Experimental methods in Hadron Spectroscopy

Spin and parity determination procedure

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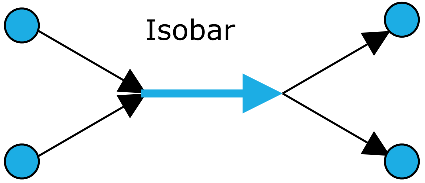
Overview

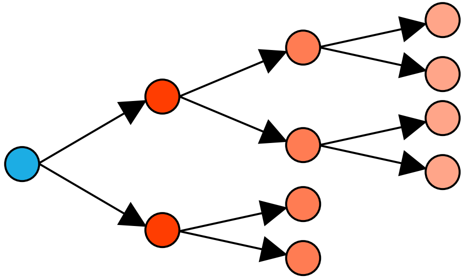
- Partial wave amplitudes
  - Isobar model and intermediate states
  - Partial wave decomposition
  - Initial state descriptions
  - At rest vs in-flight (energy dependent) interaction amplitudes

- The angular dependent part of the amplitude: spin formalisms

The isobar model (Watson)

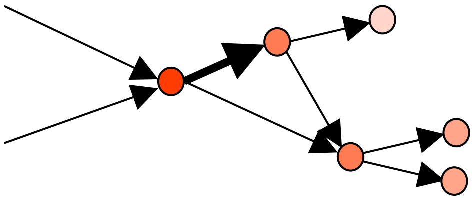
- **Assumption:** the overall reaction proceeds via intermediate two-body processes
  - The many-body system is built through a tree of subsequent two-body decays
  - The two-body systems have the same behaviour in each reaction step
  - Different initial states may interfere





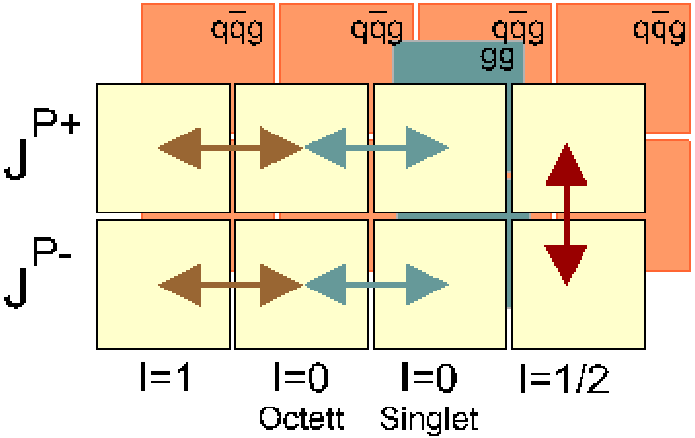
- **Ingredients:**
  - Two-body spin algebra
  - Two-body scattering formalism

Not suitable for rescattering processes



Particles in intermediate states can mix and/or interfere...

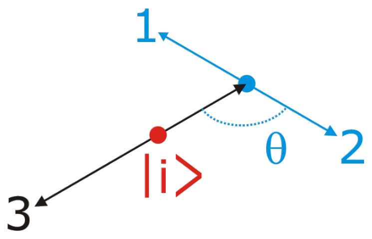
- Many intermediate states can contribute to the same final state
  - **Isoscalar mixing**
    - same I^G & J^{PC}: η-η', f_2-f_2'
  - **I=0/I=1 mixing**
    - ρ-ω
  - **Kaon mixing**
    - same I^G & J^P (no C defined)



- Interferences and spin effects determine particular patterns in the scattering amplitudes

On the road to write an amplitude: step 1

- For each node of the decay tree an amplitude can be written as a function of:
    - Isospin: total and third component
    - Spin *J* of the mother particle
    - Angles in 3D space: $\Omega = (\theta,\varphi)$
    - Relative angular momentum ℓ between the daughter pair and the mother particle
    - Energy (s) of the decaying mother



$$f(I, I_3, J, \ell, s, \Omega) = I_\ell(I, I_3) T_{J\ell}(s) R_{J\ell}(\Omega)$$

Partial wave amplitude

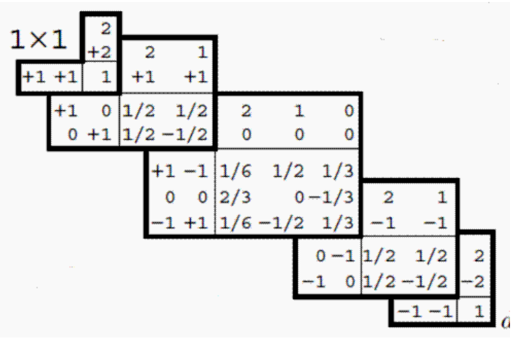
- The full amplitude is the sum over all the nodes in the final state
- Other observables are summed over

The isospin dependence

- Intermediate states with identical particles must be weighted by proper **Clebsch-Gordan coefficients** stemming from the isospin composition

- Example: $\bar{p}p \rightarrow \rho\pi$
  - Initial states: p̄p in **$^1S_0$**: $I^G=1^-$, **$^3S_1$**: $I^G=0^+$ ($I_3=0$)
  - Final states: $\rho^0\pi^0$, $\rho^\pm\pi^\mp$, $\rho^\mp\pi^\pm \rightarrow \pi^+\pi^-\pi^0$

1x1	$^1S_0$ (I=1)	$^3S_1$ (I=0)
$\rho^0\pi^0$	$\langle 10\ 10 \vert 10\rangle = 0$	$\langle 10\ 10 \vert 00\rangle = -1/\sqrt{3}$
$\rho^\pm\pi^\mp$	$\langle 1\pm1\ 1\mp1 \vert 10\rangle = \pm 1/\sqrt{2}$	$\langle 1\pm1\ 1\mp1 \vert 00\rangle = 1/\sqrt{3}$



