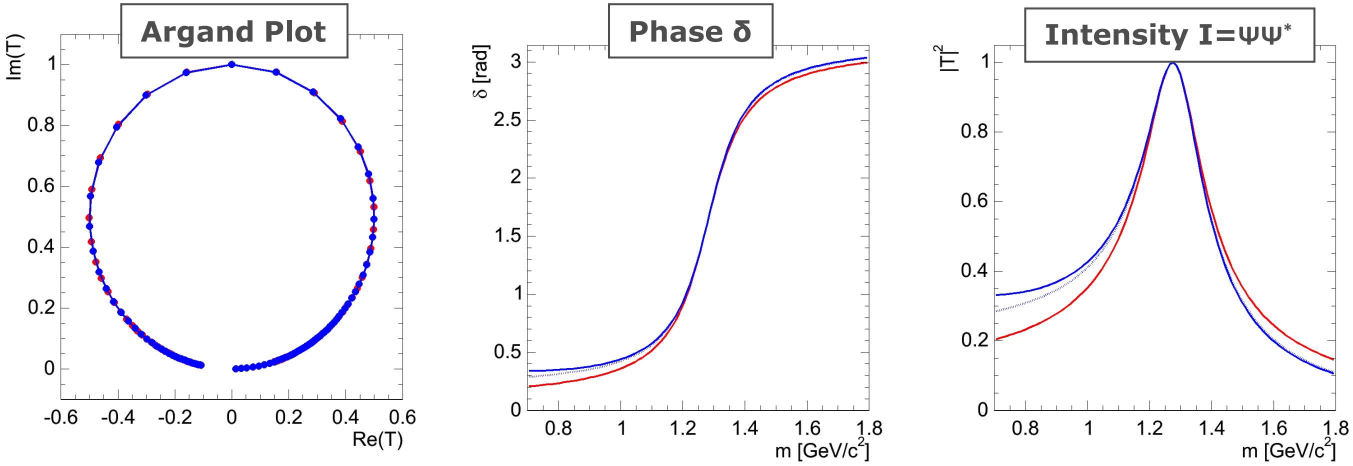
- Formulation to be used in any meson spectroscopy experiment
- From the optical theorem:

$$\boxed{\Im m(T(s,0)) = \frac{q}{4\pi}\frac{\sqrt{s}}{2}\sigma_T} \quad \Rightarrow \quad T^{-1}(s) = \frac{q}{\sqrt{s}}(\cot\delta - i) = k(s) - i\frac{2q}{\sqrt{s}}$$

- *k(s)* is any real function
- $T^{-1}$ has a peak at *k(s)=0*
- Simplest covariant form for *k(s)*: $(s_R - s)/\gamma$

$$\boxed{T(s) = \frac{\gamma}{s_R - s - i\left(\frac{2q\gamma}{\sqrt{s}}\right)} \Rightarrow \frac{\Gamma}{m_R^2 - m^2 - i\rho m_R \Gamma}}$$

# Non-relativistic vs relativistic Breit-Wigner




- Argand plot & phase motion
  - (almost) equal: non-relativistic vs relativistic BW
- Intensity
  - Narrower line-shape in the relativistic case

# Resonances in K-matrix formalism

- **Formation** of resonances: sum of poles of the ***K***-matrix
- If the amplitude is dominated by resonance formation the ***K***-matrix elements (Lorentz-invariant) may be written as

$$\hat{K}_{ij} = \sum_{\alpha} \frac{g^*_{\alpha i}(m) g_{\alpha j}(m)}{(m_\alpha^2 - m^2)\sqrt{\rho_i \rho_j}} + \hat{c}_{ij}$$

- The $g_i$'s are residual functions, real above the $i^{th}$ channel threshold and proportional to the $i^{th}$ channel partial width

$$g^2_{\alpha i}(m) = m_\alpha \Gamma_{\alpha i}(m)$$

$$\Gamma_\alpha(m) = \sum_i \Gamma_{\alpha i}(m)$$

The total width of a resonance is a sum of partial widths over different decay channels

# Partial widths & centrifugal barriers

- The widths of the resonances, in a covariant formulation of the decay amplitudes, are mass-dependent and proportional to $q^{2l+1}$
  - centrifugal barrier factor
  - Taking out the phase space factor: $\Gamma \sim q^{2l}$
    - The lifetime is related to the centrifugal barrier
  - Valid only at low energies and close to thresholds
  - Motivation: suppression of the cross-sections in L≠0 waves when the impact parameter b is large (or the break-up momentum is small)
    - Damping factors needed
      - Blatt-Weisskopf formulation

$$b = \frac{\sqrt{L(L+1)}}{q}$$

# Blatt-Weisskopf centrifugal barrier factors

- Derived from the solution of the radial differential equation
  - Proportional to spherical Hankel functions

