One-loop gravitational action and the cosmological constant problem

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Introduction

Quantum Gravity as a QFT: perturbatively non-renormalizable

- EFT valid up to the physical cutoff scale Λ
- Non-perturbatively renormalizable: UV fixed point

For the RG analysis of QG see Contino talk

* Contribution to vacuum energy **from quantum fluctuations** $\sim M_P^4$

* Value inferred **from observed accelerated expansion** of the universe

$\rho_{\rm vac} \sim 10^{-123} M_P^4$

CC problem: most severe naturalness problem in physics

One-loop effective action

One-loop Effective Action

* Euclidean action - Einstein-Hilbert truncation

$$S_{\rm grav} = \frac{1}{16\pi G} \int \mathrm{d}^4x \sqrt{g}\,(-R + 2\Lambda_{\rm cc})$$

* Gauge-invariant one-loop effective action, $\Gamma^{1l}_{\rm grav} = S_{\rm grav} + \delta S^{1l}_{\rm grav}$
  geometrical approach, Vilkovisky-DeWitt

* Strategy put forward by Fradkin and Tseytlin / Taylor and Veneziano

* Background field method: $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$

* When $\bar{g}_{\mu\nu}$ has spherical symmetry:
one-loop VDW effective action = the standard one calculated with gauge-fixing term

$$S_{\rm gf} = \frac{1}{32\pi G\xi} \int \mathrm{d}^4x \sqrt{\bar{g}} \left[\nabla_\mu \left(h^\mu_\nu - \frac{1}{2}\delta^\mu_\nu\, h^\sigma_\sigma\right)\right]^2$$

after taking the limit $\xi \to 0$ at the end of the calculation

One-loop EA: spherical background

We calculate the 1-loop correction $\delta S^{1l}_{\rm grav}$

Take spherical background $\bar{g}_{\mu\nu} = g^{(a)}_{\mu\nu}$ (a radius of the sphere)

(coordinates x angles ; $g^{(a)}_{\mu\nu}$ goes like a^2 ; $\int \mathrm{d}^4x \sqrt{g^{(a)}} = \frac{8\pi^2}{3} a^4$, $R(g^{(a)}) = \frac{12}{a^2}$)

Classical Action $\boxed{S^{(a)}_{\rm grav} = \frac{\pi \Lambda_{\rm cc}}{3G} a^4 - \frac{2\pi}{G} a^2}$ and dS solution $\boxed{a_{\rm dS} = \sqrt{\frac{3}{\Lambda_{\rm cc}}}}$

Add to $S_{\rm grav} + S_{\rm gf}$ the corresponding ghost action (v_μ vector ghost fields)

$$S_{\rm ghost} = \frac{1}{32\pi G} \int \mathrm{d}^4x \sqrt{g^{(a)}}\, g^{(a)\,\mu\nu}\, v^*_\mu \left(-\nabla_\rho \nabla^\rho - \frac{3}{a^2} \right) v_\nu$$

Identify 1-loop corrections to $\frac{\Lambda_{\rm cc}}{G}$ and $\frac{1}{G}$ with coefficients of a^4 and a^2 in $\delta S^{1l}_{\rm grav}$

($\Gamma^{1l}_{\rm grav} = S_{\rm grav} + \delta S^{1l}_{\rm grav}$)

One-loop EA: spherical background

One-loop correction $\delta S^{1l}_{\rm grav}$ to $S^{(a)}_{\rm grav}$ given by

$$e^{-\delta S^{1l}_{\rm grav}} = \lim_{\xi \to 0} \int \mathcal{D}\mu \, e^{-\delta S^{(2)}}$$

where

$$\delta S^{(2)} \equiv S_2 + S_{\rm gf} + S_{\rm ghost}$$

S_2 quadratic term in the expansion of $S_{\rm grav}[\, g^{(a)}_{\mu\nu} + h_{\mu\nu}]$

$$S_2 \equiv \frac{1}{32\pi G} \int d^4x \, \sqrt{g^{(a)}} \left[\frac{1}{2} \widetilde{h}^{\mu\nu} \left(-\nabla_\rho \nabla^\rho - 2\Lambda_{\rm cc} + \frac{8}{a^2} \right) h_{\mu\nu} + \frac{h^2}{a^2} - \nabla^\rho \widetilde{h}_{\rho\mu} \nabla^\sigma \widetilde{h}^\mu_\sigma \right]$$

$h \equiv g^{(a)}_{\mu\nu} h^{\mu\nu}$; $\widetilde{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2} g^{(a)}_{\mu\nu} h$; indices raised with $g^{(a)\,\mu\nu}$; covariant derivatives in terms of $g^{(a)}_{\mu\nu}$

A delicate point ... the measure $\mathcal{D}\mu$

Recent discussion: C.Branchina, V. Branchina, F. Contino, R. Gandolfo, A. P. / Bonanno, Ferrero, Falls

Path integral measure : Fradkin-Vilkovisky

$$\mathcal{D}\mu \equiv \prod_x \left[g^{(a)\,00}(x) \, \left(g^{(a)}(x)\right)^{-1} \left(\prod_{\alpha \leq \beta} \mathrm{d}h_{\alpha\beta}(x) \right) \left(\prod_\rho \mathrm{d}v^*_\rho(x) \right) \left(\prod_\sigma \mathrm{d}v_\sigma(x) \right) \right]$$

Liouville in phase space $\Rightarrow$ integrate over conj. momenta $\Rightarrow g^{(a)\,00}(x) \, \left(g^{(a)}(x)\right)^{-1}$

g^{00}: dangerous? Diffeomorphism invariance?

Subtle point: $\prod_x \leftrightarrow$ discretization (given coordinate system) Leutwyler / Fradkin, Vilkovisky

- Transition amplitude $\langle \mathrm{in},\, t_1 | \mathrm{out},\, t_2 \rangle$ requires a time ordering parameter
- Lattice $\mathcal{E}$: hypersurfaces $x^0 = \mathrm{cte} \cap$ curves $x^i = \mathrm{cte}$

Diffeomorphism invariance of the measure

Under a general coordinate transformation $(\hat{x} \to x)$

- change of time ordering parameter $(\hat{x}^0 \to x^0)$
- change of lattice $(\mathcal{E}_2 \to \mathcal{E}_1)$

$$\Longrightarrow \quad \prod_{\hat{x}} (\mathcal{E}_2) \to \prod_x (\mathcal{E}_1) \text{ is } \textbf{NOT} \text{ a trivial reshuffling}$$

Careful calculation shows that (scalar example)

$$e^{i\hat{\Gamma}[\hat{g}]} = \int e^{i\hat{S}[\hat{\phi},\hat{g}]} \left[\prod_{\hat{x}} M(\hat{g}(\hat{x}))\right] \left[\prod_{\hat{x}} \mathrm{d}\hat{\phi}(\hat{x})\right] = \int e^{i\hat{S}[\hat{\phi},\hat{g}]} \left[B \prod_{x} M(\hat{g}(\hat{x}(x)))\right] \left[A \prod_{x} \mathrm{d}\hat{\phi}(\hat{x}(x))\right]$$

NON-trivial factors **A** and **B** **beyond** those due to **fields transformation**

Accounting for this

- The measure turns out to be invariant
- Another typically used measure: Fujikawa[1]. No g^{00}, not invariant
- g^{00} is crucial to ensure diffeomorphism invariance

[1] Not from Liouville measure in phase space.