  - Destructive interference in $^3S_1$
  - **$^1S_0 \rightarrow \rho^0\pi^0$ forbidden**
  - All isospin combinations must be summed over

Wave-optical approach of hadron scattering: partial wave amplitudes

- Procedure to solve the Schrödinger equation in a scattering process
- The incident wave can be expanded in terms of Legendre polynomials P_ℓ and a radial function U_ℓ

$$|i\rangle = \Psi_i = \sum_{\ell=0}^{\infty} U_\ell(r) P_\ell(\cos\theta)$$

- The scattered wave function can also be factorized in a radial x angular part product
  - The radial part U_ℓ is parameterized in terms of phases δ_ℓ and inelasticities η_ℓ

Plane wave

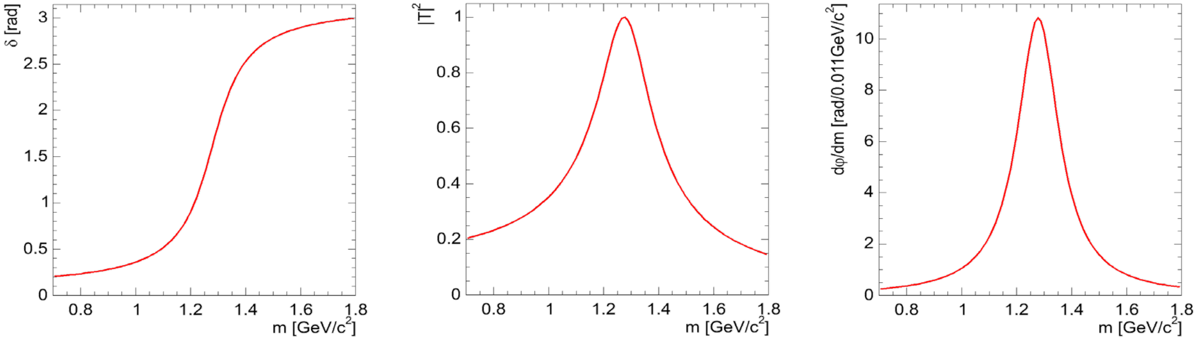
$$\Psi_s = \Psi_i - \Psi_f = \frac{1}{k}\sum_{\ell=0}^{\infty}(2\ell+1)\frac{\eta_\ell \exp(2i\delta_\ell - 1)}{2i} P_\ell(\cos\theta)\frac{\exp(ikr)}{r}$$

T_ℓ: dynamic amplitude *Angular amplitude*

The Breit-Wigner resonance formula

- **Spin ℓ Resonance**: when the elastic scattering amplitude (η_ℓ=1) reaches its maximum
  - $\delta = \pi/2 \Rightarrow T = e^{i\delta}\sin\delta = 1/(\cot\delta - i)$
  - $\cot\delta \approx -(E - E_R)\cdot\frac{2}{\Gamma}$
  - If $|(E - E_R)| \approx \Gamma \ll E_R$ the resonance is symmetric

$$T(E) = \frac{\Gamma/2}{(E_R - E) - i\Gamma/2}$$



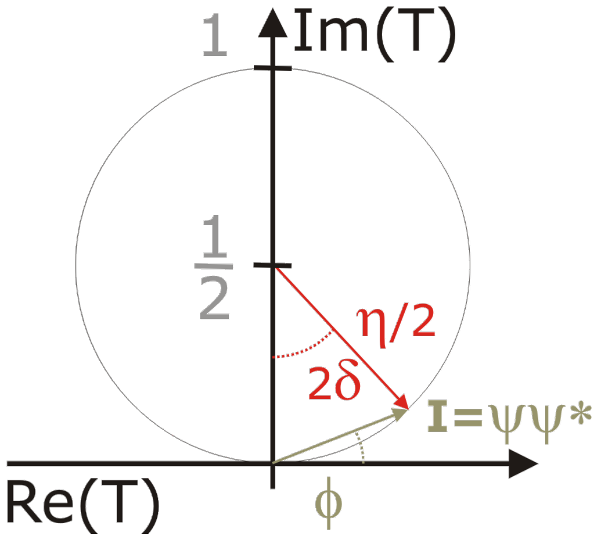
T matrix features

$$T_\ell = \frac{\eta_\ell \exp(2i\delta_\ell) - 1}{2i}$$

- The T-matrix must be **unitary** (the probability of a reaction must not exceed unity)

- Visualized via the Argand plot: $\mathrm{Im}T_\ell$ vs $\mathrm{Re}T_\ell$

- If $\eta_\ell \leq 1$:
  - δ_ℓ varies from 0 to $\pi/2$
    - If $\delta_\ell = \pi/2$: T_ℓ is purely imaginary and gets its maximum value



Resonance in ℓ wave

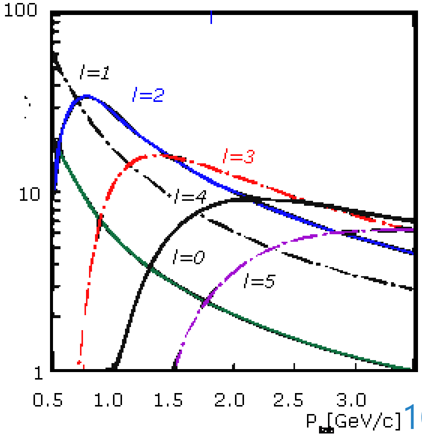
Summing up PWAs: coherent vs incoherent sum

