Efficient Representation and Symplectic Integration of Magnetic Fields on Curved Reference Frames for Improved Synchrotron Design

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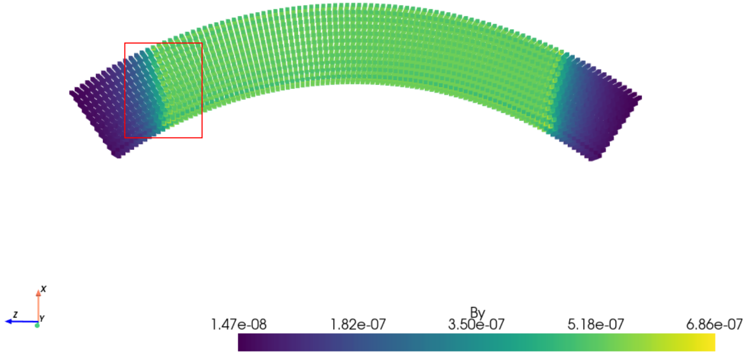
Wat are fringe fields, and why are they important?

Fringe fields

- Field smoothly changing from 0 to its design value to fulfil Maxwell equations



Example for the ELENA dipole

"The ELENA Project at CERN", W. Oelert (2017)

The impact of fringe fields on beam dynamics

- Fringe fields introduce well known (e.g. edge focusing) and less well known effects in the beam dynamics (e.g. high order non-linearities, orbit effect)

- In several cases, these less well known effects are negligible (e.g. in the LHC), but for small machines literature suggests they may be more impactful

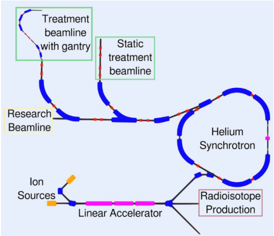
- My PhD develops and compares methods to quantify the effects:
  - Simplified tracking maps available in popular codes
  - Perturbation analysis (RDT)
  - Direct tracking through magnetic field

- Those methods can be applied with or without knowing the actual field map of the magnets

The HeLICS synchrotron

- HeLICS (Helium Light-Ion Compact Synchrotron) synchrotron facility
  - Clinical treatment center and scientific research infrastructure
    - treatment with protons
    - research and future treatment with helium ions
    - research with heavier ions (oxygen, carbon..)

- The HeLICS synchrotron
  - Triangular synchrotron with circumference of 33 m
  - 60° combined function dipole / quadrupole magnets with 30° edge angles



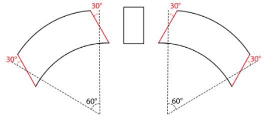
Courtesy of NIMMS collaboration (CERN)

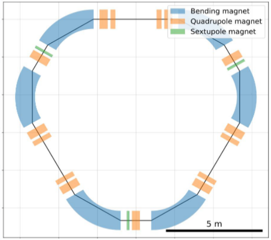
"Optics design of a compact helium synchrotron for advanced cancer therapy", H. Huttunen et al. (2024)

Difficulties of the HeLICS combined-function dipoles

Why do we need to progress on fringe fields?

- Direct tracking through magnetic field
  - Machine is still in design phase: need for methods without magnet fieldmap
- Simplified tracking maps
  - Effects from dipole- and quadrupole components cannot just be added, they don't "commute"
  - Maps for curved combined-function dipoles are not avaiblabe in the literature
  - While deriving these maps might be possible, the 30° edge angles complicate the situation





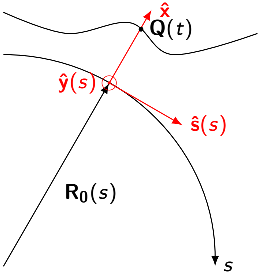
Direct tracking through magnetic field

Direct tracking

- General magnetic coefficients as derivatives of the on-axis field

$$\begin{aligned} a_n(s) &= \partial_x^{n-1} B_x(x,y,s)\big|_{x=y=0} \\ b_n(s) &= \partial_x^{n-1} B_y(x,y,s)\big|_{x=y=0} \\ b_s(s) &= B_s(x,y,s)\big|_{x=y=0} \end{aligned}$$

