Running of the Newton and cosmological constant in quantum gravity

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Talk based on:

C. Branchina, V. Branchina, **FC**, A. Pernace, PRD 111 (2025) 10, 105018

C. Branchina, V. Branchina, **FC**, A. Pernace, PRD 111 (2025) 12, 125021

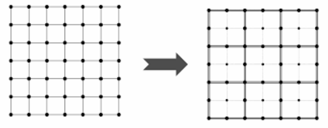
In part based on:

C. Branchina, V. Branchina, **FC**, R. Gandolfo, A. Pernace, PRD 112 (2025) 4, 045002

C. Branchina, V. Branchina, **FC**, R. Gandolfo, A. Pernace, arXiv:2505.07628

Wilson's Lesson

What is the Wilson's lesson all about?



Theory at Λ $\quad\rightarrow\quad$ Theory at $\Lambda/2$ $\quad\rightarrow\quad$ ...

$$S_\Lambda \quad\rightarrow\quad S_{\Lambda/2} \quad\rightarrow\quad \dots$$

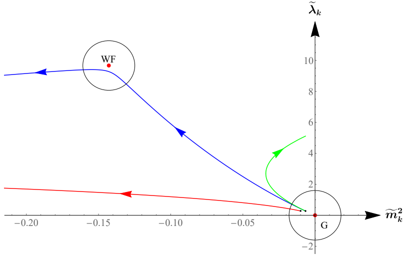
Effective Field Theory paradigm

Any QFT is an Effective Field Theory

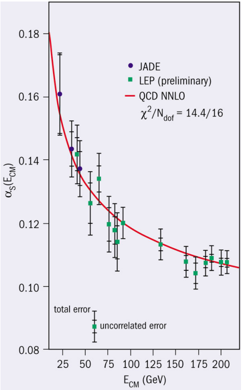
Steven Weinberg - Third Law of Progress in Theoretical Physics : you may use any degrees of freedom you like to describe a physical system, but if you use the wrong ones, you'll be sorry

... For theories in any dimesion: ..., $d = 3$, $d = 4$, ...

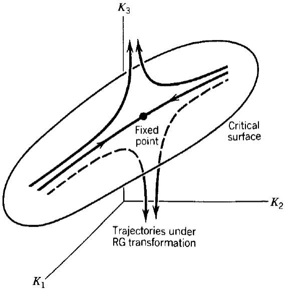
$d = 3$ dimensions : Wilson-Fisher



$d = 4$ dimensions : AF



RG flow



Renormalized theory: defined around a fixed point (critical surface)

Quantum Gravity

Perturbatively non-renormalizable

Is it **non-perturbatively renormalizable** ?

Asymptotic safety scenario: Existence of a UV-attractive fixed point

One-loop effective action - Summary

Greetings to Arcangelo

One-loop Effective Action

- Euclidean action - Einstein-Hilbert truncation with spherical background

$$S_{\rm grav} = \frac{1}{16\pi G}\int d^4x\,\sqrt{g}\,(-R + 2\Lambda_{\rm cc}) \quad \Rightarrow \quad \boxed{S^{(a)}_{\rm grav} = \frac{\pi\Lambda_{\rm cc}}{3G}a^4 - \frac{2\pi}{G}a^2}$$

- One-loop effective action $\Gamma^{1l}_{\rm grav} = S_{\rm grav} + \delta S^{1l}_{\rm grav}$

- Diffeomorphism invariant measure $\mathcal{D}\mu$: Fradkin-Vilkovisky

$$\mathcal{D}\mu \equiv \prod_x \left[g^{(a)\,00}(x)\,\left(g^{(a)}(x)\right)^{-1} \left(\prod_{\alpha\leq\beta} dh_{\alpha\beta}(x)\right)\left(\prod_\rho dv^*_\rho(x)\right)\left(\prod_\sigma dv_\sigma(x)\right)\right]$$

Related to spacetime discretization and Hamiltonian path integral

- Identify 1-loop corrections to $\frac{\Lambda_{\rm cc}}{G}$ and $\frac{1}{G}$ with coefficients of a^4 and a^2 in $\delta S^{1l}_{\rm grav}$

Taylor, Veneziano

One-loop correction to the action

$$\delta S^{1l}_{\rm grav} = -\frac{1}{2}\log\frac{\det_1[-\tilde{\Box}^{(1)} - 3]\,\det_2[-\tilde{\Box}^{(0)} - 6]}{\det_0[-\tilde{\Box}^{(2)} - 2a^2\Lambda + 8]\,\det_2[-\tilde{\Box}^{(0)} - 2a^2\Lambda]} + \dots$$

- Argument of the log is **dimensionless**
- Eigenvalues $\lambda_n^{(s)}$ of dimensionless Laplace-Beltrami operator and degeneracies $D_n^{(s)}$

$$\lambda_n^{(s)} = n^2 + 3n - s \qquad ; \qquad D_n^{(s)} = \frac{2s+1}{3}\left(n+\frac{3}{2}\right)^3 - \frac{(2s+1)^3}{12}\left(n+\frac{3}{2}\right)$$

- Regularizing with hard cut / proper time ($N \gg 1$)

$$\delta S^{1l}_{\rm grav} = -\left(\Lambda_{\rm cc}^2 \log\frac{3\Lambda_{\rm cut}^2}{\Lambda_{\rm cc}}\right) a^4 + \left(-3\Lambda_{\rm cut}^2 + 8\Lambda_{\rm cc}\log\frac{3\Lambda_{\rm cut}^2}{\Lambda_{\rm cc}}\right) a^2 + \dots$$

- $N \longrightarrow$ physical cutoff Λ through the De Sitter solution $a_{\rm dS} = \sqrt{\frac{3}{\Lambda_{\rm cc}}}$

$$\boxed{\Lambda_{\rm cut} \sim M_P = \frac{N}{a_{\rm dS}} = N\sqrt{\frac{\Lambda_{\rm cc}}{3}}}$$

One-loop corrections to $\frac{\Lambda_{cc}}{G}$ and $\frac{1}{G}$

Coefficients of a^4 and a^2 identify the one-loop corrections to $\frac{\Lambda_{cc}}{G}$ and $\frac{1}{G}$

$$\frac{\Lambda_{cc}^{1l}}{G^{1l}} = \frac{\Lambda_{cc}}{G}\left(1 - \frac{3G\Lambda_{cc}}{\pi}\log\frac{3\Lambda_{cut}^2}{\Lambda_{cc}}\right) + \text{finite}$$

$$\frac{1}{G^{1l}} = \frac{1}{G}\left[1 + \frac{G}{2\pi}\left(3\Lambda_{cut}^2 - 8\Lambda_{cc}\log\frac{3\Lambda_{cut}^2}{\Lambda_{cc}}\right)\right] + \text{finite}$$

Loop corrections $\rightarrow$ only mild (log) correction to ρ $\quad\Longrightarrow$

In pure gravity **no naturalness problem arises**:
the bare cosmological constant Λ_{cc} does not need to be $\sim M_P^2$.

We may **naturally** have $\Lambda_{cc} \ll M_P^2$, and so

$$\boxed{\Lambda_{cc}^{1l} \sim \Lambda_{cc}}$$

Renormalization group equations

Two methods to obtain RG equations for Λ_{cc} and G:

- Hard cut on sum over eigenvalues
- Proper time

Sum over eigenvalues

UV action

$$S^{\mathrm{UV}}_{\mathrm{grav}}[g_{\mu\nu}] \equiv S_N[g_{\mu\nu}] = \frac{1}{16\pi G_N}\int \mathrm{d}^4x\,\sqrt{g}\,(-R+2\Lambda_N) \qquad (N \text{ integer})$$

As before: physical cutoff $\Lambda_{\mathrm{cut}}\,(\sim M_P) = N/a_{\mathrm{dS}}$

Wilsonian action: $S_L[g^{(a)}_{\mu\nu}]$ $\qquad (L \text{ integer},\ L < N\ ;\ \delta L \ll L)$

RG equation:
$$S_{L-\delta L}[g^{(a)}_{\mu\nu}] = S_L[g^{(a)}_{\mu\nu}] + \delta S_L \equiv S_L[g^{(a)}_{\mu\nu}] + \sum_{n=L-\delta L}^{L} f_L(n)$$

where

$$\boxed{\delta S_L = -\frac{1}{2}\log\frac{\det_1[-\widetilde{\Box}^{(1)}-3]\,\det_2[-\widetilde{\Box}^{(0)}-6]}{\det_0[-\widetilde{\Box}^{(2)}-2a^2\Lambda_L+8]\,\det_2[-\widetilde{\Box}^{(0)}-2a^2\Lambda_L]}}$$

$$\Updownarrow$$

$$f_L = D^{(2)}_n\log\left(\lambda^{(2)}_n-2a^2\Lambda_L+8\right)+D^{(0)}_n\log\left(\lambda^{(0)}_n-2a^2\Lambda_L\right)-D^{(1)}_n\log\left(\lambda^{(1)}_n-3\right)-D^{(0)}_n\log\left(\lambda^{(0)}_n-6\right)$$