- **Interaction at rest:**
  - No interference among terms with different J^{PC}
    - Only possible to measure the orbital angular momentum ℓ of the initial state

$$A(m,\theta,\phi) = \sum_J \alpha_J \left|F_J(m,\theta,\phi)\right|^2$$

α_J: amplitude weigh
complex number
to be set by the fit

- **Scattering in flight:**
  - No well defined initial state (only a mixture)
  - Relative angular momentum proportional to beam energy
  - Many waves interfere with each other
  - Complex problem at high energy
    - In $\bar{p}p$ annihilations (rule of thumb): $\ell \approx p_{cms}/200$ MeV/c



How to write in-flight amplitudes

- Helicity formalism: helicities do not interfere
- Two transition amplitudes:
  - T_p: production matrix of a J^{PC} state in flight
  - T_d: decay matrix of a mother state
- Total transition amplitude for a reaction

$$a(p_1, s_1, \lambda_1) + b(p_2, s_2, \lambda_2) \to J^{PC} \to final\ state$$

$$\boxed{f_{\lambda_1\lambda_2} = \sum_{JM} \langle p_i | T_d | JM \rangle \underbrace{\langle JM | T_p^+ | p_1 p_2 \lambda_1 \lambda_2 \rangle}_{H^{*J}_{\lambda_1\lambda_2} D^{*J}_{M\lambda}(\theta,\phi,0)}} \quad (\lambda = \lambda_1 - \lambda_2)$$

- Ref. system: reaction rest frame & z axis along the projectile particle direction:

$$D^M_\lambda(0,0,0) = \delta^M_\lambda \quad \Longrightarrow \quad \boxed{\langle JM | T_p^+ | p_1 p_2 \lambda_1 \lambda_2 \rangle = H^{*J}_{\lambda_1\lambda_2} \delta_{M\lambda}}$$

In-flight amplitudes II

- If the initial state is composed by a fermion pair ($s_1=s_2=½$ and λ_1, $\lambda_2= \pm½$) one has only the $\lambda = \pm1$, 0 components, i.e.

$$\lambda = \lambda_1 - \lambda_2 = \begin{cases} +1 & H^{*J}_{+-} \\ 0 & H^{*J}_{++}, H^{*J}_{--} \\ -1 & H^{*J}_{-+} \end{cases}$$

    - Of the many |JM⟩ states produced in-flight, only the M=0,±1 projections contribute to the total amplitude
- Since the helicity (as the spin) is a measurable quantity, it does not interfere so the PWA's have to be summed incoherently
    - Average over the initial state helicities: ¼
    - Sum over the final state helicities
- Further symmetries (parity, CP, etc) can be applied to simplify

Total in-flight amplitude

$$u(\theta,\phi) = \frac{1}{4}\sum_{\lambda_1\lambda_2}\left|F_{\lambda_1\lambda_2}\right|^2 = \frac{1}{4}\sum_{\lambda_1\lambda_2}\left|\sum_J f_\lambda^J H_{\lambda_1\lambda_2}^J\right|^2$$

$$f_\lambda^J = T_\lambda^J D_{\lambda 0}^J(0,0,0) \equiv T_\lambda^J$$

- unknown parameters: helicity couplings
- selection rules help to reduce their number:
  - s+p waves in $\bar{N}N$ annihilation: 24x $H^J_{\lambda 1,\lambda 2}$ couplings
    - 4x6 states (lowest): $^1S_0$, $^3S_1$, $^1P_1$, $^3P_0$, $^3P_1$, $^3P_2$

Selection rules for helicity couplings

- Parity conservation:
  - $\mathbf{H^{J}_{-\lambda 1,-\lambda 2} = \mathit{P(-1)^{J}}\, H^{J}_{\lambda 1,\lambda 2}}$
    - $\mathbf{H_{++} = \mathit{P(-1)^{J}}\, H_{--}}$
    - $\mathbf{H_{+-} = \mathit{P(-1)^{J}}\, H_{-+}}$

- From Clebsch-Gordan properties: since

$$H^{*J}_{\lambda_1\lambda_2} = \sum_{LS} \sqrt{\frac{2L+1}{2J+1}} \langle L0S\lambda|J\lambda\rangle\langle s_1\lambda_1\ s_2-\lambda_2|S\lambda\rangle\alpha_{LS}$$

some couplings are zero, depending on *L* and *S* values
  - If $L+S-1$ = odd: $H_{++} = H_{--} = 0$
  - If J or $S = 0$ (or both): $H_{+-} = H_{-+} = 0$

Helicity couplings in fermion-antifermion initial states

$^{2S+1}L_J$	J^P	H_{++}/H_{-} (J_1)	H_{-+}/H_{+-} (J_2)
$^1S_0$	0^-	OK	0
$^3S_1$	1^-	OK	OK
$^1P_1$	1^+	OK	0
$^3P_0$	0^+	OK	0
$^3P_1$	1^+	0	OK
$^3P_2$	2^+	OK	OK
$^1D_2$	2^-	OK	0
$^3D_1$	1^-	OK	OK
$^3D_2$	2^-	0	OK
$^3D_3$	3^+	OK	OK

- $^1S_0$, $^1P_1$ and $^3P_0$ never interfere with $^3P_1$
- To write the total amplitude one has to separate initial states with ++/-- (J_1) and +-/-+ (J_2) non-zero couplings
    - Up to *P* wave:

$$u(\theta,\phi) = \left|\sum_{J_1} H^J_{++} f_{J,0}\right|^2 + \left|\sum_{J_1} P(1-)^J H^J_{++} f_{J,0}\right|^2 + \\ + \left|\sum_{J_2} H^J_{+-} f_{J,-1}\right|^2 + \left|\sum_{J_2} P(1-)^J H^J_{+-} f_{J,1}\right|^2$$

    - $\boldsymbol{H_{++}}$ and $\boldsymbol{H_{+-}}$ are *free complex parameters* in the fit