  - a_n are called normal multipole coefficients
  - b_n are called skew multipole coefficients



- Requiring to satisfy Maxwell equations allows to determine the magnetic field everywhere and construct fieldmaps based on a set of functions
- I wrote an integrator to integrate the equations of motion based on this expansion

Direct tracking

- General magnetic coefficients as derivatives of the on-axis field

$$a_n(s) = \partial_x^{n-1} B_x(x,y,s)\big|_{x=y=0}$$
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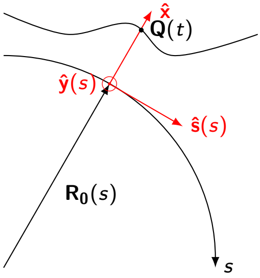

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Strengths:

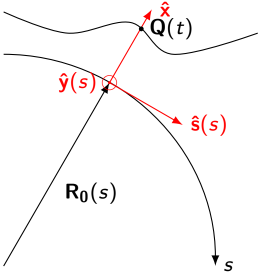
- Exact effect of a fringe field
- New method to interpolate fieldmap using meaningful functions for beam dynamics

Limitations:

- Computationally expensive
- Does not allow to identify how to improve the fieldmap

Direct tracking

- General magnetic coefficients as derivatives of the on-axis field



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$\Longrightarrow$ Will be used to benchmark results

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New Implementation for XSuite

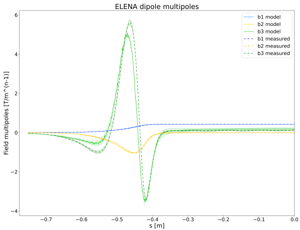
Available in my package *XSuite/bpmeth*

"Xsuite: An Integrated Beam Physics Simulation Framework", G. Iadarola et al. (2024)

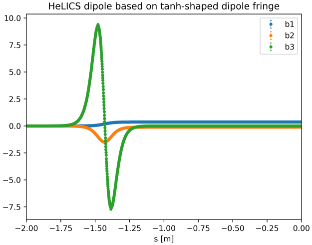
Direct tracking

General magnetic coefficients of the ELENA magnet

- General magnetic field expansion allows to model both designed (based on magnetic simulation) and measured fieldmaps



- When finished magnet design is not available, reasonable assumptions based on similar magnets can be used. Example for HeLICS



Simplified tracking maps

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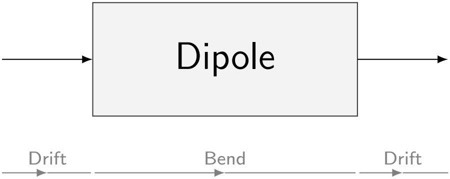

- Simplified tracking map used at CERN (XSuite, PTC, MAD-NG): E. Forest, 2005

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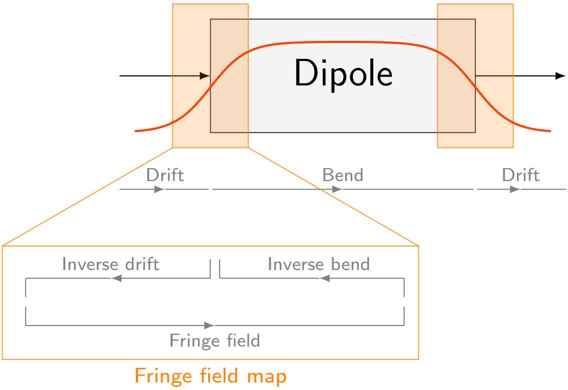

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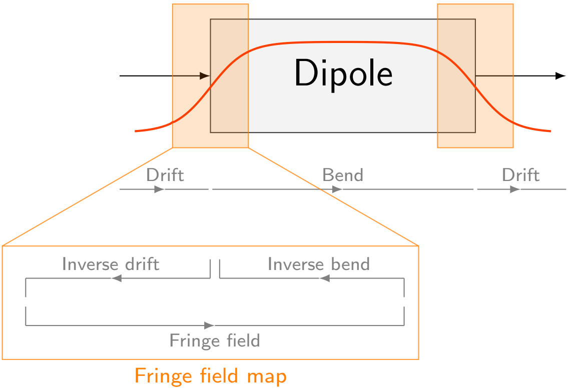