Differential form: $\frac{\partial}{\partial L}S_L = -\left(\frac{\partial}{\partial L}\sum_{n=2}^{L-2} f_L(n)\right)_{\Lambda_L,\,G_L}$ $\qquad$ (note $L_{\mathrm{min}} = 4$)

Proper time

- Proper time method

$$\det_i(-\widetilde{\Box}^{(s)} - \alpha) = e^{-\int_{1/L^2}^{1/(L-\delta L)^2} \frac{d\tau}{\tau} \mathrm{K}_i^{(s)}(\tau)} \quad ; \quad \mathrm{K_i}^{(s)}(\tau) = \sum_{n=s+i}^{+\infty} D_n^{(s)}\, e^{-\tau\left(\lambda_n^{(s)} - \alpha\right)}$$

Differential RG equation:

$$\frac{\partial}{\partial L} S_L[g_{\mu\nu}^{(a)}] = -\left(\frac{\partial}{\partial L}\delta S_L^{1l}\right)_{\Lambda_L, G_L}$$

Leads to **same result** obtained with **direct sum**

- Computing the r.h.s. (**direct sum** or **heat kernel expansion**) ($L \gg 1$):

$$L\frac{\partial}{\partial L} S_L = 2\Lambda_L^2\, a^4 + 2\Lambda_L\left(L^2 - 8\right) a^2 + \ldots$$

From which we obtain the RG equations

$$L\frac{\mathrm{d}}{\mathrm{d}L}\left(\frac{\Lambda_L}{G_L}\right) = \frac{6}{\pi}\Lambda_L^2 \qquad\qquad L\frac{\mathrm{d}\Lambda_L}{\mathrm{d}L} = \frac{G_L\Lambda_L^2}{\pi}\left(L^2 - 2\right)$$

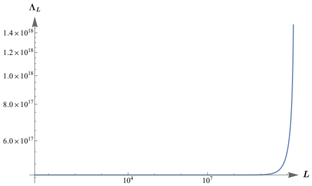
$$L\frac{\mathrm{d}}{\mathrm{d}L}\left(\frac{1}{G_L}\right) = -\frac{\Lambda_L}{\pi}\left(L^2 - 8\right) \quad \Longrightarrow \quad L\frac{\mathrm{d}G_L}{\mathrm{d}L} = \frac{G_L^2\Lambda_L}{\pi}\left(L^2 - 8\right)$$

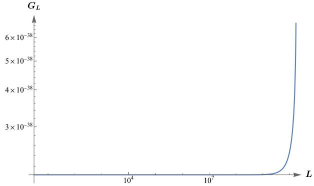
RG equations - Solutions

In the $L^2 \gg 1$ approximation we have $(\Lambda_{cc} \equiv \Lambda_N,\ G \equiv G_N)$

$$\Lambda_L = \frac{\Lambda_{cc}}{\sqrt{1 + \frac{G\Lambda_{cc}}{\pi}(N^2 - L^2)}} \qquad \xleftrightarrow{\Lambda_{cut} = N\sqrt{\frac{\Lambda_{cc}}{3}}} \qquad \Lambda_L = \frac{\Lambda_{cc}}{\sqrt{1 + \frac{G}{\pi}(3\Lambda_{cut}^2 - \Lambda_{cc}L^2)}}$$

$$G_L = \frac{G}{\sqrt{1 + \frac{G\Lambda_{cc}}{\pi}(N^2 - L^2)}} \qquad\qquad G_L = \frac{G}{\sqrt{1 + \frac{G}{\pi}(3\Lambda_{cut}^2 - \Lambda_{cc}L^2)}}$$





- Sign of Λ_L and G_L fixed
- Λ_L monotonically increasing with L

From $L \to$ physical running scale k

$$L\frac{\mathrm{d}\Lambda_L}{\mathrm{d}L} = \frac{G_L\Lambda_L^2}{\pi}\left(L^2-2\right)$$
$$L\frac{\mathrm{d}G_L}{\mathrm{d}L} = \frac{G_L^2\Lambda_L}{\pi}\left(L^2-8\right)$$

Physical running scale k

$$\boxed{k \equiv \frac{L}{\bar{a}_L} = L\sqrt{\frac{\Lambda_L}{3}}} \qquad \left(k_{\mathrm{IR}} \le k \le \Lambda_{\mathrm{cut}}\ ;\ k_{\mathrm{IR}} = \left(\tfrac{16}{3}\,\Lambda_4\right)^{1/2}\right)$$

The RG equations become $(\Lambda_k \equiv \Lambda_L,\ G_k \equiv G_L)$

$$k\frac{\mathrm{d}\Lambda_k}{\mathrm{d}k} = \frac{3G_k}{\pi}\,\frac{\Lambda_k\left(k^2-\frac{2}{3}\Lambda_k\right)}{1+\frac{3G_k}{2\pi}\left(k^2-\frac{2}{3}\Lambda_k\right)}$$
$$k\frac{\mathrm{d}G_k}{\mathrm{d}k} = \frac{3G_k^2}{\pi}\,\frac{k^2-\frac{8}{3}\Lambda_k}{1+\frac{3G_k}{2\pi}\left(k^2-\frac{2}{3}\Lambda_k\right)}$$

Solutions

$$k\frac{d\Lambda_k}{dk} = \frac{3G_k}{\pi}\frac{\Lambda_k\left(k^2-\frac{2}{3}\Lambda_k\right)}{1+\frac{3G_k}{2\pi}\left(k^2-\frac{2}{3}\Lambda_k\right)}$$

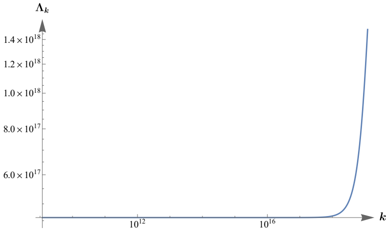
$$k\frac{dG_k}{dk} = \frac{3G_k^2}{\pi}\frac{k^2-\frac{8}{3}\Lambda_k}{1+\frac{3G_k}{2\pi}\left(k^2-\frac{2}{3}\Lambda_k\right)}$$

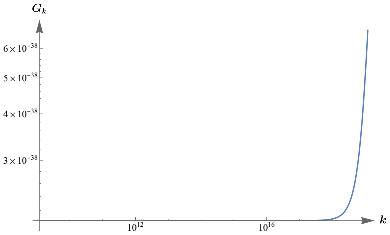
$$\xrightarrow{\frac{\Lambda_k}{k^2}=\frac{3}{L^2}\ll 1}$$

$$k\frac{d\Lambda_k}{dk} = \frac{3G_k}{\pi}\frac{\Lambda_k k^2}{1+\frac{3G_k}{2\pi}k^2}$$

$$k\frac{dG_k}{dk} = \frac{3G_k^2}{\pi}\frac{k^2}{1+\frac{3G_k}{2\pi}k^2}$$

$$\Lambda_k = \frac{k^2\Lambda_{cc}}{2\left(\Lambda_{cut}^2+\frac{\pi}{3G}\right)}\left[1+\sqrt{1+\frac{4\pi}{3G}\left(\Lambda_{cut}^2+\frac{\pi}{3G}\right)\frac{1}{k^4}}\right] \qquad G_k = \frac{k^2 G}{2\left(\Lambda_{cut}^2+\frac{\pi}{3G}\right)}\left[1+\sqrt{1+\frac{4\pi}{3G}\left(\Lambda_{cut}^2+\frac{\pi}{3G}\right)\frac{1}{k^4}}\right]$$





Naturalness

Taking for G the natural value $G \sim M_P^{-2}$

For $\sim$ any value of $\Lambda_{\rm cc}$

$$G_{\rm IR} \sim G \sim M_P^{-2}$$
$$\Lambda_{\rm IR} \sim \Lambda_{\rm cc}$$

As for the one-loop result

In pure gravity **no naturalness problem arises**:
the bare cosmological constant $\Lambda_{\rm cc}$ does not need to be $\sim M_P^2$.