Summary: Partial Waves decomposition

1. Estimate the number of partial waves you have to insert in the amplitude (depending on energy)

2. Consider the type of process (and choose the most proper reference system)
    - At rest interaction
        - Coherent sum on J^{PC} states
    - In-flight interaction
        - Incoherent sum on helicities

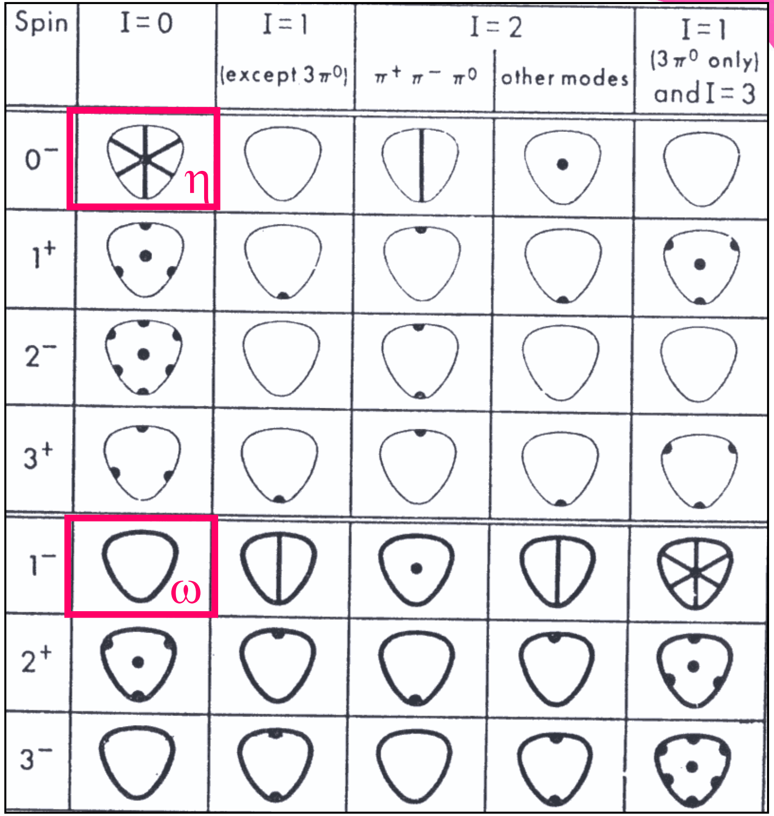
3. Define the quantum numbers of every partial wave of the initial state and how they match with final state 2-body systems

4. Go on building each partial wave amplitude with spins and energy dependent part

Three body decays of resonances

- Extension of DP features
  - two observables only needed to describe the decay
    - Analyzers:
      - Normal to the decay plane
      - Break-up momentum of a pair in the resonance c.m.s.
  - DP description useful if the decay particles have the same mass

Lines and spots: region of depletion of DP density

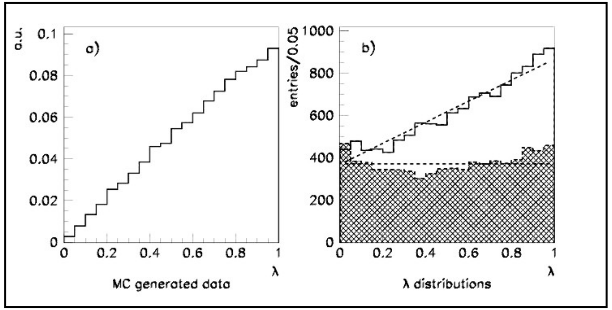


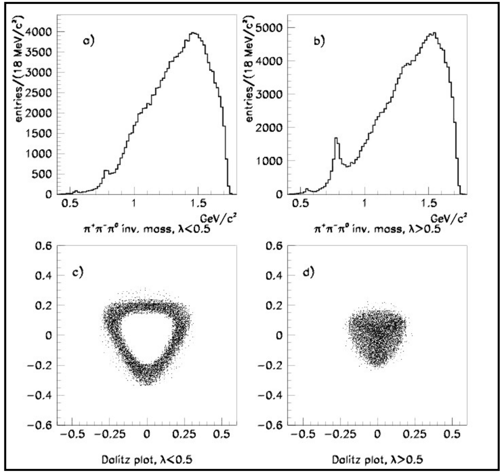
Selecting the ω meson: use of the λ parameter method

- Statistical selection of the events most likely coming from a ω decay

$$\lambda = \frac{(\vec{q_1} \times \vec{q_2})^2}{\alpha(m^2 - \sum_i \mu_i^2)^2}$$

  - **ω events: λ distribution proportional to λ**
  - **Background events: flat λ distribution**
  - Signal obtained by subtracting the λ distributions in sides bins on both ω sides, within a given interval





ω events populate the center of the Dalitz plot

Spin-Angular part of the amplitude: Spin Formalisms

- Zemach formalism
- Helicity & canonical formalisms
- Moments analysis

Spin formalisms

- Several equivalent descriptions:
  - Tensor formalisms
    - Simple and fast for small *L* and *S*
      - Non relativistic (Zemach)
      - Covariant form (Rarita-Schwinger)
  - Spin-projection formalisms
    - Quantization axis chosen + proper rotations
    - Efficient also for large *L* and *S*
      - Helicity formalism (good for half-integer spins)
      - Transversity formalism
      - Canonical (orbital) formalism

Once a formalism is chosen…

- Definition of a single particle state of given spin S
- Definition of a two-particle state in its center-of-mass system and of the relative angular momentum L between particles
- Transformation to states of given total angular momentum J
- Apply symmetry conservation and kinematic constraints to the amplitudes

