NATIONAL CENTRE FOR SCIENTIFIC RESEARCH "DEMOKRITOS"
INSTITUTE OF NUCLEAR AND PARTICLE PHYSICS

H.F.R.I.
Hellenic Foundation for Research & Innovation

On the numerical computation of dimensionally regularised QCD helicity amplitudes

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NCSR "Demokritos"

HOCTOOLS-II mini-workshop

Torino, October 23, 2025

In collaboration with C. Papadopoulos, D. Canko and A. Spourdalakis

Work in progress

Introduction

- We have presented a method for extracting coefficients of two-loop reduction at integrand level
  - The method works in either $d = 4$ or $d = 4 - 2\varepsilon$ dimensions
  - Addressing IBP reduction of Feynman integrals is the next big step

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  - The method works in either $d = 4$ or $d = 4 - 2\varepsilon$ dimensions
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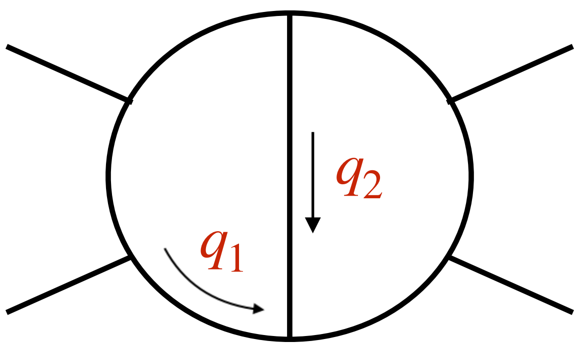
- Performing reduction in $d = 4 - 2\varepsilon$ potentially pays off
  - Solutions of cut equations get simpler structure
  - No need to compute R_2 rational terms separately



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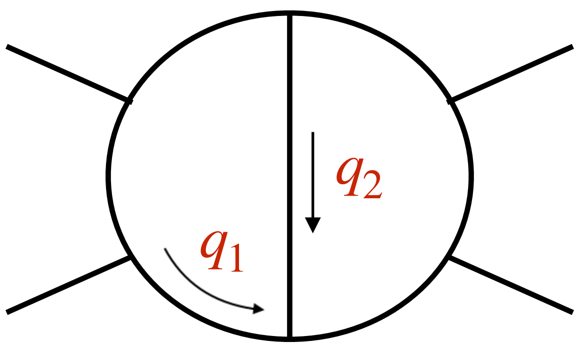
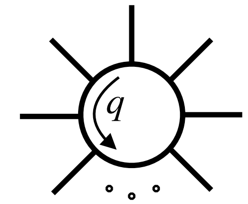


- How feasible are recursive computations of loop numerators in $d = 4 - 2\varepsilon$?

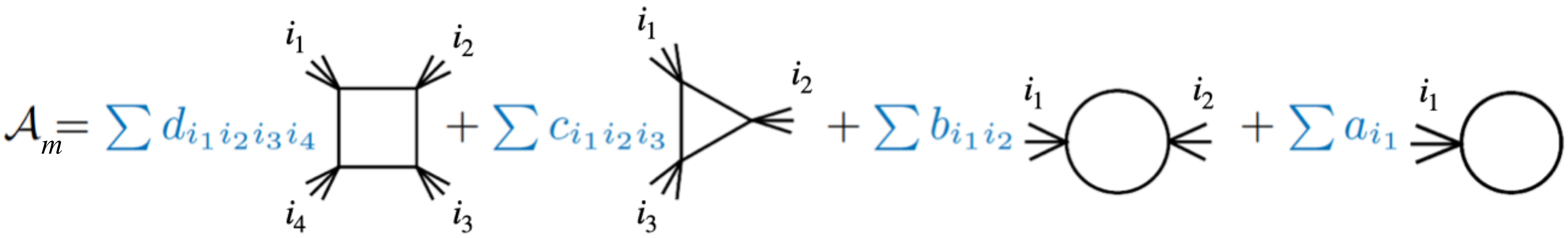
↪ Requisites:
  - a procedure to supplement *evanescent terms* at any step of the recursion;
  - a “numerator provider” working in $d = 4 - 2\varepsilon$ dimensions (possibly general-purpose)

1-loop OPP reduction in a nutshell

[Ossola, Papadopoulos and Pittau, Nucl.Phys.B 763 (2007) 147-169]



$$\rightarrow \quad \mathcal{A}_m = \int \frac{d^d q}{(2\pi)^d} \frac{\bar{N}(q)}{\bar{D}_0 \bar{D}_1 \ldots \bar{D}_{m-1}}$$



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- However in *dimensional regularisation*, the loop integrand is defined in $d = 4 - 2\varepsilon$

Recap: origin of Rational Terms

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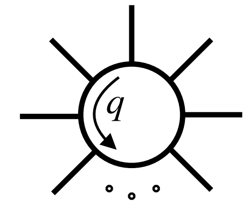

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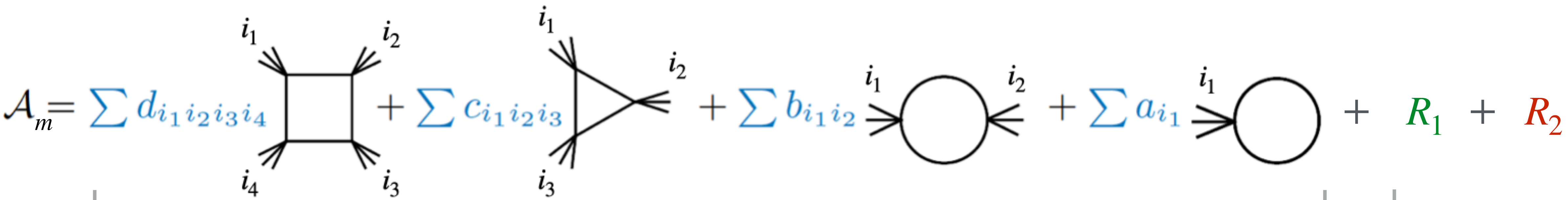

"Cut-constructible" part

"Rational Terms"

- Typically, numerical ME generators compute numerators in $d = 4$
- However in *dimensional regularisation*, the loop integrand is defined in $d = 4 - 2\varepsilon$
- The mismatch can be compensated by adding Rational Terms (R_1, R_2) to the computation

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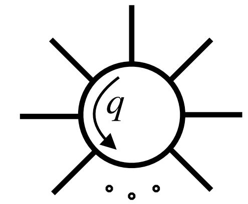

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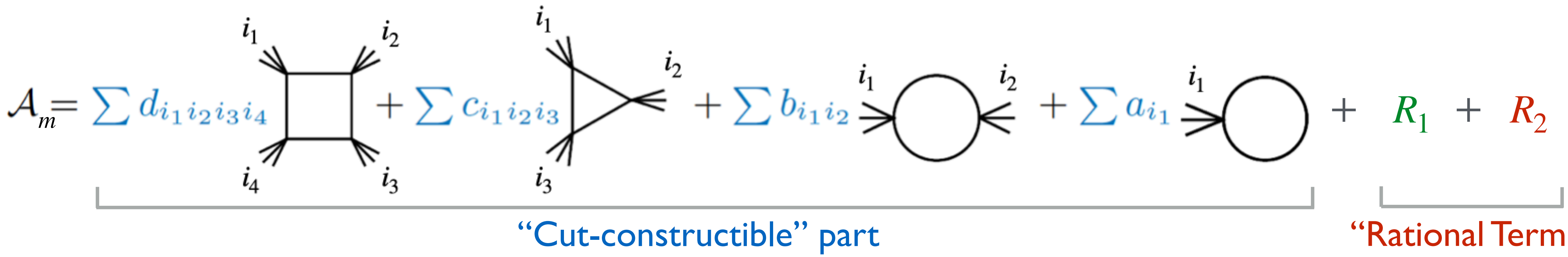
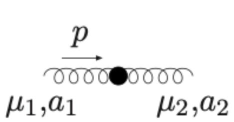


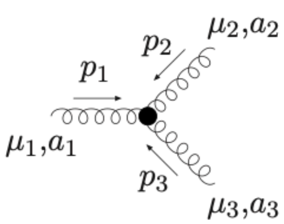
$$\mathcal{A}_m = \sum d_{i_1 i_2 i_3 i_4} \,[\text{box}] + \sum c_{i_1 i_2 i_3}\,[\text{triangle}] + \sum b_{i_1 i_2}\,[\text{bubble}] + \sum a_{i_1}\,[\text{tadpole}] + R_1 + R_2$$

- Typically, numerical ME generators compute numerators in $d = 4$
- However in *dimensional regularisation*, the loop integrand is defined in $d = 4 - 2\varepsilon$
- The mismatch can be compensated by adding Rational Terms (R_1, R_2) to the computation
- R_2 can be cast in the form of extra tree-level Feynman rules

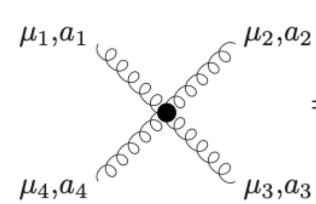
example: QCD
[Draggiotis et al, ’09]



$$= \frac{ig^2 N_{\rm col}}{48\pi^2}\delta_{a_1a_2}\left[\frac{p^2}{2}g_{\mu_1\mu_2} + \lambda_{\rm HV}\left(g_{\mu_1\mu_2}p^2 - p_{\mu_1}p_{\mu_2}\right) + \frac{N_f}{N_{\rm col}}\left(p^2 - 6\,m_q^2\right)g_{\mu_1\mu_2}\right]$$



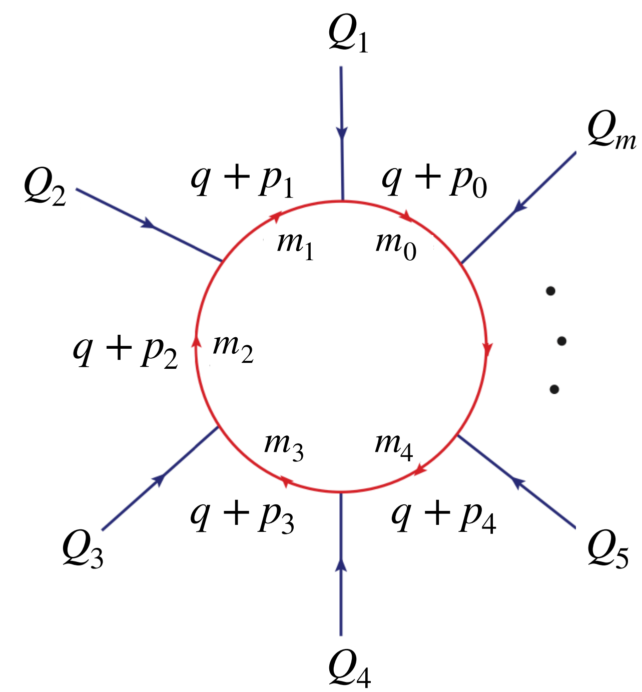
$$= -\frac{g^3 N_{\rm col}}{48\pi^2}\left(\frac{7}{4} + \lambda_{\rm HV} + 2\frac{N_f}{N_{\rm col}}\right) f^{a_1a_2a_3}\,V_{\mu_1\mu_2\mu_3}(p_1,p_2,p_3)$$



$$\begin{aligned} = -\frac{ig^4 N_{\rm col}}{96\pi^2}\sum_{P(234)}\Bigg\{\Bigg[&\frac{\delta_{a_1a_2}\delta_{a_3a_4}+\delta_{a_1a_3}\delta_{a_4a_2}+\delta_{a_1a_4}\delta_{a_2a_3}}{N_{\rm col}} \\ &+4\,Tr(t^{a_1}t^{a_3}t^{a_2}t^{a_4}+t^{a_1}t^{a_4}t^{a_2}t^{a_3})\,(3+\lambda_{\rm HV}) \\ &-Tr(\{t^{a_1}t^{a_2}\}\{t^{a_3}t^{a_4}\})\,(5+2\lambda_{\rm HV})\Bigg]\,g_{\mu_1\mu_2}g_{\mu_3\mu_4} \\ &+12\frac{N_f}{N_{\rm col}}Tr(t^{a_1}t^{a_2}t^{a_3}t^{a_4})\left(\frac{5}{3}g_{\mu_1\mu_3}g_{\mu_2\mu_4}-g_{\mu_1\mu_2}g_{\mu_3\mu_4}-g_{\mu_2\mu_3}g_{\mu_1\mu_4}\right)\Bigg\}\end{aligned}$$

Dimensional regularisation ('t Hooft-Veltman scheme)

$$\mathscr{A}_m = \int \frac{d^d\bar{q}}{(2\pi)^d} \frac{\bar{N}(\bar{q})}{\bar{D}_0\,\bar{D}_1 \ldots \bar{D}_{m-1}}$$



- physical momenta $(Q_1, Q_2, \ldots, Q_m)$ in $d = 4$
- loop momentum $(\bar{q})$ in $d = 4 - 2\varepsilon$

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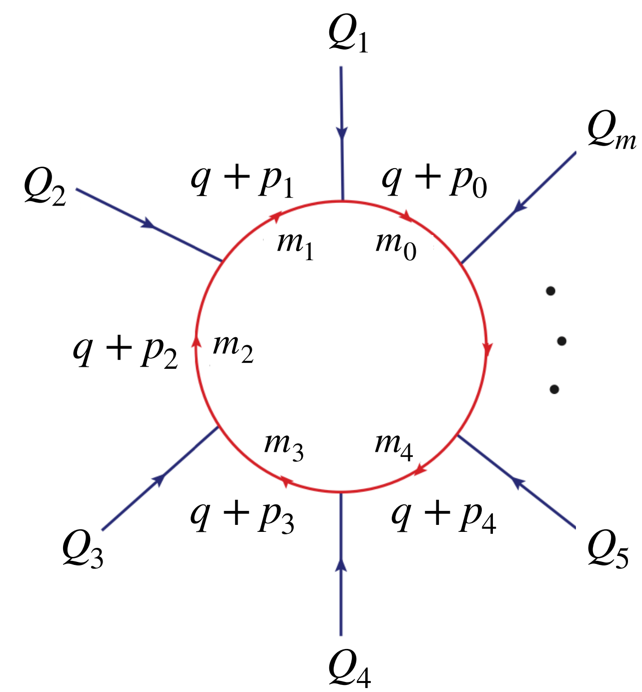