We may **naturally** have $\Lambda_{\rm cc} \ll M_P^2$, and so

$$\boxed{\boldsymbol{\Lambda_{\rm cc}^{1l} \sim \Lambda_{\rm cc}}}$$

Fixed points

Dimensionful → Dimensionless equations

$$k\frac{\mathrm{d}\Lambda_k}{\mathrm{d}k} = \frac{3G_k}{\pi}\frac{\Lambda_k\left(k^2-\frac{2}{3}\Lambda_k\right)}{1+\frac{3G_k}{2\pi}\left(k^2-\frac{2}{3}\Lambda_k\right)}$$

$$k\frac{\mathrm{d}G_k}{\mathrm{d}k} = \frac{3G_k^2}{\pi}\frac{k^2-\frac{8}{3}\Lambda_k}{1+\frac{3G_k}{2\pi}\left(k^2-\frac{2}{3}\Lambda_k\right)}$$

Now: $t = \ln\frac{k}{k_0}$; $\lambda_t = \frac{\Lambda_k}{k^2}$; $g_t = k^2 G_k$

Dimensionless RG equations

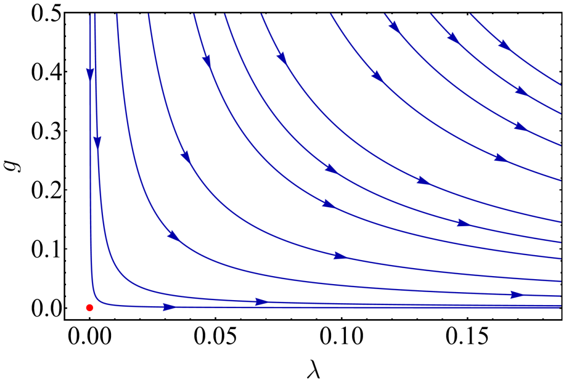
$$\frac{\mathrm{d}\lambda_t}{\mathrm{d}t} = -2\lambda_t + \frac{3g_t}{\pi}\frac{\lambda_t\left(1-\frac{2}{3}\lambda_t\right)}{1+\frac{3g_t}{2\pi}\left(1-\frac{2}{3}\lambda_t\right)} \equiv \beta_\lambda(\lambda, g)$$

$$\frac{\mathrm{d}g_t}{\mathrm{d}t} = 2g_t + \frac{3g_t^2}{\pi}\frac{1-\frac{8}{3}\lambda_t}{1+\frac{3g_t}{2\pi}\left(1-\frac{2}{3}\lambda_t\right)} \equiv \beta_g(\lambda, g)$$

Fixed points from $\beta_\lambda = 0$; $\beta_g = 0$

$$(\lambda, g)_1 = (0,0) \qquad (\lambda, g)_2 = (0, -\pi/3)$$

RG flows and Fixed points



$(\lambda, g)_1 = (0, 0)$ axes $\lambda = 0$, $g = 0$ UV-repulsive/attractive respectively

$(\lambda, g)_2 = (0, -\pi/3)$ unphysical UV-attractive fixed point

NO sign of any **physical UV-attractive** fixed point (**AS**)

What generates AS behaviour?

$$L\frac{\partial}{\partial L}S_L = 2\Lambda_L^2\, a^4 + 2\Lambda_L\left(L^2 - 8\right)a^2 + \frac{L^4}{3} - \frac{34L^2}{3} + \frac{1859}{45}$$

Identify the running scale k as $\boxed{k = L/a}$ (rather than $k = L/\bar{a}_L$)

$$k\frac{\mathrm{d}}{\mathrm{d}k}\Lambda_k = \frac{G_k}{\pi}\left[k^4 + 6\Lambda_k\left(k^2 + \Lambda_k\right) - \Lambda_k\frac{34k^2 + 48\,\Lambda_k}{6}\right] \quad ; \quad k\frac{\mathrm{d}}{\mathrm{d}k}G_k = -G_k^2\,\frac{34k^2 + 48\,\Lambda_k}{6\pi}$$

Dimless RG equations

$$\frac{\mathrm{d}\lambda_t}{\mathrm{d}t} = -2\lambda_t + \frac{g_t}{\pi}\left[1 + 6\lambda_t\left(1 + \lambda_t\right) - \lambda_t\frac{34 + 48\lambda_t}{6}\right] \equiv \beta_\lambda\left(\lambda_t, g_t\right)$$

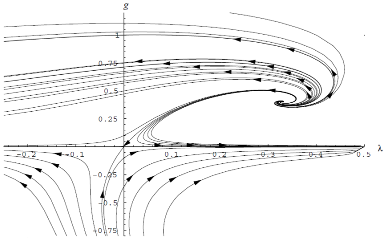
$$\frac{\mathrm{d}g_t}{\mathrm{d}t} = 2g_t - g_t^2\,\frac{34 + 48\lambda_t}{6\pi} \equiv \beta_g\left(\lambda_t, g_t\right)$$

Fixed points

$$(\lambda, g)_1 = (0, 0) \qquad (\lambda, g)_2 = \left(-\frac{8 + \sqrt{154}}{30}, \frac{2\pi}{23}\left(53 + 4\sqrt{154}\right)\right)$$

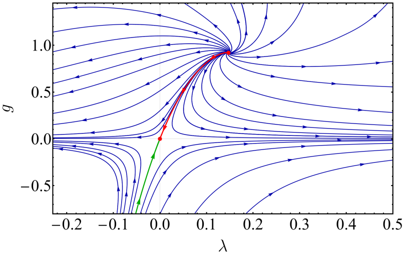
$$(\lambda, g)_3 = \left(\frac{\sqrt{154} - 8}{30}, \frac{2\pi}{23}\left(53 - 4\sqrt{154}\right)\right)$$

What generates the AS behaviour? - 2



M.Reuter, Phys.Rev.D 57 (1998) 971-985

M.Reuter and F.Saueressig, Phys.Rev.D 65 (2002) 065016



Our result with the identification $k = \frac{L}{a}$

$(\lambda, g)_3 = (0.147, 0.918)$ UV-attractive fixed point (**AS**)

Identification of k with a seems to be at the root of AS flow

... More on this point ...

Wetterich-Reuter equation

- Effective Average Action $\Gamma_k[g,\bar{g}]$

$$k\,\partial_k \Gamma_k[g,\bar{g}] = \frac{1}{2}\,\mathrm{Tr}\left[\left(\kappa^{-2}\Gamma_k^{(2)}[g,\bar{g}] + R_k^{\mathrm{grav}}[\bar{g}]\right)^{-1} k\,\partial_k R_k^{\mathrm{grav}}[\bar{g}]\right] - \mathrm{Tr}\left[\left(-\mathcal{M}[g,\bar{g}] + R_k^{\mathrm{gh}}[\bar{g}]\right)^{-1} k\,\partial_k R_k^{\mathrm{gh}}[\bar{g}]\right]$$

M. Reuter, C. Wetterich

$\mathcal{M}[g,\bar{g}]$ kinetic term of the ghosts

$$\mathcal{M}[g,\bar{g}]^{\mu}{}_{\nu} = \bar{g}^{\mu\rho}\bar{g}^{\sigma\lambda}\bar{D}_\lambda\left(g_{\rho\nu}D_\sigma + g_{\sigma\nu}D_\rho\right) - \bar{g}^{\rho\sigma}\bar{g}^{\mu\lambda}\bar{D}_\lambda g_{\sigma\nu}D_\rho$$

- Consider $\bar{g}_{\mu\nu} = g^{(a)}_{\mu\nu}$ spherical background of radius a

$R_k^{\mathrm{grav}}[\bar{g}]$ and $R_k^{\mathrm{gh}}[\bar{g}]$ have the form

$$R_k[\bar{g}] = \mathcal{Z}_k k^2 R^{(0)}(-\Box/k^2) \quad , \quad \Box \equiv \bar{g}^{\mu\nu}\bar{D}_\mu\bar{D}_\nu$$

Modes $p^2 > k^2$ integrated out ; modes $p^2 < k^2$ suppressed by R_k:

$\Longrightarrow$ "Tr" effectively restricted to eigenmodes of $-\Box$ with eigenvalues

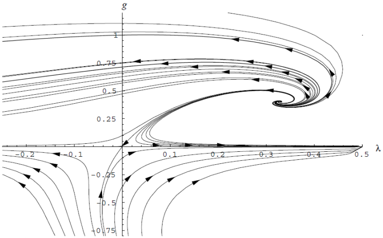
$$p^2 \sim \frac{n^2}{a^2} \sim k^2$$

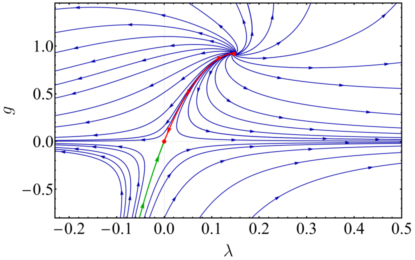
RG flows from Wetterich/Reuter equation

$$p^2 \sim \frac{n^2}{a^2} \sim k^2$$

Eigenvalues of $-\Box$ **(not** $-\widetilde{\Box}$**) used to introduce sliding scale** k $\quad \lambda_n \sim \frac{n^2}{a^2}$