Tensor formalism: non-relativistic Zemach tensors

- Zemach (non-covariant) treatment:
  - Every angular momentum L is described by a symmetric and traceless tensor of rank L in 3-D phase space
  - Written in the rest frame of every di-particle frame, without any boost to a general reference frame
    - $L = 0 \Rightarrow A^0 = 1$
    - $L = 1 \Rightarrow A^1_i = q_i$
    - $L = 2 \Rightarrow A^2_{ij} = 3/2\, q_i q_j - ½|q_i|^2\, \delta_{ij}$
  - Coupling of spins and orbital angular momenta:
    - Tensor algebra and contractions
  - These amplitudes are **not Lorentz invariant**

How to build covariant tensors

$$J^P \longrightarrow j^p + c$$
$$j^p \longrightarrow a + b$$

R

Velocity of the **R resonance** of mass $m^2 = (a+b)^2$:

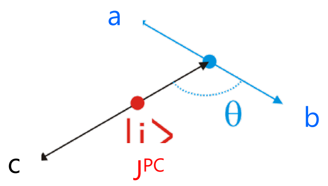
$\boldsymbol{u_\mu = (a_\mu + b_\mu)/m}$

($u^2 = 1$: *time-like vector*)

- A spin rank-1 tensor has no time component in the resonant rest frame: it only represents the behaviour of a particle at rest under rotations
  - **S_μ: space like vector** $\Rightarrow S_\mu u^\mu = 0$
- Spin-1 covariant tensor: $\boldsymbol{S_\mu = q_\mu - (qu)u_\mu}$
  - $q_\mu = a_\mu - b_\mu$ break-up momentum (in R = a+b cms)
  - S norma is negative: $S^2 = q^2 - (qu)^2 = -|\boldsymbol{q}_R|^2$
- Spin-2 covariant tensor: $\boldsymbol{T_{\mu\nu} = S_\mu S_\nu - 1/3\, S^2(g_{\mu\nu} - u_\mu u_\nu)}$
  - T is traceless and $T^2 > 0$

Properties of covariant tensors

- **General rule**: orthogonalization of the 4-velocity and the spin or the angular momentum of the particle
  - Tensor contraction
  - In the J^P rest frame:
    - $q_\mu = c_\mu - (a+b)_\mu$ break-up momentum
    - $u_\mu = (1, 0)$ cms velocity



- Spins and orbital momentum have the same representation

- For a particle of spin s: rank-s tensor
  - $(p^2+m^2)\,\Phi_{\mu 1\,\mu 2\ldots\mu s} = 0$
  - With the following symmetry, tracelessness and orthogonality properties:
    - $\Phi_{\ldots\mu i\ldots\mu j\ldots} = \Phi^{\ldots\mu j\ldots\mu i}$
    - $g_{\mu i\mu j}\,\Phi^{\ldots\mu i\ldots\mu j\ldots} = 0$
    - $p_{\mu i}\Phi^{\ldots\mu i\ldots} = 0$

Combination of covariant tensors

- Tensors are composed following the quantum-mechanical rules of angular momentum coupling:
  - $j_1 \oplus j_2 = J \quad \Rightarrow |j_1 - j_2| \leq J \leq j_1 + j_2$

$1 \oplus 1 = 0$	$S = A_\mu B^\mu$
$1 \oplus 1 = 1$	$S_\mu = \varepsilon_{\mu\nu\rho\sigma} A^\nu B^\rho u^\sigma$
$1 \oplus 1 = 2$	$T_{\mu\nu} = ½(A_\mu B_\nu + A_\nu B_\mu)$

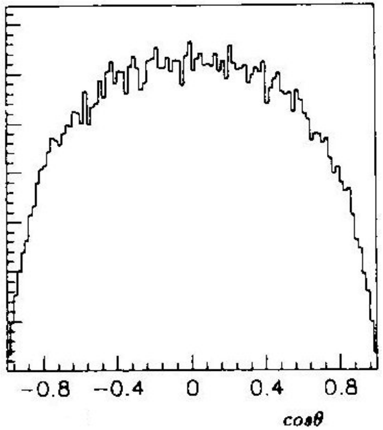
Covariant tensors vs angular distributions

- The angular distributions of the particles emitted in a decay of a resonance correspond to the square moduli of the amplitudes

- Covariant formalism takes automatically into account relativistic effects
  - More important the smaller the mass of the resonant system (larger effect of Lorentz boost)
  - Relativistic vs non-relativistic angular distributions are different
    - Important effect with large statistics

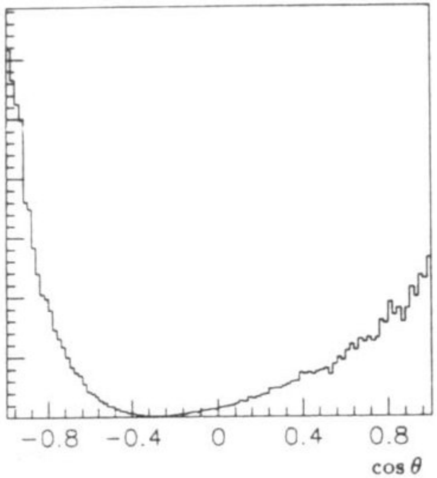
Practical application of covariant tensors – example 1

