SuperB Collaboration Meeting

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Istituto Nazionale di Fisica Nucleare

Lattice QCD in Charm Flavour Physics

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Fundamentals on Lattice QCD

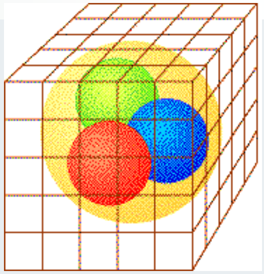
Lattice QCD:

√ **non-perturbative approach**
(path-integral method)

√ **only the QCD parameters**

√ **theory regularization**

× **discrete space and finite volume**



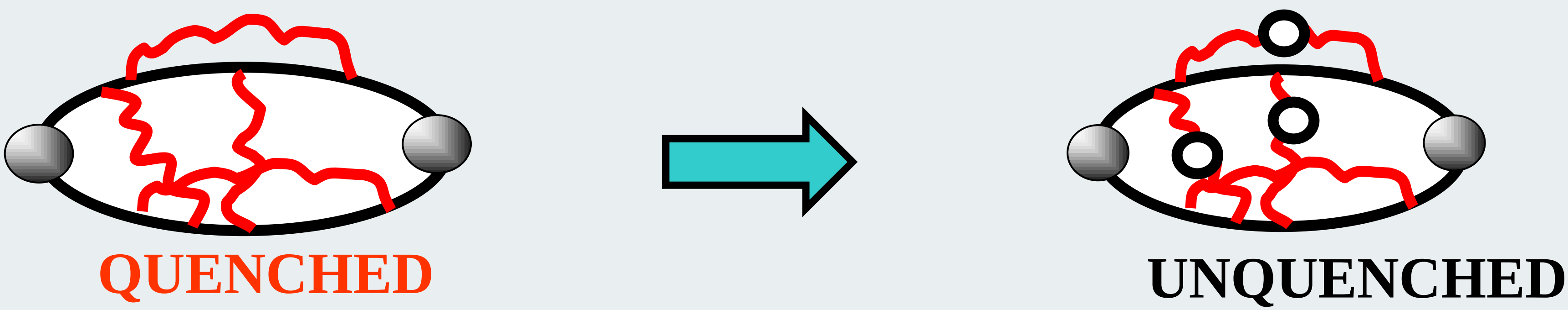
We are in the era of

"PRECISION" LATTICE QCD

1) Increasing of computational power

(Several machines of O(0.1-10 PetaFlops))

→ Unquenched simulations



2) Algorithmic improvements:

→ Light quark masses in the ChPT regime

Systematic Uncertainties:
The state of the art is evident from the color code introduced by FLAG for Pion and Kaon Physics

- Finite-volume effects:
  - ★ $M_{\pi,\mathrm{min}}L > 4$ or at least 3 volumes
  - ● $M_{\pi,\mathrm{min}}L > 3$ and at least 2 volumes
  - ■ otherwise

- Renormalization (where applicable):
  - ★ non-perturbative
  - ● 2-loop perturbation theory
  - ■ otherwise

- Chiral extrapolation:
  - ★ $M_{\pi,\mathrm{min}} < 250$ MeV
  - ● 250 MeV $\leq M_{\pi,\mathrm{min}} \leq$ 400 MeV
  - ■ $M_{\pi,\mathrm{min}} > 400$ MeV

Several recent lattice results have green stars

- Continuum extrapolation:
  - ★ 3 or more lattice spacings, at least 2 points below 0.1 fm
  - ● 2 or more lattice spacings, at least 1 point below 0.1 fm
  - ■ otherwise

...

- For Heavy Flavor Physics, in order to have the continuum limit under control, one has to be careful to $O(a^*m_h)$ discretization terms:

- Improved actions remove leading terms

- For a~ 0.1 fm↔ a^{-1}~ 2GeV, the b quark mass cannot be directly simulated on the lattice (a^*m_b~ 2)
→HQET, NRQCD, FermiLab action, step-scaling method, ratio method,…

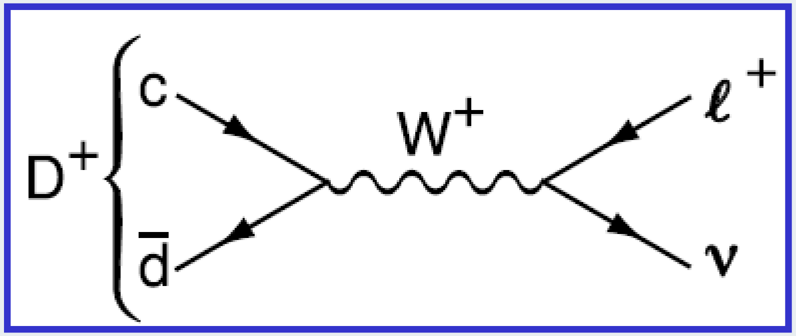
- The c quark mass, instead, can be directly simulated on the lattice (a^*m_c ~0.6, $a^2{}^*m_c^2$ ~0.36,…)