$\Longrightarrow$ spurious k^4 terms $\Longrightarrow$ "**AS flow**"





M.Reuter, Phys.Rev.D 57 (1998) 971-985

Our result with the identification $k = \frac{L}{a}$

M.Reuter and F.Saueressig, Phys.Rev.D 65 (2002) 065016

Proper time RG

With Fujikawa measure (not diffeomorphism invariant)

$$\delta S^{1l}_{\text{grav}} = -\frac{1}{2}\log\frac{\det_1[-\Box^{(1)}_a - 3/a^2]\,\det_2[-\Box^{(0)}_a - 6/a^2]}{\det_0[-\Box^{(2)}_a - 2\Lambda + 8/a^2]\,\det_2[-\Box^{(0)}_a - 2\Lambda]}$$

Wilsonian RG strategy implemented introducing an IR regulator k

$$\int_{1/\Lambda^2_{\text{cut}}}^{+\infty} \mathrm{d}s \quad\longrightarrow\quad \int_{1/\Lambda^2_{\text{cut}}}^{1/k^2} \mathrm{d}s$$

Equivalently introducing a smooth function $f_k(s)$ $\qquad (t \equiv \log\frac{k}{k_0})$

$$\partial_t\,\widehat{S}_k[g,\bar{g}] = -\frac{1}{2}\,\mathrm{Tr}\int_0^{+\infty}\frac{\mathrm{d}s}{s}\,\partial_t f_k(s)\left[e^{-s\,\widehat{S}^{(2)}_k} - 2\,e^{-s\,S^{(2)}_{\text{ghost}}}\right]$$

- $\widehat{S}^{(2)}_k$ contains the dimensionful operators $-\Box_a$: eigenvalues $\widehat{\lambda}_n \sim \frac{n^2}{a^2}$
- $\partial_t f_k(s)$ effectively selects eigenmodes with $\widehat{\lambda}_n \sim k^2$

  Again k identified through $k = L/a$

Conclusion: UV-attractive fixed point of **AS generated by $k = L/a$**

Conclusions

- Sliding cutoff scale imposed on the dimless Laplace Beltrami operator: cut on the number of modes
- Wilsonian RG equations: absence of quartic and quadratic contributions to ρ!
  $\Rightarrow$ no naturalness problem arises in pure gravity!
- When diffeomorphism invariant (FV) measure used and physical running scale properly identified

$\Longrightarrow$ **Only Gaussian fixed point**

no sign of the UV-attractive fixed point of the **AS** scenario

- AS fixed point generated if we impose the cutoff as a cut on $\sim n^2/a^2$ rather than n ... **spurious** powers of a

... Is Gravity non-perturbatively renormalizable? ...

BACKUP SLIDES

Diffeomorphism invariance of the measure

Gravitational action $S_g[g]$ in the presence of a scalar field with action $S_m[\phi, g]$ (sign. $(-,+,+,+)$)

$$S[\phi, g] = S_g[g] + S_m[\phi, g] = S_g[g] + \int d^4x\, \mathcal{L}_m(\phi(x), \partial_\mu\phi(x), g(x))$$

where $\mathcal{L}_m(\phi(x), \partial_\mu\phi(x), g(x))$ is the matter Lagrangian density

$$\mathcal{L}_m(\phi(x), \partial_\mu\phi(x), g(x)) = -\frac{1}{2}\sqrt{-g(x)}\left(g^{\mu\nu}(x)\partial_\mu\phi(x)\partial_\nu\phi(x) + m^2\phi^2(x)\right)$$

Effective gravitational action $\Gamma[g]$ given by

$$e^{i\Gamma[g]} = \int e^{iS[\phi(x),\, g(x)]} \prod_x \Big[M(g(x))\, d\phi(x)\Big]$$

$M(g(x))$ non-trivial measure term

FV measure obtained from the Liouville measure in phase space $\prod_x \Big[d\pi(x)\, d\phi(x)\Big]$

Diffeomorphism invariance of the measure - 2

Subtle point: $\prod_x \leftrightarrow$ discretization. Given a coordinate system x^μ:

Leutwyler / Fradkin, Vilkovisky

- Lattice $\mathcal{E}_1$: hypersurfaces $x^0 = \mathrm{cte} \cap$ curves $x^i = \mathrm{cte}$
- Transition amplitude $\langle \mathrm{in},\ t_1 | \mathrm{out},\ t_2 \rangle$ requires a time ordering product, that can be identified with x^0

$\prod_x$ indicates the product over all the points Q_i of the lattice $\mathcal{E}_1$

Σ reference frame with coordinates x ; $\hat{\Sigma}$ frame with coordinates $\hat{x}$. In $\hat{\Sigma}$, for the classical action

$$\hat{S}[\hat{\phi}, \hat{g}] = \hat{S}_{\mathrm{g}}[\hat{g}] + \hat{S}_{\mathrm{m}}[\hat{\phi}, \hat{g}] = \hat{S}_{\mathrm{g}}[\hat{g}] + \int \mathrm{d}^4 \hat{x}\, \mathcal{L}_{\mathrm{m}}(\hat{\phi}(\hat{x}), \hat{\partial}_\mu \hat{\phi}(\hat{x}), \hat{g}(\hat{x}))$$

for the effective action $\hat{\Gamma}[\hat{g}]$ we have

$$e^{i\hat{\Gamma}[\hat{g}]} = \int e^{i\,\hat{S}[\hat{\phi}, \hat{g}]} \prod_{\hat{x}} \left[M(\hat{g}(\hat{x}))\, \mathrm{d}\hat{\phi}(\hat{x}) \right]$$

$\prod_{\hat{x}}$ product over the points P_i of the lattice $\mathcal{E}_2$: hypersurfaces $\hat{x}^0 = \mathrm{cte} \cap$ curves $\hat{x}^i = \mathrm{cte}$. Time ordering parameter $\hat{x}^0$.

Diffeomorphism invariance of the measure - 3

Under a general coordinate transformation

- change of time ordering parameter
- change of lattice

$$\Longrightarrow \qquad \prod_x (\mathcal{E}_1) \to \prod_{\hat{x}} (\mathcal{E}_2) \text{ is } \textbf{NOT} \text{ a trivial reshuffling}$$

To realize the bridge $\hat{\Gamma}[\hat{g}]$ (in $\hat{\Sigma}$) $\to \Gamma[g]$ (in Σ):

1 - In $\hat{\Sigma}$, move from $\mathcal{E}_2 \;\to\; \mathcal{E}_1$ (from $\hat{x}^0 \to x^0$)

$$e^{i\hat{\Gamma}[\hat{g}]} = \int e^{i\,\hat{S}[\hat{\phi},\hat{g}]} \Big[\prod_{\hat{x}} M(\hat{g}(\hat{x}))\Big] \Big[\prod_{\hat{x}} \mathrm{d}\hat{\phi}(\hat{x})\Big] = \int e^{i\,\hat{S}[\hat{\phi},\hat{g}]} \Big[B \prod_{x} M(\hat{g}(\hat{x}(x)))\Big] \Big[A \prod_{x} \mathrm{d}\hat{\phi}(\hat{x}(x))\Big]$$

with $\hat{S}$ written as (change of integration variables from $\hat{x}$ to x through $\hat{x} = \hat{x}(x)$)

$$\begin{aligned}
\hat{S}[\hat{\phi},\hat{g}] &= \hat{S}_{\mathrm{g}}[\hat{g}] + \hat{S}_{\mathrm{m}}[\hat{\phi},\hat{g}] \\
&= \hat{S}_{\mathrm{g}}[\hat{g}] - \frac{1}{2}\int \mathrm{d}^4x\, J\sqrt{-\hat{g}(\hat{x}(x))}\left(\hat{g}^{\mu\nu}(\hat{x}(x))\frac{\partial x^\rho}{\partial \hat{x}^\mu}\frac{\partial x^\sigma}{\partial \hat{x}^\nu}\partial_\rho\hat{\phi}(\hat{x}(x))\partial_\sigma\hat{\phi}(\hat{x}(x)) + m^2\hat{\phi}^2(\hat{x}(x))\right) \\
&\equiv \hat{S}_{\mathrm{g}}[\hat{g}] + \int \mathrm{d}^4x\, \widetilde{\mathcal{L}}_{\mathrm{m}}(\hat{\phi}(\hat{x}(x)), \partial_\mu\hat{\phi}(\hat{x}(x)), \hat{g}(\hat{x}(x))) \qquad J \equiv \left|\det\frac{\partial \hat{x}}{\partial x}\right|
\end{aligned}$$