- $\bar{p}p(^3S_1) \rightarrow \rho^0\pi^0,\ \rho^0 \rightarrow \pi^+\pi^-$
  - Initial state quantum numbers: $0^-(1^{--})$
  - Final state:
    - ρ^0 **= 1$^+$ (1^{--})**
    - **π = 1$^-$ (0^{-+}), p_0 momentum**
      - $P(\rho^0\pi^0) = (-1)^L \Rightarrow L(\rho^0\pi^0)$ **must be odd == 1**
      - $S(\rho^0) = 1 \Rightarrow \ell(\pi^+\pi^-) = 1$
  - Tensors:
    - ρ^0 spin : $R_\mu = q_\mu - (q \cdot u)u_\mu$ (spin 1)
      - $q = p_+ - p_-,\ u = \frac{1}{2} * (p_+ + p_-)/(p_+ + p_-)$
    - Intermediate state: $L(\rho^0\pi^0)$: $L_\mu = r_\mu - (r \cdot v)v_\mu$ (relative angular momentum $L=1$)
      - $r = p_0 - p_\rho,\ v = (p_0 + p_\rho)/\sqrt{s}$
    - Spin amplitude of the reaction: $1 \oplus 1 = 1$
      - $A = (p_+ - p_-) \times (p_0 - p_\rho) = (p_+ - p_-) \times p_0$
        - $\sin^2\theta$ distribution for the angle between $(\boldsymbol{p}_+ - \boldsymbol{p}_-)$ and $\boldsymbol{p}_0$



Practical application of covariant tensors – example 2

- $\bar{p}p(^1S_0) \rightarrow \bar{K}^0 K^{*0},\ K^{*0} \rightarrow K^+\pi^-$ (particles of different masses)
  - Initial state quantum numbers: $^-$ (0^{-+})
  - Final state:
    - $\bar{K}^0$ = ½ (0$^-$)
    - **K^{0*} = ½ (1$^-$), momentum p^***
    - **π = 1$^-$ (0^{-+})**
      - ***$P(\bar{K}^0K^{0*}) = (-1)^L \Rightarrow L(\bar{K}^0K^{0*})$* must be odd == 1**
  - Tensors:
    - K^{0*} spin : $K_\mu = q_\mu - (q\cdot u)u_\mu$ (spin 1)
      - $q = p_+ - p_-,\ u = ½ * (p_+ + p_-)/(p_+ + p_-)$
    - Intermediate state: $L(\bar{K}^0K^{0*})$: $L_\mu = r_\mu - (r\cdot v)v_\mu$ (spin 1)
      - $r = p_0 - p^*,\ v = (p_0 + p^*)/\sqrt{s}$
    - Spin amplitude of the reaction: 1 ⊕ 1 = 0
      - $A = (\mathbf{p}_+ - \mathbf{p}_-) \times (\mathbf{p}_0 - \mathbf{p}^*) = (\mathbf{p}_+ - \mathbf{p}_-) \times \mathbf{p}^*$
        - $(1 + z^2)\cos^2\theta$ distribution for the angle between ($\mathbf{p}_+ - \mathbf{p}_-$) and $\mathbf{p}^*$



Practical application of covariant tensors – example 3

- $\bar{p}p(^1S_0) \rightarrow f_2(1270)\pi^0,\ f_2(1270) \rightarrow \pi^+\pi^-$
  - Initial state quantum numbers: $1^-(0^{-+})$
  - Final state:
    - $f_2 = 0^+ (2^{++})$
    - $\pi = 1^- (0^{-+})$
      - $P(f_2\pi^0) = (-1)^L \Rightarrow L(f_2\pi^0)$ **must be even = 2**
      - $S(f_2) = 2 \Rightarrow \ell(\pi^+\pi^-) = 2$
  - Tensors:
    - f_2 spin : $T_{\mu\nu} = \sqrt{3/2}\,(S_\mu S_\nu + 1/3\,(g_{\mu\nu} - w_\mu w_\nu))$ (spin 2)
      - $w = (a+b)_\mu/m_R$
    - Intermediate state $L(f_2\pi^0)$: $M_{\rho\sigma} = \sqrt{3/2}\,(N_\rho N_\sigma + 1/3\,(g_{\rho\sigma} - v_\rho v_\sigma))$ (spin 2)
      - $v = (p_a + p_b + p_c)/\sqrt{s}$
    - Spin amplitude of the reaction: $2 \oplus 2 = 0$
      - $A = \varepsilon_{\mu\nu\rho\sigma} T_{\mu\nu} M^{\rho\sigma}$
        - $1 + z^2/3 + z^2\cos^2\theta + z^4(\cos^2\theta - 1/3)^2 \approx (Y^0_2(\theta))^2$ **distribution**

Summary: angular distributions

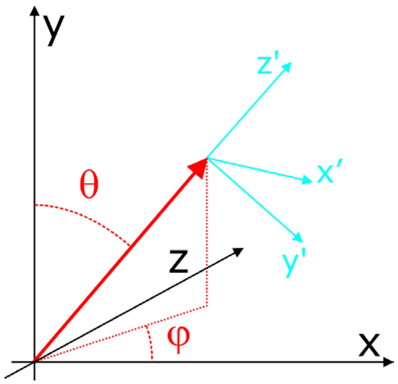
- **Spin 0 resonance**
  - Flat angular distribution, *in any wave* (defined by the relative angular momentum)

- **Spin 1 resonance**, with $z = p/\sqrt{s}$:
  - *S wave:* $(1+ z^2\cos^2\theta)$ example: $\varphi/\omega/\rho$ production from spin singlet $^1P_1$
  - *P wave:*
    - $J=0$: $(1+z^2)\cos^2\theta$
    - $J=1$: $\sin^2\theta$ example: $\varphi/\omega/\rho$ production from spin triplet $^3S_1$
    - $J=2$: $\frac{1}{2} + (1/6+ 2/3\, z^2)\cos^2\theta$
  - *D wave:*
    - $J=1$: $1/6+ (\frac{1}{2}+ 2/3\, z^2)\cos^2\theta$
    - $J=2$: $\frac{3}{4}(1 - \cos^2\theta)$