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Fringe field map for the dipole in PTC, XSuite, MAD-NG

Strenghts:

- Symplectic: suited for long-term tracking
- Only one parameter: the *fringe field integral*

$$\boxed{K = \int_{-\infty}^{+\infty} \frac{b(s)(b_0 - b(s))}{g b_0^2} ds}$$

The fringe field integral:

- Dimensionless parameter
- Linear in the range of the fringe field
- Independent of the total strength of the magnet
- Ranges between 0 (hard edge) and ∞

How to obtain its value?

- From the magnetic field map
- If not available: good guess based on similar models

"Fringe Effects in MAD PART I, Second Order Fringe in MAD-X for the Module PTC", E. Forest (2005)

Fringe field map for the dipole in PTC, XSuite, MAD-NG

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Limitations:

- Only physical effect in p_x, p_y, other coordinates follow from symplectification
- Does not include the closed orbit effect (but could be added)

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Perturbation analysis

Perturbation analysis: what are resonance driving terms?

- Unperturbed optics: the linear lattice

  $\Longrightarrow$ only dipoles, quadrupoles, expanded drifts
- All other effects as perturbation
- To quantify deformation of phase space, coupling between two planes, detuning with amplitude
- Identify specific imperfection sources from experimental data or compare the impact of different imperfections

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Strenghts:

- Allows to identify the effects of fringe fields in a perturbative way from the general magnetic field expansion

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- No direct relation between the values of the RDTs and their effect on stability

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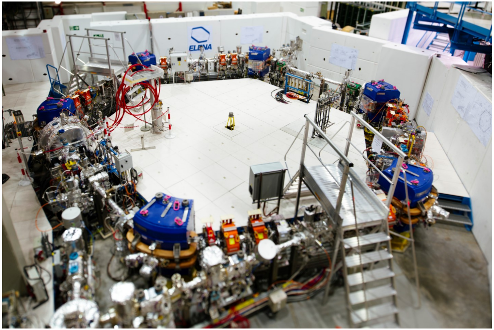
$\Longrightarrow$ I determined the resonance driving terms for fringe fields

New
First time using RDTs
with fringe fields

Measurements with ELENA

Extra Low Energy Antiproton Ring (ELENA)

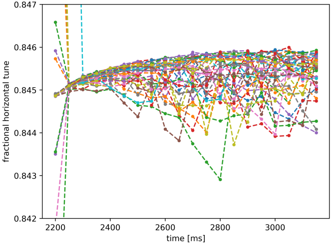
- 30 m circumference
- Decelerates antiprotons from AD at CERN
- Typical small machine that can be used to compare theory and experiment by performing measurements

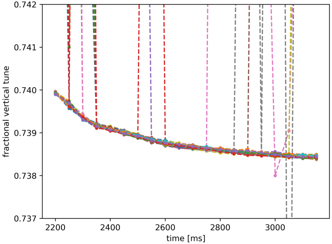


CERN

Tune measurements

- ELENA is dominated by fringe fields: possible to turn off all quadrupoles in the machine
- Reproducible horizontal and vertical tune measurements
- Influence of hysteresis throughout cycle





- Best results achieved with direct tracking through fieldmap, but remaining discrepancy unexplained