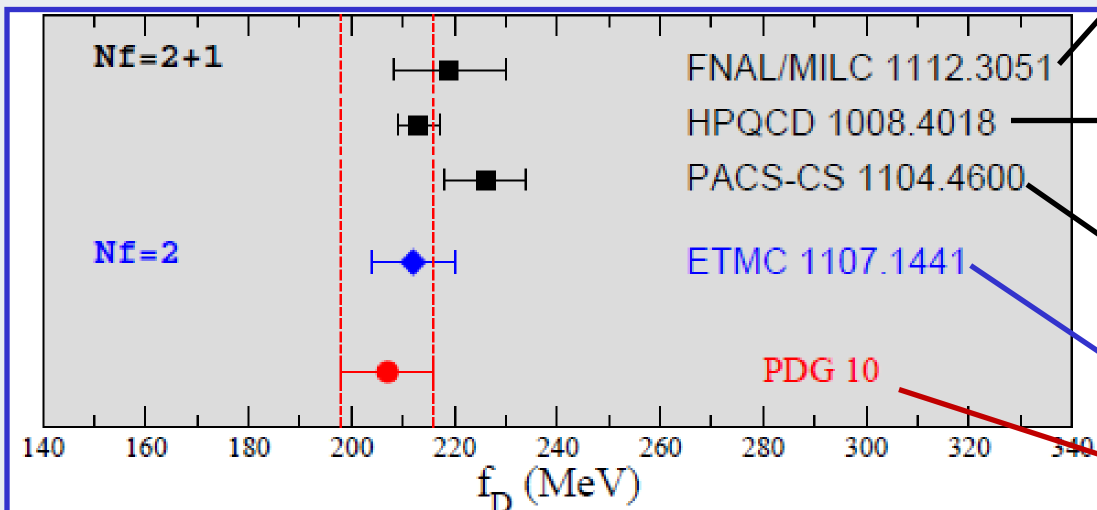
Charm Physics has favorable properties for Lattice QCD

Recent and expected experimental progresses are motivating Lattice Collaborations to perform charm studies

$D_{(s)}$ leptonic decays: f_D and f_{Ds}

$$\Gamma(P \to \ell\nu) = \frac{G_F^2}{8\pi} f_P^2\, m_\ell^2 m_P \left(1 - \frac{m_\ell^2}{m_P^2}\right)^2 |V_{q_1 q_2}|^2$$





Asqtad-impr. light, FermiLab charm, $O(\alpha_s\ \Lambda/m_c)$, $O(\Lambda/m_c)^2$ and $O(\alpha_s\ a\ \Lambda)$, $O(a\ \Lambda)^2$ discretiz. effects, 2 *a* values + 1 for check, mixed non-pert. and pert. renormal.

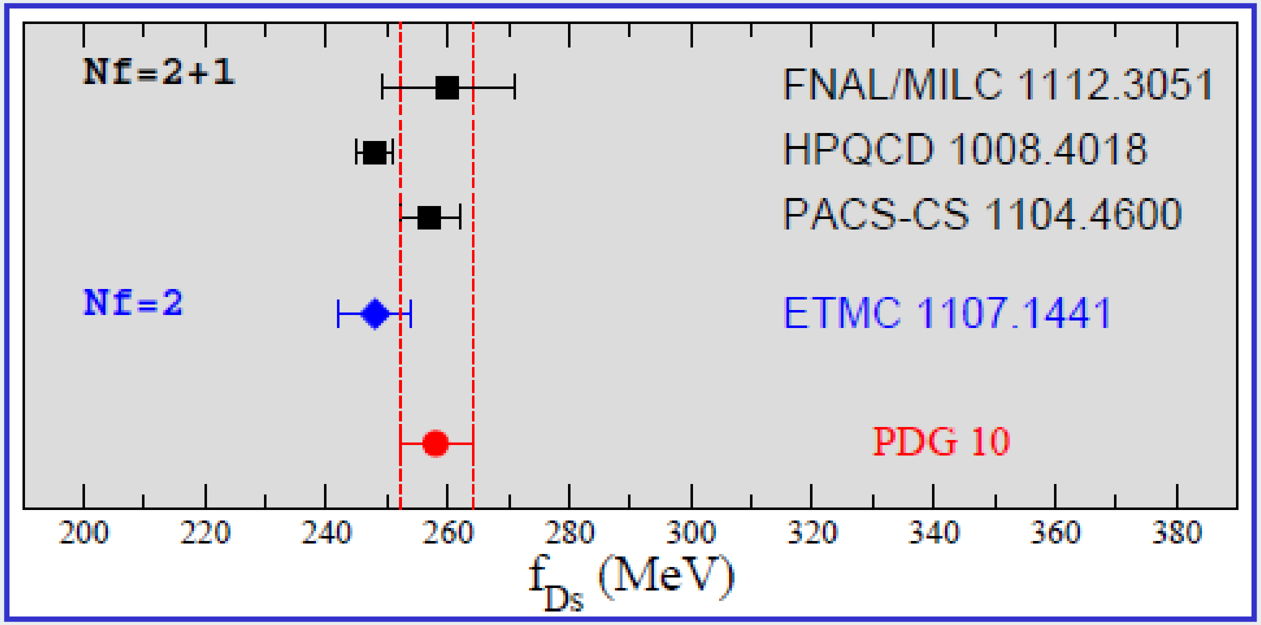
HISQ light and charm, $O(\alpha_s\ a^2\ mc^2)$, Update w.r.t. 2007 for improved estimate of r1 and 2 finer lattices, increased result by 1.5σ