Diffeomorphism invariance of the measure - 4

2 - Move from $\hat{\phi}(\hat{x}(x))$ and $\hat{g}_{\mu\nu}(\hat{x}(x))$ to $\phi(x)$ and $g_{\mu\nu}(x)$

$$e^{i\hat{\Gamma}[\hat{g}]} = \int e^{i\,\hat{S}[\hat{\phi},\hat{g}]}\left[B\prod_x M(\hat{g}(\hat{x}(x)))\right]\left[A\prod_x \mathrm{d}\hat{\phi}(\hat{x}(x))\right]$$

$$= \int e^{i\,S[\phi,g]}\left[B\,C\prod_x M(g(x))\right]\left[A\,E\prod_x \mathrm{d}\phi(x)\right]$$

* Factor C arises when we express $M(\hat{g}(\hat{x}(x)))$ in terms of $g_{\mu\nu}(x)$

* E Jacobian of $\hat{\phi}(\hat{x}(x)) \to \phi(x)$

To have $\hat{\Gamma}[\hat{g}] = \Gamma[g]$, it is necessary that $\boldsymbol{A\,B\,C\,E = 1}$

Transformation factors

Factor E - ϕ scalar field, $\hat{\phi}(\hat{x}(x)) = \phi(x)$; E Jacobian of $\hat{\phi}(\hat{x}(x)) \to \phi(x) \quad \Longrightarrow \quad E = 1$.

Factor A - In phase space

$$e^{i\Gamma_{\mathcal{E}_1}[g]} = e^{iS_g[g]} \int e^{i\int \mathrm{d}^4x[\partial_0\phi(x)\pi(x) - \mathcal{H}(\pi(x),\, \phi(x))]} \left[\prod_x f(g(x))\, \mathrm{d}\pi(x)\, \mathrm{d}\phi(x) \right]$$

with $\mathcal{H}(\pi(x),\, \phi(x))$ the Hamiltonian density, $\pi(x)$ the momentum conjugate to $\phi(x)$

$$\pi(x) = \frac{\partial \mathcal{L}_\mathrm{m}(\phi(x), \partial_\mu\phi(x), g(x))}{\partial(\partial_0\phi(x))}\,,$$

and $\mathcal{L}_\mathrm{m}(\phi(x), \partial_\mu\phi(x), g(x))$ the Lagrangian density in Σ

$f(g(x)) = 1$ FV measure ; $f(g(x)) = (g^{00}(x))^{-1/2}$ Fujikawa measure.

$\prod_x \Longleftrightarrow$ lattice $\mathcal{E}_1$ and time ordering $x^0 \quad \Longrightarrow \quad \pi(x)$ correctly defined above

Transformation factors

In $\hat{\Sigma}$ (lattice $\mathcal{E}_2$)

$$e^{i\hat{\Gamma}_{\mathcal{E}_2}[\hat{g}]} = e^{i\hat{S}_g[\hat{g}]} \int e^{i\int \mathrm{d}^4\hat{x}\left[\hat{\partial}_0\hat{\phi}(\hat{x})\hat{\pi}(\hat{x}) - \mathcal{H}(\hat{\pi}(\hat{x}),\, \hat{\phi}(\hat{x}))\right]} \left[\prod_{\hat{x}} f(\hat{g}(\hat{x}))\; \mathrm{d}\hat{\pi}(\hat{x})\, \mathrm{d}\hat{\phi}(\hat{x})\right]$$

with $\mathcal{H}(\hat{\pi}(\hat{x}),\, \hat{\phi}(\hat{x}))$ the Hamiltonian density, $\hat{\pi}(\hat{x})$ the momentum conjugate to $\hat{\phi}(\hat{x})$

$$\hat{\pi}(\hat{x}) = \frac{\partial \mathcal{L}_{\mathrm{m}}(\hat{\phi}(\hat{x}), \hat{\partial}_\mu \hat{\phi}(\hat{x}), \hat{g}(\hat{x}))}{\partial(\hat{\partial}_0 \hat{\phi}(\hat{x}))}\,,$$

and $\mathcal{L}_{\mathrm{m}}(\hat{\phi}(\hat{x}), \hat{\partial}_\mu \hat{\phi}(\hat{x}), \hat{g}(\hat{x}))$ the Lagrangian density in $\hat{\Sigma}$

$\prod_{\hat{x}} \Longleftrightarrow$ lattice $\mathcal{E}_2$ and time ordering parameter $\hat{x}^0 \quad \Longrightarrow \quad \hat{\pi}(\hat{x})$ correctly given above

Using in $\hat{\Sigma}$ the lattice $\mathcal{E}_1$ (and x^0 as time ordering parameter), for the effective action we have

$$e^{i\hat{\Gamma}_{\mathcal{E}_1}[\hat{g}]}$$

$$= e^{i\hat{S}_g[\hat{g}]} \int e^{i\int \mathrm{d}^4x\left[\partial_0\hat{\phi}(\hat{x}(x))\widetilde{\pi}(\hat{x}(x)) - \widetilde{\mathcal{H}}(\widetilde{\pi}(\hat{x}(x)),\, \hat{\phi}(\hat{x}(x)))\right]} \left[F\prod_{x} f(\hat{g}(\hat{x}(x)))\right] \left[\prod_{x} \mathrm{d}\widetilde{\pi}(\hat{x}(x))\, \mathrm{d}\hat{\phi}(\hat{x}(x))\right]$$

$\widetilde{\mathcal{H}}(\widetilde{\pi}(\hat{x}(x)),\, \hat{\phi}(\hat{x}(x)))$ Hamiltonian density corresponding to $\widetilde{\mathcal{L}}_{\mathrm{m}}(\hat{\phi}(\hat{x}(x)), \partial_\mu \hat{\phi}(\hat{x}(x)), \hat{g}(\hat{x}(x)))$ and $\widetilde{\pi}(\hat{x}(x))$ momentum conjugate to $\hat{\phi}(\hat{x}(x))$

$$\widetilde{\pi}(\hat{x}(x)) = \frac{\partial \widetilde{\mathcal{L}}_{\mathrm{m}}(\hat{\phi}(\hat{x}(x)), \partial_\mu \hat{\phi}(\hat{x}(x)), \hat{g}(\hat{x}(x)))}{\partial(\partial_0 \hat{\phi}(\hat{x}(x)))}\,.$$

Transformation factors

* Integrating over $\hat{\pi}(\hat{x})$ for $\hat{\Gamma}_{\mathcal{E}_2}[\hat{g}]$ and writing $\prod_{\hat{x}} \mathrm{d}\hat{\phi}(\hat{x}) = A \prod_x \mathrm{d}\hat{\phi}(\hat{x}(x))$

$$e^{i\hat{\Gamma}_{\mathcal{E}_2}[\hat{g}]} = e^{i\hat{S}_{\mathrm{g}}[\hat{g}]} \left[\prod_{\hat{x}} f(\hat{g}(\hat{x})) \right] \int e^{i\hat{S}_{\mathrm{m}}[\hat{\phi},\,\hat{g}]} \left[\prod_{\hat{x}} W(\hat{g}(\hat{x})) \right] \left[A \prod_x \mathrm{d}\hat{\phi}(\hat{x}(x)) \right]$$

with $\quad W(\hat{g}(\hat{x})) \equiv (-\hat{g}(\hat{x}))^{1/4} (-\hat{g}^{00}(\hat{x}))^{1/2}$

* Integrating over $\widetilde{\pi}(\hat{x}(x))$ for $\hat{\Gamma}_{\mathcal{E}_1}[\hat{g}]$

$$e^{i\hat{\Gamma}_{\mathcal{E}_1}[\hat{g}]} = e^{i\hat{S}_{\mathrm{g}}[\hat{g}]} F \left[\prod_x f(\hat{g}(\hat{x}(x))) \right] \int e^{i\hat{S}_{\mathrm{m}}[\hat{\phi},\,\hat{g}]} \left[\prod_x Y(\hat{g}(\hat{x}(x))) \right] \prod_x \mathrm{d}\hat{\phi}(\hat{x}(x))$$

with $\quad Y(\hat{g}(\hat{x}(x))) \equiv J^{1/2} (-\hat{g}(\hat{x}(x)))^{1/4} \left[-\hat{g}^{\mu\nu}(\hat{x}(x))\, \hat{\partial}_\mu x^0\, \hat{\partial}_\nu x^0 \right]^{1/2}$