Back to the calculation of $\delta S^{1l}_{\rm grav}$

Now : $g^{(a)}_{\mu\nu} = a^2 \widetilde{g}_{\mu\nu}$ $\widetilde{g}_{\mu\nu}$ dimensionless and a-independent metric

convenient (though not necessary) way to dig out the a-dependence from the background metric

Redefinition : $\widehat{h}_{\mu\nu} = (32\pi G)^{-1/2}\, a^{-1} h_{\mu\nu}$, $v_\mu \to (32\pi G)^{\frac{1}{2}}\, v_\mu$

convenient but not necessary

$$\Longrightarrow \quad \left[\mathcal{D}\mu\right] \sim \prod_x \left[\left(\prod_{\alpha\leq\beta} \mathrm{d}\widehat{h}_{\alpha\beta}(x)\right)\left(\prod_\rho \mathrm{d}v^*_\rho(x)\right)\left(\prod_\sigma \mathrm{d}v_\sigma(x)\right)\right]$$

The radius "a" does not appear in the measure

In terms of $\widehat{h}_{\mu\nu}$ and v_μ

$$S_2 + S_{\rm gf} = \int \mathrm{d}^4x\, \sqrt{g^{(1)}}\left[\frac{1}{2}\overline{h}^{\mu\nu}\left(-\nabla_\rho\nabla^\rho - 2a^2\Lambda + 8\right)\widehat{h}_{\mu\nu} + \widehat{h}^2 - \left(1-\frac{1}{\xi}\right)\nabla^\rho\overline{h}_{\rho\mu}\nabla^\sigma\overline{h}^\mu_\sigma\right]$$

$\widehat{h} \equiv g^{(1)}_{\mu\nu}\,\widehat{h}^{\mu\nu}$, $\overline{h}_{\mu\nu} \equiv \widehat{h}_{\mu\nu} - \frac{1}{2} g^{(1)}_{\mu\nu}\,\widehat{h}$; indexes raised with $g^{(1)\,\mu\nu}$; covariant derivatives in terms of $g^{(1)}_{\mu\nu}$

$$S_{\rm ghost} = \int \mathrm{d}^4x\, \sqrt{g^{(1)}}\, g^{(1)\,\mu\nu}\, v^*_\mu \left(-\nabla_\rho\nabla^\rho - 3\right) v_\nu$$

$\Longrightarrow$ **$\delta S^{(2)} = S_2 + S_{\rm gf} + S_{\rm ghost}$ contains only dimensionless operators** (no arbitrary scale μ)

Decomposing onto the eigenbasis

Consider the basis constructed with the eigenfunctions of

Dimensionless Laplace-Beltrami operator $-\widetilde{\Box}^{(s)} \equiv -a^2\,\Box_a^{(s)}$

Dimensionless eigenvalues $\lambda_n^{(s)}$ and corresponding degeneracies $D_n^{(s)}$

$$\lambda_n^{(s)} = n^2 + 3n - s\,; \quad D_n^{(s)} = \frac{2s+1}{3}\left(n+\frac{3}{2}\right)^3 - \frac{(2s+1)^3}{12}\left(n+\frac{3}{2}\right) \qquad n = s, s+1, \ldots$$

Decomposing $\widehat{h}_{\mu\nu}$, v^*_ρ and v_σ onto this basis we get

backup slides

$$\delta S^{1l}_{\mathrm{grav}} = -\frac{1}{2}\log\frac{\det_1[-\widetilde{\Box}^{(1)} - 3]\,\det_2[-\widetilde{\Box}^{(0)} - 6]}{\det_0[-\widetilde{\Box}^{(2)} - 2a^2\Lambda + 8]\,\det_2[-\widetilde{\Box}^{(0)} - 2a^2\Lambda]} + \ldots$$

* index i in $\det_i$ indicates that product of eigenvalues starts from $\lambda^{(s)}_{\mathrm{s+i}}$

Calculation of the fluctuation determinants

Two different strategies

1 - product of eigenvalues of Laplace-Beltrami ops.

2 - proper-time

1 - Calculation in terms of the eigenvalues $\lambda_n^{(s)}$

$$\delta S^{1l}_{\text{grav}} = \frac{1}{2}\sum_{n=2}^{N-2}\left[D_n^{(2)}\log\left(\lambda_n^{(2)} - 2a^2\Lambda + 8\right) + D_n^{(0)}\log\left(\lambda_n^{(0)} - 2a^2\Lambda\right) - D_n^{(1)}\log\left(\lambda_n^{(1)} - 3\right) - D_n^{(0)}\log\left(\lambda_n^{(0)} - 6\right)\right] + \ldots$$

N: numerical UV cutoff on the number of eigenvalues

De Sitter solution for the classical action $a_{\text{dS}} = \sqrt{\frac{3}{\Lambda_{\text{cc}}}}$

a_{dS} size of the universe $\Rightarrow$ connection between N and physical cutoff scale Λ given by

$$\Lambda \sim M_P = \frac{N}{a_{\text{dS}}} = N\sqrt{\frac{\Lambda_{\text{cc}}}{3}}$$

Expanding for $N \gg 1$

$$\delta S^{1l}_{\text{grav}} = -\left(\Lambda_{\text{cc}}^2 \log N^2\right) a^4 + \Lambda_{\text{cc}}\left(-N^2 + 8\log N^2\right) a^2 + \frac{N^4}{24}\left(-1 + 2\log N^2\right) + \frac{N^2}{36}\left(203 - 75\log N^2\right) - \frac{779}{90}\log N^2$$

$$\delta S^{1l}_{\text{grav}} = -\left(\Lambda_{\text{cc}}^2 \log\frac{3\Lambda^2}{\Lambda_{\text{cc}}}\right) a^4 + \left(-3\Lambda^2 + 8\Lambda_{\text{cc}}\log\frac{3\Lambda^2}{\Lambda_{\text{cc}}}\right) a^2 + \ldots$$

2 - Calculation with proper-time

$(-\widetilde{\Box}^{(s)} - \alpha)$ dimensionless $\Longrightarrow$ $\det_i$ regularized with a dimensionless proper-time τ (lower cut: $N \gg 1$) with kernel kernel $\mathrm{K}_i^{(s)}(\tau)$

$$\det_i(-\widetilde{\Box}^{(s)} - \alpha) = e^{-\int_{1/N^2}^{+\infty} \frac{d\tau}{\tau} \mathrm{K}_i^{(s)}(\tau)} \quad ; \quad \mathrm{K}_i^{(s)}(\tau) = \sum_{n=s+i}^{+\infty} D_n^{(s)} \, e^{-\tau\left(\lambda_n^{(s)} - \alpha\right)}$$

After integration over τ, sum over n with EML sum formula. Expanding for $N \gg 1$

$$\delta S^{1l}_{\rm grav} = -\left(\Lambda_{\rm cc}^2 \log N^2\right) a^4 + \Lambda_{\rm cc}\left(-N^2 + 8\log N^2\right) a^2 - \frac{N^4}{12} + \frac{17}{3}N^2 - \frac{1859}{90}\log N^2 + \ldots$$

$$\text{With} \quad \Lambda \equiv \frac{N}{a_{\rm dS}} = \sqrt{\frac{\Lambda_{\rm cc}}{3}}\, N \quad \Longrightarrow$$

$$\delta S^{1l}_{\rm grav} = -\left(\Lambda_{\rm cc}^2 \log\frac{3\Lambda^2}{\Lambda_{\rm cc}}\right) a^4 + \left(-3\Lambda^2 + 8\Lambda_{\rm cc}\log\frac{3\Lambda^2}{\Lambda_{\rm cc}}\right) a^2 + \ldots$$