Physical quark masses through reweighting, $O(\alpha_s^2\ a\Lambda)f(am_c)$, 1 value of a

4 values of a, $O(a^2\ mc^2)$ discr. terms thanks to TM

From the exp. BR (CLEO) and V_{cd} from CKM unitarity

f_{Ds}



From the exp. BR (CLEO+BaBar+Belle) and V_{cs} from CKM unitarity

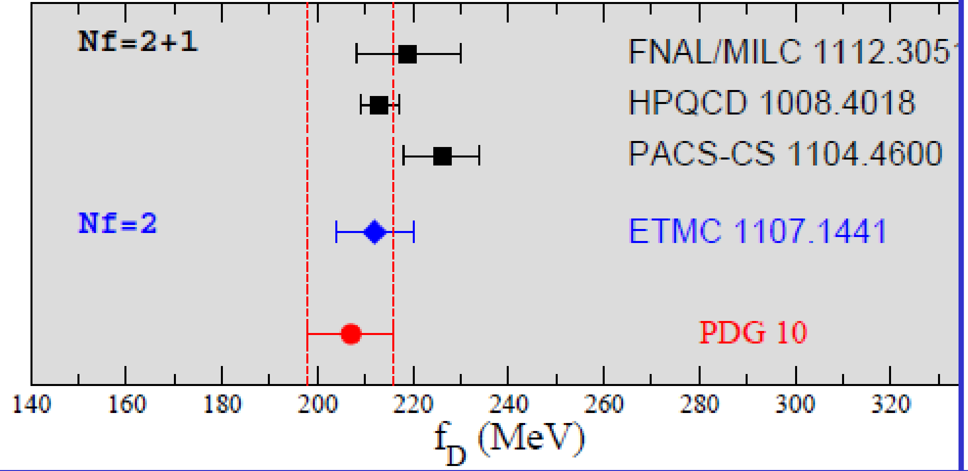
New preliminary result from Belle @ CHARM2012

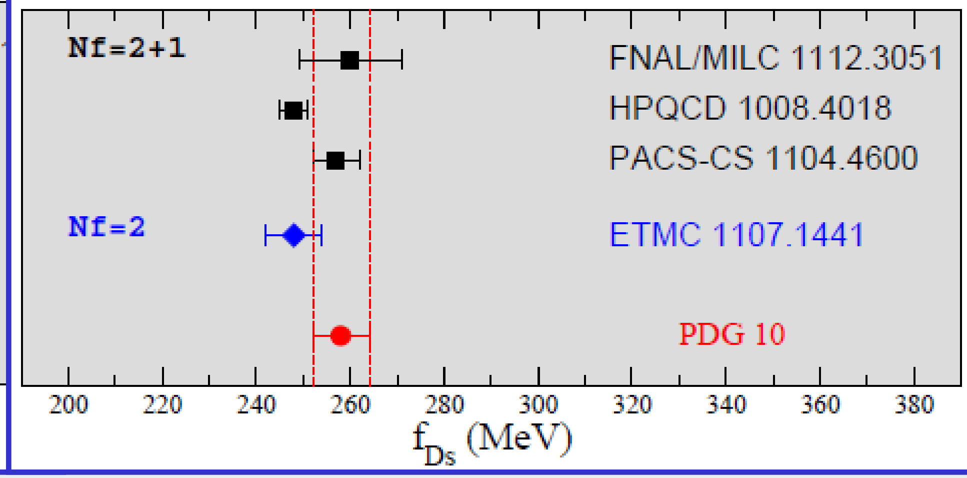
The past (2008) f_{Ds} puzzle has been solved!

Tension between lattice determination and experimental measurement, mainly due to the 3 σ deviation between:

HPQCD 2007 f_{Ds} = 241± 3 MeV ↑ (by 2.3 σ)

PDG 2008 f_{Ds} = 273 ±10 MeV ↓ (by 1.5 σ)





- **Comparable uncertainty between the PDG indirect determination and lattice results, i.e. between the experimental uncertainty on the BR and the lattice uncertainty**
- **Both experimental and theoretical improvements are feasible and looked forward for and accurate determination of V_{cd} and V_{cs}**

$$|V_{cq}| \propto \frac{\sqrt{Br(D_q \to l\nu)}}{f_{Dq}}$$

With present lattice and experimental uncertainties:
V_{cd} is known at the 6% level
V_{cs} is known at the 4% level