- For $\ell$=*0* up to 3: with r ~ 1 fm

$$F_0(q) = 1$$

$$F_1(q) = \sqrt{\frac{2(qr)^2}{qr+1}}$$

$$F_2(q) = \sqrt{\frac{13(qr)^4}{((qr)^2-3)^2+9(qr)^2}}$$

$$B_l(q, q_R) = \frac{F_l(q)}{F_l(q_R)}$$

$$T_l(s) = \frac{B_l^2(q)\Gamma}{m_R^2 - m^2 - i\rho B_l^2(q) m_0 \Gamma}$$

Relativistic Breit-Wigner amplitude
with damping factors

# Resonances coupled to many channels

- The partial widths are unknown and can be expressed through complex parameters to be adjusted by the fits
  - The residual functions read as

Phase space

$$g_{\alpha i}(m) = \gamma_{\alpha i}\sqrt{m_\alpha \Gamma_\alpha^0}\; B_{\alpha i}^l(q, q_\alpha)\sqrt{\rho_i}$$

$\sum_i \gamma_{\alpha i}^2 = 1$

Fit parameters (normalized to 1)

Centrifugal barrier function

- The ***K***-matrix widths do not have to be identical to the observed widths nor to ***T***-matrix poles widths
  - Only if the masses of the decay particles are much smaller than the mass of the resonance: $\Gamma_\alpha\,(m_\alpha) \approx \Gamma_\alpha^0$

$$\boxed{\hat{K}_{ij}(m) = \sum_\alpha \frac{\gamma_{\alpha i}\gamma_{\alpha j} m_\alpha \Gamma_\alpha^0\; B_{\alpha i}^l(q, q_\alpha)\; B_{\alpha j}^l(q, q_\alpha)}{m_\alpha^2 - m^2}}$$

# Example 1: one resonance, one decay channel, spin 1 (e.g., a ρ)

- **K matrix:**

$$\boxed{K = \frac{m_0 \Gamma(m)}{m_0^2 - m^2} = \tan\delta}$$

- **Mass-dependent width:**

$$\boxed{\Gamma(m) = \Gamma_0 \rho [B_1(q, q_0)]^2 = \Gamma_0^{obs} \left(\frac{m_0}{m}\right)\left(\frac{q}{q_0}\right)^3}$$

- **T matrix:**

$$\boxed{T = e^{i\delta}\sin\delta = \frac{m_0 \Gamma_0^{obs}}{m_0^2 - m^2 - i m_0 \Gamma(m)} \underbrace{\left(\frac{q}{q_0}\right)^2}_{\text{P-wave factor}} \left[\left(\frac{m_0}{m}\right)\left(\frac{q}{q_0}\right)\right]}$$

2-body phase space

P-wave factor

# Example II: two resonances (a,b), one decay channel, spin zero

- **(2 x 1) K-matrix:**

$$K = \frac{m_a\Gamma_a(m)}{m_a^2 - m^2} + \frac{m_b\Gamma_b(m)}{m_b^2 - m^2}$$

- If $m_a$ and $m_b$ are far apart relative to their width (so they do not overlap), the ***K***-matrix is dominated by one of them depending on the $m$ value
  - In this case the transition amplitude can be written as a sum (but this is an approximation because unitarity is violated!)

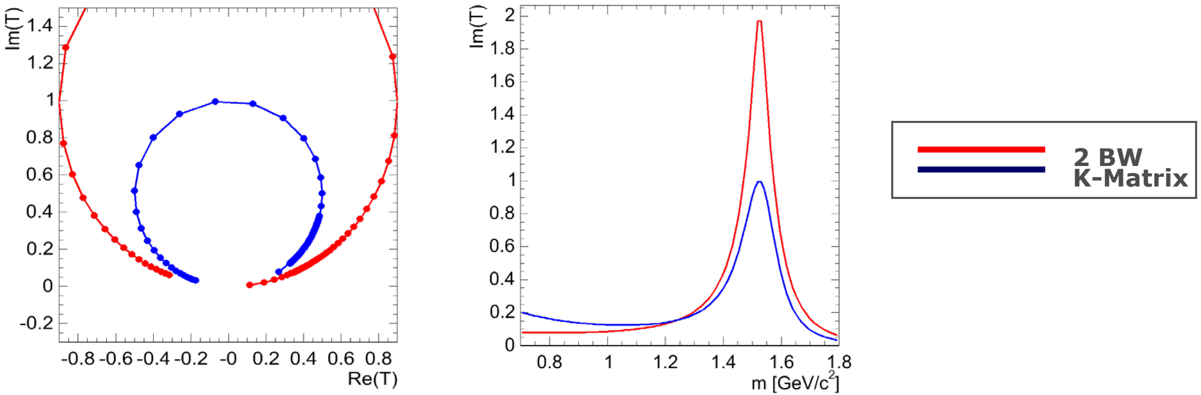
$$T \cong \frac{m_a\Gamma_a^{obs}}{m_a^2 - m^2 - i m_a\Gamma_a(m)}\left[\left(\frac{m_a}{m}\right)\left(\frac{q}{q_a}\right)\right] + \frac{m_b\Gamma_b^{obs}}{m_b^2 - m^2 - i m_b\Gamma_b(m)}\left[\left(\frac{m_b}{m}\right)\left(\frac{q}{q_b}\right)\right]$$