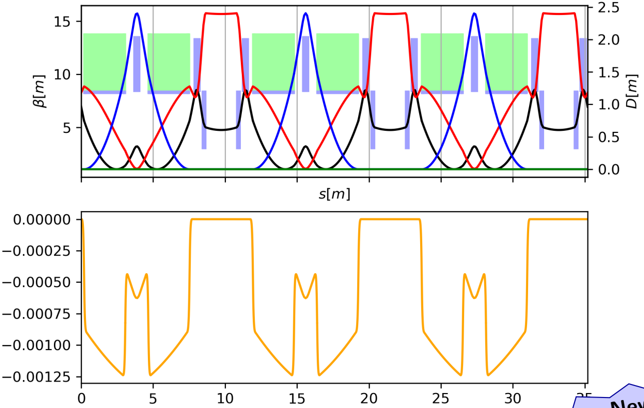
New
Slight improvement in tune by optimizing fringe field integral

	Q_x		Q_y	
Design parameters	1.7897		1.7724	
Currently used model	1.8366		1.7295	
Simplified map with fitted parameters	1.8379		1.7420	
Direct tracking with fieldmap	1.8393		1.7399	
Measurement in machine	x.8453	$\pm$ 0.0005 (stat)	x.7386	$\pm$ 0.0002 (stat)

First results for the HeLICS machine

Impact on the HeLICS lattice of the dipole component: closed orbit effect

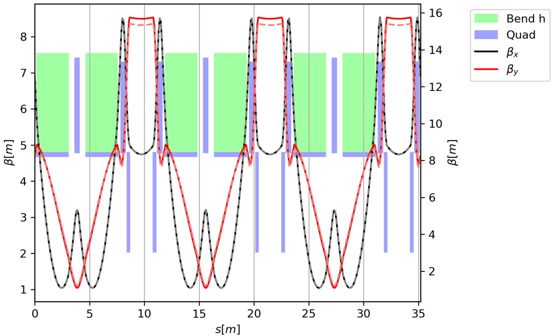
- Estimated closed orbit effect from resonance driving terms is under control



New
Fringe field closed orbit distortion for the HeLICS lattice

Impact on the HeLICS lattice of the dipole component: beta beating

- Estimated beta-beating using simplified tracking in XSuite with ELENA fringe field integral is under control



Solid lines: without fringe fields. Dashed lines: with fringe fields

Conclusions

Conclusions

- Gained an understanding of the available fringe field maps in the literature
- Wrote package *bpmeth* compatible with XSuite to directly track through fieldmaps, also when magnet design is not available
- Theoretically derived resonance driving terms
- Measured impact of fringe field on linear optics in ELENA machine
- Estimated the impact of fringe fields on the linear optics on the HeLICS machine - a machine in the design phase

Open issues

- Focus on non-linear behaviour of fringe fields in ELENA
  - Large discrepancies observed in chromaticity of ELENA between different models
  - Measurements of chromaticity in ELENA will allow to draw conclusions about the models
  - Compare conclusions with measured and design fieldmap to estimate unavoidable uncertainty
- Conclusions of measurements in ELENA will allow to study the non-linear effects in the HeLICS machine
  - Investigate effects of combined function magnet fringe fields on beam dynamics
  - Sextupole-like effects might be important for resonant extraction
- Focus on the resonance driving terms
  - Comparison between the theoretical values and lumped tracking map
  - Identify which RDTs are important for the HeLICS machine compared to other sources

Publications and presentations

- S. Van der Schueren et al., *"Magnetic field modelling and symplectic integration of magnetic fields on curved reference frames for improved synchrotron design: first steps"*, in Proc. IPAC'24, Nashville, TN, May 2024, pp. 1649-1652. doi:10.18429/JACoW-IPAC2024-TUPS09. `https://accelconf.web.cern.ch/ipac2024/doi/jacow-ipac2024-tups09/`
- *Symplectic maps for fringe fields*, internal meeting on Fringe fields, 30th of September 2024. `https://indico.cern.ch/event/1462429/`
- *Maps for dipole fringe fields*, ABP-LNO Section Meeting, 1st of November 2024. `https://indico.cern.ch/event/1468109/`
- *Resonant driving terms for dipole fringe fields*, ABP-CAP Section Meeting, 7th of February 2025. `https://indico.cern.ch/event/1504506/`
- *Impact of fringe fields on beam dynamics*, NIMMS Collaboration Meeting #105, 9th of May 2025. `https://indico.cern.ch/event/1544534/`
- *NIMMS studies*, ABP-CAP Section Meeting, 4th of July 2025. `https://indico.cern.ch/event/1533185/`