Reducing these uncertainties will be interesting for testing unitarity

$$V_{ud} = 1 - \frac{1}{2}\lambda^2 - \frac{1}{8}\lambda^4 + \mathcal{O}(\lambda^6)$$
$$V_{us} = \lambda + \mathcal{O}(\lambda^7)$$
$$V_{cd} = -\lambda + \frac{1}{2}A^2\lambda^5[1 - 2(\varrho + i\eta)] + \mathcal{O}(\lambda^7)$$
$$V_{cs} = 1 - \frac{1}{2}\lambda^2 - \frac{1}{8}\lambda^4(1 + 4A^2) + \mathcal{O}(\lambda^6)$$
$$V_{cb} = A\lambda^2 + \mathcal{O}(\lambda^8)$$

Neglecting $O(\lambda^5)$ terms:
$|V_{cd}| \approx |V_{us}| = 0.225(2)$
$|V_{cs}| \approx |V_{ud}| - |V_{cb}|^2/2 = 0.9737(5)$

<1% accuracy

- **Calculations with N_f=2+1+1 are being performed by FNAL/MILC and ETMC, see:**

- **Elvira Gamiz's talk @ *''Beautiful Mesons and Baryons on the Lattice''*, Trento, March 2012**

- **ETMC *PoS @ Lattice 2011*, F. Farchioni et al.**

D semileptonic decays: form factors

$D \to K\, l\, \nu$ and $D \to \pi\, l\, \nu$

e.g. $D \to K\, l\, \nu$

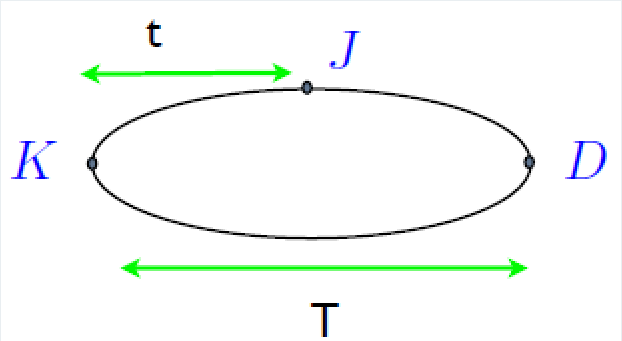
$q = p_D - p_K$

$$\frac{d\Gamma}{dq^2} = \frac{G_F^2 p_K^3}{24\pi^3} |V_{cs}|^2 |f_+(q^2)|^2$$

$$\langle K|V^\mu|D\rangle = f_+(q^2)\left[p_D^\mu + p_K^\mu - \frac{m_D^2 - m_K^2}{q^2} q^\mu\right] + f_0(q^2)\frac{m_D^2 - m_K^2}{q^2} q^\mu$$

f_0 disappers as e and μ masses can be neglected

3-point correlator on the Lattice



- On the lattice the most precise form factor value is obtained at $q^2_{max} = (m_D - m_K)^2$
- Experimental errors are best for $q^2 \approx 0$ where the rate is larger

Some useful tools to compute $f_+(q^2)$ on the Lattice

- **Double Ratio method for a precise determination of the form factors:**
  some statistical and systematic uncertainties cancel out [ETMC]
  [S.Hashimoto et al. hep-ph/9906376, D.Becirevic et al. hep-ph/0403217 & 0710.1741]

- **From the PCVC relation** $\partial_\mu V^\mu = (m_1 - m_2) S$ **the form factor can be more precisely determined from the scalar matrix element [HPQCD]**

- **Twisted boundary conditions allow for arbitrary quark momenta in order to cover the physical q^2 range [P.F.Bedaque nucl-th/0402051]**

- **Sudy of the q^2-dependence** **(Becirevic-Kaidalov parametrization, …)**
  taking into account the (outside physical region) poles

$$t_+ = (m_D + m_K)^2$$

- **The fit may be stabilized if performed versus the variable** $z = \frac{\sqrt{t_+ - q^2} - \sqrt{t_+ - t_0}}{\sqrt{t_+ - q^2} + \sqrt{t_+ - t_0}}$
  (in the physical region form factors are described by a simple power series in z)
  [FNAL/MILC, HPQCD]

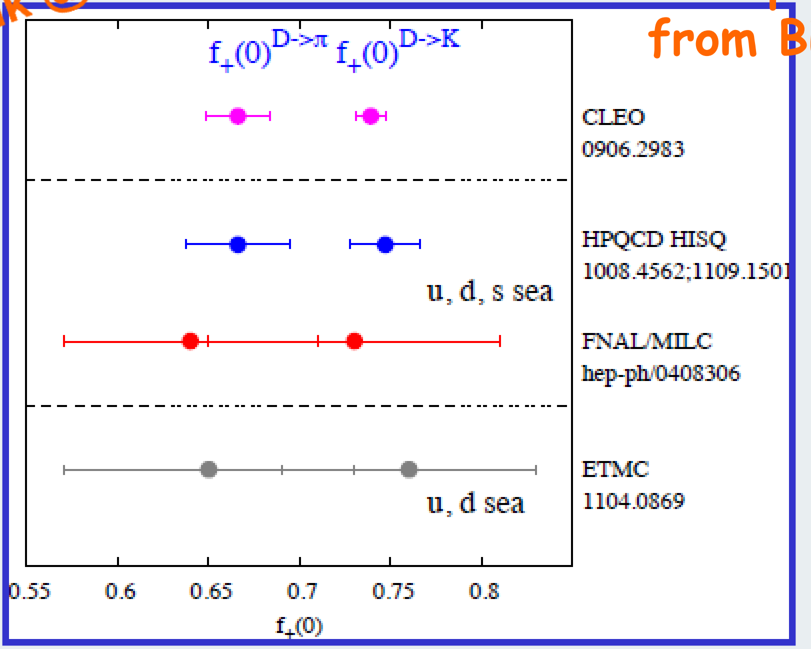
- **Combined chiral and continuum extrapolations [FNAL/MILC, HPQCD, ETMC]**