# Sum of resonances: same mass

- If two resonances have the same mass (and in this case only!) one can sum their widths: $m_a = m_b = m_0$

$$T \cong \frac{m_0[\Gamma_a(m) + \Gamma_b(m)]}{m_0^2 - m^2 - i m_0[\Gamma_a(m) + \Gamma_b(m)]}$$

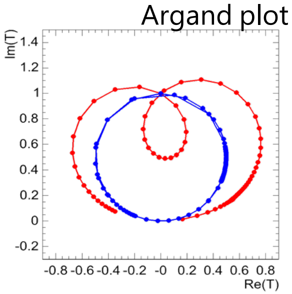
- Remember: the simple sum of amplitudes violates unitarity

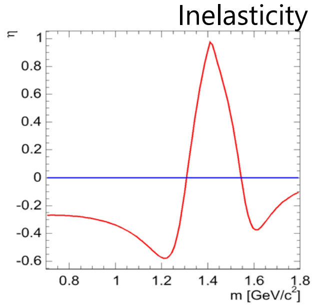


# Sum of resonances with nearby masses

- Example: two nearby resonances decaying into $\pi\pi$, with the same spin
  - $f_2(1275)$: $m_a$ = 1275 MeV/$c^2$, $\Gamma_a$ = 185 MeV
  - $f_2(1565)$: $m_b$ = 1565 MeV/$c_2$, $\Gamma_b$ = 150 MeV

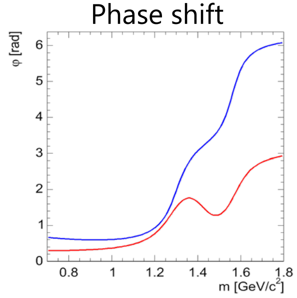
- The sum of Breit Wigner amplitudes violates unitarity

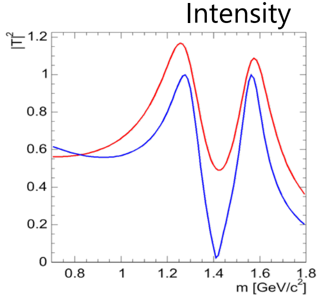



2 Breit Wigner's
K-matrix

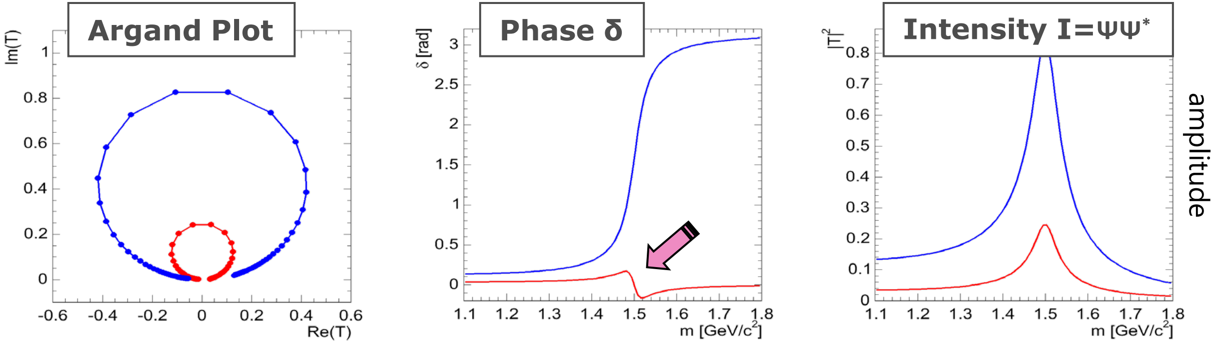
- Phase shift: move backwards
- Intensity: exceeds unity