- physical momenta $(Q_1, Q_2, \ldots, Q_m)$ in $d = 4$
- loop momentum $(\bar{q})$ in $d = 4 - 2\varepsilon$

$$\boxed{d = 4 - 2\varepsilon}$$

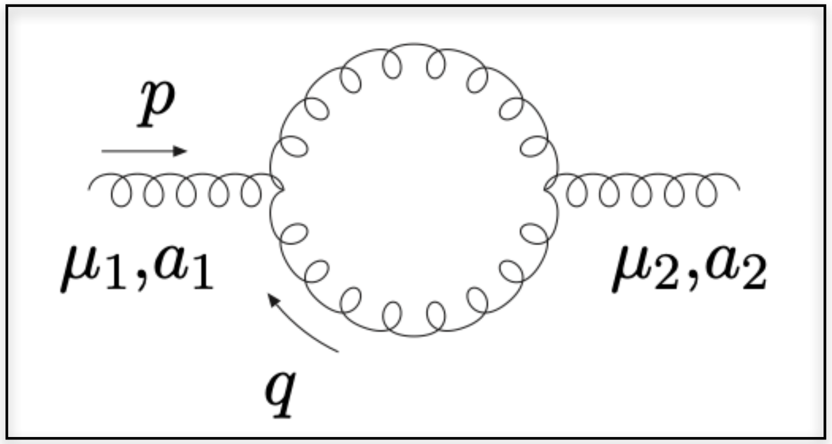
$$\bar{q}^2 = q^2 + \tilde{q}^2, \quad \tilde{q}^2 \equiv \mu$$

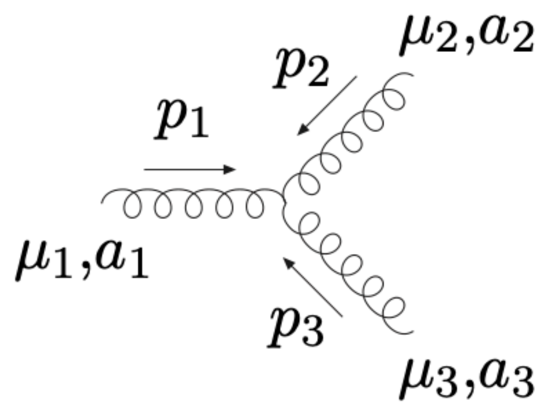
$$\bar{\gamma}^\mu = \gamma^\mu + \tilde{\gamma}^\mu \quad \hookrightarrow \quad \bar{\gamma}^\mu \bar{\gamma}_\mu = d = 4 - 2\epsilon$$

$$\bar{g}^{\mu\nu} = g^{\mu\nu} + \tilde{g}^{\mu\nu} \quad \hookrightarrow \quad \bar{g}^{\mu\nu} \bar{g}_{\mu\nu} = d = 4 - 2\epsilon$$

$$\{\gamma^\alpha, \gamma^\beta\} = 2\, g^{\alpha\beta} I_4 \quad | \quad \{\tilde{\gamma}^\alpha, \tilde{\gamma}^\beta\} = 2\, \tilde{g}^{\alpha\beta} I_4 \quad | \quad \{\gamma^\alpha, \tilde{\gamma}^\beta\} = 0$$

Origin of evanescent terms: a simple example





$$= g\, f^{a_1 a_2 a_3}\, \underbrace{V_{\mu_1\mu_2\mu_3}(p_1,p_2,p_3)}_{\substack{\parallel \\ g_{\mu_1\mu_2}(p_2-p_1)_{\mu_3} \\ +\, g_{\mu_2\mu_3}(p_3-p_2)_{\mu_1} \\ +\, g_{\mu_3\mu_1}(p_1-p_3)_{\mu_2}}}$$

- Loop numerator in $d = 4 - 2\varepsilon$ has a genuine 4-dimensional and an *evanescent* component:

$$\bar{N}(\bar{q}) \;=\; \bar{N}^{\mu_1\mu_2}(\bar{q})\; \varepsilon_{\mu_1}(p)\; \varepsilon_{\mu_2}(p)$$

$$\hookrightarrow \quad \bar{N}^{\mu_1\mu_2}(\bar{q}) \;\to\; V^{\mu_1}{}_{\bar{\beta}\bar{\gamma}}(p,\, -\bar{q}-p,\, \bar{q})\; V^{\mu_2\,\bar{\gamma}\bar{\beta}}(-p,\, -\bar{q},\, \bar{q}+p) \;=$$

$$\boxed{\begin{aligned} = \;& \left(5p^2 + 2p\cdot q + 2q^2\right) g^{\mu_1\mu_2} - 2\,p^{\mu_1}p^{\mu_2} + 5\,p^{\mu_1}q^{\mu_2} + 5\,q^{\mu_1}p^{\mu_2} + 10\,q^{\mu_1}q^{\mu_2} \\ & - \epsilon\left(2p^{\mu_1}p^{\mu_2} + 4p^{\mu_1}q^{\mu_2} + 4q^{\mu_1}p^{\mu_2} + 8q^{\mu_1}q^{\mu_2}\right) + 2\,\mu\, g^{\mu_1\mu_2} \end{aligned}}$$

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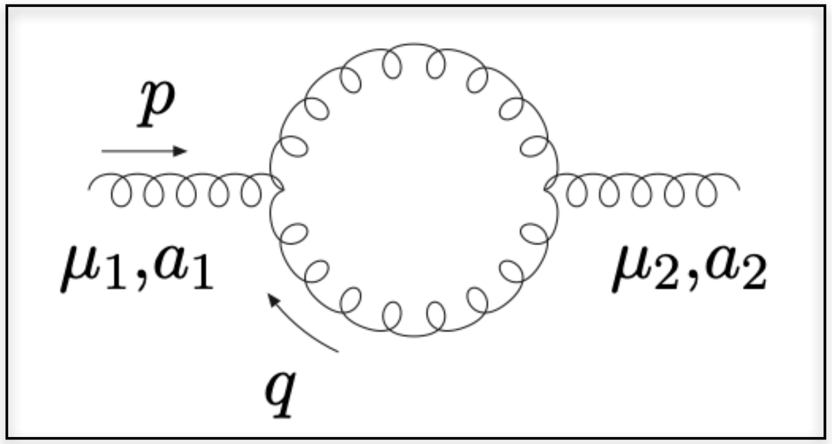

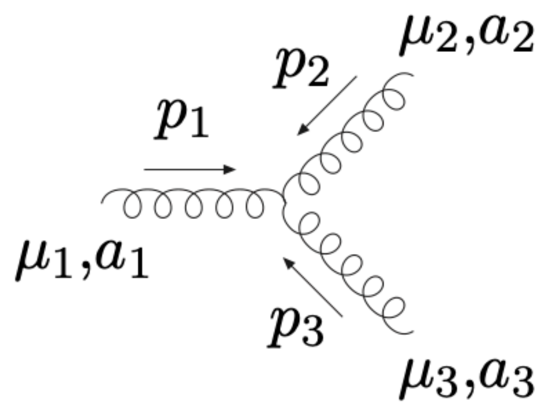

$$= g\, f^{a_1 a_2 a_3}\, \underbrace{V_{\mu_1\mu_2\mu_3}(p_1,p_2,p_3)}_{\substack{\equiv \\ g_{\mu_1\mu_2}(p_2-p_1)_{\mu_3} \\ +\, g_{\mu_2\mu_3}(p_3-p_2)_{\mu_1} \\ +\, g_{\mu_3\mu_1}(p_1-p_3)_{\mu_2}}}$$

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$$\bar{N}(\bar{q}) \;=\; \bar{N}^{\mu_1\mu_2}(\bar{q})\; \varepsilon_{\mu_1}(p)\; \varepsilon_{\mu_2}(p) \;\equiv\; N(q) \;+\; \tilde{N}(q)$$

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$$\boxed{\begin{aligned} =\;& \left(5p^2+2p\cdot q+2q^2\right) g^{\mu_1\mu_2} - 2\,p^{\mu_1}p^{\mu_2} + 5\,p^{\mu_1}q^{\mu_2} + 5\,q^{\mu_1}p^{\mu_2} + 10\,q^{\mu_1}q^{\mu_2} \\ & - \epsilon\left(2p^{\mu_1}p^{\mu_2} + 4p^{\mu_1}q^{\mu_2} + 4q^{\mu_1}p^{\mu_2} + 8q^{\mu_1}q^{\mu_2}\right) + 2\,\mu\, g^{\mu_1\mu_2} \end{aligned}} \quad \begin{aligned} &\longrightarrow\; N(q) \\ &\longrightarrow\; \tilde{N}(q) \end{aligned}$$

The evanescent component of the numerator, $\tilde{N}(q)$, is made of terms $\propto \mu, \epsilon$

$$\mathcal{A}_m = \int \frac{d^d\bar{q}}{(2\pi)^d} \frac{\bar{N}(\bar{q}) \quad \rightarrow \; N(q) + \tilde{N}(q)}{\bar{D}_0 \, \bar{D}_1 \ldots \bar{D}_{m-1}}$$

- Can we compute $\tilde{N}(q)$ using a *numerical, recursive* framework (built in $d = 4$)?

schematically: $\boxed{\tilde{N}(q) = \mathcal{E}[\, N(q) \,]}$

Computing evanescent terms

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- Possible solution: *Four-Dimensional Formulation* (FDF)

[Fazio, Mastrolia, Mirabella and Torres Bobadilla, Eur.Phys.J.C 74 (2014) 12, 3197]

↪ • developed for 1-loop calculations

• 4-dimensional d.o.f of gauge bosons carried out by μ

• $(d-4)$ dimensional d.o.f carried out by *scalar* particles $\Rightarrow$ *extra Feynman rules*

Computing evanescent terms

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- Can we com … in $d = 4$)?

- Possible …

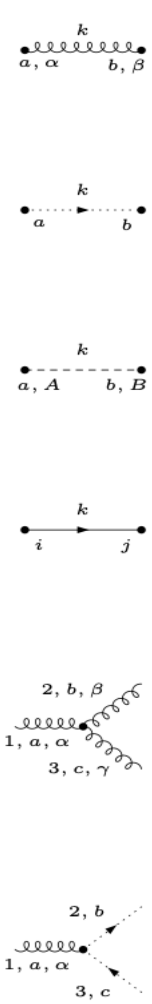
 … 14) 12, 3197]

↪ • deve …

• 4-dir …

• $(d -$ … *man rules*

Fazio, Mastrolia, Mirabella and Torres Bobadilla, Eur.Phys.J.C 74 (2014) 12, 3197



$$= -i\,\delta^{ab}\,\frac{g^{\alpha\beta}}{k^2-\mu^2+i0} \quad \text{(gluon)},$$

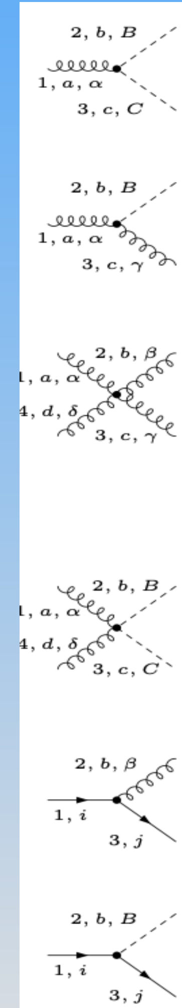
$$= i\,\delta^{ab}\,\frac{1}{k^2-\mu^2+i0} \quad \text{(ghost)},$$

$$= -i\,\delta^{ab}\,\frac{G^{AB}}{k^2-\mu^2+i0}\,, \quad \text{(scalar)},$$

$$= i\,\delta^{ij}\,\frac{\not{k}+i\mu\gamma^5+m}{k^2-m^2-\mu^2+i0}\,, \quad \text{(fermion)},$$

$$= -g\,f^{abc}\left[(k_1-k_2)^\gamma g^{\alpha\beta} + (k_2-k_3)^\alpha g^{\beta\gamma} + (k_3-k_1)^\beta g^{\gamma\alpha}\right],$$

$$= -g\,f^{abc}\,k_2^\alpha\,,$$



$$= -g\,f^{abc}\,(k_2-k_3)^\alpha\,G^{BC}\,,$$

$$= \mp g\,f^{abc}\,(i\mu)\,g^{\gamma\alpha}\,Q^B\,, \quad (\tilde{k}_1 = 0, \quad \tilde{k}_3 = \pm\tilde{\ell})$$

$$= -ig^2\left[+ f^{xad}f^{xbc}\left(g^{\alpha\beta}g^{\delta\gamma}-g^{\alpha\gamma}g^{\beta\delta}\right) + f^{xac}f^{xbd}\left(g^{\alpha\beta}g^{\delta\gamma}-g^{\alpha\delta}g^{\beta\gamma}\right) + f^{xab}f^{xdc}\left(g^{\alpha\delta}g^{\beta\gamma}-g^{\alpha\gamma}g^{\beta\delta}\right)\right],$$

$$= 2ig^2\,g^{\alpha\delta}\left(f^{xab}f^{xcd} + f^{xac}f^{xbd}\right)G^{BC}\,,$$

$$= -ig\,(t^b)_{ji}\,\gamma^\beta\,,$$

$$= -ig\,(t^b)_{ji}\,\gamma^5\,\Gamma^B\,.$$

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$$\tilde{N}(q) = \mathscr{E}[\, N(q)\,]$$

- Possible solution (at 1-loop): *Four-Dimensional Formulation* (FDF)

[Fazio, Mastrolia, Mirabella and Torres Bobadilla, Eur.Phys.J.C 74 (2014) 12, 3197]