- **Lattice and experimental data at several q^2 can be fitted simultaneously to extract the CKM element [FNAL/MILC]**

Results for D→ π and D→ K form factors

C.Davies' plenary talk @ Lattice2011

New preliminary results from BESIII @ CHARM2012

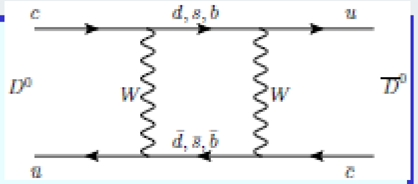


- V_{cd} and V_{cs} can be determined at present with a bit less accuracy than from the leptonic decays (~10%)
- In this case the lattice uncertainty dominates over the experimental uncertainty
- New lattice analysis are in progress:

-FNAL/MILC HISQ light valence and Fermilab charm on Nf=2+1 Asqtad confs. and on Nf=2+1+1 HISQ confs.
-HPQCD HISQ light and charm valence with Nf=2+1 (on more lattice ensembles)
-ETMC on Nf=2+1+1 twisted mass confs.

First preliminary results exist for the form factors entering $D_{(s)} \to \eta^{(\prime)} l\nu$ [QCDSF] $D_s \to \phi l\nu$ [HPQCD]

$D-\bar{D}$ mixing: B_D parameters



- At variance with K and B systems, the first evidence for $D-\bar{D}$ mixing is quite recent, 2007 (BaBar & Belle)

- It is sensitive to a different sector of New Physics (NP) with respect to K and B, being the charm an up-type quark

- In some NP models, like SUSY with alignement, sizable effects can be expected in the up-type sector

- $D-\bar{D}$ mixing is affected by large long-distance effects (internal d and s quarks) which dominate over the short-distance contribution

- Only order of magnitude estimates exist for the long-distance contributions and are at the level of the experimental constraints preventing from revealing and unambiguous sign of NP

Donoghue&Uraltsev 1986,
Colangelo et al. 1990
Bigi et al. 2000,
Falk et al. 2001-2004

- Still, barring accidental cancellations between SM and NP contributions, significant constraints can be put on the NP parameter space

- NP contributions are short-distance and can be accurately computed. Five four-fermion operators are involved whose matrix elements may be computed on the Lattice

D-$\bar{D}$ mixing: B_D parameters from Lattice QCD

$$O_1 = \bar{u}^i \gamma_\mu (1-\gamma_5) c^i \bar{u}^j \gamma_\mu (1-\gamma_5) c^j$$

$$O_2 = \bar{u}^i (1-\gamma_5) c^i \bar{u}^j (1-\gamma_5) c^j$$

$$O_3 = \bar{u}^i (1-\gamma_5) c^j \bar{u}^j (1-\gamma_5) c^i$$

$$O_4 = \bar{u}^i (1-\gamma_5) c^i \bar{u}^j (1+\gamma_5) c^j$$

$$O_5 = \bar{u}^i (1-\gamma_5) c^j \bar{u}^j (1+\gamma_5) c^i$$

- In the SM there is O_1 only
- In NP models all 5 operators may be present

B_D-parameters are defined as the deviation from the VIA (like for K and B)

Only quenched results existed so far:
D. Becirevic et al. hep-lat/0110091
H.W.Lin et al. hep-lat/0607035 (B_1 only)

NEW Preliminary unquenched (N_f=2) Results by ETMC

[N. Carrasco, P. Dimopoulos, R. Frezzotti, V. Gimenez, V. Lubicz, G. Martinelli, F. Mescia, M. Papinutto, G.C. Rossi, S. Simula, C. T., A. Vladikas]

Statistical+fitting error

Systematic uncertainty due to chiral extrapolation and different non-pert. renormalization procedures

$B_D[\overline{MS}(2\ \mathrm{GeV})]$	
B_1	0.74(02)(04)
B_2	0.71(01)(03)
B_3	1.19(03)(10)
B_4	0.93(02)(02)
B_5	0.97(02)(24)

First accurate results:
unquenched, improved operators, non-perturbative renormalization, continuum limit, chiral extrapolation with $m_\pi \geq 260$ MeV

PRELIMINARY

Update of the D-$\bar{D}$ mixing analysis of M.Ciuchini et al. hep-ph/0703204

http://www.utfit.org/UTfit/DDbarMixing

$$A = A_{SM} + A_{NP} e^{i\phi_{NP}}$$

With A_{SM}, due to large long-distance uncertainties, taken as flatly distributed in [-0.01,0.01] ps^{-1}