Additional content

Magnetic field expansion

The magnetic scalar potential $\phi(x, y, s)$ as general expansion in y

$$\phi(x, y, s) = \sum_{i=0}^{\infty} \phi_i(x, s) \frac{y^i}{i!}$$

$$\Big\downarrow \text{ Laplace equation } \nabla^2 \phi = 0$$

$$\phi_{i+2} = -\frac{1}{1+hx} \left(\partial_x \left((1+hx) \partial_x \phi_i \right) + \partial_s \left(\frac{1}{1+hx} \partial_s \phi_i \right) \right)$$

S. D. Fartoukh, 1997; C. Wang and L. Teng, 2001

Magnetic field expansion

Expansion of the initial functions $\phi_0(x, s)$ and $\phi_1(x, s)$

- Two initial functions $\phi_0(x, s)$ and $\phi_1(x, s)$ can be independently chosen

$$\phi_0(x, s) = -a_0(s) - \sum_{n=1}^{\infty} a_n(s) \frac{x^n}{n!}$$

$$\phi_1(x, s) = -\sum_{n=1}^{\infty} b_n(s) \frac{x^{n-1}}{(n-1)!}$$

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$$\phi_1(x,s) = -\sum_{n=1}^{\infty} b_n(s)\frac{x^{n-1}}{(n-1)!}$$

- Apply the formula to determine ϕ_2

$$\phi_2(x,s) = -\frac{1}{1+hx}\left(\partial_x\left((1+hx)\partial_x\phi_0\right) + \partial_s\left(\frac{1}{1+hx}\partial_s\phi_0\right)\right)$$

$$= \tfrac{h}{1+hx}\textstyle\sum_{n=1}^{\infty} a_n \tfrac{x^{n-1}}{(n-1)!} + \sum_{n=1}^{\infty} a_n \tfrac{x^{n-2}}{(n-2)!} - \tfrac{h'x}{(1+hx)^3}\sum_{n=0}^{\infty} a_n'(s)\tfrac{x^n}{n!} + \tfrac{1}{(1+hx)^2}\sum_{n=0}^{\infty} a_n''(s)\tfrac{x^n}{n!}$$

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- Successively determine next ϕ_i and calculate ϕ

S. D. Fartoukh, 1997; C. Wang and L. Teng, 2001

Magnetic field expansion

Relation with the magnetic field $\vec{B} = -\vec{\nabla}\phi$

$$B_x(x, y=0, s) = -\partial_x \phi_0(x,s) = \sum_{n=1}^{\infty} a_n(s) \frac{x^{n-1}}{(n-1)!}$$

$$B_y(x, y=0, s) = -\phi_1(x,s) = \sum_{n=1}^{\infty} b_n(s) \frac{x^{n-1}}{(n-1)!}$$

$$B_s(x=0, y=0, s) = -\frac{1}{1+hx} \partial_s \phi_0(x,s)\Big|_{x=0} = b_s(s)$$

$$\boxed{\begin{aligned} a_n(s) &= \partial_x^{n-1} B_x(x,y,s)\big|_{x=y=0} \\ b_n(s) &= \partial_x^{n-1} B_y(x,y,s)\big|_{x=y=0} \end{aligned}}$$

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Magnetic field expansion
First terms

	1	x	y	x^2	xy	y^2
B_x	a_1	a_2	b_2	$\frac{a_3}{2}$	b_3	$\frac{-a_3-h(a_2-a_1h-2b_s')+b_sh'-a_1''}{2}$
B_y	b_1	b_2	$-b_s'-a_1h-a_2$	$\frac{b_3}{2}$	$-a_3-h(a_2-a_1h-2b_s')+b_sh'-a_1''$	$-\frac{b_3+hb_2+b_1''}{2}$
B_s	b_s	$-b_sh+a_1'$	b_1'	$b_sh^2-a_1'h+\frac{a_2'}{2}$	$-hb_1'+b_2'$	$-\frac{a_1h'+ha_1'+b_s''+a_2'}{2}$