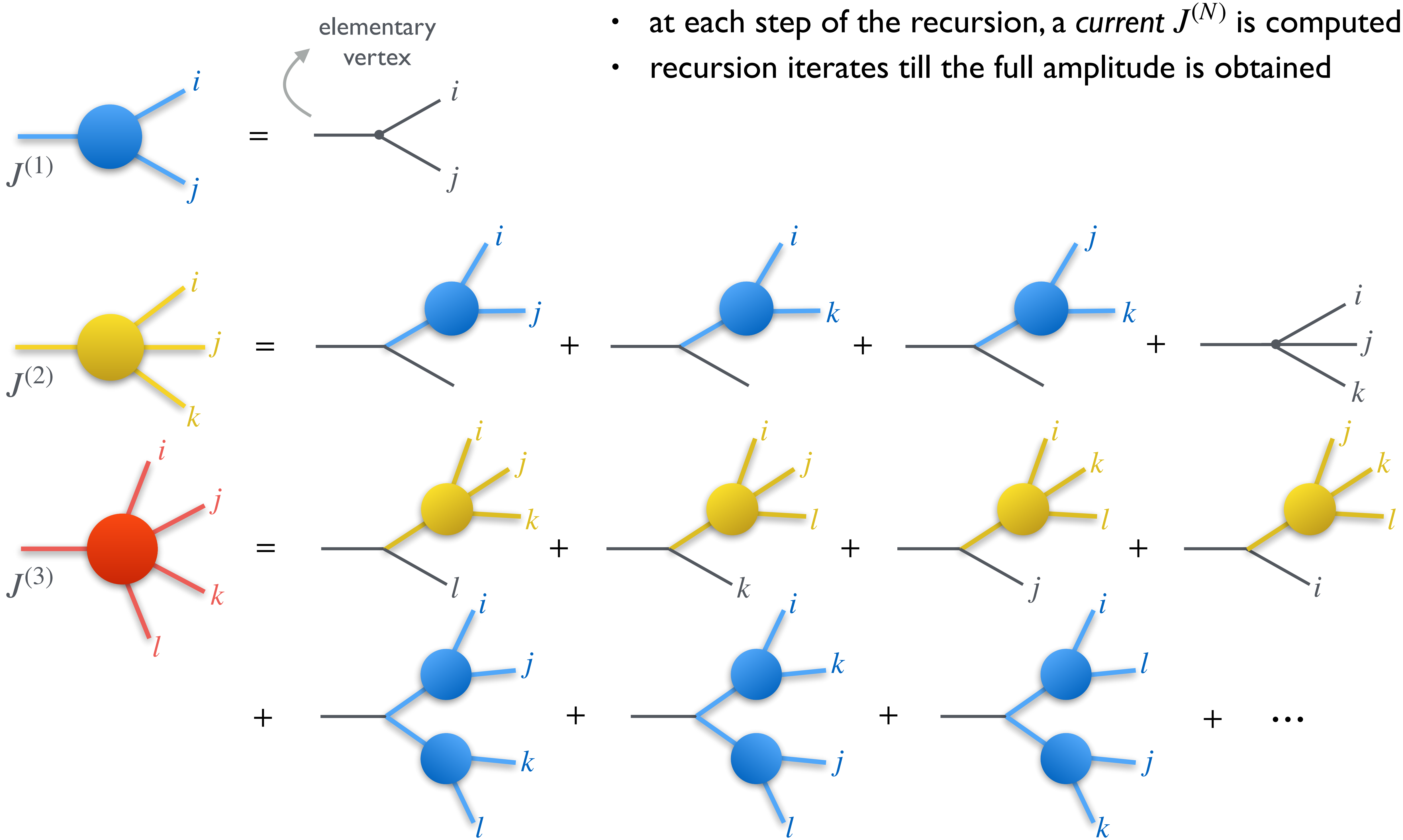
- We are exploring an *alternative* formulation (not requiring extra Feynman rules) *and* we want to extend it to 2-loop applications

I. Basic formalism

1-loop construction

Sketch of recursive amplitude computation

- at each step of the recursion, a *current* $J^{(N)}$ is computed
- recursion iterates till the full amplitude is obtained



- Cutting loop propagator → Tree-level process with two extra particles

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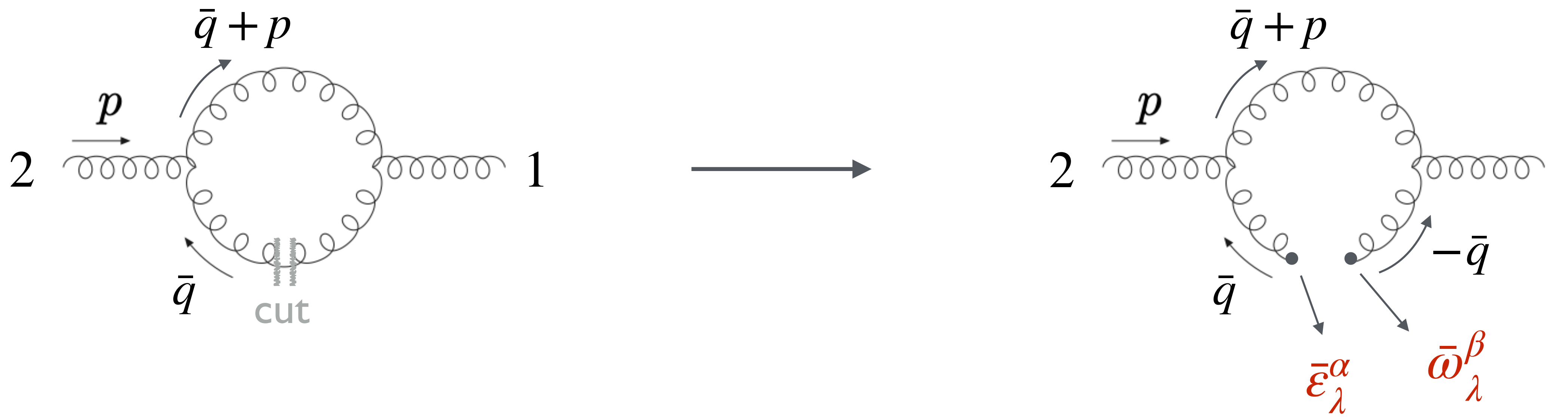


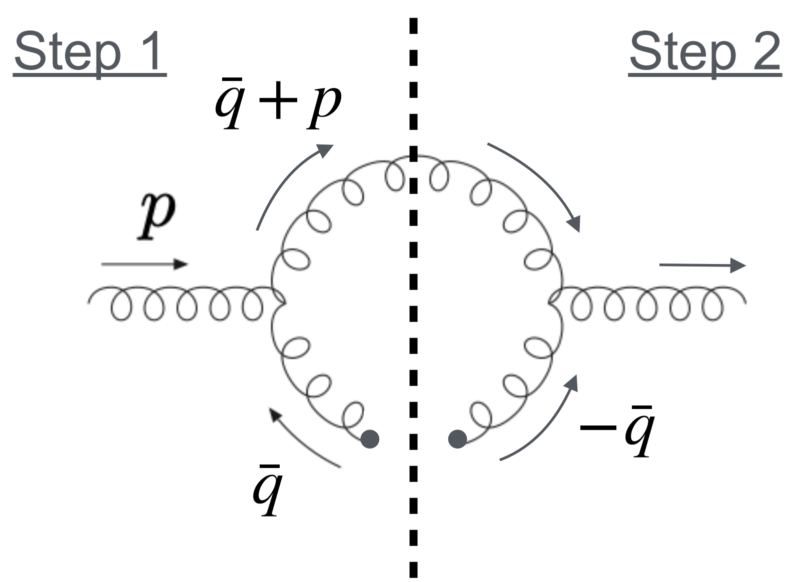
$$\hookrightarrow \quad \boxed{\sum_{\lambda} \bar{\varepsilon}^{\alpha}_{\lambda}\, \bar{\omega}^{\beta}_{\lambda} \rightarrow \bar{g}^{\alpha\beta}}$$

Computing loop numerators recursively

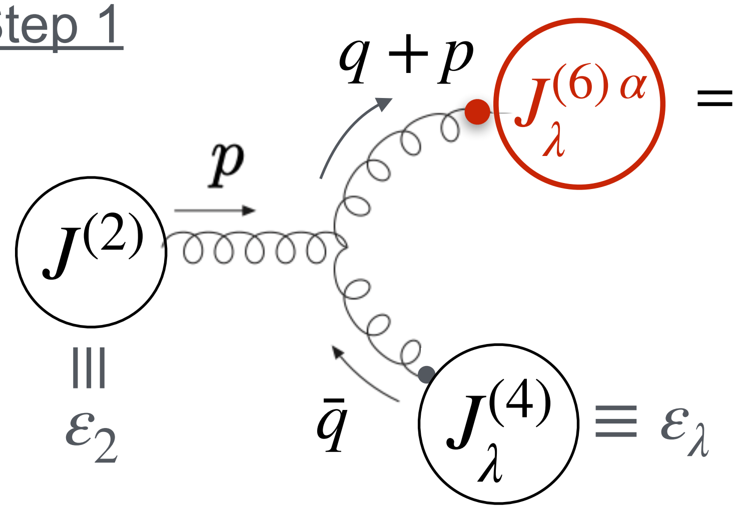
- Cutting loop propagator $\rightarrow$ Tree-level process with two extra particles

$\hookrightarrow$ $\boxed{\sum_{\lambda} \bar{\varepsilon}^{\alpha}_{\lambda}\, \bar{\omega}^{\beta}_{\lambda} \rightarrow \bar{g}^{\alpha\beta}}$

- Recursive calculation of tree-level process

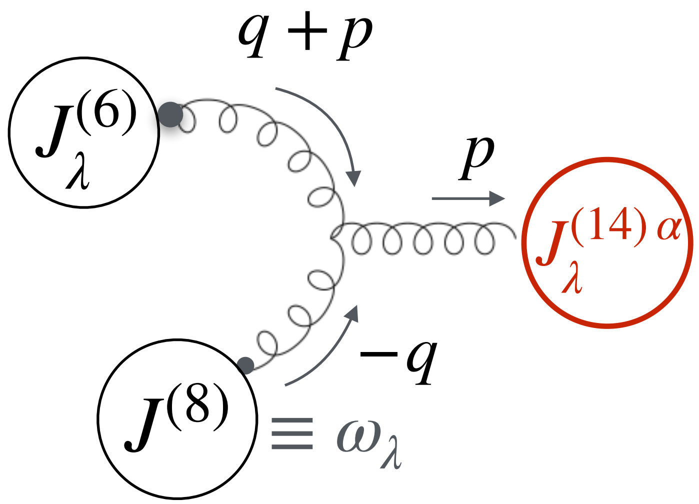


Step 1



$$J^{(6)\,\alpha}_\lambda = \left((-q - 2p)\cdot\varepsilon_\lambda\right) J^{(2)\,\alpha} + \left(-p\cdot J^{(2)}\right)\varepsilon^\alpha_\lambda + \left(J^{(2)}\cdot\varepsilon_\lambda\right)(p-q)^\alpha$$

Step 2



$$J^{(14)\,\alpha}_\lambda = \left((-q - 2p)\cdot\omega_\lambda\right) J^{(6)\,\alpha} + \left((-q-p)\cdot J^{(6)}\right)\omega^\alpha_\lambda + \left(J^{(6)}\cdot\omega_\lambda\right)(p+2q)^\alpha$$

$$\hookrightarrow \quad \boxed{N(q) = \sum_\lambda \left(J^{(14)}_\lambda \cdot J^{(1)}\right)} \quad \to \quad \text{Numerator in } d = 4$$

Moving to d dimensions

- Obtaining the full d-dimensional expression, $\bar{N}(\bar{q})$, from $N(q)$ is straightforward *analytically*. Just use the following replace rules at any step of the recursion:

$$q^2 X \to (q^2 + \mu) X \qquad \sum_\lambda (\varepsilon_\lambda \cdot \omega_\lambda)\, X \to \sum_\lambda (\varepsilon_\lambda \cdot \omega_\lambda)\, X + (d-4)\, X$$

$$\sum_\lambda (\varepsilon_\lambda \cdot q)\,(\omega_\lambda \cdot q)\, X \to \sum_\lambda (\varepsilon_\lambda \cdot q)\,(\omega_\lambda \cdot q)\, X + \mu\, X$$

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- This approach is simple if one uses *analytic* expressions. It is not as simple when the computation is performed *numerically*
  - ∵ Numerical recursion keeps "memory" of the previous step only, not of the full recursion history!
- We need to track the *coefficients* of all critical *Lorentz structures* which can originate μ or $(d-4)$ terms during the recursion

Moving to d dimensions

- Obtaining the full d-dimensional expression, $\bar{N}(\bar{q})$, from $N(q)$ is straightforward *analytically*. Just use the following replace rules at any step of the recursion:

$$q^2 X \to (q^2 + \mu) X \qquad \sum_\lambda (\varepsilon_\lambda \cdot \omega_\lambda)\, X \to \sum_\lambda (\varepsilon_\lambda \cdot \omega_\lambda)\, X + (d-4)\, X$$

$$\sum_\lambda (\varepsilon_\lambda \cdot q)\,(\omega_\lambda \cdot q)\, X \to \sum_\lambda (\varepsilon_\lambda \cdot q)\,(\omega_\lambda \cdot q)\, X + \mu\, X$$

- This approach is simple if one uses *analytic* expressions. It is not as simple when the computation is performed *numerically*

  ∵ Numerical recursion keeps "memory" of the previous step only, not of the full recursion history!

- We need to track the *coefficients* of all critical *Lorentz structures* which can originate μ or $(d-4)$ terms during the recursion

For pure YM at 1-loop:

$\tilde{q}^\alpha$	$\tilde{\varepsilon}^\alpha$	$(\tilde{\varepsilon} \cdot \tilde{q})$	$(\tilde{\varepsilon} \cdot \tilde{q})\, \tilde{q}^\alpha$

Moving to d dimensions

- Introduce formal operator $\mathscr{E}$ which generates all *evanescent contributions* to an input $d=4$ current $J^{(N)}$ at any step of the recursion:

$$\mathscr{E}[J^{(N)\,\alpha}_{\text{gluon}}] = \boxed{C^{(N)}_{q}}\,\tilde{q}^{\alpha} + \boxed{C^{(N)}_{\varepsilon}}\,\tilde{\varepsilon}_{\lambda}{}^{\alpha} + \boxed{J^{(N)\,\alpha}_{q\varepsilon}}(\tilde{q}\cdot\tilde{\varepsilon}_{\lambda}) + \boxed{C^{(N)}_{q\varepsilon,q}}(\tilde{q}\cdot\tilde{\varepsilon}_{\lambda})\,\tilde{q}^{\alpha} + \boxed{J^{(N)\,\alpha}_{\mu}} + \boxed{J^{(N)\,\alpha}_{d-4}}$$

pure YM @ 1-loop

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pure YM @ 1-loop

- All *coefficients* above satisfy *recursion relations* which can be derived from the analytic structure of QCD vertices

    ↪ The coefficients "evolve" during the recursion, in the same spirit of the standard current $J^{(N)\,\alpha}$ in $d = 4$

Moving to d dimensions

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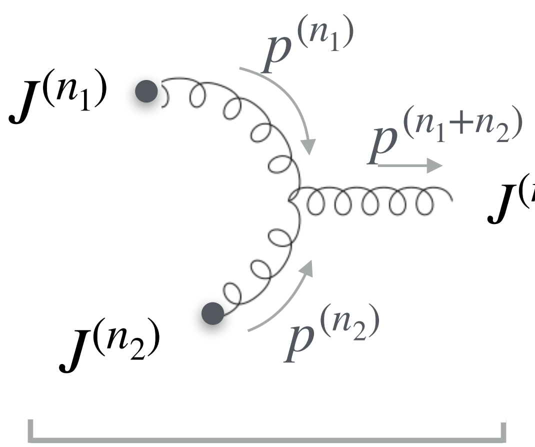
- All *coefficients* above satisfy *recursion relations* which can be derived from the analytic structure of QCD vertices

  ↪ The coefficients "evolve" during the recursion, in the same spirit of the standard current $J^{(N)\,\alpha}$ in $d=4$