By using the experimental results

Observable	Value	Correlation Coeff.				Reference
y_{CP}	$(0.866 \pm 0.155)\%$					[1–10]
A_Γ	$(0.022 \pm 0.161)\%$					[4, 5, 8–11]
x	$(0.811 \pm 0.334)\%$	1	-0.007	-0.255α	0.216	[12]
y	$(0.309 \pm 0.281)\%$	-0.007	1	-0.019α	-0.280	[12]
$\lvert q/p\rvert$	$(0.95 \pm 0.22 \pm 0.10)\%$	-0.255α	-0.019α	1	-0.128 α	[12]
ϕ	$(-0.035 \pm 0.19 \pm 0.09)$	0.216	-0.280	-0.128 α	1	[12]
x	$(0.16 \pm 0.23 \pm 0.12 \pm 0.08)\%$	1	0.0615			[13]
y	$(0.57 \pm 0.20 \pm 0.13 \pm 0.07)\%$	0.0615	1			[13]
R_M	$(0.0130 \pm 0.0269)\%$					[14–18]
$(x'_+)_{K\pi\pi}$	$(2.48 \pm 0.59 \pm 0.39)\%$	1	-0.69			[19]
$(y'_+)_{K\pi\pi}$	$(-0.07 \pm 0.65 \pm 0.50)\%$	-0.69	1			[19]
$(x'_-)_{K\pi\pi}$	$(3.50 \pm 0.78 \pm 0.65)\%$	1	-0.66			[19]
$(y'_-)_{K\pi\pi}$	$(-0.82 \pm 0.68 \pm 0.41)\%$	-0.66	1			[19]
R_D	$(0.3030 \pm 0.0189)\%$	1	0.77	-0.87		[20]
$(x'_+)^2$	$(-0.024 \pm 0.052)\%$	0.77	1	-0.94		[20]
y'_+	$(0.98 \pm 0.78)\%$	-0.87	-0.94	1		[20]
A_D	$(-2.1 \pm 5.4)\%$	1	0.77	-0.87		[20]
$(x'_-)^2$	$(-0.020 \pm 0.050)\%$	0.77	1	-0.94		[20]
y'_-	$(0.96 \pm 0.75)\%$	-0.87	-0.94	1		[20]
R_D	$(0.364 \pm 0.018)\%$	1	0.655	-0.834		[21]
$(x'_+)^2$	$(0.032 \pm 0.037)\%$	0.655	1	-0.909		[21]
y'_+	$(-0.12 \pm 0.58)\%$	-0.834	-0.909	1		[21]
A_D	$(2.3 \pm 4.7)\%$	1	0.655	-0.834		[21]
$(x'_-)^2$	$(0.006 \pm 0.034)\%$	0.655	1	-0.909		[21]
y'_-	$(0.20 \pm 0.54)\%$	-0.834	-0.909	1		[21]

TABLE I. Experimental data used in the analysis of D mixing, from ref. [22]. $\alpha = (1+|q/p|)^2/2$. Asymmetric errors have been symmetrized. We do not use measurements that do not allow for CP violation in mixing, except for ref. [13].ª

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Including new (preliminary) results by BaBar and Belle presented @ Charm2012

Strong constraints can be put on the parameter space of some NP models

In the MSSM with a general Flavour Structure

It is useful to work in the SuperCKM basis where **gluino couplings** are flavour diagonal and to expand **(non-diagonal)** sfermion mass matrices

Mass Insertion Approximation

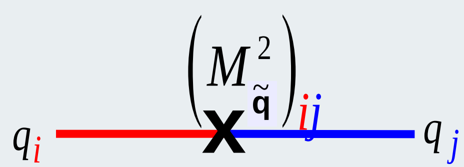
$$M^2_{\tilde{u}} = \begin{pmatrix} (m^2_U)_{LL} & (m^2_U)_{LR} \\ (m^2_U)^{\dagger}_{LR} & (m^2_U)_{RR} \end{pmatrix}$$

3x3 non-diagonal flavour matrices

expanded in small off-diagonal entries:

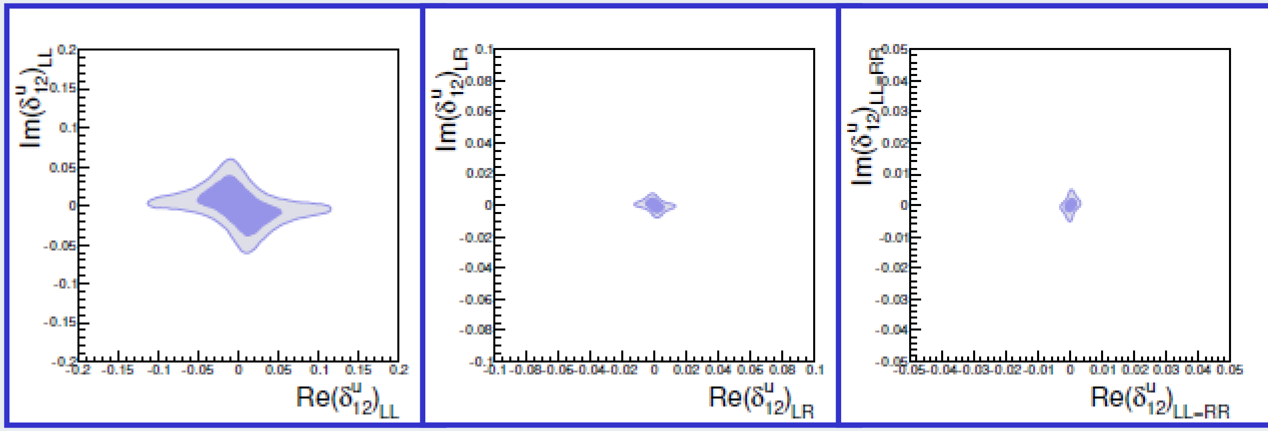
e.g., $(\delta^U_{LL})_{ij} \equiv (m^2_U)^{ij}_{LL} \,/\, \tilde{m}^2$