The two methods give the same result

One-loop corrections to $\frac{\Lambda_{cc}}{G}$ and $\frac{1}{G}$

Coefficients of a^4 and a^2 identify the one-loop corrections to $\frac{\Lambda_{cc}}{G}$ and $\frac{1}{G}$

$$\frac{\Lambda_{cc}^{1l}}{G^{1l}} = \frac{\Lambda_{cc}}{G}\left(1 - \frac{3G\Lambda_{cc}}{\pi}\log\frac{3\Lambda^2}{\Lambda_{cc}}\right) \qquad \frac{1}{G^{1l}} = \frac{1}{G}\left[1 + \frac{G}{2\pi}\left(3\Lambda^2 - 8\Lambda_{cc}\log\frac{3\Lambda^2}{\Lambda_{cc}}\right)\right]$$

* Taking $G \sim M_P^{-2}$: quantum corrections **do not spoil the naturalness** of this relation $G \sim G^{1l}$

* **Usual result**: $\rho \sim M_P^4 \quad \Longrightarrow \quad$ bare value of $\rho \sim M_P^4$ with coefficient that must be enormously fine-tuned to cancel the one-loop M_P^4 correction

* **Our result**: only mild logarithmic corrections to $\rho = \frac{\Lambda_{cc}}{8\pi G}$: **unexpected!**

In pure gravity **no naturalness problem arises**:
the bare cosmological constant Λ_{cc} does not need to be $\sim M_P^2$.

We may **naturally** have $\Lambda_{cc} \ll M_P^2$, and so

$$\boxed{\Lambda_{cc}^{1l} \sim \Lambda_{cc}}$$

Peeking behind the curtain: what is going on

...Usually we have power-law "divergences" in ρ...

Why don't we see them? What is going on?

- Consider $\delta S^{1l}_{\rm grav}$

$$\delta S^{1l}_{\rm grav} = -\left(\Lambda_{\rm cc}^2 \log N^2\right) a^4 + \Lambda_{\rm cc}\left(-N^2 + 8\log N^2\right) a^2 - \frac{N^4}{12} + \frac{17}{3}N^2 - \frac{1859}{90}\log N^2$$

- Connect N and Λ through $\Lambda = \frac{N}{a}$ (rather than $\Lambda = \frac{N}{a_{\rm dS}}$):

*Corresponds to identify Λ with the maximal eigenvalue $\widetilde{\lambda}^{(s)}_{\rm max}$ of $-\Box^{(s)}_a$

$$\delta S^{1l}_{\rm grav} = -\left[\frac{\Lambda^4}{12} + \Lambda_{\rm cc}\Lambda^2 + \Lambda_{\rm cc}^2 \log\left(\Lambda^2 a^2\right)\right] a^4 + \left[\frac{17}{3}\Lambda^2 + 8\Lambda_{\rm cc}\log\left(\Lambda^2 a^2\right)\right] a^2 - \frac{1859}{90}\log\left(\Lambda^2 a^2\right)$$

known result found with proper time regularization using heat-kernel expansion

Dependence on $g^{(a)}_{\mu\nu}$ **artificially** altered in $\delta S^{1l}_{\rm grav}$

Usual calculations implement $\Lambda = N/a$

* Using Fujikawa measure (not diffeomorphism invariant)

$$\delta S^{1l}_{\text{grav}} = -\frac{1}{2}\log\frac{\det_1[-\Box^{(1)}_a - 3/a^2]\,\det_2[-\Box^{(0)}_a - 6/a^2]}{\det_0[-\Box^{(2)}_a - 2\Lambda + 8/a^2]\,\det_2[-\Box^{(0)}_a - 2\Lambda]} + \dots$$

* Trace usually regulated as

$$\mathrm{Tr}\log\left(\frac{-\Box^{(s)}_a + \alpha}{\mu^2}\right) \to -\int_{1/\Lambda^2}^{+\infty}\frac{d\tau}{\tau}\,\mathrm{Tr}\,e^{-\tau\left(-\Box^{(s)}_a+\alpha\right)} = -\int_{1/\Lambda^2}^{+\infty}\frac{d\tau}{\tau}\mathrm{K}(\tau)$$

* τ **dimensionful** proper-time $(\text{mass})^{-2}$ and

$$\mathrm{K}(\tau) = \sum_{n=0}^{+\infty} D_n\, e^{-\tau\left(\widetilde{\lambda}^{(s)}_n+\alpha\right)}$$

with $\widetilde{\lambda}^{(s)}_n$ the *a-dependent* eigenvalues of $-\Box^{(s)}_a$ and D_n the corresponding degeneracies

$$\widetilde{\lambda}^{(s)}_n = \frac{n^2+3n-s}{a^2} \quad ; \quad D^{(s)}_n = \frac{2s+1}{3}\left(n+\frac{3}{2}\right)^3 - \frac{(2s+1)^3}{12}\left(n+\frac{3}{2}\right)$$

* Cut $1/\Lambda^2$ in the proper-time integral effectively restricts the sum to the N eigenvalues of $-\Box^{(s)}_a$ such that $\widetilde{\lambda}^{(s)}_n \lesssim \Lambda^2 \implies$ highest eigenvalue $\widetilde{\lambda}^{(s)}_N \sim \frac{N^2}{a^2} \sim \Lambda^2$

Scalar field on spherical background

Free real scalar field ϕ of mass m_ϕ on gravitational background $g^{(a)}_{\mu\nu}$

see Gandolfo talk for the interacting case

$$S = \frac{\pi \Lambda_{\rm cc}}{3G} a^4 - \frac{2\pi}{G} a^2 + \int \mathrm{d}^4 x \sqrt{g^{(a)}} \left[\frac{1}{2} g^{(a)\,\mu\nu} \partial_\mu \phi \partial_\nu \phi + \frac{1}{2} m_\phi^2 \phi^2 \right]$$

Integrating $\phi(x)$ $\implies$ Effective gravitational action $S^{\rm eff}_{\rm grav}$

$$S^{\rm eff}_{\rm grav} = \frac{\pi \Lambda_{\rm cc}}{3G} a^4 - \frac{2\pi}{G} a^2 + \delta S_{\rm grav}$$

with $\delta S_{\rm grav}$ given by

$$e^{-\delta S_{\rm grav}} = \int \prod_x \left[\left(g^{(a)\,00}(x) \right)^{\frac{1}{2}} \left(g^{(a)}(x) \right)^{\frac{1}{4}} \mathrm{d}\phi(x) \right] e^{-\int \mathrm{d}^4 x \sqrt{g^{(a)}} \left[-\frac{1}{2} \phi \Box \phi + \frac{1}{2} m_\phi^2 \phi^2 \right]}$$

$\left(g^{(a)\,00}(x) \right)^{\frac{1}{2}} \left(g^{(a)}(x) \right)^{\frac{1}{4}}$ from integration over conjugate momenta Fradkin, Vilkovisky

Using $g^{(a)}_{\mu\nu} = a^2 \widetilde{g}_{\mu\nu}$ and introducing $\widehat{\phi} = a\phi$ (convenient but not necessary)

$$e^{-\delta S_{\rm grav}} = \mathcal{N} \int \prod_x \left[\mathrm{d}\, \widehat{\phi}(x) \right] e^{-\int \mathrm{d}^4 x \sqrt{\widetilde{g}} \left[-\frac{1}{2} \widehat{\phi} \left(\widetilde{\Box}^{(0)} \right) \widehat{\phi} + \frac{1}{2} a^2 m_\phi^2 \widehat{\phi}^2 \right]}$$