$\hat{\Gamma}_{\mathcal{E}_2}[\hat{g}]$ and $\hat{\Gamma}_{\mathcal{E}_1}[\hat{g}]$ same effective action in $\hat{\Sigma}$

$$\Longrightarrow \qquad A = \frac{\prod_x Y(\hat{g}(\hat{x}(x)))}{\prod_{\hat{x}} W(\hat{g}(\hat{x}))}$$

Transformation factors

$$A = \frac{\prod_x Y(\hat{g}(\hat{x}(x)))}{\prod_{\hat{x}} W(\hat{g}(\hat{x}))}$$

To write A in a convenient form, consider an infinitesimal coordinate transformation (only first order in ε)

$$\hat{x}^\mu = x^\mu + \varepsilon^\mu(x)\,,$$

and switch from $\prod_x$ to $\prod_{\hat{x}}$ in the numerator. Observing that ($\delta^{(4)}(0)$ from $\sum_x \to \int \mathrm{d}^4 x$)

$$\prod_x J^{1/2} = \exp\left(\frac{\delta^{(4)}(0)}{2}\int \mathrm{d}^4 x \log\left(1 + \partial_\mu \varepsilon^\mu\right)\right) = \exp\left(\frac{\delta^{(4)}(0)}{2}\int \mathrm{d}^4 x\, \partial_\mu \varepsilon^\mu\right) = 1$$

where we have used $\mathrm{d}^4\hat{x} = \mathrm{d}^4 x\, J \equiv \mathrm{d}^4 x \left|\det \frac{\partial \hat{x}}{\partial x}\right| = \mathrm{d}^4 x\,(1 + \partial_\mu \varepsilon^\mu(x))$, and observing also that

$$-\hat{g}^{\mu\nu}(\hat{x}(x))\,\hat{\partial}_\mu x^0\,\hat{\partial}_\nu x^0 = -\hat{g}^{00} + 2\,\hat{g}^{0\mu}\,\partial_\mu \varepsilon^0 = -\hat{g}^{00}\left(1 - 2\,\frac{\hat{g}^{0\mu}}{\hat{g}^{00}}\,\partial_\mu \varepsilon^0\right)$$

we have

$$\prod_x Y(\hat{g}(\hat{x}(x))) = \left[\prod_{\hat{x}} W(\hat{g}(\hat{x}))\right] \exp\left(-\frac{\delta^{(4)}(0)}{2}\int \mathrm{d}^4\hat{x}\left[\partial_\mu \varepsilon^\mu \log\left((-\hat{g}(\hat{x}))^{1/2}\left(-\hat{g}^{00}(\hat{x})\right)\right) + 2\,\frac{\hat{g}^{0\mu}(\hat{x})}{\hat{g}^{00}(\hat{x})}\,\partial_\mu \varepsilon^0\right]\right)$$

$$\Longrightarrow A = \exp\left(-\frac{\delta^{(4)}(0)}{2}\int \mathrm{d}^4 x\left[\partial_\mu \varepsilon^\mu \log\left((-g(x))^{1/2}\left(-g^{00}(x)\right)\right) + 2\,\frac{g^{0\mu}(x)}{g^{00}(x)}\,\partial_\mu \varepsilon^0\right]\right)$$

Transformation factors

Factor B - We begin by observing that $M(\hat{g}(\hat{x}))$ is nothing but the product $f(\hat{g}(\hat{x}))W(\hat{g}(\hat{x}))$. We then have

$$\prod_{\hat{x}} M(\hat{g}(\hat{x})) = \prod_{\hat{x}} \left[f(\hat{g}(\hat{x}))W(\hat{g}(\hat{x})) \right] = \exp\left(\delta^{(4)}(0) \int \mathrm{d}^4\hat{x} \log(f(\hat{g}(\hat{x}))W(\hat{g}(\hat{x}))) \right)$$

$$= \exp\left(\delta^{(4)}(0) \int \mathrm{d}^4x \, \partial_\mu \varepsilon^\mu \log(f(g(x))W(g(x))) \right) \prod_x \left[f(\hat{g}(\hat{x}(x)))W(\hat{g}(\hat{x}(x))) \right]$$

$$= \exp\left(\delta^{(4)}(0) \int \mathrm{d}^4x \, \partial_\mu \varepsilon^\mu \log(M(g(x))) \right) \prod_x M(\hat{g}(\hat{x}(x))) \, .$$

Recalling that $\prod_{\hat{x}} M(\hat{g}(\hat{x})) = B \prod_x M(\hat{g}(\hat{x}(x)))$

$$B = \exp\left(\delta^{(4)}(0) \int \mathrm{d}^4x \, \partial_\mu \varepsilon^\mu \log(M(g(x))) \right)$$

Transformation factors

Factor C - Defined by $\prod_x M(\hat{g}(\hat{x}(x))) = C \prod_x M(g(x))$. At first order in ε

$$M(\hat{g}^{\mu\nu}(\hat{x}(x))) = M\Big(g^{\mu\nu}(x) + g^{\mu\sigma}(x)\partial_\sigma\varepsilon^\nu(x) + g^{\nu\sigma}(x)\partial_\sigma\varepsilon^\mu(x)\Big) =$$

$$= M(g) + \frac{\partial M(g)}{\partial g^{\alpha\beta}(x)}\left(g^{\alpha\sigma}(x)\partial_\sigma\varepsilon^\beta(x) + g^{\beta\sigma}(x)\partial_\sigma\varepsilon^\alpha(x)\right)$$

where we used

$$\hat{g}^{\mu\nu}(\hat{x}) = \frac{\partial\hat{x}^\mu(x)}{\partial x^\rho}\frac{\partial\hat{x}^\nu(x)}{\partial x^\sigma}g^{\rho\sigma}(x) = g^{\mu\nu}(x) + g^{\mu\sigma}(x)\partial_\sigma\varepsilon^\nu(x) + g^{\nu\sigma}(x)\partial_\sigma\varepsilon^\mu(x)$$

We then get

$$\prod_x M(\hat{g}(\hat{x}(x))) = \exp\left(\delta^{(4)}(0)\int \mathrm{d}^4x \log\left[M(g) + \frac{\partial M(g)}{\partial g^{\alpha\beta}(x)}\left(g^{\alpha\sigma}(x)\partial_\sigma\varepsilon^\beta(x) + g^{\beta\sigma}(x)\partial_\sigma\varepsilon^\alpha(x)\right)\right]\right)$$

$$= \exp\left(\delta^{(4)}(0)\int \mathrm{d}^4x\,[M(g)]^{-1}\frac{\partial M(g)}{\partial g^{\alpha\beta}(x)}\left(g^{\alpha\sigma}(x)\partial_\sigma\varepsilon^\beta(x) + g^{\beta\sigma}(x)\partial_\sigma\varepsilon^\alpha(x)\right)\right)\prod_x M(g(x))$$

$$\Longrightarrow C = \exp\left(\delta^{(4)}(0)\int \mathrm{d}^4x\,[M(g)]^{-1}\frac{\partial M(g)}{\partial g^{\alpha\beta}(x)}\left(g^{\alpha\sigma}(x)\partial_\sigma\varepsilon^\beta(x) + g^{\beta\sigma}(x)\partial_\sigma\varepsilon^\alpha(x)\right)\right)$$

Fradkin - Vilkovisky versus Fujikawa measure

Note: B and C depend on the specific form of $M(g(x))$, A does not and $E = 1$. We have then to calculate B and C for $M_{\mathrm{FV}}(g(x))$ and $M_{\mathrm{Fuji}}(g(x))$, respectively.

Fradkin - Vilkovisky measure - Using $M(g(x)) = M_{\mathrm{FV}}(g(x)) = \left(-g^{00}(x)\right)^{1/2}(-g(x))^{1/4}$ we get

$$B_{\mathrm{FV}} = \exp\left(\frac{\delta^{(4)}(0)}{2}\int \mathrm{d}^4x\, \partial_\mu \varepsilon^\mu \log[(-g(x))^{1/2}(-g^{00}(x))]\right)$$

$$C_{\mathrm{FV}} = \exp\left(\delta^{(4)}(0)\int \mathrm{d}^4x\, \frac{g^{0\mu}(x)}{g^{00}(x)}\, \partial_\mu \varepsilon^0\right)$$

from which

$$B_{\mathrm{FV}}\, C_{\mathrm{FV}} = A^{-1}$$

which shows that for the FV measure

$$A\, B_{\mathrm{FV}}\, C_{\mathrm{FV}}\, E = 1$$

and therefore that in this case: $\hat{\Gamma}[\hat{g}] = \Gamma[g]$.