Flavour non-diagonality is brought by squark propagators



Constraints on the δs from D-$\overline{\text{D}}$ mixing

e.g. $m_{\tilde{q}} = m_{\tilde{g}} = 1TeV$



Assuming a dominant LL mass insertion

Assuming a dominant LR mass insertion (more strongly constrained as chirality-flipping operators are generated, which are RG-enhanced)

Allowing for (equal) LL and RR mass insertions (strongly constrained as chirality-flipping operators are generated)

Mass insertions turn out to be more constrained than in hep-ph/0703204 by ≈20-30% mainly due to the increased lattice accuracy on f_D and m_c

SUSY models with quark-squark alignement generically predict $(\delta^u_{12})_{LL} \approx 0.2$ [Nir&Raz2002], to be phenomenologically viable they need squark and gluino masses above 2-3 TeV

In view of the next future experimental progress a further increase in the Lattice accuracy is desired

Summary of the SuperB physics program 1109.5028

Observable/mode	Current now	LHCb (2017) 5 fb^{-1}	Super$B$ (2021) 75 ab^{-1}	Belle II (2021) 50 ab^{-1}	LHCb upgrade (10 years of running) 50 fb^{-1}	theory now
x	$(0.63 \pm 0.20\%$	0.06%	0.02%	0.04%	0.02%	$\sim 10^{-2}$ [k]
y	$(0.75 \pm 0.12)\%$	0.03%	0.01%	0.03%	0.01%	$\sim 10^{-2}$ (see above).
y_{CP}	$(1.11 \pm 0.22)\%$	0.02%	0.03%	0.05%	0.01%	$\sim 10^{-2}$ (see above).
$\lvert q/p \rvert$	$(0.91 \pm 0.17)\%$	8.5%	2.7%	3.0%	3%	$\sim 10^{-3}$ (see above).
$\arg\{q/p\}$ (°)	-10.2 ± 9.2	4.4	1.4	1.4	2.0	$\sim 10^{-3}$ (see above).

Conclusions

- Charm Physics has favorable properties for Lattice QCD:
  the charm quark mass has the "right" value to be directly simulated

- The charm sector is less explored than strange and and beauty sectors.
  Experimental and theoretical interest is significantly increasing

- Latticists are aware of that, important progresses are expected

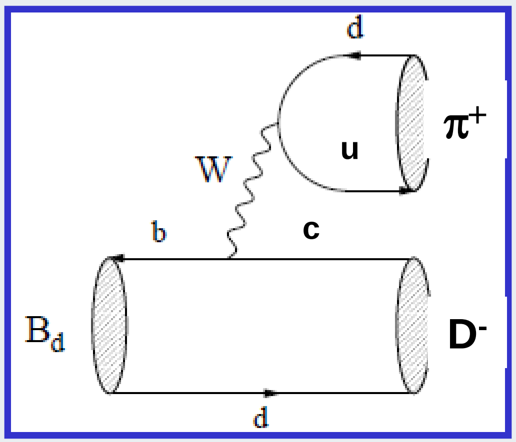
Backup

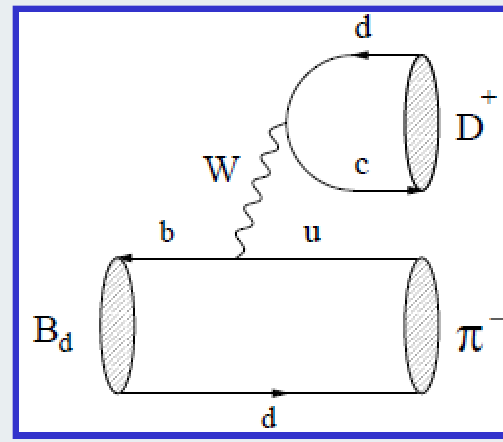
Vector meson decay constants: testing factorization

$$\langle 0|\bar{c}(0)\gamma_\mu\gamma_5 q(0)|D_q(p)\rangle = f_{D_q} p_\mu \,, \quad \rightarrow \textbf{pseudoscalar}$$

$$\langle 0|\bar{c}(0)\gamma_\mu q(0)|D_q^*(p,\lambda)\rangle = f_{D_q^*} m_{D_q^*} e_\mu^\lambda \,, \quad \rightarrow \textbf{vector}$$