We'll show the recursion relations for the coefficients and how to combine them to obtain a numerator in d dimensions

[G.B, Canko, Papadopoulos and Spourdalakis, in progress]

Color-stripped vertex function:



$$
\begin{aligned}
J^{(n_1+n_2)\,\alpha} \;\equiv\; & \left(J^{(n_2)}\cdot\left(2p^{(n_1)}+p^{(n_2)}\right)\right) J^{(n_1)\,\alpha} \\
- & \left(J^{(n_1)}\cdot\left(p^{(n_1)}+2\,p^{(n_2)}\right)\right) J^{(n_2)\,\alpha} \\
+ & \left(J^{(n_1)}\cdot J^{(n_2)}\right)\left(p^{(n_2)}-p^{(n_1)}\right)^{\alpha}
\end{aligned}
$$

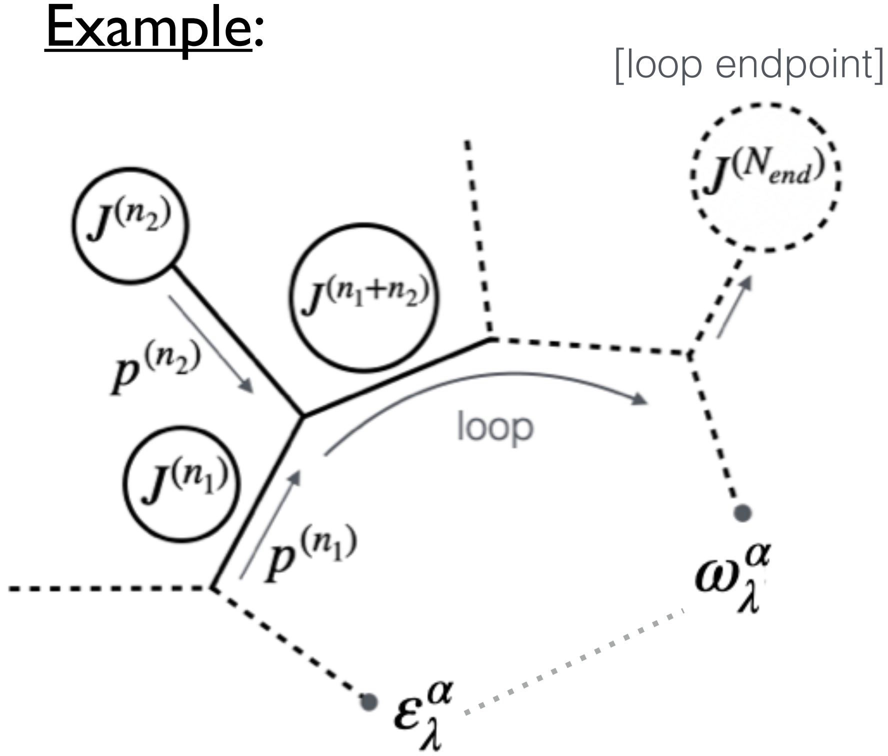
$$\equiv\; \mathrm{V}^{\alpha}_{3g}\left(J^{(n_1)},J^{(n_2)}\right)$$

<u>Initial conditions:</u>

$$C_q^{(1)} = 0 \qquad C_\varepsilon^{(1)} = 1$$

$$C_{\varepsilon q,q}^{(1)} = 0 \qquad J_{\varepsilon q}^{(1)\,\alpha} = 0$$

<u>Example</u>:



$$C_q^{(n_1+n_2)} = C_q^{(n_1)}\left[J^{(n_2)}\cdot\left(2\,p^{(n_1)}+p^{(n_2)}\right)\right] - \left(1+\delta_{(n_1+n_2)\,N_{\mathrm{end}}}\right)\left(J^{(n_1)}\cdot J^{(n_2)}\right)$$

$$C_\varepsilon^{(n_1+n_2)} = C_\varepsilon^{(n_1)}\left[J^{(n_2)}\cdot\left(2\,p^{(n_1)}+p^{(n_2)}\right)\right]$$

$$J_{\varepsilon q}^{(n_1+n_2)\,\alpha} = \mathrm{V}^{\alpha}_{3g}\left(J_{\varepsilon q}^{(n_1)},J^{(n_2)}\right) - \left(C_\varepsilon^{(n_1)} + \mu\, C_{\varepsilon q,q}^{(n_1)}\right) J^{(n_2)\,\alpha}$$

$$C_{\varepsilon q,q}^{(n_1+n_2)} = C_{\varepsilon q,q}^{(n_1)}\left[J^{(n_2)}\cdot\left(2\,p^{(n_1)}+p^{(n_2)}\right)\right] - \left(1+\delta_{(n_1+n_2)\,N_{\mathrm{end}}}\right)\left(J_{q\varepsilon}^{(n_1)}\cdot J^{(n_2)}\right)$$

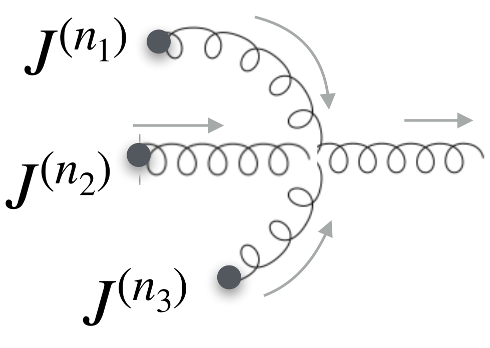
$$J_{\mu}^{(n_1+n_2)\,\alpha} = \mu\left[\left(-1+2\,\delta_{(n_1+n_2)N_{\mathrm{end}}} C_q^{(n_1)} J^{(n_2)\,\alpha}\right) + \delta_{(n_1+n_2)N_{\mathrm{end}}}\left(J_{\varepsilon q}^{(n_1)\,\alpha} + C_{\varepsilon q,q}^{(n_1)}\left(p^{(n_2)}-p^{(n_1)}\right)^{\alpha}\right)\right]$$

$$J_{d-4}^{(n_1+n_2)\,\alpha} = (d-4)\left[\delta_{(n_1+n_2)N_{end}}\, C_\varepsilon^{(n_1)}\left(p^{(n_2)}-p^{(n_1)}\right)^{\alpha}\right]$$

1-loop recursion relations: 4-gluon vertex

[G.B, Canko, Papadopoulos and Spourdalakis, in progress]

Color-stripped vertex function:



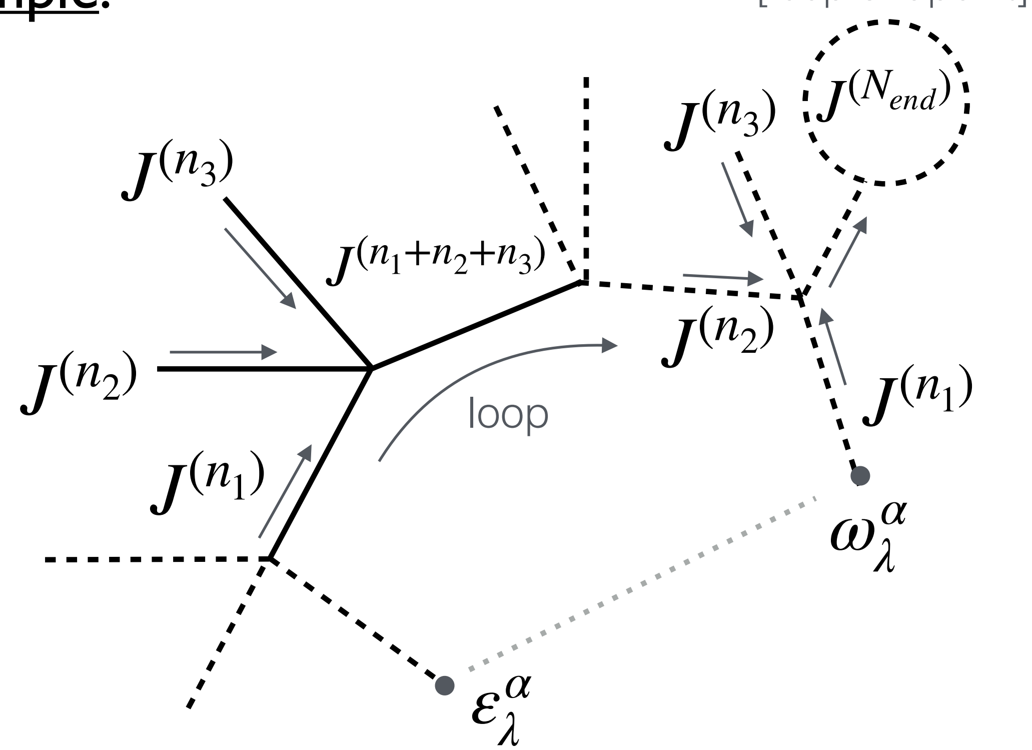
$$J^{(n_1+n_2+n_3)\,\alpha} \equiv 2\left(J^{(n_2)}\cdot J^{(n_3)}\right)J^{(n_1)\,\alpha} - \left(J^{(n_1)}\cdot J^{(n_2)}\right)J^{(n_3)\,\alpha} - \left(J^{(n_1)}\cdot J^{(n_3)}\right)J^{(n_2)\,\alpha}$$

$$\equiv \mathrm{V}^{\alpha}_{4g}\left(J^{(n_1)}, J^{(n_2)}, J^{(n_3)}\right)$$

<u>Initial conditions:</u>

$$C_q^{(1)} = 0 \qquad C_\varepsilon^{(1)} = 1$$
$$C_{\varepsilon q,q}^{(1)} = 0 \qquad J_{\varepsilon q}^{(1)\,\alpha} = 0$$

<u>**Example**</u>:



$$C_q^{(n_1+n_2+n_3)} = C_q^{(n_1)}\left[2\left(J^{(n_2)}\cdot J^{(n_3)}\right)\right]$$

$$C_\varepsilon^{(n_1+n_2+n_3)} = C_\varepsilon^{(n_1)}\left[2\left(J^{(n_2)}\cdot J^{(n_3)}\right)\right]$$

$$J_{\varepsilon q}^{(n_1+n_2+n_3)\,\alpha} = \mathrm{V}^{\alpha}_{4g}\left(J_{\varepsilon q}^{(n_1)}, J^{(n_2)}, J^{(n_3)}\right)$$

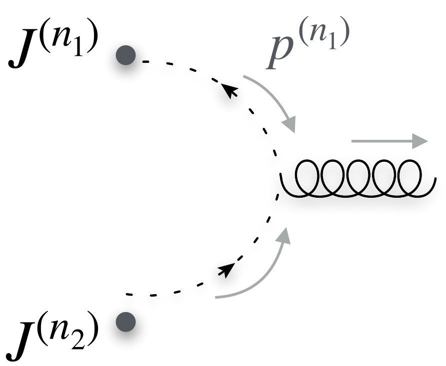
$$C_{\varepsilon q,q}^{(n_1+n_2+n_3)} = C_{\varepsilon q,q}^{(n_1)}\left[2\left(J^{(n_2)}\cdot J^{(n_3)}\right)\right]$$

$$J_\mu^{(n_1+n_2+n_3)\,\alpha} = \mu\left[-\delta_{(n_1+n_2+n_3)N_{\rm end}}\,C_{\varepsilon q,q}^{(n_2)}\,J^{(n_3)}\right]$$

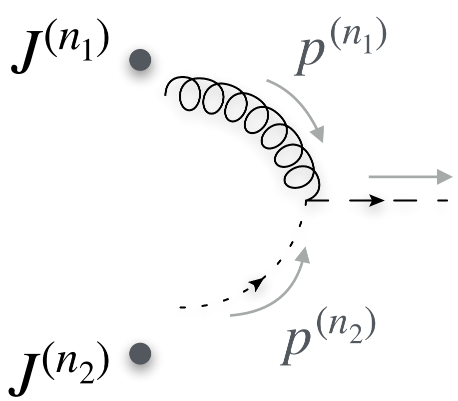
$$J_{d-4}^{(n_1+n_2+n_3)\,\alpha} = (d-4)\left[-\delta_{(n_1+n_2+n_3)N_{\rm end}}\,C_{\varepsilon}^{(n_2)}\,J^{(n_3)}\right]$$

[G.B, Canko, Papadopoulos and Spourdalakis, in progress]

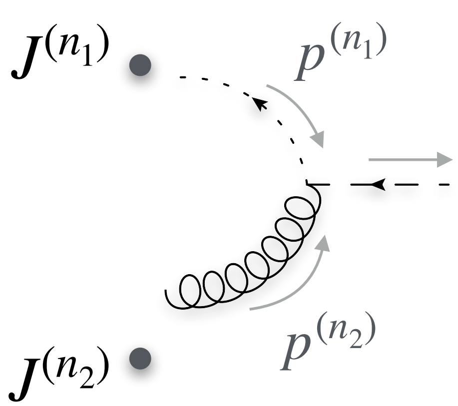
Color-stripped vertex functions:



$$J^{(n_1+n_2)\,\alpha} \equiv -\,J^{(n_1)}\,J^{(n_2)}\,p^{(n_1)\,\alpha}$$



$$J^{(n_1+n_2)\,\alpha} \equiv \left(J^{(n_1)}\cdot\left(p^{(n_1)}+p^{(n_2)}\right)\right)\,J^{(n_2)}$$



$$J^{(n_1+n_2)\,\alpha} \equiv \left(J^{(n_2)}\cdot\left(p^{(n_1)}\right)\right)\,J^{(n_1)\,\alpha}$$

$$\mathscr{E}[J^{(N)}_{\text{ghost}}] = \overset{=0}{\boxed{\cancel{J^{(N)}_{\mu}}}} + \overset{=0}{\boxed{\cancel{J^{(N)}_{d-4}}}}$$

QCD @ 1-loop

No evanescent terms are generated by 1-loop gluon-ghost vertices!