# Example III: one resonance, two decay channels: *coupled channel analysis*

- **(1x2) K-matrix**
- Practical case: $f_0(1500) \rightarrow \pi\pi, \bar{K}K, \; \Gamma = 100$ MeV
  - Test of different couplings:
    - $\pi\pi$ dominated resonance ($\Gamma_{\pi\pi} = 80$ MeV)
    - $\bar{K}K$ dominated resonance ($\Gamma_{\pi\pi} = 20$ MeV)




If the $\bar{K}K$ coupling is large the measurement of the $\pi\pi$ phase shift is not enough to claim for the existence of a resonance

# Example IV: one resonance, two decay channels, one close to threshold

- **(2x2) K-matrix**

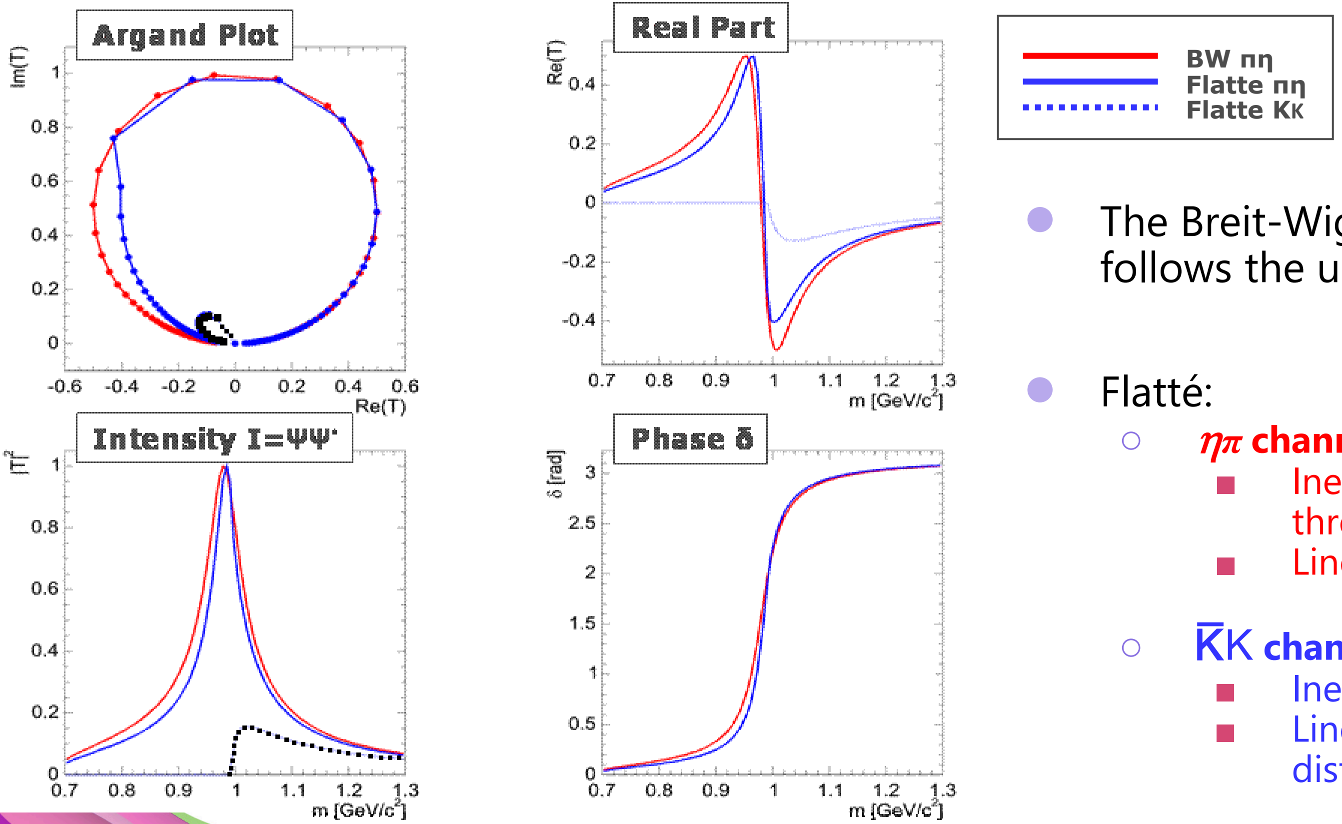
$$\hat{K} = \begin{pmatrix} \frac{\gamma_1^2 m_0 \Gamma_0}{m_0^2 - m^2} & \frac{\gamma_1 \gamma_2 m_0 \Gamma_0}{m_0^2 - m^2} \\ \frac{\gamma_1 \gamma_2 m_0 \Gamma_0}{m_0^2 - m^2} & \frac{\gamma_2^2 m_0 \Gamma_0}{m_0^2 - m^2} \end{pmatrix}$$