Fluctuation determinant

Performing Gaussian integrations

$$S^{\rm eff}_{\rm grav} = \frac{\pi\Lambda_{\rm cc}}{3G}\,a^4 - \frac{2\pi}{G}\,a^2 + \frac{1}{2}\,\mathrm{Tr}\log\left(-\widetilde{\Box}^{(0)} + a^2 m_\phi^2\right)$$

To evaluate the "Tr log"

$$\text{Proper time } \int_{1/N^2}^{\infty} \frac{d\tau}{\tau}\mathrm{K}^{(0)}(\tau), \qquad \mathrm{K}^{(0)}(\tau) = \sum_{n=0}^{+\infty} D_n^{(0)}\, e^{-\tau\left(\lambda_n^{(0)}+a^2 m_\phi^2\right)}$$

$$\oplus$$

Expansion for $N \gg 1$:

$$S^{\rm eff}_{\rm grav} = \frac{\pi}{3}\left(\frac{\Lambda_{\rm cc}}{G} - \frac{m_\phi^4}{8\pi}\log N^2\right)a^4 - 2\pi\left[\frac{1}{G} - \frac{m_\phi^2}{24\pi}\left(N^2 + 2\log N^2\right)\right]a^2 + \dots$$

* Similarity with the result obtained in the pure gravity case evident

* Once again : if N *properly* related to $\Lambda \sim M_P$ through $\Lambda \sim \frac{N}{a_{\rm dS}} = N\sqrt{\frac{\Lambda_{\rm cc}}{3}}$

$\Longrightarrow \quad \rho = \frac{\Lambda_{\rm cc}}{8\pi G}$ receives only mild logarithmically divergent correction

$$\delta\left(\frac{\Lambda_{\rm cc}}{G}\right) = -\frac{m_\phi^4}{8\pi}\log\frac{3\Lambda^2}{\Lambda_{\rm cc}}$$

Usual result

Again: if we were to perform the mapping $N(\Lambda)$ as $N = a\,\Lambda$

$\Longrightarrow$ spurious quartically and quadratically divergent terms $\Longleftarrow$

For instance:

$$-\frac{N^4}{48} \quad\longrightarrow\quad -\frac{\Lambda^4}{48}\,a^4$$

Quartically divergent contribution to $\frac{\Lambda_{\mathrm{cc}}}{G}$

$$\frac{N^2}{12}\,m_\phi^2\,a^2 \quad\longrightarrow\quad \frac{\Lambda^2}{12}\,m_\phi^2\,a^4$$

Quadratically divergent contribution to $\frac{\Lambda_{\mathrm{cc}}}{G}$

Conclusions

We have seen that when

- quantum fluctuations are calculated in curved spacetime
- diffeomorphism invariant measure (FV) used
- physical cutoff properly identified

Power-law divergences are **absent in the vacuum energy**

Same is true even in the presence of matter fields

THANKS FOR YOUR ATTENTION

ADDITIONAL SLIDES

Diffeomorphism invariance of the measure

Gravitational action $S_{\rm g}[g]$ in the presence of a scalar field with action $S_{\rm m}[\phi, g]$ (sign. $(-,+,+,+)$)

$$S[\phi, g] = S_{\rm g}[g] + S_{\rm m}[\phi, g] = S_{\rm g}[g] + \int \mathrm{d}^4x\, \mathcal{L}_{\rm m}(\phi(x), \partial_\mu \phi(x), g(x))$$

where $\mathcal{L}_{\rm m}(\phi(x), \partial_\mu \phi(x), g(x))$ is the matter Lagrangian density

$$\mathcal{L}_{\rm m}(\phi(x), \partial_\mu \phi(x), g(x)) = -\frac{1}{2}\sqrt{-g(x)}\left(g^{\mu\nu}(x)\partial_\mu \phi(x)\partial_\nu \phi(x) + m^2\phi^2(x)\right)$$

Effective gravitational action $\Gamma[g]$ given by

$$e^{i\Gamma[g]} = \int e^{iS[\phi(x),\, g(x)]} \prod_x \left[M(g(x))\, \mathrm{d}\phi(x)\right]$$

$M(g(x))$ non-trivial measure term

FV measure obtained from the Liouville measure in phase space $\prod_x \left[\mathrm{d}\pi(x)\, \mathrm{d}\phi(x)\right]$

Diffeomorphism invariance of the measure - 2

Subtle point: $\prod_x \leftrightarrow$ discretization. Given a coordinate system x^μ: Leutwyler / Fradkin, Vilkovisky

- Lattice $\mathcal{E}_1$: hypersurfaces $x^0 = \mathrm{cte} \cap$ curves $x^i = \mathrm{cte}$
- Transition amplitude $\langle \mathrm{in},\ t_1 | \mathrm{out},\ t_2 \rangle$ requires a time ordering product, that can be identified with x^0

$\prod_x$ indicates the product over all the points Q_i of the lattice $\mathcal{E}_1$

Σ reference frame with coordinates x ; $\hat{\Sigma}$ frame with coordinates $\hat{x}$. In $\hat{\Sigma}$, for the classical action

$$\hat{S}[\hat{\phi}, \hat{g}] = \hat{S}_{\mathrm{g}}[\hat{g}] + \hat{S}_{\mathrm{m}}[\hat{\phi}, \hat{g}] = \hat{S}_{\mathrm{g}}[\hat{g}] + \int \mathrm{d}^4\hat{x}\, \mathcal{L}_{\mathrm{m}}(\hat{\phi}(\hat{x}), \hat{\partial}_\mu \hat{\phi}(\hat{x}), \hat{g}(\hat{x}))$$

for the effective action $\hat{\Gamma}[\hat{g}]$ we have

$$e^{i\hat{\Gamma}[\hat{g}]} = \int e^{i\,\hat{S}[\hat{\phi},\hat{g}]} \prod_{\hat{x}} \left[M(\hat{g}(\hat{x}))\, \mathrm{d}\hat{\phi}(\hat{x}) \right]$$

$\prod_{\hat{x}}$ product over the points P_i of the lattice $\mathcal{E}_2$: hypersurfaces $\hat{x}^0 = \mathrm{cte} \cap$ curves $\hat{x}^i = \mathrm{cte}$. Time ordering parameter $\hat{x}^0$.

Diffeomorphism invariance of the measure - 3

Under a general coordinate transformation

- change of time ordering parameter
- change of lattice

$$\Longrightarrow \qquad \prod_x (\mathcal{E}_1) \to \prod_{\hat{x}} (\mathcal{E}_2) \text{ is **NOT** a trivial reshuffling}$$

To realize the bridge $\hat{\Gamma}[\hat{g}]$ (in $\hat{\Sigma}$) $\to \Gamma[g]$ (in Σ):