The vector meson decay constants $f_{D^*_{(s)}}$ enter in some BRs of non-leptonic B decays, computed in the factorization approximation





The spectator quark goes into the heavy meson

→ **Factorization is exact in the static limit**

[M.Beneke et al., hep-ph/0006124]

$$\boxed{A_{FACT} \propto \left\langle \pi^+ \left| \bar{u}\gamma_L^\mu d \right| 0 \right\rangle \left\langle D^- \left| \bar{b}\gamma_L^\mu c \right| B^0 \right\rangle}$$

Spectator quark in the light meson

→ **Factorization is just an assumption**

- **Similarly for** $\boxed{B^0 \to D_{(s)}^{(*)+}\pi^-}$
- **It is interesting to test it**
- **$f_{D^*_{(s)}}$ are needed**

D. Becirevic et al., 1201.4039
Unquenched (N_f=2) results

$$f_{D^*} = 278 \pm 13 \pm 10 \text{ MeV}\,, \quad f_{D_s^*} = 311 \pm 9 \text{ MeV}$$

$$\frac{f_{D_s^*}}{f_{D_s}} = 1.26 \pm 0.03\,, \quad \frac{f_{D^*}}{f_D} = 1.28 \pm 0.06\,, \quad \frac{f_{D_s^*}}{f_{D^*}} = 1.16 \pm 0.02 \pm 0.06\,.$$

Significant deviations from the static limit 1 (heavy quark spin symmetry)

Assuming factorization, and being ratios of form factors ≈1,
the non-perturbative contribution is enclosed in ratios of decay constants

$$R_1 = \frac{B(B^0 \to D_s^+\pi^-)}{B(B^0 \to D^+\pi^-)} = \left(\frac{V_{cs}}{V_{cd}}\right)^2 \left[\frac{\lambda(m_B, m_{D_s}, m_\pi)}{\lambda(m_B, m_D, m_\pi)}\right]^{1/2} \left[\frac{F_0^{B\to\pi}(m_{D_s}^2)}{F_0^{B\to\pi}(m_D^2)}\right]^2 \left(\frac{f_{D_s}}{f_D}\right)^2$$

$$R_2 = \frac{B(B^0 \to D_s^+ D^-)}{B(B^0 \to D^+ D^-)} = \left(\frac{V_{cs}}{V_{cd}}\right)^2 \left[\frac{\lambda(m_B, m_{D_s}, m_D)}{\lambda(m_B, m_D, m_D)}\right]^{1/2} \left[\frac{F_0^{B\to D}(m_{D_s}^2)}{F_0^{B\to D}(m_D^2)}\right]^2 \left(\frac{f_{D_s}}{f_D}\right)^2$$

$$R_3 = \frac{B(B^0 \to D_s^{*+}\pi^-)}{B(B^0 \to D^{*+}\pi^-)} = \left(\frac{V_{cs}}{V_{cd}}\right)^2 \left[\frac{\lambda(m_B, m_{D_s^*}, m_\pi)}{\lambda(m_B, m_{D^*}, m_\pi)}\right]^{3/2} \left[\frac{F_+^{B\to\pi}(m_{D_s^*}^2)}{F_+^{B\to\pi}(m_{D^*}^2)}\right]^2 \left(\frac{f_{D_s^*}}{f_{D^*}}\right)^2$$

$$R_4 = \frac{B(B^0 \to D_s^{*+} D^-)}{B(B^0 \to D^{*+} D^-)} = \left(\frac{V_{cs}}{V_{cd}}\right)^2 \left[\frac{\lambda(m_B, m_{D_s^*}, m_D)}{\lambda(m_B, m_{D^*}, m_D)}\right]^{3/2} \left[\frac{F_+^{B\to D}(m_{D_s^*}^2)}{F_+^{B\to D}(m_{D^*}^2)}\right]^2 \left(\frac{f_{D_s^*}}{f_{D^*}}\right)^2$$

PDG:
R1 is not a direct measurement but assumes factorization (and it was corrected in D. Becirevic et al.)

$$R_1^{(\text{fact.})} = 26.0 \pm 0.4 \pm 2.6\,, \quad \text{vs.} \quad R_1^{(\text{exp.})} = 27.7 \pm 5.0\,,$$
$$R_2^{(\text{fact.})} = 25.7 \pm 0.4 \pm 2.6\,, \quad \text{vs.} \quad R_2^{(\text{exp.})} = 34.1 \pm 6.3\,,$$
$$R_3^{(\text{fact.})} = 23.8 \pm 0.8 \pm 2.5\,, \quad \text{vs.} \quad R_3^{(\text{exp.})} = \text{N.A.}\,,$$
$$R_4^{(\text{fact.})} = 22.7 \pm 0.8 \pm 2.4\,, \quad \text{vs.} \quad R_4^{(\text{exp.})} = 12.1 \pm 4.0\,.$$

Factorization turns out to be a reasonable assumption