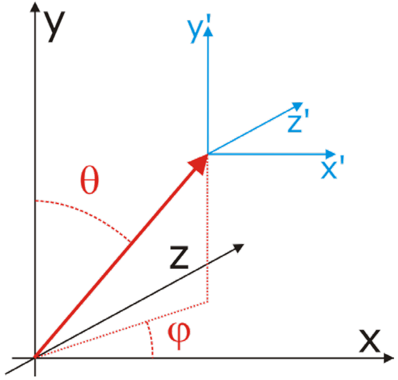
- **Spin 2 resonance**
  - *S wave:*
    - $J=2$: $1+z^2/3+ z^2\cos^2\theta + z^4(\cos^2\theta - 1/3)^2$

Spin-Projection formalisms

- Different choices of the quantization axis
- All single particle states are derived from the basic states through a Lorentz transformation and a Wigner rotation



Helicity:
new axis z′ parallel to momentum

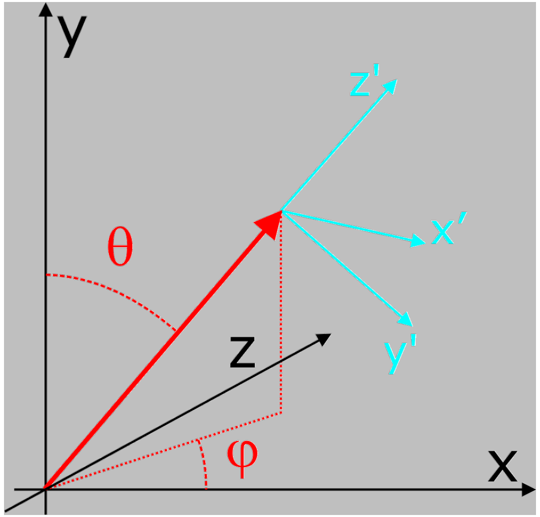


Canonical:
new axis z′ rotated

Single particle state: Helicity

$$|\vec{p},\lambda\rangle$$

- 1) z-axis rotated around the intermediate state momentum direction
- 2) Lorentz boost along the momentum



$$\hat{R}|\vec{p},\lambda\rangle = \left|\hat{R}\vec{p},\lambda\right\rangle$$

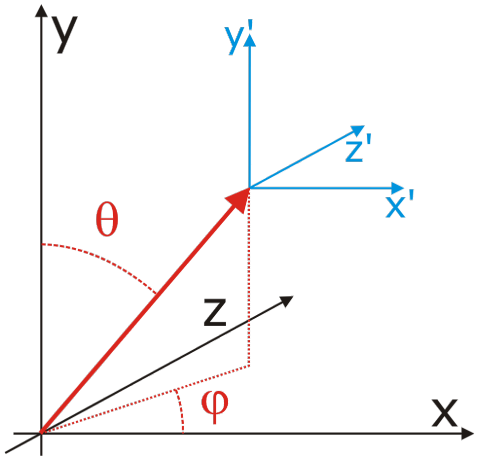
$$\hat{R}_0 = \hat{R}_0(\theta,\phi,0)$$

$$|\vec{p},\lambda\rangle = \sum_m D^j_{m\lambda}(R_0)\,|\vec{p},\lambda\rangle$$

Single particle state: Canonical formalism

$$|\vec{p}, jm\rangle \equiv L\vec{p}|jm\rangle = \hat{R}_0 L_z \hat{R}_0^{-1}|jm\rangle$$

- 1) rotation of the momentum vector around the z-axis
- 2) Lorentz boost along z
- 3) z-axis rotated back to the momentum direction



$$\hat{R}|\vec{p}, m\rangle = \sum_{m'} D^j_{m'm} \left|\hat{R}\vec{p}, m\right\rangle$$

$$\vec{e}_{\vec{p}} = \hat{R}_0(\theta, \phi, 0)\vec{e}_z$$

Two-particle state: Helicity

- Built from single particle states (back-to-back)

$$|\Omega_s s\lambda_s t\lambda_t\rangle \equiv \frac{1}{4\pi}\sqrt{\frac{p_s}{m}}\hat{R}_0[|L_z p_s|s\lambda_s\rangle|L_z p_t|t\lambda_t\rangle]$$

$$|JM\lambda_s\lambda_t\rangle \equiv \sqrt{\frac{2J+1}{4\pi}}\int d\Omega\ \ D^{J*}_{M,\lambda_s-\lambda_t}|\Omega, s\lambda_s t\lambda_t\rangle$$

Wigner rotation functions

- Intrinsically non-covariant
  - *D* functions expressed in each resonance rest frame
  - *p/m* dependence

Two-particle state: Canonical

- Built from single particle states (back-to-back)
- $s \oplus t = S$
- $L \oplus S = J$

$$|\Omega_s^0 s m_s t m_t\rangle \equiv \frac{1}{4\pi}\sqrt{\frac{p_s}{m}}\left[|L\vec{p}_s|sm_s\rangle|L\vec{p}_t|tm_t\rangle\right]$$

$$|\Omega, s m_s\rangle = \sum_{m_s, m_t} \langle s m_s t m_t | S m_s\rangle |\Omega, s m_s t m_t\rangle$$

$$\left|L m_L S m_S\right\rangle = \int d\Omega \; Y^L_{m_L}(\Omega)\; |\Omega, S m_S\rangle$$

$$|JMLS\rangle = \sum_{m_L, m_S, m_s, m_t} \langle L m_L S m_S | JM\rangle \langle s m_s t m_t | S m_S\rangle \int d\Omega \; Y^L_{m_L}(\Omega)\; \left|\Omega_S^0, s m_s t m_t\right\rangle$$

Clebsch-Gordan coefficients

Spherical harmonics

Example 1: $f_2(1270) \to \pi^+\pi^-$ decay in helicity formalism

- Helicity amplitude:

$$A^{JM}_{\lambda_1\lambda_2} = \frac{4\pi}{\rho_s}\langle \Omega_s, s\lambda_s, t\lambda_t | \mathrm{M} | JM\rangle = N_J f_{\lambda_1\lambda_2} D^{J*}_{M\lambda}(\Omega_s)$$