1 - In $\hat{\Sigma}$, move from $\mathcal{E}_2 \;\to\; \mathcal{E}_1$ (from $\hat{x}^0 \to x^0$)

$$e^{i\hat{\Gamma}[\hat{g}]} = \int e^{i\,\hat{S}[\hat{\phi},\hat{g}]}\Big[\prod_{\hat{x}} M(\hat{g}(\hat{x}))\Big]\Big[\prod_{\hat{x}} \mathrm{d}\hat{\phi}(\hat{x})\Big] = \int e^{i\,\hat{S}[\hat{\phi},\hat{g}]}\Big[B\prod_{x} M(\hat{g}(\hat{x}(x)))\Big]\Big[A\prod_{x} \mathrm{d}\hat{\phi}(\hat{x}(x))\Big]$$

with $\hat{S}$ written as (change of integration variables from $\hat{x}$ to x through $\hat{x} = \hat{x}(x)$)

$$\begin{aligned}
\hat{S}[\hat{\phi},\hat{g}] &= \hat{S}_{\mathrm{g}}[\hat{g}] + \hat{S}_{\mathrm{m}}[\hat{\phi},\hat{g}] \\
&= \hat{S}_{\mathrm{g}}[\hat{g}] - \frac{1}{2}\int \mathrm{d}^4x\, J\sqrt{-\hat{g}(\hat{x}(x))}\left(\hat{g}^{\mu\nu}(\hat{x}(x))\frac{\partial x^\rho}{\partial \hat{x}^\mu}\frac{\partial x^\sigma}{\partial \hat{x}^\nu}\partial_\rho\hat{\phi}(\hat{x}(x))\partial_\sigma\hat{\phi}(\hat{x}(x)) + m^2\hat{\phi}^2(\hat{x}(x))\right) \\
&\equiv \hat{S}_{\mathrm{g}}[\hat{g}] + \int \mathrm{d}^4x\, \widetilde{\mathcal{L}}_{\mathrm{m}}(\hat{\phi}(\hat{x}(x)), \partial_\mu\hat{\phi}(\hat{x}(x)), \hat{g}(\hat{x}(x))) \qquad\qquad J \equiv \left|\det\frac{\partial\hat{x}}{\partial x}\right|
\end{aligned}$$

Diffeomorphism invariance of the measure - 4

2 - Move from $\hat{\phi}(\hat{x}(x))$ and $\hat{g}_{\mu\nu}(\hat{x}(x))$ to $\phi(x)$ and $g_{\mu\nu}(x)$

$$e^{i\hat{\Gamma}[\hat{g}]} = \int e^{i\,\hat{S}[\hat{\phi},\hat{g}]} \left[B \prod_x M(\hat{g}(\hat{x}(x))) \right] \left[A \prod_x \mathrm{d}\hat{\phi}(\hat{x}(x)) \right]$$

$$= \int e^{i\,S[\phi,g]} \left[B\,C \prod_x M(g(x)) \right] \left[A\,E \prod_x \mathrm{d}\phi(x) \right]$$

* Factor C arises when we express $M(\hat{g}(\hat{x}(x)))$ in terms of $g_{\mu\nu}(x)$

* E Jacobian of $\hat{\phi}(\hat{x}(x)) \to \phi(x)$

To have $\hat{\Gamma}[\hat{g}] = \Gamma[g]$, it is necessary that $\boldsymbol{A\,B\,C\,E = 1}$

Transformation factors

Factor E - ϕ scalar field, $\hat{\phi}(\hat{x}(x)) = \phi(x)$; E Jacobian of $\hat{\phi}(\hat{x}(x)) \to \phi(x) \quad \Longrightarrow \quad E = 1$.

Factor A - In phase space

$$e^{i\Gamma_{\mathcal{E}_1}[g]} = e^{iS_g[g]} \int e^{i\int \mathrm{d}^4x[\partial_0\phi(x)\pi(x) - \mathcal{H}(\pi(x),\,\phi(x))]} \left[\prod_x f(g(x))\,\mathrm{d}\pi(x)\,\mathrm{d}\phi(x)\right]$$

with $\mathcal{H}(\pi(x),\,\phi(x))$ the Hamiltonian density, $\pi(x)$ the momentum conjugate to $\phi(x)$

$$\pi(x) = \frac{\partial\mathcal{L}_{\mathrm{m}}(\phi(x), \partial_\mu\phi(x), g(x))}{\partial(\partial_0\phi(x))}\,,$$

and $\mathcal{L}_{\mathrm{m}}(\phi(x), \partial_\mu\phi(x), g(x))$ the Lagrangian density in Σ

$f(g(x)) = 1$ FV measure ; $f(g(x)) = (g^{00}(x))^{-1/2}$ Fujikawa measure.

$\prod_x \Longleftrightarrow$ lattice $\mathcal{E}_1$ and time ordering $x^0 \quad \Longrightarrow \quad \pi(x)$ correctly defined above

Transformation factors

In $\hat{\Sigma}$ (lattice $\mathcal{E}_2$)

$$e^{i\hat{\Gamma}_{\mathcal{E}_2}[\hat{g}]} = e^{i\hat{S}_g[\hat{g}]} \int e^{i\int d^4\hat{x}\left[\hat{\partial}_0\hat{\phi}(\hat{x})\hat{\pi}(\hat{x}) - \mathcal{H}(\hat{\pi}(\hat{x}),\,\hat{\phi}(\hat{x}))\right]} \left[\prod_{\hat{x}} f(\hat{g}(\hat{x}))\,d\hat{\pi}(\hat{x})\,d\hat{\phi}(\hat{x})\right]$$

with $\mathcal{H}(\hat{\pi}(\hat{x}),\,\hat{\phi}(\hat{x}))$ the Hamiltonian density, $\hat{\pi}(\hat{x})$ the momentum conjugate to $\hat{\phi}(\hat{x})$

$$\hat{\pi}(\hat{x}) = \frac{\partial\mathcal{L}_m(\hat{\phi}(\hat{x}),\hat{\partial}_\mu\hat{\phi}(\hat{x}),\hat{g}(\hat{x}))}{\partial(\hat{\partial}_0\hat{\phi}(\hat{x}))},$$

and $\mathcal{L}_m(\hat{\phi}(\hat{x}),\hat{\partial}_\mu\hat{\phi}(\hat{x}),\hat{g}(\hat{x}))$ the Lagrangian density in $\hat{\Sigma}$

$\prod_{\hat{x}} \Longleftrightarrow$ lattice $\mathcal{E}_2$ and time ordering parameter $\hat{x}^0 \quad \Longrightarrow \quad \hat{\pi}(\hat{x})$ correctly given above

Using in $\hat{\Sigma}$ the lattice $\mathcal{E}_1$ (and x^0 as time ordering parameter), for the effective action we have

$$e^{i\hat{\Gamma}_{\mathcal{E}_1}[\hat{g}]}$$

$$= e^{i\hat{S}_g[\hat{g}]} \int e^{i\int d^4x\left[\partial_0\hat{\phi}(\hat{x}(x))\widetilde{\pi}(\hat{x}(x)) - \widetilde{\mathcal{H}}(\widetilde{\pi}(\hat{x}(x)),\,\hat{\phi}(\hat{x}(x)))\right]} \left[F\prod_x f(\hat{g}(\hat{x}(x)))\right]\left[\prod_x d\widetilde{\pi}(\hat{x}(x))\,d\hat{\phi}(\hat{x}(x))\right]$$

$\widetilde{\mathcal{H}}(\widetilde{\pi}(\hat{x}(x)),\,\hat{\phi}(\hat{x}(x)))$ Hamiltonian density corresponding to $\widetilde{\mathcal{L}}_m(\hat{\phi}(\hat{x}(x)),\partial_\mu\hat{\phi}(\hat{x}(x)),\hat{g}(\hat{x}(x)))$ and $\widetilde{\pi}(\hat{x}(x))$ momentum conjugate to $\hat{\phi}(\hat{x}(x))$

$$\widetilde{\pi}(\hat{x}(x)) = \frac{\partial\widetilde{\mathcal{L}}_m(\hat{\phi}(\hat{x}(x)),\partial_\mu\hat{\phi}(\hat{x}(x)),\hat{g}(\hat{x}(x)))}{\partial(\partial_0\hat{\phi}(\hat{x}(x)))}.$$