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Magnetic field expansion

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B_y	b_1	b_2	$-b_s'-a_1h-a_2$	$\frac{b_3}{2}$	$-a_3-h(a_2-a_1h-2b_s')+b_sh'-a_1''$	$-\frac{b_3+hb_2+b_1''}{2}$
B_s	b_s	$-b_sh+a_1'$	b_1'	$b_sh^2-a_1'h+\frac{a_2'}{2}$	$-hb_1'+b_2'$	$-\frac{a_1h'+ha_1'+b_s''+a_2'}{2}$

Limit to a straight reference frame with s-independent fields

$$h = 0,\ \partial_s = 0$$

	1	x	y	x^2	xy	y^2
B_x	a_1	a_2	b_2	$\frac{a_3}{2}$	b_3	$-\frac{a_3}{2}$
B_y	b_1	b_2	$-a_2$	$\frac{b_3}{2}$	$-a_3$	$-\frac{b_3}{2}$
B_s	b_s	0	0	0	0	0

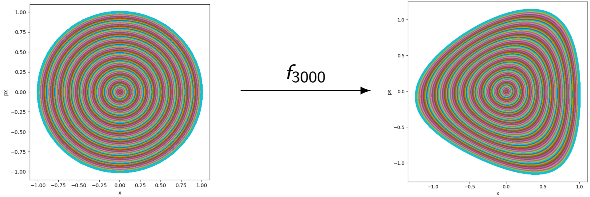
The coefficients a_n, b_n fall back to the usual multipole expansion for straight magnets with transverse-only fields

S. D. Fartoukh, 1997; C. Wang and L. Teng, 2001

Perturbation analysis: what are resonance driving terms?

Resonance driving terms f_{pqrt}

- Order $n = p + q + r + t$, typically comes from a 2n-pole
- p, q are related to the horizontal plane
- r, t are related to the vertical plane
- For example: f_{3000} originates from a sextupole, and describes a deformation of phase space in the horizontal plane



Resonance driving terms with dipole fringe fields

In a straight reference frame orthogonal to the edge

- $f_{1000}(s)$, $f_{0100}(s)$ $\Longrightarrow$ **Closed orbit** distortion
- f_{10kl} and f_{01kl} with $k + l = 2n$
  - $n = 1$: f_{1020}, f_{1011}, f_{1002}, f_{0120}, f_{0111}, f_{0102} $\Longrightarrow$ "Sextupole-like" effect: **chromaticity**
  - $n = 2$: f_{1040}, f_{1031}, f_{1022}, f_{1013}, f_{1004}, f_{0140}, f_{0131}, f_{0122}, f_{0113}, f_{0104}
  - ...
- f_{00kl} with $k + l = 2(m + n)$
  - $m, n = 1$: f_{0040}, f_{0031}, f_{0013}, f_{0004}, $\Longrightarrow$ "Octupole-like": **second-order chromaticity**
    h_{0022} $\Longrightarrow$ **detuning with amplitude**
  - ...
- Edge angle and curvature will contribute to additional RDTs

Resonance driving terms with dipole fringe fields

In a straight reference frame orthogonal to the edge

Not all sextupole RDTs are present: f_{3000}, f_{0300} important for resonant extraction are absent

- $f_{1000}(s)$, $f_{0100}(s)$ $\Longrightarrow$ **Closed orbit** distortion
- f_{10kl} and f_{01kl} with $k+l=2n$
  - $n=1$: f_{1020}, f_{1011}, f_{1002}, f_{0120}, f_{0111}, f_{0102} $\Longrightarrow$ "Sextupole-like" effect: **chromaticity**
  - $n=2$: f_{1040}, f_{1031}, f_{1022}, f_{1013}, f_{1004}, f_{0140}, f_{0131}, f_{0122}, f_{0113}, f_{0104}
  - ...
- f_{00kl} with $k+l=2(m+n)$
  - $m,n=1$: f_{0040}, f_{0031}, f_{0013}, f_{0004}, $\Longrightarrow$ "Octupole-like": **second-order chromaticity**
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New
First time using RDTs with fringe fields