$\because$ The gluon is never associated to a loop propagator

Evanescent terms for *fermionic* currents

$$\mathcal{E}[J^{(N)}_{\text{fermion}}] = \tilde{\not{q}}\,\boxed{\psi^{(N)}_{q}} + \tilde{\not{q}}\,\boxed{\psi^{(N)}_{q,R}} + \tilde{\not{\varepsilon}}_{\lambda}\,\boxed{\psi^{(N)}_{\varepsilon}}$$
$$+ (\tilde{q}\cdot\tilde{\varepsilon}_{\lambda})\,\boxed{\psi^{(N)}_{q\varepsilon}} + (\tilde{q}\cdot\tilde{\varepsilon}_{\lambda})\tilde{\not{q}}\,\boxed{\psi^{(N)}_{q\varepsilon,q}} + \boxed{\tilde{J}^{(N)}_{R}}$$
$$+ \boxed{J^{(N)}_{\mu}} + \boxed{J^{(N)}_{d-4}}$$

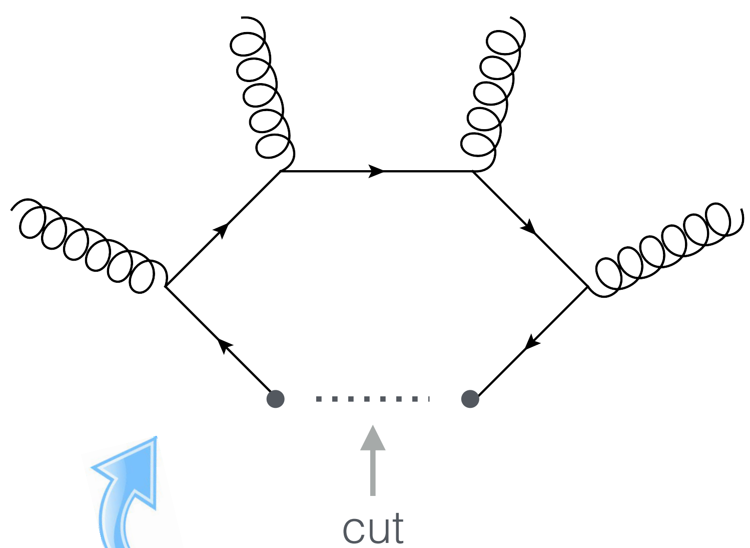
QCD @ 1-loop

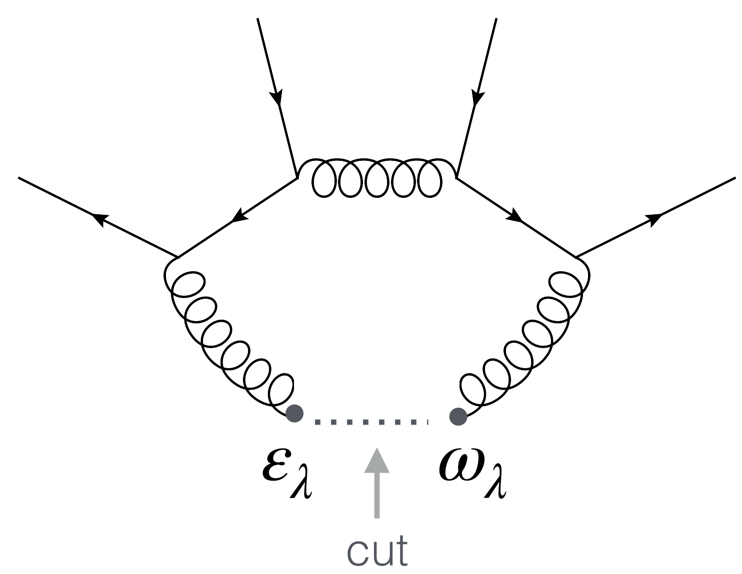
Steps to full QCD @ 1-loop

Evanescent terms for *fermionic* currents

$$
\begin{aligned}
\mathscr{E}[J^{(N)}_{\text{fermion}}] = {} & \tilde{\not{q}}\,\boxed{\psi^{(N)}_{q}} + \tilde{\not{q}}\,\boxed{\psi^{(N)}_{q,R}} + \tilde{\not{\varepsilon}}_{\lambda}\,\boxed{\psi^{(N)}_{\varepsilon}} \\
& + (\tilde{q}\cdot\tilde{\varepsilon}_{\lambda})\,\boxed{\psi^{(N)}_{q\varepsilon}} + (\tilde{q}\cdot\tilde{\varepsilon}_{\lambda})\,\tilde{\not{q}}\,\boxed{\psi^{(N)}_{q\varepsilon,q}} + \boxed{\tilde{J}^{(N)}_{R}} \\
& + \boxed{J^{(N)}_{\mu}} + \boxed{J^{(N)}_{d-4}}
\end{aligned}
$$

QCD @ 1-loop



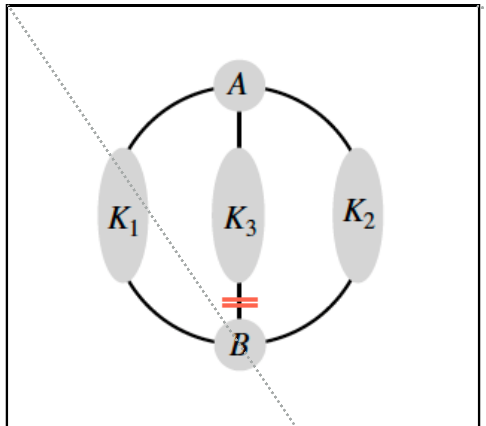


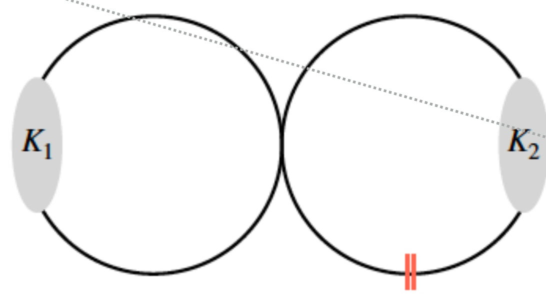
Checked for *closed-fermion loop* topologies so far. General case under study

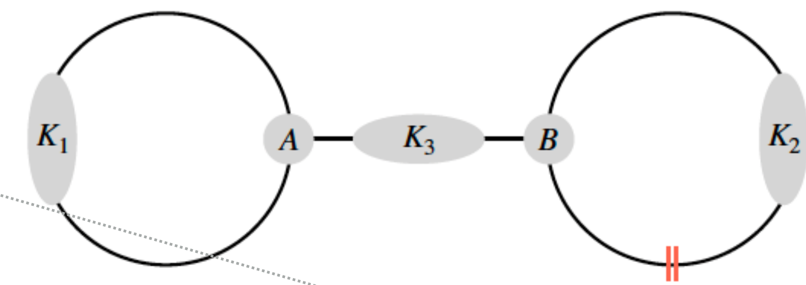
II. Beyond 1-loop

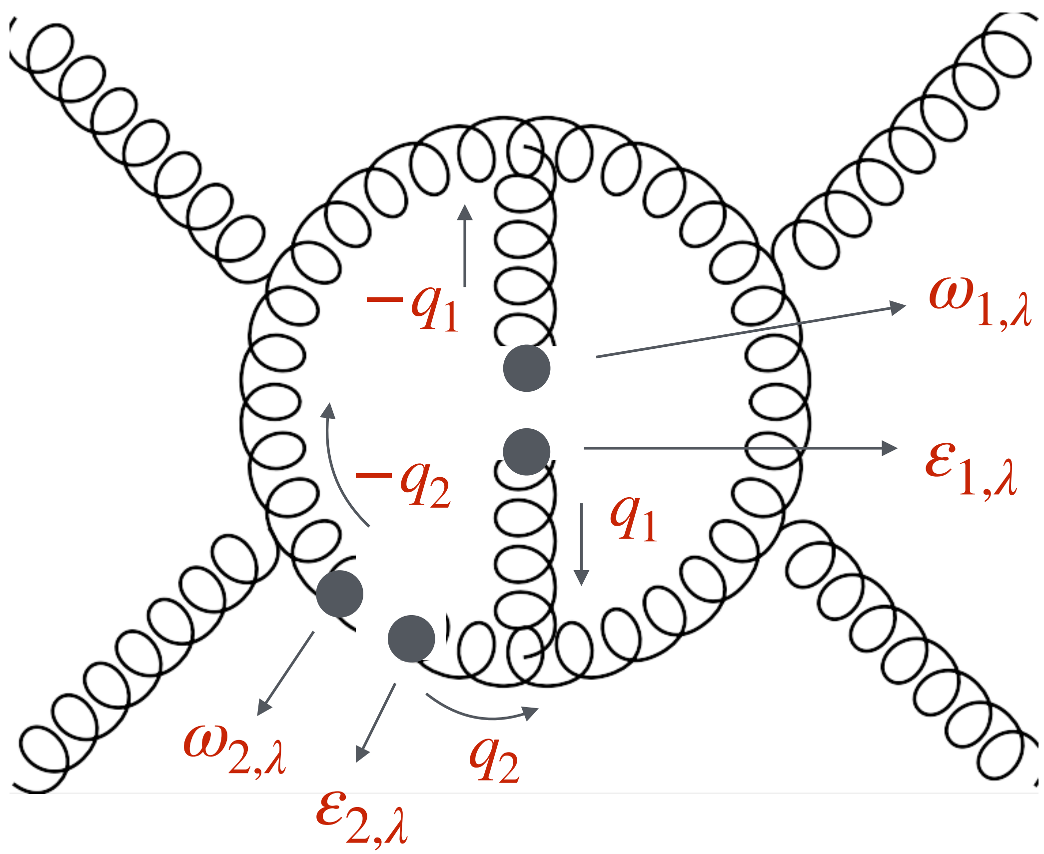
Steps to 2-loop construction

Basic notation at 2-loop





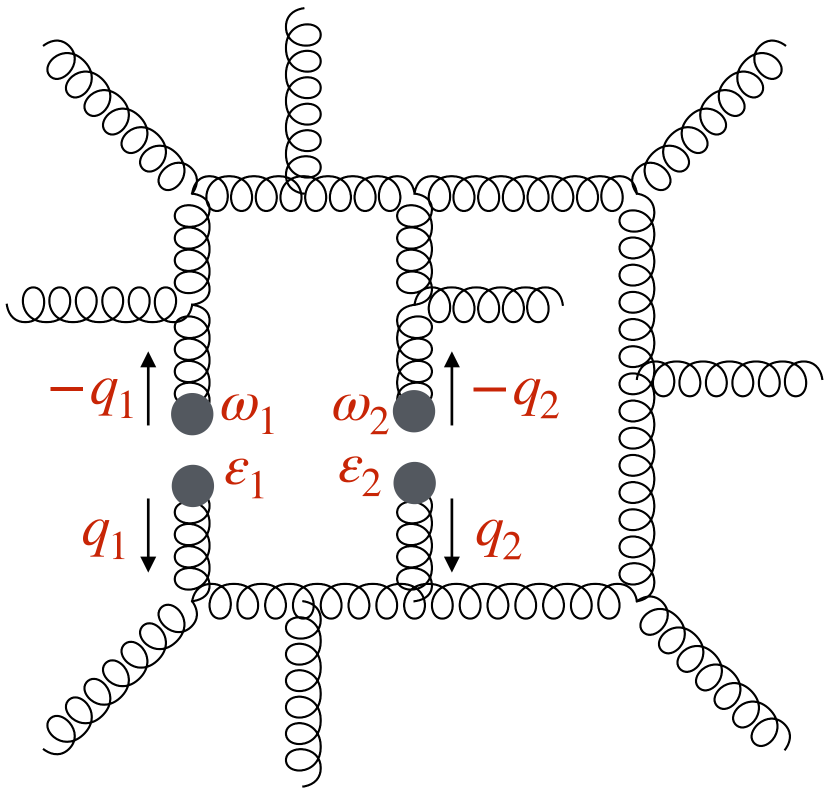




$$\sum_{\lambda} \varepsilon^{\alpha}_{1,\lambda}\, \omega^{\beta}_{1,\lambda} \rightarrow g^{\alpha\beta}$$

$$\sum_{\lambda} \varepsilon^{\alpha}_{2,\lambda}\, \omega^{\beta}_{2,\lambda} \rightarrow g^{\alpha\beta}$$

Extension to the 2-loop case



Origin of evanescent terms at 2-loops:

$$\mathscr{E}[\,(q_i \cdot q_j)\, X\,] = \mu_{ij}\, X \qquad [i,j=1{,}2]$$

$$\mathscr{E}[\,\sum_{\lambda} (q_i \cdot \varepsilon_{k,\lambda})\,(q_j \cdot \omega_{k,\lambda})\; X\,] = \mu_{ij}\, X \qquad [i,j=1{,}2]$$

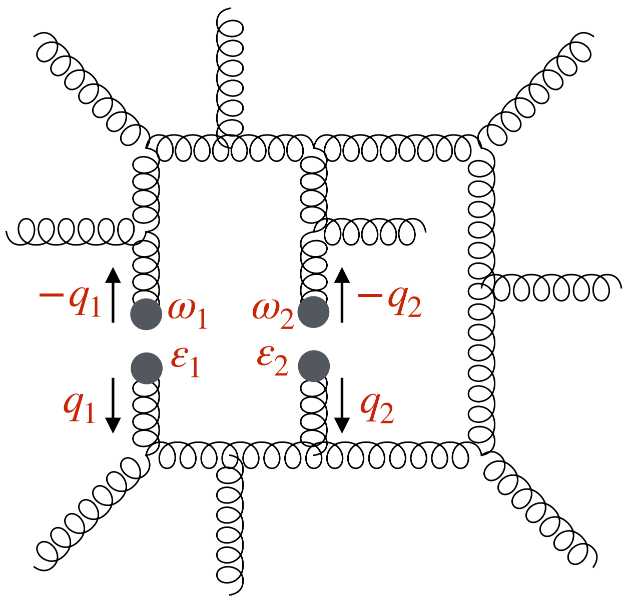
$$\mathscr{E}[\,\sum_{\lambda} (\varepsilon_{i,\lambda} \cdot \omega_{i,\lambda})\; X\,] = (d-4)\, X \qquad [i,j=1{,}2]$$

$$\mathscr{E}[\,\sum_{\lambda_1,\lambda_2} (\varepsilon_{1,\lambda_1} \cdot \omega_{1,\lambda_1})\,(\varepsilon_{2,\lambda_2} \cdot \omega_{2,\lambda_2})\; X\,] = (d-4)\, X$$

$$\mathscr{E}[\,\sum_{\lambda_1,\lambda_2} (\varepsilon_{1,\lambda_1} \cdot \omega_{2,\lambda_2})\,(\varepsilon_{2,\lambda_2} \cdot \omega_{1,\lambda_1})\; X\,] = (d-4)\, X$$

$$\mathscr{E}[\,\sum_{\lambda_1,\lambda_2} (\varepsilon_{1,\lambda_1} \cdot \varepsilon_{2,\lambda_2})\,(\omega_{1,\lambda_1} \cdot \omega_{2,\lambda_2})\; X\,] = (d-4)\, X$$