Transformation factors

* Integrating over $\hat{\pi}(\hat{x})$ for $\hat{\Gamma}_{\mathcal{E}_2}[\hat{g}]$ and writing $\prod_{\hat{x}} \mathrm{d}\hat{\phi}(\hat{x}) = A \prod_x \mathrm{d}\hat{\phi}(\hat{x}(x))$

$$e^{i\hat{\Gamma}_{\mathcal{E}_2}[\hat{g}]} = e^{i\hat{S}_{\mathrm{g}}[\hat{g}]} \left[\prod_{\hat{x}} f(\hat{g}(\hat{x}))\right] \int e^{i\hat{S}_{\mathrm{m}}[\hat{\phi},\,\hat{g}]} \left[\prod_{\hat{x}} W(\hat{g}(\hat{x}))\right] \left[A \prod_x \mathrm{d}\hat{\phi}(\hat{x}(x))\right]$$

with $\quad W(\hat{g}(\hat{x})) \equiv (-\hat{g}(\hat{x}))^{1/4}(-\hat{g}^{00}(\hat{x}))^{1/2}$

* Integrating over $\widetilde{\pi}(\hat{x}(x))$ for $\hat{\Gamma}_{\mathcal{E}_1}[\hat{g}]$

$$e^{i\hat{\Gamma}_{\mathcal{E}_1}[\hat{g}]} = e^{i\hat{S}_{\mathrm{g}}[\hat{g}]}\, F \left[\prod_x f(\hat{g}(\hat{x}(x)))\right] \int e^{i\hat{S}_{\mathrm{m}}[\hat{\phi},\,\hat{g}]} \left[\prod_x Y(\hat{g}(\hat{x}(x)))\right] \prod_x \mathrm{d}\hat{\phi}(\hat{x}(x))$$

with $\quad Y(\hat{g}(\hat{x}(x))) \equiv J^{1/2}(-\hat{g}(\hat{x}(x)))^{1/4}\left[-\hat{g}^{\mu\nu}(\hat{x}(x))\,\hat{\partial}_\mu x^0\,\hat{\partial}_\nu x^0\right]^{1/2}$

$\hat{\Gamma}_{\mathcal{E}_2}[\hat{g}]$ and $\hat{\Gamma}_{\mathcal{E}_1}[\hat{g}]$ same effective action in $\hat{\Sigma}$

$$\Longrightarrow \quad A = \frac{\prod_x Y(\hat{g}(\hat{x}(x)))}{\prod_{\hat{x}} W(\hat{g}(\hat{x}))}$$

Transformation factors

$$A = \frac{\prod_x Y(\hat{g}(\hat{x}(x)))}{\prod_{\hat{x}} W(\hat{g}(\hat{x}))}$$

To write A in a convenient form, consider an infinitesimal coordinate transformation (only first order in ε)

$$\hat{x}^\mu = x^\mu + \varepsilon^\mu(x)\,,$$

and switch from $\prod_x$ to $\prod_{\hat{x}}$ in the numerator. Observing that ($\delta^{(4)}(0)$ from $\sum_x \to \int \mathrm{d}^4 x$)

$$\prod_x J^{1/2} = \exp\left(\frac{\delta^{(4)}(0)}{2}\int \mathrm{d}^4 x \log\left(1+\partial_\mu \varepsilon^\mu\right)\right) = \exp\left(\frac{\delta^{(4)}(0)}{2}\int \mathrm{d}^4 x\, \partial_\mu \varepsilon^\mu\right) = 1$$

where we have used $\mathrm{d}^4\hat{x} = \mathrm{d}^4 x\, J \equiv \mathrm{d}^4 x \left|\det \frac{\partial \hat{x}}{\partial x}\right| = \mathrm{d}^4 x\,(1+\partial_\mu \varepsilon^\mu(x))$, and observing also that

$$-\hat{g}^{\mu\nu}(\hat{x}(x))\,\hat{\partial}_\mu x^0\,\hat{\partial}_\nu x^0 = -\hat{g}^{00} + 2\,\hat{g}^{0\mu}\,\partial_\mu \varepsilon^0 = -\hat{g}^{00}\left(1 - 2\,\frac{\hat{g}^{0\mu}}{\hat{g}^{00}}\,\partial_\mu \varepsilon^0\right)$$

we have

$$\prod_x Y(\hat{g}(\hat{x}(x))) = \left[\prod_{\hat{x}} W(\hat{g}(\hat{x}))\right] \exp\left(-\frac{\delta^{(4)}(0)}{2}\int \mathrm{d}^4\hat{x}\left[\partial_\mu \varepsilon^\mu \log\left((-\hat{g}(\hat{x}))^{1/2}\left(-\hat{g}^{00}(\hat{x})\right)\right) + 2\,\frac{\hat{g}^{0\mu}(\hat{x})}{\hat{g}^{00}(\hat{x})}\,\partial_\mu \varepsilon^0\right]\right)$$

$$\Longrightarrow A = \exp\left(-\frac{\delta^{(4)}(0)}{2}\int \mathrm{d}^4 x\left[\partial_\mu \varepsilon^\mu \log\left((-g(x))^{1/2}\left(-g^{00}(x)\right)\right) + 2\,\frac{g^{0\mu}(x)}{g^{00}(x)}\,\partial_\mu \varepsilon^0\right]\right)$$

Transformation factors

Factor B - We begin by observing that $M(\hat{g}(\hat{x}))$ is nothing but the product $f(\hat{g}(\hat{x}))W(\hat{g}(\hat{x}))$. We then have

$$\prod_{\hat{x}} M(\hat{g}(\hat{x})) = \prod_{\hat{x}} \left[f(\hat{g}(\hat{x}))W(\hat{g}(\hat{x})) \right] = \exp\left(\delta^{(4)}(0) \int \mathrm{d}^4\hat{x} \log(f(\hat{g}(\hat{x}))W(\hat{g}(\hat{x}))) \right)$$

$$= \exp\left(\delta^{(4)}(0) \int \mathrm{d}^4 x\, \partial_\mu \varepsilon^\mu \log(f(g(x))W(g(x))) \right) \prod_x \left[f(\hat{g}(\hat{x}(x)))W(\hat{g}(\hat{x}(x))) \right]$$

$$= \exp\left(\delta^{(4)}(0) \int \mathrm{d}^4 x\, \partial_\mu \varepsilon^\mu \log(M(g(x))) \right) \prod_x M(\hat{g}(\hat{x}(x)))\,.$$

Recalling that $\prod_{\hat{x}} M(\hat{g}(\hat{x})) = B \prod_x M(\hat{g}(\hat{x}(x)))$

$$B = \exp\left(\delta^{(4)}(0) \int \mathrm{d}^4 x\, \partial_\mu \varepsilon^\mu \log(M(g(x))) \right)$$