- **Practical case: *$a_0$(980)* $\rightarrow \eta\pi$ || *$a_0$(980)* $\rightarrow \bar{\mathrm{K}}\mathrm{K}$ with *J=0***
  - *$a_0$(980)* close to $\bar{\mathrm{K}}\mathrm{K}$ threshold

$$\boxed{\hat{T} = \frac{m_0 \Gamma_0}{m_0^2 - m^2 - i m_0 \Gamma_0 (\rho_1 \gamma_1^2 + \rho_2 \gamma_2^2)} \begin{pmatrix} \gamma_1^2 & \gamma_1 \gamma_2 \\ \gamma_1 \gamma_2 & \gamma_2^2 \end{pmatrix}}$$

**Flatté formula**

# Flatté vs Breit-Wigner amplitudes




- The Breit-Wigner parameterization follows the unitarity circle

- Flatté:
  - ***ηπ* channel:**
    - Inelasticity: drop at the $\bar{K}$K threshold
    - Line-shape: OK
  - **$\bar{K}$K channel:**
    - Inelasticity: smaller circle
    - Line shape: dramatic distortion

# Production of resonances: *P*-vector

- Extension of ***K***-matrix formalism to more complex reactions (resonances not formed in *s*-channel)
  - Effect of Final State Interactions

- **Assumptions:**
  - The two-body system in the final state is isolated
  - The two particles do not interact simultaneously with the rest of the particles in the final state

- The production of a resonance is described by the ***P*-vector**

$$\boldsymbol{T} = (\boldsymbol{1} - i\boldsymbol{K})^{-1}\boldsymbol{K}$$

$$\boldsymbol{F} = (\boldsymbol{1} - i\boldsymbol{K})^{-1}\boldsymbol{P} = \boldsymbol{T}\boldsymbol{K}^{-1}\boldsymbol{P}$$

# *P*-vector properties

- ***P*** contains the same set of poles of **K** and both are built the same way

Information on the resonance production

$$\hat{P}_i = \sum_R \frac{\beta_R^0 g_{Ri}(m)}{(m_R^2 - m^2)\sqrt{\rho_i}} + d_i$$

Information on resonance width/decay

Constant term or polynomial

- A linear term can be introduced to account for non resonant contributions to the final state
- For a single decay channel and a single resonance:

$$\boxed{\hat{F} = e^{i\delta} \cos\delta \, \hat{P}}$$

  - The final state interaction brings in a factor $e^{i\delta}$
  - ***P*** is a **real** function

# Production of resonances: *Q*-vector

- The ***P***-vector has the same singularities (poles, "left hand cuts" due to threshold opening) of ***K***-matrix, depending on the reaction

- In a limited energy range ***P*** can be considered as a constant vector ***Q***
  - ***Q*** depends only on $s=m^2$
  - ***Q*** has not threshold singularities
  - ***Q*** does not contain poles

$$\boxed{\hat{F} = \hat{T}\hat{Q}}$$

$$\boxed{\hat{Q} = \hat{K}^{-1}P}$$

- ***P*** or ***Q***: two equivalent approximations
  - ***P***: (***1***-*i**K***) propagator × a constant (to be determined by the fit)
  - ***Q***: ***T*** × a constant (to be determined by the fit)

# N/D approach

- Derived from dispersion relations
  - Maximum analiticity
  - Unitarity

- The amplitude $\boldsymbol{T_\ell}$ may be expressed by dispersion relations derived from the optical theorem $\Rightarrow$ *integral equations*

- $\boldsymbol{T_\ell}$ can always be expressed by a ratio of functions, correlated by a solvable system of integral equations

$$\boxed{T_\ell(s) = \frac{N_\ell(s)}{D_\ell(s)}}$$

Contains only left-hand singularities

Contains only right-hand singularities

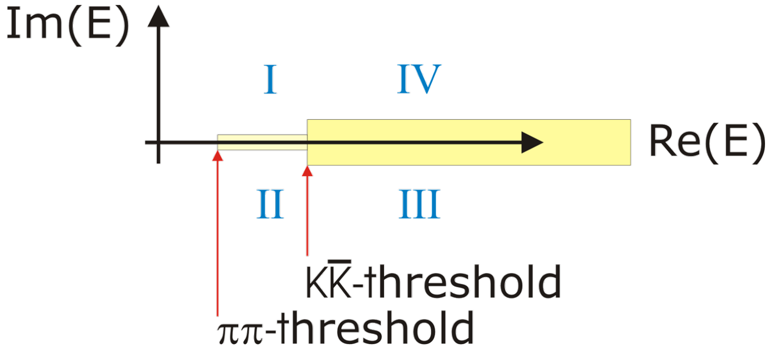
# Amplitude singularities vs resonances

- **Poles: zeros of propagators**
  - ***K***-matrix and ***T***-matrix poles are located in different positions
    - They are similar if:
      - Resonances are very far apart
      - The coupling to non-dominant channels is very small
      - They are far from thresholds
    - If these conditions do not hold:
      - Interference mechanisms move the pole positions