Resonance driving terms with dipole fringe fields

Applied to the ELENA machine

- For example: closed orbit distortion from dipole component

$$\delta x(s) = -\frac{\sqrt{\beta_x(s)}}{2} 2i(f_{0100}(s) - f_{1000}(s))$$

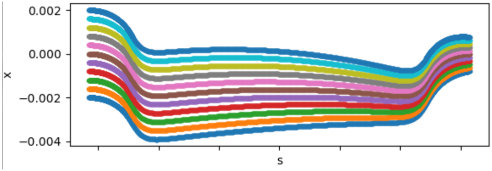
- Applied to fringe fields:

$$f_{1000}(s) = \int_s^{s+L} ds' \, b_1(s') \frac{\sqrt{\beta_x(s')}}{2} \frac{e^{i\Delta\mu_x(s,s')}}{1 - e^{2\pi i Q_x}}$$

$$f_{0100}(s) = \int_s^{s+L} ds' \, b_1(s') \frac{\sqrt{\beta_x(s')}}{2} \frac{e^{-i\Delta\mu_x(s,s')}}{1 - e^{-2\pi i Q_x}}$$

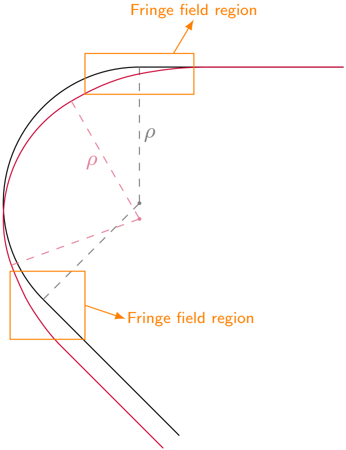
Closed orbit distortion: some intuition

- Dipole field changes gradually instead of a hard edge
- Does not cancel between entrance and exit fringe
- Not included in Forest fringe field map (PTC, MAD-NG, XSuite)



Closed orbit effect in ELENA dipole with rescaled field such that $x = 0, p_x = 0 \rightarrow x = 0, p_x = 0$

(neglecting quadrupole component from edge angles)



Fringe field map for the dipole in PTC, XSuite, MAD-NG - Etienne Forest

- Derive the kick caused by an inverse drift - fringe - inverse bend on the particle
  - Lowest order $\Delta p_x = 0$
  - First order contribution Δp_y

$$\Delta p_{y,1} = -\frac{x'}{1+y'^2} b_0 y$$

  - Second order contribution Δp_y

$$\Delta p_{y,2} = \underbrace{\int_{-\infty}^{+\infty} b(s)(b_0 - b(s))ds}_{gb_0^2 K} \left(\frac{(1+\delta)^2 - p_y^2}{p_s^3} + \frac{p_x^2}{p_s^2} \frac{(1+\delta)^2 - p_x^2}{p_s^3} \right) y$$

- Construct a generating function that results in this kick

$$F = p_x x_f + p_y y_f - \delta \ell_f - \frac{1}{2} \psi(p_x, p_y, \delta) {y_f}^2$$

$$\psi = b_0 \tan \left[\arctan \left(\frac{x'}{1+y'^2} \right) - g b_0 K \left(1 + \frac{p_x^2}{p_s^2} \left(2 + \frac{p_y^2}{p_s^2} \right) \right) p_s \right]$$