Fradkin-Vilkovisky versus Fujikawa measure

Fujikawa measure - Using $M(g(x)) = M_{\mathrm{Fuji}}(g(x)) = \mu\,(-g(x))^{1/4}$ (μ arbitrary mass scale) we get

$$
\begin{aligned}
B_{\mathrm{Fuji}} &= \exp\left(\delta^{(4)}(0)\int \mathrm{d}^4x\,\partial_\mu\varepsilon^\mu \log\left((-g(x))^{1/4}\mu\right)\right)\\
C_{\mathrm{Fuji}} &= 1
\end{aligned}
$$

so that

$$
A\,B_{\mathrm{Fuji}}\,C_{\mathrm{Fuji}}\,E = \exp\left(\delta^{(4)}(0)\int \mathrm{d}^4x\left[\partial_\mu\varepsilon^\mu \log\left(\frac{\mu}{(-g^{00})^{1/2}}\right) - \frac{g^{0\mu}(x)}{g^{00}(x)}\,\partial_\mu\varepsilon^0\right]\right)
$$

which means that in this case: $\hat{\Gamma}[\hat{g}] \neq \Gamma[g]$.

Contrary to what is claimed in JHEP 05 (2025), 164, the **effective action** calculated using the **Fujikawa measure *is* not diffeomorphism invariant**

Comparison with Becker-Reuter, PRD 102 (2020) 12 - PRD 104 (2021) 12

and Ferrero-Percacci, e-Print: 2404.12357

Consider the modified Einstein equation at one-loop   Becker, Reuter PRD 104 (2021) 12

$$\frac{3G}{\pi}\, a \frac{\mathrm{d}}{\mathrm{d}a}\Gamma^{1l}_{\mathrm{grav}}(a) = 4\Lambda a^4 - 12a^2 + \frac{3G}{\pi}\, a \frac{\mathrm{d}}{\mathrm{d}a}\delta S^{1l}_{\mathrm{grav}}(a) = 0 \qquad (g_{\mu\nu} = g^{(a)}_{\mu\nu})$$

* **BRST-invariant Fujikawa measure** (or measure not fully considered)

$$\left[\mathcal{D}u(h)\mathcal{D}v^*_\rho\, \mathcal{D}v_\sigma\right] \equiv \prod_x \Bigg[\underbrace{\left(g^{(a)}(x)\right)^{-2}}_{\propto\, a^{-16}} \Big(\prod_{\alpha\leq\beta} \mathrm{d}h_{\alpha\beta}(x)\Big) \Big(\prod_\rho \mathrm{d}v^*_\rho(x)\Big) \Big(\prod_\sigma \mathrm{d}v_\sigma(x)\Big)\Bigg]$$

$\Longrightarrow \quad \delta S^{1l}_{\mathrm{grav}}(a) \sim N^4 \log(a\mu) \quad \Longrightarrow \quad$ **solution** $\quad \bar{a}_N \sim N\sqrt{M_P^{-1}\Lambda_{\mathrm{cc}}^{-1}}$

* **Fradkin-Vilkovisky measure**

$$\left[\mathcal{D}u(h)\mathcal{D}v^*_\rho\, \mathcal{D}v_\sigma\right] \equiv \prod_x \Bigg[\underbrace{g^{(a)\,00}(x)\, \left(g^{(a)}(x)\right)^{-1}}_{\propto\, a^{-10}} \Big(\prod_{\alpha\leq\beta} \mathrm{d}h_{\alpha\beta}(x)\Big) \Big(\prod_\rho \mathrm{d}v^*_\rho(x)\Big) \Big(\prod_\sigma \mathrm{d}v_\sigma(x)\Big)\Bigg]$$

$\Longrightarrow \quad \delta S^{1l}_{\mathrm{grav}}(a)$ does **not** contain N^4 **log** $(a\mu) \quad \Longrightarrow \quad$ **NO** $\bar{a}_N$ **solution**

Similar considerations apply to the results found in Becker-Reuter, PRD 102 (2020) 12, and Ferrero-Percacci, JHEP 09 (2024) 074

Calculation with EML in proper time regularization

$(-\widetilde{\Box}^{(s)} - \alpha)$ dimensionless $\Longrightarrow$ determinants regularized in terms of a **dimensionless** proper-time τ (lower cut: **number** $N \gg 1$)

$$\det_i(-\widetilde{\Box}^{(s)} - \alpha) = e^{-\int_{1/N^2}^{+\infty} \frac{d\tau}{\tau} \mathrm{K}_i^{(s)}(\tau)} \quad ; \quad \mathrm{K}_i^{(s)}(\tau) = \sum_{n=s+i}^{+\infty} D_n^{(s)} \, e^{-\tau\left(\lambda_n^{(s)} - \alpha\right)}$$

After integration over τ, sum over n performed with EML sum formula

$$\sum_{n=n_i}^{n_f} f(n) = \int_{n_i}^{n_f} \mathrm{d}x \, f(x) + \frac{f(n_f) + f(n_i)}{2} + \sum_{k=1}^{p} \frac{B_{2k}}{(2k)!} \left(f^{(2k-1)}(n_f) - f^{(2k-1)}(n_i)\right) + R_{2p}$$

p is an integer, B_m are Bernoulli numbers, R_{2p} is the rest given by

$$R_{2p} = \sum_{k=p+1}^{\infty} \frac{B_{2k}}{(2k)!} \left(f^{(2k-1)}(n_f) - f^{(2k-1)}(n_i)\right) = \frac{(-1)^{2p+1}}{(2p)!} \int_{n_i}^{n_f} \mathrm{d}x \, f^{(2p)}(x) B_{2p}(x - [x])$$

$B_n(x)$ are the Bernoulli polynomials, $[x]$ the integer part of x, and $f^{(i)}$ the i-th derivative of f with respect to its argument

Expansion of $\widehat{h}_{\mu\nu}$, v^*_ρ and v_σ

We indicate with $h_n^{\mu\nu(i)}$ (transverse-traceless), $\xi_n^{\mu(i)}$ (transverse) and $\phi_n^{(i)}$ the pure spin-2, spin-1 and spin-0 eigenfunctions of the Laplace-Beltrami operator on the sphere of unitary radius that are normalized as

$$\delta^{ij}\delta_{nm} = \int \mathrm{d}^4x\,\sqrt{\widetilde{g}}\,h_n^{\mu\nu(i)}(x)h_{\mu\nu}^{m(j)}(x) = \int \mathrm{d}^4x\,\sqrt{\widetilde{g}}\,\xi_n^{\mu(i)}(x)\xi_\mu^{m(j)}(x) = \int \mathrm{d}^4x\,\sqrt{\widetilde{g}}\,\phi_n^{(i)}(x)\,\phi_m^{(j)}(x)\,,$$

corresponding to the eigenvalues $\lambda_n^{(2)}$, $\lambda_n^{(1)}$ and $\lambda_n^{(0)}$ respectively. The modes $\{h_n^{\mu\nu}, v_n^{\mu\nu}, w_n^{\mu\nu}, z_n^{\mu\nu}\}$, with

$$\begin{aligned}
v_n^{\mu\nu} &= \left[\frac{1}{2}\left(\lambda_n^{(1)} - 3\right)\right]^{-\frac{1}{2}} \nabla^{(\mu}\xi_n^{\nu)}\,, \quad n = 2, \ldots\,, \\
w_n^{\mu\nu} &= \left[\lambda_n^{(0)}\left(\frac{3}{4}\lambda_n^{(0)} - 3\right)\right]^{-\frac{1}{2}} \left(\nabla^\mu\nabla^\nu - \frac{1}{4}\widetilde{g}^{\mu\nu}\Box\right)\phi_n\,, \quad n = 2, \ldots\,, \\
z_n^{\mu\nu} &= \frac{1}{2}\widetilde{g}^{\mu\nu}\phi_n\,, \quad n = 0, 1, 2, \ldots\,,
\end{aligned}$$

of which we do not write explicitly the degeneracy indexes form the orthonormal basis for symmetric tensors.

Moreover, defining the longitudinal vector modes

$$l_n^\mu = \left(\lambda_n^{(0)}\right)^{-\frac{1}{2}} \nabla^\mu \phi_n\,, \quad n = 1, 2, \ldots\,,$$

the latter, together with the transverse modes ξ_n^μ, form the orthonormal basis for vectors.