- **Cuts: opening of thresholds**
  - $s$ channel: right hand cut
  - $t$, $u$ channels: left hand cut

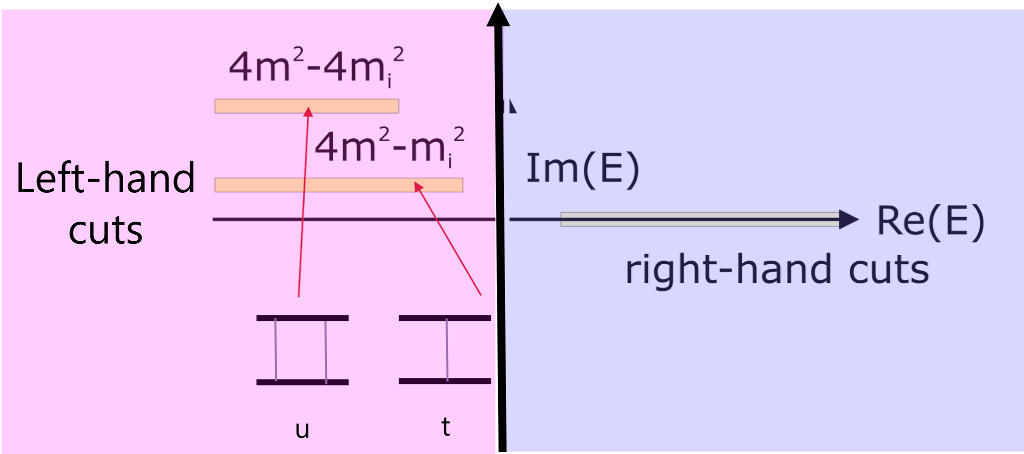
# Cuts and Riemann sheets: Right Hand Cuts

- *ρ(m)* is a square root: ***ρ(m) = √(2q/m)***
  - Below threshold (*q=0*) *ρ(m)* gets complex
    - there are 4 solutions, every pair of roots lie π rad apart
  - For each threshold a pair of Riemann sheets *Im(E)* vs *Re(E)* opens (each for one of the roots)
  - The cut between sheets is taken (by convention) on the real axis and starts at the channel threshold
  - ***s* channel cut: right hand cut *(RHC)***



# Cuts and Riemann sheets: Left Hand Cuts

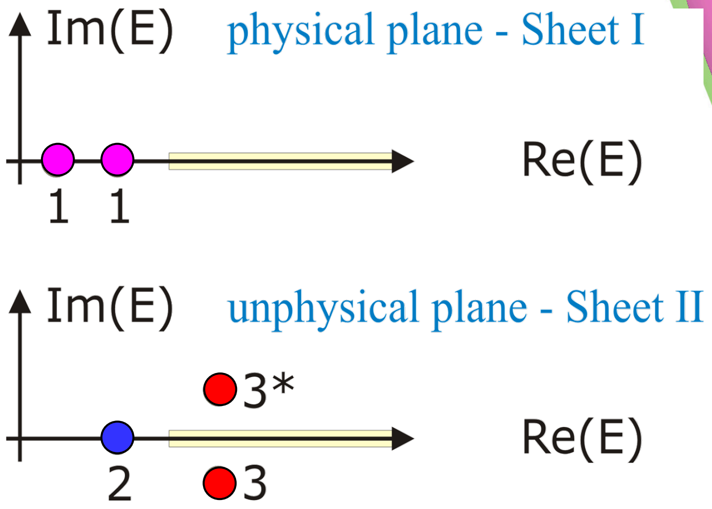
- **Singularities in *t*- and *u*- channels:**
  - Usually they appear below threshold
  - Not taken into account in ***K***-matrix
  - They can imitate resonances and influence the amplitude in the physical region
  - ***t*-, *u*- channel cuts: left hand cuts (*LHC*)**



# Resonances? Properties of *T* poles in the complex E plane

- ***T* matrix pole: zero of the complex denominator D**
  - D(E + $i\,\Gamma/2$) = 0
  - **Mass: real part of the pole**
  - **Width: 2x imaginary part**