Transformation factors

Factor C - Defined by $\prod_x M(\hat{g}(\hat{x}(x))) = C \prod_x M(g(x))$. At first order in ε

$$M(\hat{g}^{\mu\nu}(\hat{x}(x))) = M\Big(g^{\mu\nu}(x) + g^{\mu\sigma}(x)\partial_\sigma \varepsilon^\nu(x) + g^{\nu\sigma}(x)\partial_\sigma \varepsilon^\mu(x)\Big) =$$

$$= M(g) + \frac{\partial M(g)}{\partial g^{\alpha\beta}(x)} \left(g^{\alpha\sigma}(x)\partial_\sigma \varepsilon^\beta(x) + g^{\beta\sigma}(x)\partial_\sigma \varepsilon^\alpha(x)\right)$$

where we used

$$\hat{g}^{\mu\nu}(\hat{x}) = \frac{\partial \hat{x}^\mu(x)}{\partial x^\rho}\frac{\partial \hat{x}^\nu(x)}{\partial x^\sigma} g^{\rho\sigma}(x) = g^{\mu\nu}(x) + g^{\mu\sigma}(x)\partial_\sigma \varepsilon^\nu(x) + g^{\nu\sigma}(x)\partial_\sigma \varepsilon^\mu(x)$$

We then get

$$\prod_x M(\hat{g}(\hat{x}(x))) = \exp\left(\delta^{(4)}(0)\int \mathrm{d}^4x \log\left[M(g) + \frac{\partial M(g)}{\partial g^{\alpha\beta}(x)}\left(g^{\alpha\sigma}(x)\partial_\sigma \varepsilon^\beta(x) + g^{\beta\sigma}(x)\partial_\sigma \varepsilon^\alpha(x)\right)\right]\right)$$

$$= \exp\left(\delta^{(4)}(0)\int \mathrm{d}^4x \, [M(g)]^{-1}\frac{\partial M(g)}{\partial g^{\alpha\beta}(x)}\left(g^{\alpha\sigma}(x)\partial_\sigma \varepsilon^\beta(x) + g^{\beta\sigma}(x)\partial_\sigma \varepsilon^\alpha(x)\right)\right)\prod_x M(g(x))$$

$$\Longrightarrow C = \exp\left(\delta^{(4)}(0)\int \mathrm{d}^4x \, [M(g)]^{-1}\frac{\partial M(g)}{\partial g^{\alpha\beta}(x)}\left(g^{\alpha\sigma}(x)\partial_\sigma \varepsilon^\beta(x) + g^{\beta\sigma}(x)\partial_\sigma \varepsilon^\alpha(x)\right)\right)$$

Fradkin-Vilkovisky versus Fujikawa measure

Note: B and C depend on the specific form of $M(g(x))$, A does not and $E = 1$. We have then to calculate B and C for $M_{\text{FV}}(g(x))$ and $M_{\text{Fuji}}(g(x))$, respectively.

Fradkin-Vilkovisky measure - Using $M(g(x)) = M_{\text{FV}}(g(x)) = \left(-g^{00}(x)\right)^{1/2}(-g(x))^{1/4}$ we get

$$B_{\text{FV}} = \exp\left(\frac{\delta^{(4)}(0)}{2}\int \mathrm{d}^4x\, \partial_\mu \varepsilon^\mu \log[(-g(x))^{1/2}(-g^{00}(x))]\right)$$

$$C_{\text{FV}} = \exp\left(\delta^{(4)}(0)\int \mathrm{d}^4x\, \frac{g^{0\mu}(x)}{g^{00}(x)}\, \partial_\mu \varepsilon^0\right)$$

from which

$$B_{\text{FV}}\, C_{\text{FV}} = A^{-1}$$

which shows that for the FV measure

$$A\, B_{\text{FV}}\, C_{\text{FV}}\, E = 1$$

and therefore that in this case: $\hat{\Gamma}[\hat{g}] = \Gamma[g]$.

Fradkin - Vilkovisky versus Fujikawa measure

Fujikawa measure - Using $M(g(x)) = M_{\rm Fuji}(g(x)) = \mu\,(-g(x))^{1/4}$ (μ arbitrary mass scale) we get

$$
\begin{aligned}
B_{\rm Fuji} &= \exp\left(\delta^{(4)}(0)\int \mathrm{d}^4x\,\partial_\mu\varepsilon^\mu \log\left((-g(x))^{1/4}\mu\right)\right)\\
C_{\rm Fuji} &= 1
\end{aligned}
$$

so that

$$
A\,B_{\rm Fuji}\,C_{\rm Fuji}\,E = \exp\left(\delta^{(4)}(0)\int \mathrm{d}^4x\left[\partial_\mu\varepsilon^\mu \log\left(\frac{\mu}{(-g^{00})^{1/2}}\right) - \frac{g^{0\mu}(x)}{g^{00}(x)}\,\partial_\mu\varepsilon^0\right]\right)
$$

which means that in this case: $\hat\Gamma[\hat g] \neq \Gamma[g]$.

Contrary to what is claimed in JHEP 05 (2025), 164, the **effective action** calculated using the **Fujikawa measure** *is* **not diffeomorphism invariant**

Comparison with Becker-Reuter, PRD 102 (2020) 12 - PRD 104 (2021) 12

and Ferrero-Percacci, e-Print: 2404.12357

Consider the modified Einstein equation at one-loop Becker, Reuter PRD 104 (2021) 12

$$\frac{3G}{\pi}\, a \frac{\mathrm{d}}{\mathrm{d}a}\Gamma^{1l}_{\mathrm{grav}}(a) = 4\Lambda a^4 - 12a^2 + \frac{3G}{\pi}\, a\frac{\mathrm{d}}{\mathrm{d}a}\delta S^{1l}_{\mathrm{grav}}(a) = 0 \qquad (g_{\mu\nu} = g^{(a)}_{\mu\nu})$$

* **BRST-invariant Fujikawa measure** (or measure not fully considered)

$$\left[\mathcal{D}u(h)\mathcal{D}v^*_\rho\, \mathcal{D}v_\sigma\right] \equiv \prod_x \Big[\underbrace{\left(g^{(a)}(x)\right)^{-2}}_{\propto\, a^{-16}}\Big(\prod_{\alpha\leq\beta} \mathrm{d}h_{\alpha\beta}(x)\Big)\Big(\prod_\rho \mathrm{d}v^*_\rho(x)\Big)\Big(\prod_\sigma \mathrm{d}v_\sigma(x)\Big)\Big]$$

$$\Longrightarrow \quad \delta S^{1l}_{\mathrm{grav}}(a) \sim N^4 \log{(a\mu)} \quad \Longrightarrow \quad \textbf{solution} \quad \bar{a}_N \sim N\sqrt{M_P^{-1}\Lambda_{cc}^{-1}}$$

* **Fradkin-Vilkovisky measure**

$$\left[\mathcal{D}u(h)\mathcal{D}v^*_\rho\, \mathcal{D}v_\sigma\right] \equiv \prod_x \Big[\underbrace{g^{(a)\,00}(x)\,\left(g^{(a)}(x)\right)^{-1}}_{\propto\, a^{-10}}\Big(\prod_{\alpha\leq\beta} \mathrm{d}h_{\alpha\beta}(x)\Big)\Big(\prod_\rho \mathrm{d}v^*_\rho(x)\Big)\Big(\prod_\sigma \mathrm{d}v_\sigma(x)\Big)\Big]$$

$\Longrightarrow$ $\delta S^{1l}_{\mathrm{grav}}(a)$ does **not** contain N^4 **log** $(a\mu)$ $\Longrightarrow$ **NO** $\bar{a}_N$ **solution**

Similar considerations apply to the results found in Becker-Reuter, PRD 102 (2020) 12, and Ferrero-Percacci, JHEP 09 (2024) 074