Expand the graviton field $\widehat{h}^{\mu\nu}$ as

Taylor, Veneziano

$$\widehat{h}^{\mu\nu} = \sum_{n=2}^{\infty} a_n h_n^{\mu\nu} + \sum_{n=2}^{\infty} b_n v_n^{\mu\nu} + \sum_{n=2}^{\infty} c_n w_n^{\mu\nu} + \sum_{n=0}^{\infty} e_n z_n^{\mu\nu}$$

$$\widehat{h} \equiv \widetilde{g}_{\mu\nu} \widehat{h}^{\mu\nu} = 2 \sum_{n=0}^{\infty} e_n \phi_n\,,$$

and the ghost field v^μ as

$$v^\mu = \sum_{n=1}^{\infty} g_n\, \xi_n^\mu + \sum_{n=1}^{\infty} f_n\, l_n^\mu$$

so that we have

$$
\begin{aligned}
64\pi G\,(S_2+S_{\rm gf}) = & \sum_{n=2}^{\infty} a_n^2\left[\lambda_n^{(2)} - 2a^2\Lambda + 8\right] \\
& + \sum_{n=2}^{\infty} b_n^2\left[\xi^{-1}\left(\lambda_n^{(1)} - 3\right) - 2a^2\Lambda + 6\right] \\
& + \sum_{n=2}^{\infty} c_n^2\left[\xi^{-1}\left(\frac{3}{4}\lambda_n^{(0)} - 6\right) - \frac{\lambda_n^{(0)}}{2} - 2a^2\Lambda + 6\right] \\
& + \sum_{n=0}^{\infty} e_n^2\left[\frac{-3+\xi^{-1}}{2}\lambda_n^{(0)} + 2a^2\Lambda\right] \\
& + \sum_{n=2}^{\infty} 2e_n c_n(\xi^{-1} - 1)\left[\lambda_n^{(0)}\left(\frac{3}{4}\lambda_n^{(0)} - 3\right)\right]^{\frac{1}{2}}
\end{aligned}
$$

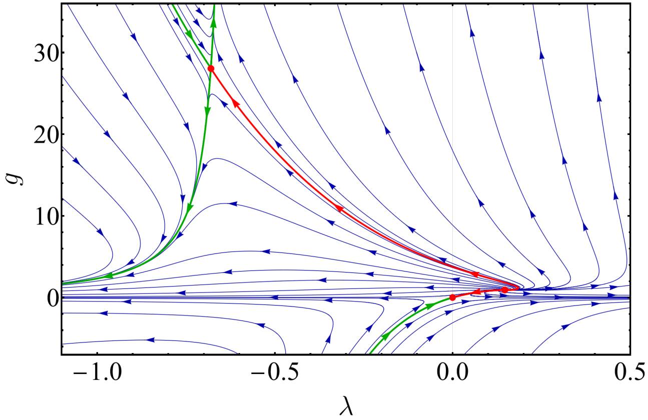
$$
32\pi G\, S_{\rm ghost} = \sum_{n=1}^{\infty} g_n^*\, g_n\left(\lambda_n^{(1)} - 3\right) + \sum_{n=1}^{\infty} f_n^*\, f_n\left(\lambda_n^{(0)} - 6\right) \; .
$$

Therefore, the functional measure can be written as (defined as)

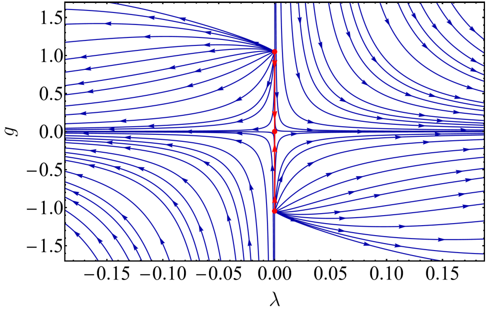
$$\mathcal{D}\widehat{h}_{\mu\nu}\,\mathcal{D}v^*_\rho\,\mathcal{D}v_\sigma \equiv \frac{1}{V_{SO(5)}}\prod_{n=2}^{\infty}\mathrm{d}a_n\prod_{n=2}^{\infty}\mathrm{d}b_n\prod_{n=2}^{\infty}\mathrm{d}c_n\prod_{n=0}^{\infty}\mathrm{d}e_n\prod_{n=2}^{\infty}\mathrm{d}g^*_n\prod_{n=2}^{\infty}\mathrm{d}g_n\prod_{n=1}^{\infty}\mathrm{d}f^*_n\prod_{n=1}^{\infty}\mathrm{d}f_n\,,$$

Notice that there is no integration over the zero modes g^*_1 and g_1 of S_{ghost}. The corresponding ghost fields are proportional to the ten Killing vectors ξ^μ_1. These zero eigenmodes correspond to residual gauge invariances which are not eliminated by gauge fixing in the presence of an $SO(5)$ spherical symmetry. Overcounting has been compensated by inserting the explicit group-volume factor $V_{SO(5)}$

$K = L/a$ full



Flow - all quadrants



$\lambda > 0$

$$\frac{d\lambda}{dt} = -2\lambda + \frac{2g\lambda\,(3-2\lambda)}{2\pi + g\,(3-2\lambda)}$$

$$\frac{dg}{dt} = 2g + \frac{2g^2\,(3-8\lambda)}{2\pi + g\,(3-2\lambda)}$$

$\lambda < 0$

$$\frac{d\lambda}{dt} = -2\lambda - \frac{2g\lambda\,(3+2\lambda)}{2\pi - g\,(3+2\lambda)}$$

$$\frac{dg}{dt} = 2g - \frac{2g^2\,(3+8\lambda)}{2\pi - g\,(3+2\lambda)}$$

Fixed points

$(\lambda, g)_1 = (0, -\pi/3)$ $\quad (\lambda, g)_2 = (0, 0)$ $\quad (\lambda, g)_3 = (0, \pi/3)$

RG: The case $\Lambda = 0$

$$L\frac{\partial}{\partial L}S_L = 2\Lambda_L^2\, a^4 + 2\Lambda_L \left(L^2 - 8\right) a^2 + \frac{L^4}{3} - \frac{34L^2}{3} + \frac{1859}{45}$$

Using our equations with the identification $k = L/a$

$$\rightarrow k\frac{\partial}{\partial k}S_k = \left[\frac{k^4}{3} + 2\Lambda_k\left(k^2 + \Lambda_k\right)\right] a^4 - \left(\frac{34k^2}{3} + 16\Lambda_k\right) a^2 + \frac{1859}{45}$$

$$\frac{\mathrm{d}g_t}{\mathrm{d}t} = 2g_t - g_t^2\frac{34}{6\pi} \qquad \longrightarrow \qquad g^* = \frac{12\pi}{34} \qquad \text{UV-attractive fixed point}$$

Match results in the literature

Souma

Generation of g^*: $2\Lambda_L L^2 a^2 \rightarrow 2\Lambda_k k^2 a^4$, $-34/3L^2 \rightarrow -34/3k^2 a^2$

Change sign of a^2 term!

$$\delta S_L = -\frac{1}{2}\log\frac{\det_1[-\widetilde{\Box}^{(1)} - 3]\det_2[-\widetilde{\Box}^{(0)} - 6]}{\det_0[-\widetilde{\Box}^{(2)} - 2a^2\Lambda_L + 8]\det_2[-\widetilde{\Box}^{(0)} - 2a^2\Lambda_L]}$$

When $\Lambda_L = 0 \rightarrow$ no powers of a in fluctuation determinants

Renormalization of G_L must be $\propto \Lambda_L$: g^* cannot exist

The case $\Lambda = 0$ - 2

$$
\begin{aligned}
\partial_t g_k = \beta_g &= [d - 2 + \eta(k)] g_k, \\
\partial_t \lambda_k = \beta_\lambda &= -(2-\eta)\lambda_k + \frac{1}{2} g_k (4\pi)^{1-d/2} \Big\{ 2d(d+1)\Phi^1_{d/2}(-2\lambda_k) \\
&\quad - d(d+1)\eta(k)\widetilde{\Phi}^1_{d/2}(-2\lambda_k) - 8d\Phi^1_{d/2}(0) \Big\} .
\end{aligned}
$$

Souma

$$
\begin{aligned}
\frac{\mathrm{d}\lambda_t}{\mathrm{d}t} &= -2\lambda_t + \frac{g_t}{\pi}\left[1 + 6\lambda_t(1+\lambda_t) - \lambda_t \frac{34 + 48\lambda_t}{6}\right] \equiv \beta_\lambda(\lambda_t, g_t) \\
\frac{\mathrm{d}g_t}{\mathrm{d}t} &= 2g_t - g_t^2 \frac{34 + 48\lambda_t}{6\pi} \equiv \beta_g(\lambda_t, g_t)
\end{aligned}
$$

Our equations with the identification $k = L/a$

λ is generated ... inconsistent to study g equation alone