- The position of the pole in each Riemann sheet characterizes the resonance




- Possible singularities:
  1. Bound states
  2. Anti-bound states
  3. Resonances
  4. Spurious singularities (wrong model)

# Poles positions and resonances

- **Real resonances are located in the II and III Riemann sheets**
  - The poles lie on the unphysical plane since *Im*($q_R$)<0
  - They lie symmetrically around the real axis of the energy plane
  - At the boundary between II-III sheets

$$\Gamma_R^{BW} \approx \frac{1}{2}(\Gamma_R^{II} + \Gamma_R^{III})$$

- **Cross sections** are **real** and mainly influenced by the nearest pole
  - The nearest pole fixes the resonance candidate

- Without thresholds: the poles on the two sheets are identical

- Close to threshold: shadow poles in two sheets
  - A resonance exists if the poles match

# Summary: energy dependent part of the amplitude

- **Always use relativistic functions**
- Use **Breit-Wigner functions** with
  - Mass dependent width
  - Centrifugal barrier functions (ℓ dependence + phase space)
- **Single channel resonances:**
  - If far apart: single BW's OK
  - If overlapping with the same mass: sum of BW's is OK
  - If nearby: use ***K***-matrix, avoid BW's
- **Resonances decaying to several channels**
  - Use ***K***-matrix or ***Q***-vector
  - Special cases:
    - $\eta\pi/\bar{K}K$ : Flatté function
    - $\pi\pi/\bar{K}K$ **I=0, S wave**: phenomenological parameterizations exist taking into account the overlap of $f_0(400\text{-}1200)$, $f_0(1300)$, $f_0(1500)$, $f_0(1700)$

# Fitting methods

- Max. likelihood methods
- $\chi^2$ minimization
- Channel likelihood method

# Free parameters in amplitudes

$$a(\theta,\phi) = \sum_J \alpha_J \left|f_J(\theta,\phi)\right|^2$$

At rest amplitude

$$u(\theta,\phi) = \frac{1}{4}\sum_{\lambda_1\lambda_2}\left|\sum_J f_\lambda^J H^J_{\lambda_1\lambda_2}\right|^2$$

In flight amplitude

- Couplings (real): final to initial state: $\boldsymbol{\alpha_J}$ or $\boldsymbol{H^J}_{\lambda 1\lambda 2}$
- Isobar production rates (complex): weight of different isobars in PW $\boldsymbol{w_i}$

$$f_J = \sum_i w_i A_J^i$$

$$A_J^i = \sum_r Z_{ir}^{J^{PC}}(p,q)F_{ir}(q)$$

- Dynamical parameters: masses, partial and total widths (decay BR's)

# Once the amplitude is written… now?

- Several tens of free parameters need to be estimated

- A proper minimization package must be used to adapt the amplitude to the experimental data
  - Log(likelihood) or direct $\chi^2$ evaluation
  - **For each event a real number is obtained**
    - Used as **weight** for the entry in a Dalitz plot cell or histogram bin
    - A theoretical (modelled) distribution is prepared, according to a given hypothesis
    - Events to be weighted: Monte Carlo generated events with the physical cuts (acceptance distorted)
  - Comparison between the experimental plot and the Montecarlo model through statistical estimators

# -log($\mathcal{L}$) minimization

- For each experimental event a global likelihood probability density is obtained, for any set $\Theta$ of free parameters
  - $\mu_i$ = event-by-event intensity (*i.e.* weight)
    - Normalized to a large number of Montecarlo generated events which take into account the apparatus efficiency and acceptance

$$\mathcal{L} = \prod_{i=1}^{n} \frac{\mu_i(\Theta)}{\int \mu(\Theta) d\Omega}$$

- The set $\Theta$ which minimizes the function is found, then the fit quality is checked

- Advantages:
  - Works with any statistics (also few events)
  - Uncorrelated to data distribution lineshapes and binning
  - Acceptance effects automatically taken into account
  - Inclusion of background contribution possible

# $\chi^2$ minimization

- A $\chi^2$ can be obtained as goodness-of-fit estimator after having determined the best-fit $\Theta$ set of parameters, OR a $\chi^2(\Theta)$ function can be minimized comparing the experimental distributions to the theoretical ones

- Over a Dalitz plot volume:

$$\boxed{\chi^2 = \sum_{cells} \frac{\left(N_i^{\exp} - N_i^{BG} - N_i^{th}\right)^2}{\sigma^2_{N_i^{\exp}} + \sigma^2_{N_i^{BG}} + \sigma^2_{N_i^{th}}}}$$