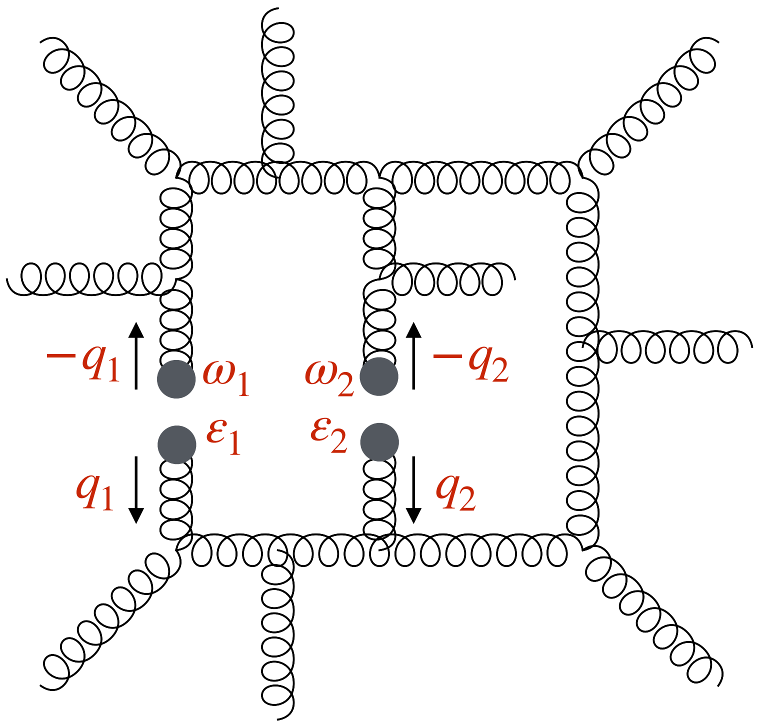
Extension to the 2-loop case



At two loops, more critical structures (⇒ coefficients) need to be tracked during the recursion

$$
\begin{aligned}
\mathscr{E}[J^{(N)\,\alpha}_{\text{gluon}}] \equiv\; & \sum_{i=1}^{2} C_{q_i}{}^{(N)}\, \tilde{q}_i^{\alpha} + \sum_{i=1}^{2} C_{\varepsilon_i}{}^{(N)}\, \tilde{\varepsilon}_i^{\alpha} + C_{\omega_2}{}^{(N)}\, \tilde{\omega}_2^{\alpha} \\
& + \sum_{i,j=1}^{2} J^{(N)\,\alpha}_{\varepsilon_i q_j}\, (\tilde{\varepsilon}_i \cdot \tilde{q}_j) + J^{(N)\,\alpha}_{\omega_2 q_2}\, (\tilde{\omega}_2 \cdot \tilde{q}_2) + J^{(N)\,\alpha}_{\varepsilon_1 \varepsilon_2}\, (\tilde{\varepsilon}_1 \cdot \tilde{\varepsilon}_2) + \sum_{i,j=1}^{2} J^{(N)\,\alpha}_{\varepsilon_1 q_i \varepsilon_2 q_j}\, (\tilde{\varepsilon}_1 \cdot \tilde{q}_i)(\tilde{\varepsilon}_2 \cdot \tilde{q}_j) \\
& + \sum_{i,j,k=1}^{2} C^{(N)}_{\varepsilon_i q_j, q_k}\, (\tilde{\varepsilon}_i \cdot \tilde{q}_j)\, \tilde{q}_k^{\alpha} + C^{(N)}_{\omega_2 q_2, q_2}\, (\tilde{\omega}_2 \cdot \tilde{q}_2)\, \tilde{q}_2^{\alpha} + \sum_{\substack{i,j,k=1 \\ k \neq i}}^{2} C^{(N)}_{\varepsilon_i q_j, \varepsilon_k}\, (\tilde{\varepsilon}_i \cdot \tilde{q}_j)\, \tilde{\varepsilon}_k^{\alpha} \\
& + \sum_{i,j=1}^{2} C^{(N)}_{\varepsilon_1 \varepsilon_2, q_i}\, (\tilde{\varepsilon}_1 \cdot \tilde{\varepsilon}_2)\, \tilde{q}_i^{\alpha} + \sum_{i,j,k=1}^{2} C^{(N)}_{\varepsilon_1 q_i \varepsilon_2 q_j, q_k}\, (\tilde{\varepsilon}_1 \cdot \tilde{q}_i)\, (\tilde{\varepsilon}_2 \cdot \tilde{q}_j)\, \tilde{q}_k^{\alpha} \\
& + J^{(N)\,\alpha}_{\mu} + J^{(N)\,\alpha}_{(d-4)}
\end{aligned}
$$

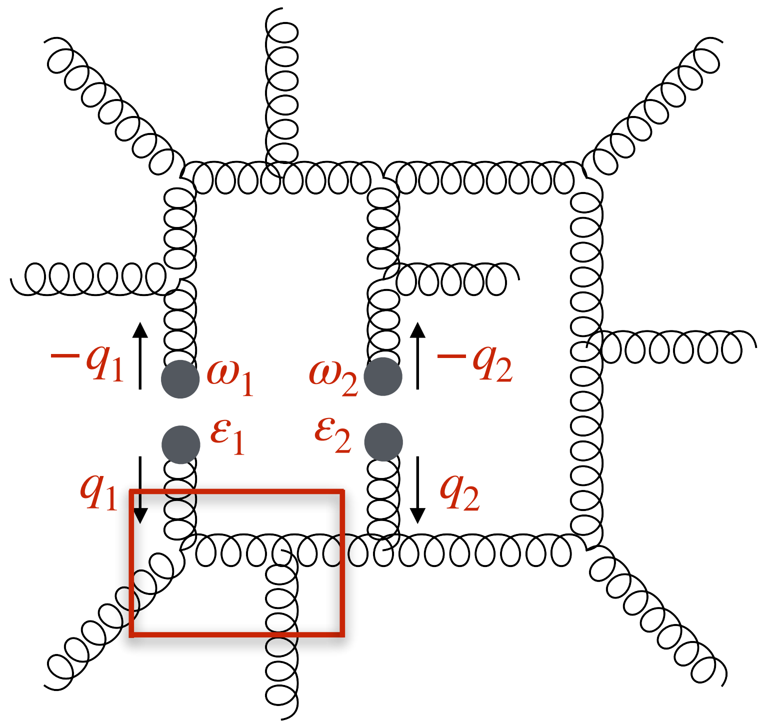
pure YM @ 2-loop



Not *all* coefficients are involved at each step of the recursion!

$$
\begin{aligned}
\mathcal{E}[J^{(N)\,\alpha}_{\text{gluon}}] \equiv\; & \sum_{i=1}^{2} C_{q_i}{}^{(N)}\, \tilde{q}_i^{\alpha} + \sum_{i=1}^{2} C_{\varepsilon_i}{}^{(N)}\, \tilde{\varepsilon}_i^{\alpha} + C_{\omega_2}{}^{(N)}\, \tilde{\omega}_2^{\alpha} \\
& + \sum_{i,j=1}^{2} J^{(N)\,\alpha}_{\varepsilon_i q_j}\, (\tilde{\varepsilon}_i \cdot \tilde{q}_j) + J^{(N)\,\alpha}_{\omega_2 q_2}\, (\tilde{\omega}_2 \cdot \tilde{q}_2) + J^{(N)\,\alpha}_{\varepsilon_1 \varepsilon_2}\, (\tilde{\varepsilon}_1 \cdot \tilde{\varepsilon}_2) + \sum_{i,j=1}^{2} J^{(N)\,\alpha}_{\varepsilon_1 q_i \varepsilon_2 q_j}\, (\tilde{\varepsilon}_1 \cdot \tilde{q}_i)(\tilde{\varepsilon}_2 \cdot \tilde{q}_j) \\
& + \sum_{i,j,k=1}^{2} C^{(N)}_{\varepsilon_i q_j, q_k}\, (\tilde{\varepsilon}_i \cdot \tilde{q}_j)\, \tilde{q}_k^{\alpha} + C^{(N)}_{\omega_2 q_2, q_2} (\tilde{\omega}_2 \cdot \tilde{q}_2)\, \tilde{q}_2^{\alpha} + \sum_{\substack{i,j,k=1 \\ k \neq i}}^{2} C^{(N)}_{\varepsilon_i q_j, \varepsilon_k}\, (\tilde{\varepsilon}_i \cdot \tilde{q}_j)\, \tilde{\varepsilon}_k^{\alpha} \\
& + \sum_{i,j=1}^{2} C^{(N)}_{\varepsilon_1 \varepsilon_2, q_i} (\tilde{\varepsilon}_1 \cdot \tilde{\varepsilon}_2)\, \tilde{q}_i^{\alpha} + \sum_{i,j,k=1}^{2} C^{(N)}_{\varepsilon_1 q_i \varepsilon_2 q_j, q_k} (\tilde{\varepsilon}_1 \cdot \tilde{q}_i)\, (\tilde{\varepsilon}_2 \cdot \tilde{q}_j)\, \tilde{q}_k^{\alpha} \\
& + J^{(N)\,\alpha}_{\mu} + J^{(N)\,\alpha}_{(d-4)}
\end{aligned}
$$

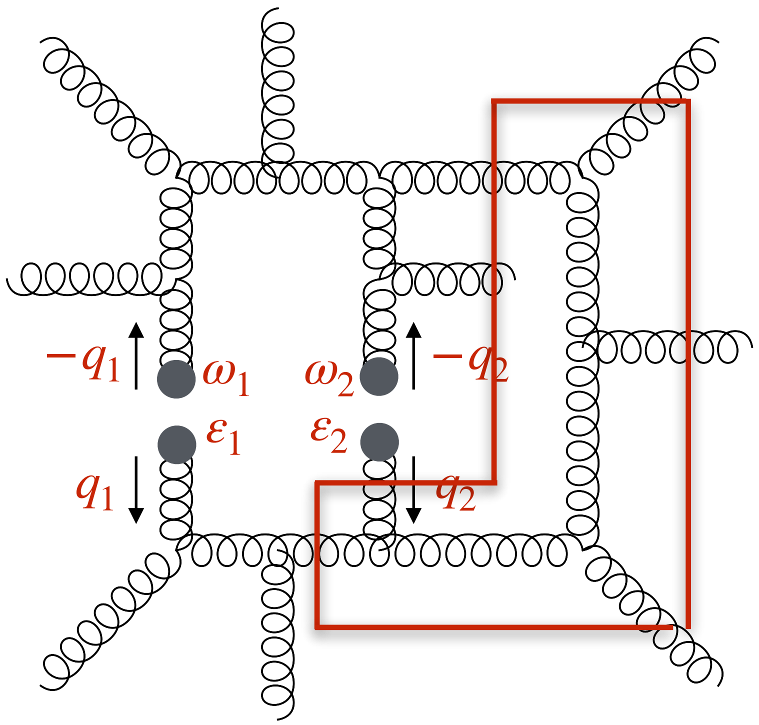
Strategy of calculation



Not *all* coefficients are involved at each step of the recursion!

$$
\begin{aligned}
\mathcal{E}[J^{(N)\,\alpha}_{\text{gluon}}] \equiv\; & \sum_{i=1}^{1} C_{q_i}^{(N)}\, \tilde{q}_i^{\alpha} + \sum_{i=1}^{1} C_{\varepsilon_i}^{(N)}\, \tilde{\varepsilon}_i^{\alpha} + C_{\omega_2}^{(N)}\, \tilde{\omega}_2^{\alpha} \\
& + \sum_{i,j=1}^{1} J^{(N)\,\alpha}_{\varepsilon_i q_j} (\tilde{\varepsilon}_i \cdot \tilde{q}_j) + J^{(N)\,\alpha}_{\omega_2 q_2} (\tilde{\omega}_2 \cdot \tilde{q}_2) + J^{(N)\,\alpha}_{\varepsilon_1 \varepsilon_2} (\tilde{\varepsilon}_1 \cdot \tilde{\varepsilon}_2) + \sum_{i,j=1}^{2} J^{(N)\,\alpha}_{\varepsilon_1 q_i \varepsilon_2 q_j} (\tilde{\varepsilon}_1 \cdot \tilde{q}_i)(\tilde{\varepsilon}_2 \cdot \tilde{q}_j) \\
& + \sum_{i,j,k=1}^{2} C^{(N)}_{\varepsilon_i q_j, q_k} (\tilde{\varepsilon}_i \cdot \tilde{q}_j)\, \tilde{q}_k^{\alpha} + C^{(N)}_{\omega_2 q_2, q_2} (\tilde{\omega}_2 \cdot \tilde{q}_2)\, \tilde{q}_2^{\alpha} + \sum_{\substack{i,j,k=1 \\ k \neq i}}^{2} C^{(N)}_{\varepsilon_i q_j, \varepsilon_k} (\tilde{\varepsilon}_i \cdot \tilde{q}_j)\, \tilde{\varepsilon}_k^{\alpha} \\
& + \sum_{i,j=1}^{2} C^{(N)}_{\varepsilon_1 \varepsilon_2, q_i} (\tilde{\varepsilon}_1 \cdot \tilde{\varepsilon}_2)\, \tilde{q}_i^{\alpha} + \sum_{i,j,k=1}^{2} C^{(N)}_{\varepsilon_1 q_i \varepsilon_2 q_j, q_k} (\tilde{\varepsilon}_1 \cdot \tilde{q}_i)(\tilde{\varepsilon}_2 \cdot \tilde{q}_j)\, \tilde{q}_k^{\alpha} \\
& + J^{(N)\,\alpha}_{\mu} + J^{(N)\,\alpha}_{(d-4)}
\end{aligned}
$$

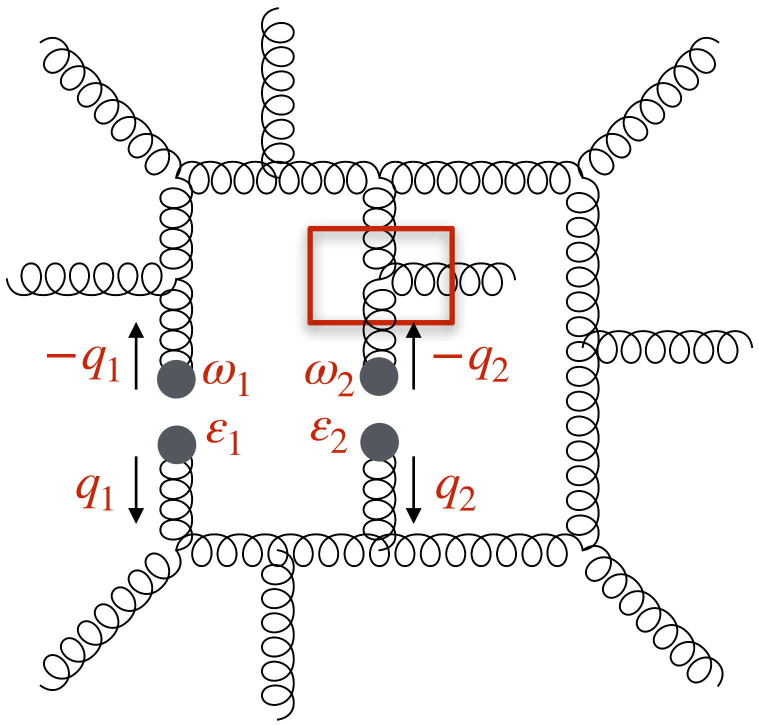
Strategy of calculation



Not *all* coefficients are involved at each step of the recursion!