Calculation with EML in proper time regularization

$(-\widetilde{\Box}^{(s)} - \alpha)$ dimensionless $\Longrightarrow$ determinants regularized in terms of a **dimensionless** proper-time τ (lower cut: **number** $N \gg 1$)

$$\det_i(-\widetilde{\Box}^{(s)} - \alpha) = e^{-\int_{1/N^2}^{+\infty} \frac{\mathrm{d}\tau}{\tau} \mathrm{K}_i^{(s)}(\tau)} \quad ; \quad \mathrm{K}_i^{(s)}(\tau) = \sum_{n=s+i}^{+\infty} D_n^{(s)}\, e^{-\tau\left(\lambda_n^{(s)} - \alpha\right)}$$

After integration over τ, sum over n performed with EML sum formula

$$\sum_{n=n_i}^{n_f} f(n) = \int_{n_i}^{n_f} \mathrm{d}x\, f(x) + \frac{f(n_f) + f(n_i)}{2} + \sum_{k=1}^{p} \frac{B_{2k}}{(2k)!} \left(f^{(2k-1)}(n_f) - f^{(2k-1)}(n_i)\right) + R_{2p}$$

p is an integer, B_m are Bernoulli numbers, R_{2p} is the rest given by

$$R_{2p} = \sum_{k=p+1}^{\infty} \frac{B_{2k}}{(2k)!} \left(f^{(2k-1)}(n_f) - f^{(2k-1)}(n_i)\right) = \frac{(-1)^{2p+1}}{(2p)!} \int_{n_i}^{n_f} \mathrm{d}x\, f^{(2p)}(x) B_{2p}(x - [x])$$

$B_n(x)$ are the Bernoulli polynomials, $[x]$ the integer part of x, and $f^{(i)}$ the i-th derivative of f with respect to its argument

Expansion of $\widehat{h}_{\mu\nu}$, v^*_ρ and v_σ

We indicate with $h_n^{\mu\nu(i)}$ (transverse-traceless), $\xi_n^{\mu(i)}$ (transverse) and $\phi_n^{(i)}$ the pure spin-2, spin-1 and spin-0 eigenfunctions of the Laplace-Beltrami operator on the sphere of unitary radius that are normalized as

$$\delta^{ij}\delta_{nm} = \int \mathrm{d}^4x\,\sqrt{\widetilde{g}}\,h_n^{\mu\nu(i)}(x)h_{\mu\nu}^{m(j)}(x) = \int \mathrm{d}^4x\,\sqrt{\widetilde{g}}\,\xi_n^{\mu(i)}(x)\xi_\mu^{m(j)}(x) = \int \mathrm{d}^4x\,\sqrt{\widetilde{g}}\,\phi_n^{(i)}(x)\,\phi_m^{(j)}(x)\,,$$

corresponding to the eigenvalues $\lambda_n^{(2)}$, $\lambda_n^{(1)}$ and $\lambda_n^{(0)}$ respectively. The modes $\{h_n^{\mu\nu},\, v_n^{\mu\nu},\, w_n^{\mu\nu},\, z_n^{\mu\nu}\}$, with

$$\begin{aligned}
v_n^{\mu\nu} &= \left[\frac{1}{2}\left(\lambda_n^{(1)}-3\right)\right]^{-\frac{1}{2}} \nabla^{(\mu}\xi_n^{\nu)}\,, \quad n=2,\dots\,,\\
w_n^{\mu\nu} &= \left[\lambda_n^{(0)}\left(\frac{3}{4}\lambda_n^{(0)}-3\right)\right]^{-\frac{1}{2}}\left(\nabla^\mu\nabla^\nu-\frac{1}{4}\widetilde{g}^{\mu\nu}\Box\right)\phi_n\,, \quad n=2,\dots\,,\\
z_n^{\mu\nu} &= \frac{1}{2}\widetilde{g}^{\mu\nu}\phi_n\,, \quad n=0,1,2,\dots\,,
\end{aligned}$$

of which we do not write explicitly the degeneracy indexes form the orthonormal basis for symmetric tensors.

Moreover, defining the longitudinal vector modes

$$l_n^\mu = \left(\lambda_n^{(0)}\right)^{-\frac{1}{2}} \nabla^\mu \phi_n \,, \quad n = 1, 2, \dots \,,$$

the latter, together with the transverse modes ξ_n^μ, form the orthonormal basis for vectors.

Expand the graviton field $\widehat{h}^{\mu\nu}$ as

Taylor, Veneziano

$$\widehat{h}^{\mu\nu} = \sum_{n=2}^{\infty} a_n h_n^{\mu\nu} + \sum_{n=2}^{\infty} b_n v_n^{\mu\nu} + \sum_{n=2}^{\infty} c_n w_n^{\mu\nu} + \sum_{n=0}^{\infty} e_n z_n^{\mu\nu}$$

$$\widehat{h} \equiv \widetilde{g}_{\mu\nu} \widehat{h}^{\mu\nu} = 2 \sum_{n=0}^{\infty} e_n \phi_n \,,$$

and the ghost field v^μ as

$$v^\mu = \sum_{n=1}^{\infty} g_n \, \xi_n^\mu + \sum_{n=1}^{\infty} f_n \, l_n^\mu$$

so that we have

$$
\begin{aligned}
64\pi G\,(S_2 + S_{\mathrm{gf}}) = & \sum_{n=2}^{\infty} a_n^2 \left[\lambda_n^{(2)} - 2a^2\Lambda + 8\right] \\
& + \sum_{n=2}^{\infty} b_n^2 \left[\xi^{-1}\left(\lambda_n^{(1)} - 3\right) - 2a^2\Lambda + 6\right] \\
& + \sum_{n=2}^{\infty} c_n^2 \left[\xi^{-1}\left(\frac{3}{4}\lambda_n^{(0)} - 6\right) - \frac{\lambda_n^{(0)}}{2} - 2a^2\Lambda + 6\right] \\
& + \sum_{n=0}^{\infty} e_n^2 \left[\frac{-3+\xi^{-1}}{2}\lambda_n^{(0)} + 2a^2\Lambda\right] \\
& + \sum_{n=2}^{\infty} 2e_n c_n (\xi^{-1} - 1)\left[\lambda_n^{(0)}\left(\frac{3}{4}\lambda_n^{(0)} - 3\right)\right]^{\frac{1}{2}}
\end{aligned}
$$

$$
32\pi G\,S_{\mathrm{ghost}} = \sum_{n=1}^{\infty} g_n^* \, g_n \left(\lambda_n^{(1)} - 3\right) + \sum_{n=1}^{\infty} f_n^* \, f_n \left(\lambda_n^{(0)} - 6\right) .
$$

Therefore, the functional measure can be written as (defined as)

$$\mathcal{D}\widehat{h}_{\mu\nu}\,\mathcal{D}v^*_\rho\,\mathcal{D}v_\sigma \equiv \frac{1}{V_{SO(5)}}\prod_{n=2}^{\infty}\mathrm{d}a_n\prod_{n=2}^{\infty}\mathrm{d}b_n\prod_{n=2}^{\infty}\mathrm{d}c_n\prod_{n=0}^{\infty}\mathrm{d}e_n\prod_{n=2}^{\infty}\mathrm{d}g^*_n\prod_{n=2}^{\infty}\mathrm{d}g_n\prod_{n=1}^{\infty}\mathrm{d}f^*_n\prod_{n=1}^{\infty}\mathrm{d}f_n\,,$$

Notice that there is no integration over the zero modes g_1^* and g_1 of S_{ghost}. The corresponding ghost fields are proportional to the ten Killing vectors ξ_1^μ. These zero eigenmodes correspond to residual gauge invariances which are not eliminated by gauge fixing in the presence of an $SO(5)$ spherical symmetry. Overcounting has been compensated by inserting the explicit group-volume factor $V_{SO(5)}$