- $N^{th}_{j}$: *ij*-cell content, obtained weighting events of the acceptance Dalitz plot with the squared amplitude from the phase-space Monte Carlo events
    - $\sigma^2_{exp} = N^{exp}_i = n_{ij}$
    - $\sigma^2_{th} = \Sigma w_{ij} = t^2_{ij}/p_{ij}$ (acceptance/phase space DP cell content)

# Channel likelihood fit I

- Extension of max likelihood method
- Useful for events with many particles in the final state
- Purpose: separation of the data sample in several resonant contributions on a per-event basis
  - Determine the contribution of each channel to the total data sample
  - Identify the channel an event belongs to

- For each *j* channel:
  - Density function containing the dynamics × angular info, normalized over phase space: $f_j$
  - Weight of event i in the channel *j*: $w_{ij}$
- For all *i* events:

$$\boxed{\sum_{j=1}^{M} w_{ij} = 1}$$

# Channel likelihood fit II

- Probability that a $i^{th}$ event belongs to the $j^{th}$ channel:

$$w_{ji} = \frac{N_j f_{ji}}{\int_{LIPS} f_j(\Omega) d\Omega} \left[ \sum_{j=1}^{M} \frac{N_j f_{ji}}{\int_{LIPS} f_j(\Omega) d\Omega} \right]^{-1}$$

- A new number of events per channel $N'$ may be iteratively found solving the $M$ coupled equations system

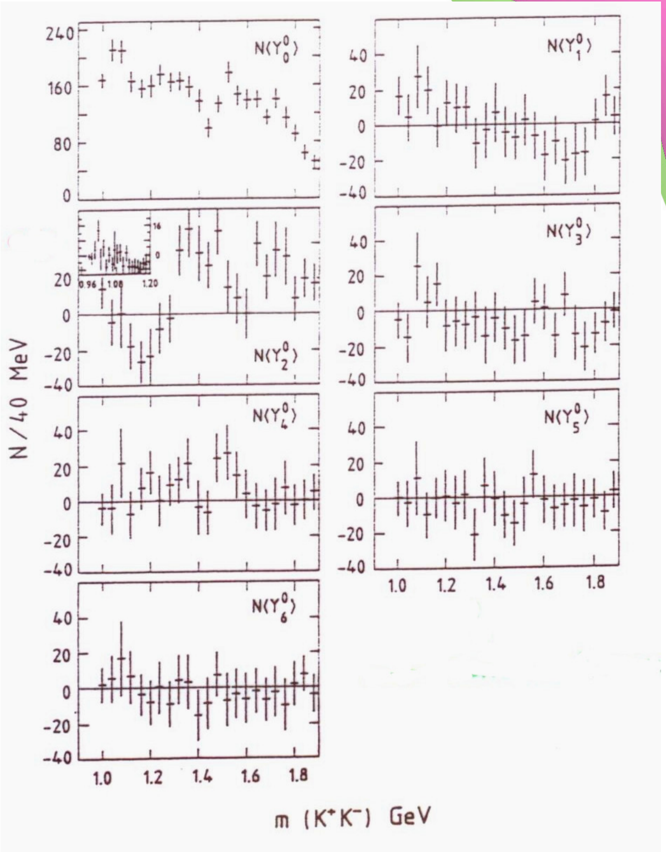
$$N_j = \sum_{i=1}^{N} w_{ji} \qquad j = 1, \dots M$$

- Once the solutions are found the $w_{ij}$ $N \times M$ numbers are used to weight the experimental data in the control plots

  - Recursive method
  - minimization

# Example of channel likelihood fit results

- **WA76 experiment @CERN:**
  - Central pp, $\pi^+$p collisions @ 85 GeV/c
    - $pp \rightarrow p(K^+K^-)p$
    - $\pi^+ p \rightarrow \pi^+ (K^+K^-)p$
  - Analysis of momenta:
    - $Y^0{}_2$ shows activity over the full mass range: $\theta(1720)$?
    - $Y^0{}_5$ & $Y^0{}_6$ consistent with zero



# Summary: fitting methods

- The amplitude is written according to a given hypothesis for the configuration of the two-body intermediate states set

- A minimization procedure must be used to find the set of parameters which reproduces at best the shape of the experimental distributions
    - Likelihood methods (binned/unbinned)
        - Difficult to compare different best fits obtained in different hypotheses
    - $\chi^2$ minimization

# Overall summary

- Data con be interpreted resorting to several hypotheses for the production of intermediate states
  - Each hypothesis has its own formulation depending on particle spins, relative angular momentum and features of the energy dependent part

- A best-fit series of parameters must be obtained for each hypothesis

- Best fit solutions must be compared to estimate the best description of the data

- The procedure is long (and boring), and must proceed through gradual (little) improvements of the amplitude description, always keeping under control the effect on the fit quality