$$
\begin{aligned}
\mathcal{E}[J^{(N)\,\alpha}_{\text{gluon}}] \equiv & \sum_{i=1}^{2} C_{q_i}{}^{(N)}\,\tilde{q}_i^\alpha + \sum_{i=1}^{2} C_{\varepsilon_i}{}^{(N)}\,\tilde{\varepsilon}_i^\alpha + C_{\omega_2}{}^{(N)}\,\tilde{\omega}_2^\alpha \\
& + \sum_{i,j=1}^{2} J^{(N)\,\alpha}_{\varepsilon_i q_j}\,(\tilde{\varepsilon}_i\cdot\tilde{q}_j) + J^{(N)\,\alpha}_{\omega_2 q_2}\,(\tilde{\omega}_2\cdot\tilde{q}_2) + J^{(N)\,\alpha}_{\varepsilon_1\varepsilon_2}\,(\tilde{\varepsilon}_1\cdot\tilde{\varepsilon}_2) + \sum_{i,j=1}^{2} J^{(N)\,\alpha}_{\varepsilon_1 q_i \varepsilon_2 q_j}\,(\tilde{\varepsilon}_1\cdot\tilde{q}_i)(\tilde{\varepsilon}_2\cdot\tilde{q}_j) \\
& + \sum_{i,j,k=1}^{2} C^{(N)}_{\varepsilon_i q_j, q_k}\,(\tilde{\varepsilon}_i\cdot\tilde{q}_j)\,\tilde{q}_k^\alpha + C^{(N)}_{\omega_2 q_2, q_2}(\tilde{\omega}_2\cdot\tilde{q}_2)\,\tilde{q}_2^\alpha + \sum_{\substack{i,j,k=1\\ k\neq i}}^{2} C^{(N)}_{\varepsilon_i q_j, \varepsilon_k}\,(\tilde{\varepsilon}_i\cdot\tilde{q}_j)\,\tilde{\varepsilon}_k^\alpha \\
& + \sum_{i,j=1}^{2} C^{(N)}_{\varepsilon_1\varepsilon_2, q_i}\,(\tilde{\varepsilon}_1\cdot\tilde{\varepsilon}_2)\,\tilde{q}_i^\alpha + \sum_{i,j,k=1}^{2} C^{(N)}_{\varepsilon_1 q_i \varepsilon_2 q_j, q_k}\,(\tilde{\varepsilon}_1\cdot\tilde{q}_i)\,(\tilde{\varepsilon}_2\cdot\tilde{q}_j)\,\tilde{q}_k^\alpha \\
& + J^{(N)\,\alpha}_{\mu} + J^{(N)\,\alpha}_{(d-4)}
\end{aligned}
$$

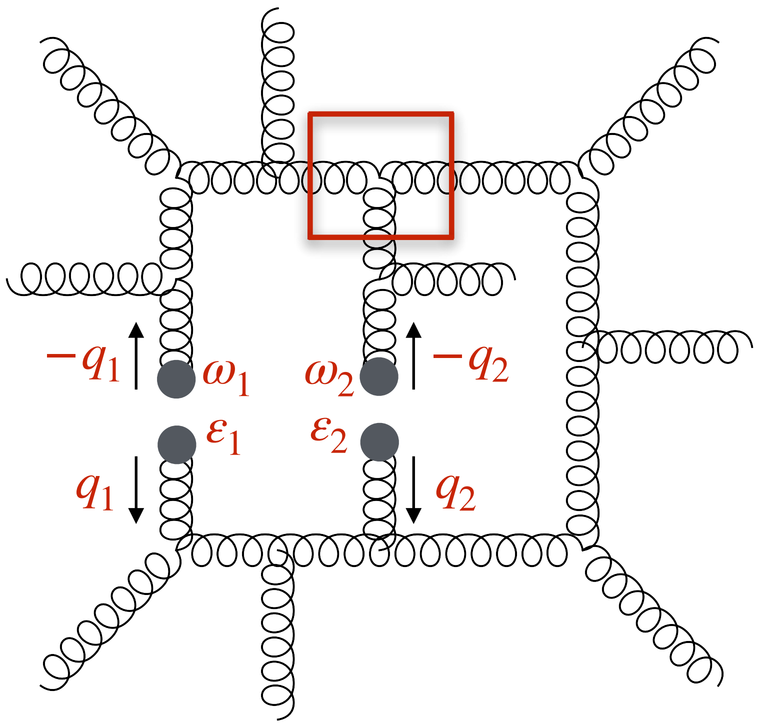
Strategy of calculation



Not *all* coefficients are involved at each step of the recursion!

$$
\begin{aligned}
\mathcal{E}[J^{(N)\,\alpha}_{\text{gluon}}] \equiv\ & \sum_{i=2}^{2} C_{q_i}{}^{(N)}\,\tilde{q}_i^{\alpha} + \sum_{i,j=1}^{2} C_{\varepsilon_i}{}^{(N)}\,\tilde{\varepsilon}_i^{\alpha} + C_{\omega_2}{}^{(N)}\,\tilde{\omega}_2^{\alpha} \\
& + \sum_{i,j=1}^{2} J^{(N)\,\alpha}_{\varepsilon_i q_j}\,(\tilde{\varepsilon}_i\cdot\tilde{q}_j) + J^{(N)\,\alpha}_{\omega_2 q_2}\,(\tilde{\omega}_2\cdot\tilde{q}_2) + J^{(N)\,\alpha}_{\varepsilon_1\varepsilon_2}\,(\tilde{\varepsilon}_1\cdot\tilde{\varepsilon}_2) + \sum_{i,j=1}^{2} J^{(N)\,\alpha}_{\varepsilon_1 q_i \varepsilon_2 q_j}\,(\tilde{\varepsilon}_1\cdot\tilde{q}_i)(\tilde{\varepsilon}_2\cdot\tilde{q}_j) \\
& + \sum_{i,j,k=1}^{2} C^{(N)}_{\varepsilon_i q_j, q_k}\,(\tilde{\varepsilon}_i\cdot\tilde{q}_j)\,\tilde{q}_k^{\alpha} + C^{(N)}_{\omega_2 q_2, q_2}(\tilde{\omega}_2\cdot\tilde{q}_2)\,\tilde{q}_2^{\alpha} + \sum_{\substack{i,j,k=1\\ k\neq i}}^{2} C^{(N)}_{\varepsilon_i q_j, \varepsilon_k}\,(\tilde{\varepsilon}_i\cdot\tilde{q}_j)\,\tilde{\varepsilon}_k^{\alpha} \\
& + \sum_{i,j=1}^{2} C^{(N)}_{\varepsilon_1\varepsilon_2, q_i}\,(\tilde{\varepsilon}_1\cdot\tilde{\varepsilon}_2)\,\tilde{q}_i^{\alpha} + \sum_{i,j,k=1}^{2} C^{(N)}_{\varepsilon_1 q_i \varepsilon_2 q_j, q_k}\,(\tilde{\varepsilon}_1\cdot\tilde{q}_i)\,(\tilde{\varepsilon}_2\cdot\tilde{q}_j)\,\tilde{q}_k^{\alpha} \\
& + J^{(N)\,\alpha}_{\mu} + J^{(N)\,\alpha}_{(d-4)}
\end{aligned}
$$

Strategy of calculation



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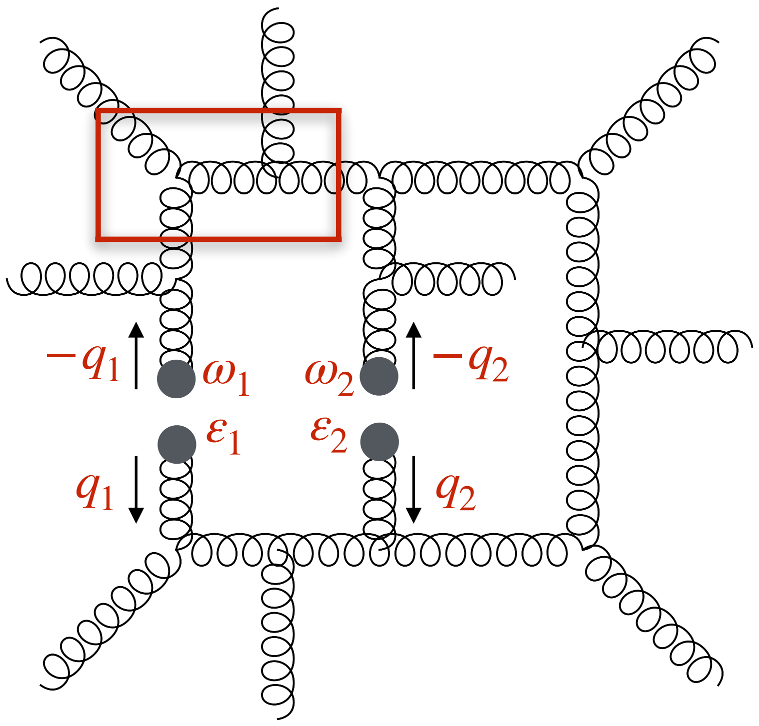
$$
\begin{aligned}
\mathscr{E}[J^{(N)\,\alpha}_{\text{gluon}}] \equiv\ & \sum_{i=1}^{2} C_{q_i}^{(N)}\, \tilde{q}_i^{\alpha} + \sum_{i=1}^{1} C_{\varepsilon_i}^{(N)}\, \tilde{\varepsilon}_i^{\alpha} + C_{\omega_2}^{(N)}\, \tilde{\omega}_2^{\alpha} \\
& + \sum_{i,j=1}^{2} J^{(N)\,\alpha}_{\varepsilon_i q_j}\, (\tilde{\varepsilon}_i \cdot \tilde{q}_j) + J^{(N)\,\alpha}_{\omega_2 q_2}\, (\tilde{\omega}_2 \cdot \tilde{q}_2) + J^{(N)\,\alpha}_{\varepsilon_1 \varepsilon_2}\, (\tilde{\varepsilon}_1 \cdot \tilde{\varepsilon}_2) + \sum_{i,j=1}^{2} J^{(N)\,\alpha}_{\varepsilon_1 q_i \varepsilon_2 q_j}\, (\tilde{\varepsilon}_1 \cdot \tilde{q}_i)(\tilde{\varepsilon}_2 \cdot \tilde{q}_j) \\
& + \sum_{i,j,k=1}^{2} C^{(N)}_{\varepsilon_i q_j, q_k}\, (\tilde{\varepsilon}_i \cdot \tilde{q}_j)\, \tilde{q}_k^{\alpha} + C^{(N)}_{\omega_2 q_2, q_2} (\tilde{\omega}_2 \cdot \tilde{q}_2)\, \tilde{q}_2^{\alpha} + \sum_{\substack{i,j,k=1 \\ k \neq i}}^{2} C^{(N)}_{\varepsilon_i q_j, \varepsilon_k}\, (\tilde{\varepsilon}_i \cdot \tilde{q}_j)\, \tilde{\varepsilon}_k^{\alpha} \\
& + \sum_{i,j=1}^{2} C^{(N)}_{\varepsilon_1 \varepsilon_2, q_i}\, (\tilde{\varepsilon}_1 \cdot \tilde{\varepsilon}_2)\, \tilde{q}_i^{\alpha} + \sum_{i,j,k=1}^{2} C^{(N)}_{\varepsilon_1 q_i \varepsilon_2 q_j, q_k} (\tilde{\varepsilon}_1 \cdot \tilde{q}_i)\, (\tilde{\varepsilon}_2 \cdot \tilde{q}_j)\, \tilde{q}_k^{\alpha} \\
& + J^{(N)\,\alpha}_{\mu} + \boxed{J^{(N)\,\alpha}_{(d-4)}}
\end{aligned}
$$

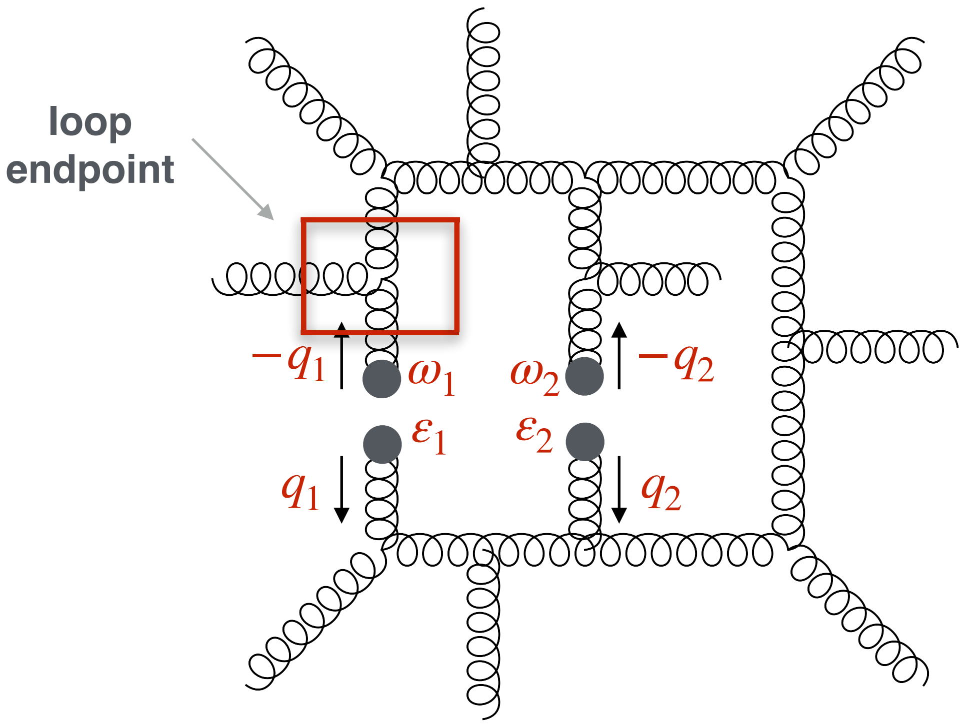
Strategy of calculation



Not *all* coefficients are involved at each step of the recursion!