- A map that is derived from a generating function is by definition symplectic

Fringe field map for the dipole in PTC, XSuite, MAD-NG - Etienne Forest

To be applied at the magnet edge

$$x_f = x + \frac{1}{2}\frac{\partial\psi}{\partial p_x}{y_f}^2 \qquad p_{x,f} = p_x$$

$$y_f = \frac{2y}{1+\sqrt{1-2\frac{\partial\psi}{\partial p_y}y}} \qquad p_{y,f} = p_y - \psi y_f - \frac{b_0^2}{9Kg(1+\delta)}\, y_f^3$$

$$\ell_f = \ell - \frac{1}{2}\frac{\partial\psi}{\partial\delta}{y_f}^2 \qquad \delta_f = \delta$$

with

$$\psi = b_0 \tan\left[\arctan\left(\frac{x'}{1+y'^2}\right) - gb_0K\left(1+\frac{p_x^2}{p_s^2}\left(2+\frac{p_y^2}{p_s^2}\right)\right)p_s\right]$$

First order effect: quadrupole $\Longrightarrow$ **beta-beating**

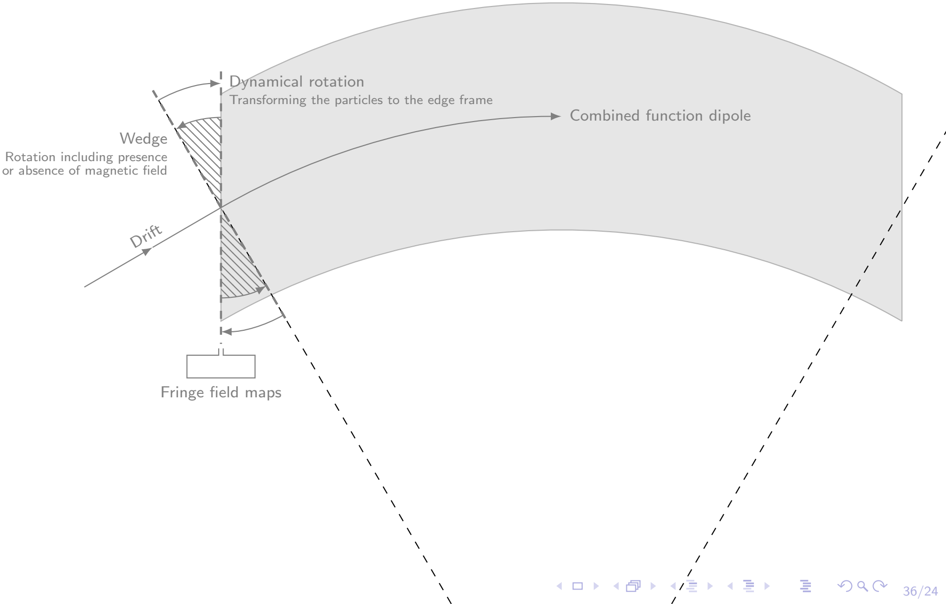
Third order effect: octupole-like $\Longrightarrow$ **second-order chromaticity**

Second order effect: sextupole-like $\Longrightarrow$ **chromaticity**

Dipole with edge angle: quadrupole $\Longrightarrow$ **beta-beating**

Forest, Leemann, Schmidt - 2006

Fringe field maps with edge angles and curvature: rotation - fringe - wedge



Fringe field map for multipoles in PTC, XSuite, MAD-NG

- Hard edge multipole fringes: generating function

$$F(\mathbf{x}, \mathbf{p^f}) = \mathbf{x} \cdot \mathbf{p^f} \mp \mathcal{R} \frac{c_{n,body} r^n e^{in\phi}}{4(n+1)D} \left(r p_r^f + i \frac{n+2}{n} r p_\phi^f \right)$$

  - The Lee-Whiting quadrupole fringe

$$p_x = p_x^f \pm \frac{b_2}{12(1+\delta)} (3(x^2+y^2)\, p_x^f - 6xy\, p_y^f)$$

$$p_y = p_y^f \pm \frac{b_2}{12(1+\delta)} (6xy\, p_x^f - 3(x^2+y^2)\, p_y^f)$$

$\Longrightarrow$ Octupolar effect
$\Longrightarrow$ Reduces to sextupolar effect if there is an edge angle

- Quadrupole SAD soft fringe

- Missing: beta-beating originating from the finite extent of the fringe field