- Since $\lambda = \lambda_1 - \lambda_2 = 0$ and J=2:

$$A^{2M}_{00}(\theta,\phi) = \sqrt{5}\langle 2000|00\rangle\langle 0000|00\rangle f_{00} D^{2*}_{M0}(\theta,\phi)$$

- The relevant D Wigner functions are 5:

$$D^{2*}_{M0}(\theta,\phi) = \begin{pmatrix} d^2_{-20}(\theta)\; e^{-2i\phi} \\ d^2_{-10}(\theta)\; e^{-i\phi} \\ d^2_{00}(\theta) \\ d^2_{10}(\theta)\; e^{i\phi} \\ d^2_{20}(\theta)\; e^{2i\phi} \end{pmatrix}$$

Example 1: $f_2(1270) \to \pi^+\pi^-$ decay in helicity formalism (cont.'ed)

- The intensity is given by

$$I(\theta) = \sum_{M,M'} A_{00}^{2M}(\theta,\phi)\,\rho_{MM'} A_{00}^{2M'*}(\theta,\phi)$$

- Being the spin density matrix

$$\rho = \frac{1}{5}\begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix}$$

$$I(\theta) = |f_{00}|^2 \left(\frac{15}{4}\sin^2\theta + 15\sin^2\theta\cos^2\theta + 5\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)^2\right)$$

- The amplitude is a constant:

$$I(\theta) = |f_{00}|^2$$

Example 2: $\bar{p}p(0^{-+}) \to f_2(1270)\pi^0 \to \pi^+\pi^- \pi^0$ in helicity formalism

- Helicity amplitude for each of the angular momenta (f_2 spin and $L(f_2\,\pi^0)=2$):

$$A^{JM}_{\lambda_1\lambda_2} = N_J f_{\lambda_1\lambda_2} D^{J*}_{M\lambda}(\Omega_s)$$

- For the π^0 spin: $|f_{00}|^2 = const.$
- For the relative angular momentum between f_2 and π^0:

$$A^{00}_{00}(\Omega_{f_2\pi^0}) = \langle 2020|00\rangle\langle 2000|00\rangle f_{00} D^{0*}_{00}(\theta,\phi) = \sqrt{\frac{1}{5}} \cdot 1 \cdot 1$$

- For the f_2 spin (zero helicity):

$$A^{20}_{00}(\theta,\phi) = \sqrt{5}\langle 2000|00\rangle\langle 0000|00\rangle f_{00} D^{2*}_{00}(\theta,\phi) = f_{00} Y^2_0$$

Example 2: $\bar{p}\,p(0^{-+}) \rightarrow f_2(1270)\pi^0 \rightarrow \pi^+\pi^-\pi^0$ in helicity formalism (cont.'ed)

- Put everything together to get the total intensity:

$$
\begin{aligned}
I(\theta) &= \left|A_{00}^{00}(\theta,\phi)\rho_{00}A_{20}^{00}(\theta,\phi)\right|^2 \\
&= 5|f_{00}f_{20}|^2\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)^2
\end{aligned}
$$

to be compared with the example for the application of tensor formalism

Covariant tensors vs helicity

- The two approaches are equivalent if:
  - The helicity amplitude is written in covariant form as well
  - The covariant tensor is properly normalized

- Tensor moduli:
  - Kinematic factors depending on p and q
    - ***Centrifugal barrier functions***
    - Almost constant for sharp (narrow) resonances
    - Slowly varying functions of the momenta
    - They do not affect largely angular distributions

    $\Rightarrow$ inserted in the dynamical part of the amplitude

Summary: key points of spin formalisms

- **Tensor formalism**
  - Easy to handle and to code for INTEGER angular momenta
    - Half-integers: more complicated tensors
    - Use of non-normalized tensors allows centrifugal barrier functions to be omitted in the energy-dependent parte of the amplitude

- **Helicity formalism**
  - based on Wigner rotation functions + helicity couplings
  - Half integer spin treated in a simpler way

- **Canonical formalism**
  - based on spherical armonics series + Clebsch-Gordan factors for momentum couplings

Always use covariant formulations (whenever possible)

Moments analysis I

- Based on a Fourier decomposition of the final state, in each invariant mass bin of the final state system
  - *d* functions can be expanded in spherical harmonics
- Spherical harmonics moments: directly related to
  - amplitudes
  - helicity partial waves and relative phases
- The density matrix gets absorbed in a spherical moment
- *z* axis in the production plane:
  - Angular distribution of a 2-body system: sum of the real parts of spherical harmonic moments

$$I(t, M, \theta, \phi) = N \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \langle \mathrm{Re}\, Y_l^m \rangle\, \mathrm{Re}\, Y_l^m\, (\theta, \phi)$$

Moments analysis II

- A new set of real coefficients t_l^m can be obtained inverting the last equation
  - They depend on:
    - Momentum transfer *t*
    - Mass of the 2-body system *M*

- t_l^m related to spherical harmonics by

$$\langle \mathrm{Re}\, Y_l^m \rangle = \frac{1}{N} \varepsilon_l^m t_l^m$$

$$\varepsilon_l^m = \begin{cases} 0 & if \;\; m = 0 \\ 1/2 & if \;\; m \neq 0 \end{cases}$$

$$t_l^m = \left\langle D_{M0}^{L}(\theta, \phi, 0) \right\rangle$$

- t_l^m obtained by a fit $\Rightarrow$ $\langle Y_l^m \rangle$ $\Rightarrow$ total amplitudes

Example: application of moments analysis

- Useful tool to gather further information, especially when
  - No evident signal in the Dalitz Plot
  - No crossing bands