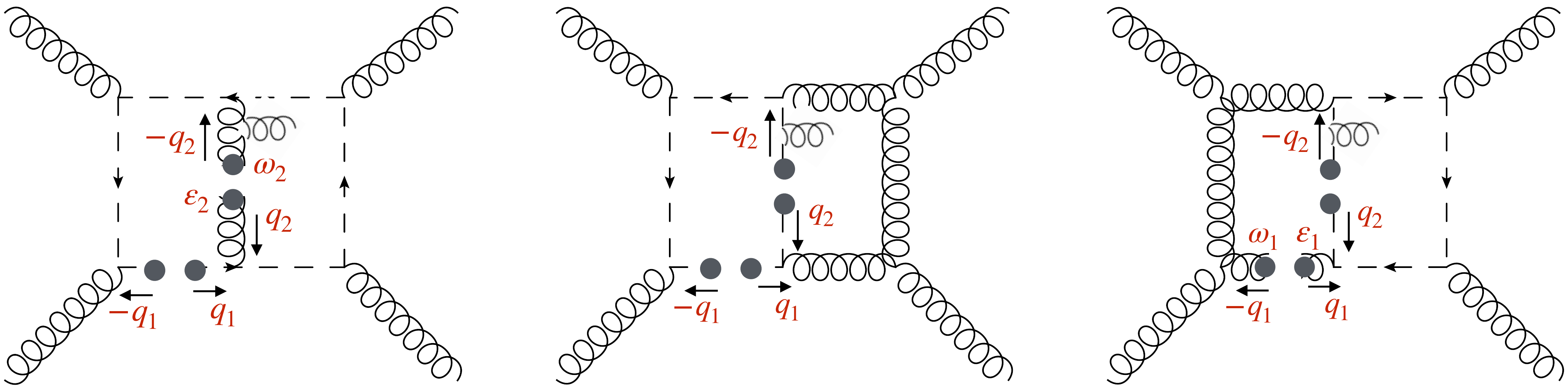
$$
\begin{aligned}
\mathscr{E}[J^{(N)\,\alpha}_{\text{gluon}}] \equiv & \sum_{i=1}^{2} C_{q_i}{}^{(N)}\, \tilde{q}_i^\alpha + \sum_{i=1}^{1} C_{\varepsilon_i}{}^{(N)}\, \tilde{\varepsilon}_i^\alpha + C_{\omega_2}{}^{(N)}\, \tilde{\omega}_2^\alpha \\
& + \sum_{j=1}^{2} J^{(N)\,\alpha}_{\varepsilon_i q_j}\, (\tilde{\varepsilon}_1 \cdot \tilde{q}_j) + J^{(N)\,\alpha}_{\omega_2 q_2}\, (\tilde{\omega}_2 \cdot \tilde{q}_2) + J^{(N)\,\alpha}_{\varepsilon_1 \varepsilon_2}\, (\tilde{\varepsilon}_1 \cdot \tilde{\varepsilon}_2) + \sum_{i,j=1}^{2} J^{(N)\,\alpha}_{\varepsilon_1 q_i \varepsilon_2 q_j}\, (\tilde{\varepsilon}_1 \cdot \tilde{q}_i)(\tilde{\varepsilon}_2 \cdot \tilde{q}_j) \\
& + \sum_{j,k=1}^{2} C^{(N)}_{\varepsilon_1 q_j, q_k}\, (\tilde{\varepsilon}_1 \cdot \tilde{q}_j)\, \tilde{q}_k^\alpha + C^{(N)}_{\omega_2 q_2, q_2} (\tilde{\omega}_2 \cdot \tilde{q}_2)\, \tilde{q}_2^\alpha + \sum_{\substack{i,j,k=1 \\ k \neq i}}^{2} C^{(N)}_{\varepsilon_i q_j, \varepsilon_k}\, (\tilde{\varepsilon}_i \cdot \tilde{q}_j)\, \tilde{\varepsilon}_k^\alpha \\
& + \sum_{i,j=1}^{2} C^{(N)}_{\varepsilon_1 \varepsilon_2, q_i}\, (\tilde{\varepsilon}_1 \cdot \tilde{\varepsilon}_2)\, \tilde{q}_i^\alpha + \sum_{i,j,k=1}^{2} C^{(N)}_{\varepsilon_1 q_i \varepsilon_2 q_j, q_k}\, (\tilde{\varepsilon}_1 \cdot \tilde{q}_i)\, (\tilde{\varepsilon}_2 \cdot \tilde{q}_j)\, \tilde{q}_k^\alpha \\
& + J^{(N)\,\alpha}_{\mu} + J^{(N)\,\alpha}_{(d-4)}
\end{aligned}
$$

Strategy of calculation



Not *all* coefficients are involved at each step of the recursion!

$$
\begin{aligned}
\mathcal{E}[J^{(N)\,\alpha}_{\text{gluon}}] \equiv\; & \sum_{i,j=1}^{2} C_{q_i}^{(N)}\,\tilde{q}_i^{\alpha} + \sum_{i,j=1}^{2} C_{\varepsilon_i}^{(N)}\,\tilde{\varepsilon}_i^{\alpha} + C_{\omega_2}^{(N)}\,\tilde{\omega}_2^{\alpha} \\
& + \sum_{i,j=1}^{2} J^{(N)\,\alpha}_{\varepsilon_i q_j}\,(\tilde{\varepsilon}_i\cdot\tilde{q}_j) + J^{(N)\,\alpha}_{\omega_2 q_2}\,(\tilde{\omega}_2\cdot\tilde{q}_2) + J^{(N)\,\alpha}_{\varepsilon_1\varepsilon_2}\,(\tilde{\varepsilon}_1\cdot\tilde{\varepsilon}_2) + \sum_{i,j=1}^{2} J^{(N)\,\alpha}_{\varepsilon_1 q_i \varepsilon_2 q_j}\,(\tilde{\varepsilon}_1\cdot\tilde{q}_i)(\tilde{\varepsilon}_2\cdot\tilde{q}_j) \\
& + \sum_{i,j,k=1}^{2} C^{(N)}_{\varepsilon_i q_j, q_k}\,(\tilde{\varepsilon}_i\cdot\tilde{q}_j)\,\tilde{q}_k^{\alpha} + C^{(N)}_{\omega_2 q_2, q_2}(\tilde{\omega}_2\cdot\tilde{q}_2)\,\tilde{q}_2^{\alpha} + \sum_{\substack{i,j,k=1\\ k\neq i}}^{2} C^{(N)}_{\varepsilon_i q_j, \varepsilon_k}\,(\tilde{\varepsilon}_i\cdot\tilde{q}_j)\,\tilde{\varepsilon}_k^{\alpha} \\
& + \sum_{i,j=1}^{2} C^{(N)}_{\varepsilon_1\varepsilon_2, q_i}\,(\tilde{\varepsilon}_1\cdot\tilde{\varepsilon}_2)\,\tilde{q}_i^{\alpha} + \sum_{i,j,k=1}^{2} C^{(N)}_{\varepsilon_1 q_i \varepsilon_2 q_j, q_k}\,(\tilde{\varepsilon}_1\cdot\tilde{q}_i)\,(\tilde{\varepsilon}_2\cdot\tilde{q}_j)\,\tilde{q}_k^{\alpha} \\
& + J_{\mu}^{(N)\,\alpha} + \boxed{J^{(N)\,\alpha}_{(d-4)}}
\end{aligned}
$$



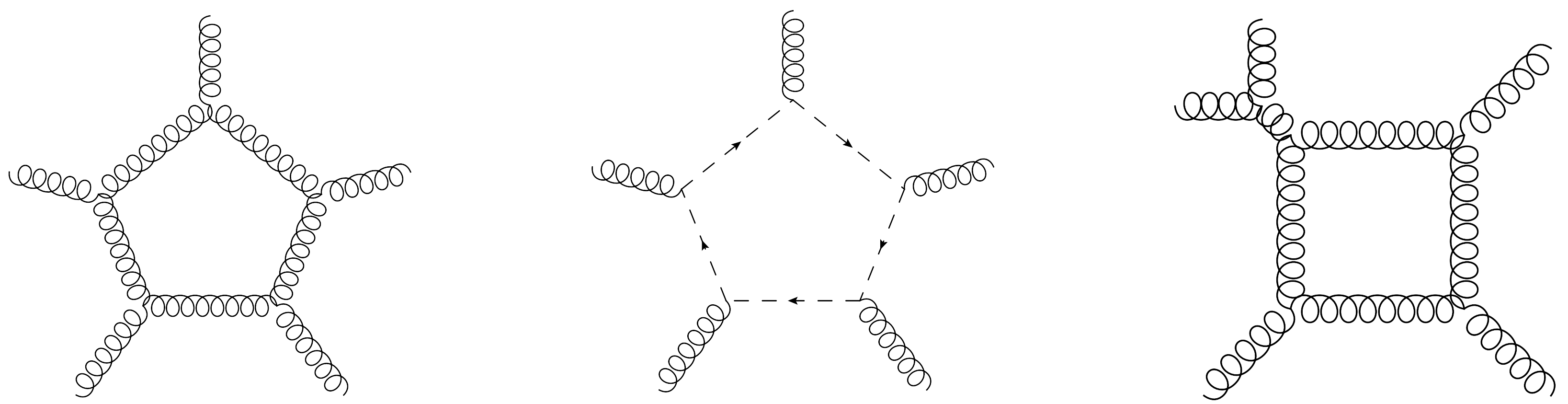
$$\mathscr{E}[J^{(N)}_{\text{ghost}}] \equiv \sum_{i,j=1}^{2} C^{(N)}_{\varepsilon_i q_j} \, (\tilde{\varepsilon}_i \cdot \tilde{q}_j) + J^{(N)\,\alpha}_{\mu} + \cancel{J^{(N)\,\alpha}_{(d-4)}} = 0$$

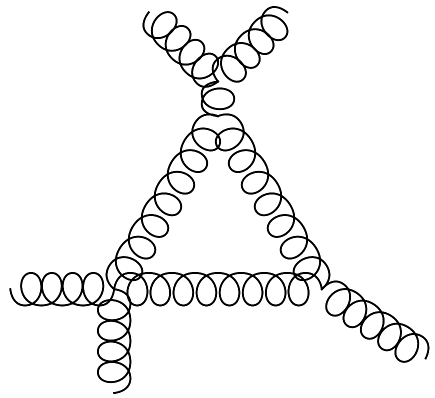
pure YM @ 2-loop

Starting from 2-loops, also ghost vertices develop μ_{ij} terms through the recursion

1. Completed numerical validation for pure YM in HELAC-1LOOP:

- $gg \to gg$
- $gg \to ggg$

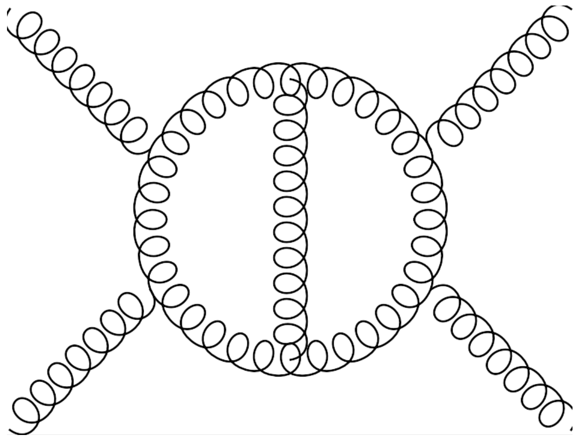




- Cut Constructible (CC) + Rational Terms computed at *integrand* level

- CC+R_1 at integrand level, R_2 via effective Feynman rules (standard approach)

II. Analytic proof of concept of recursion @ 2-loop $\rightarrow$ *double-box* topology



μ_{ij} and $(d-4)$ terms generated recursively

$$
\begin{aligned}
\mathcal{E}[J^{(N)\,\alpha}_{\text{gluon}}] \equiv\; & \sum_{i=1}^{2} C_{q_i}{}^{(N)}\, \tilde{q}_i^{\alpha} + \sum_{i=1}^{2} C_{\varepsilon_i}{}^{(N)}\, \tilde{\varepsilon}_i^{\alpha} + C_{\omega_2}{}^{(N)}\, \tilde{\omega}_2^{\alpha} \\
& + \sum_{i,j=1}^{2} J^{(N)\,\alpha}_{\varepsilon_i q_j}\, (\tilde{\varepsilon}_i \cdot \tilde{q}_j) + J^{(N)\,\alpha}_{\omega_2 q_2}\, (\tilde{\omega}_2 \cdot \tilde{q}_2) + J^{(N)\,\alpha}_{\varepsilon_1 \varepsilon_2}\, (\tilde{\varepsilon}_1 \cdot \tilde{\varepsilon}_2) + \sum_{i,j=1}^{2} J^{(N)\,\alpha}_{\varepsilon_1 q_i \varepsilon_2 q_j}\, (\tilde{\varepsilon}_1 \cdot \tilde{q}_i)(\tilde{\varepsilon}_2 \cdot \tilde{q}_j) \\
& + \sum_{i,j,k=1}^{2} C^{(N)}_{\varepsilon_i q_j, q_k}\, (\tilde{\varepsilon}_i \cdot \tilde{q}_j)\, \tilde{q}_k^{\alpha} + C^{(N)}_{\omega_2 q_2, q_2}\, (\tilde{\omega}_2 \cdot \tilde{q}_2)\, \tilde{q}_2^{\alpha} + \sum_{\substack{i,j,k=1 \\ k \neq i}}^{2} C^{(N)}_{\varepsilon_i q_j, \varepsilon_k}\, (\tilde{\varepsilon}_i \cdot \tilde{q}_j)\, \tilde{\varepsilon}_k^{\alpha} \\
& + \sum_{i,j=1}^{2} C^{(N)}_{\varepsilon_1 \varepsilon_2, q_i}\, (\tilde{\varepsilon}_1 \cdot \tilde{\varepsilon}_2)\, \tilde{q}_i^{\alpha} + \sum_{i,j,k=1}^{2} C^{(N)}_{\varepsilon_1 q_i \varepsilon_2 q_j, q_k}\, (\tilde{\varepsilon}_1 \cdot \tilde{q}_i)\, (\tilde{\varepsilon}_2 \cdot \tilde{q}_j)\, \tilde{q}_k^{\alpha} \\
& + J^{(N)\,\alpha}_{\mu} + J^{(N)\,\alpha}_{(d-4)}
\end{aligned}
$$

Numerical implementation in HELAC-2LOOP in progress

Summary

- Work is underway to enable HELAC framework to perform numerical computations of loop numerators in $d = 4 - 2\varepsilon$ dimensions

- Interesting applications to integrand-level reduction of 2-loop amplitudes

- We are formulating a new method for computing μ_{ij} and $(d-4)$ contributions in the context of *recursive* numerator computations

- The method has been established for pure YM at 2-loops

- Extension to quarks and numerical implementation in HELAC-2LOOP in progress

Thank you for your attention