SSM
Scuola Superiore Meridionale

The Maldacena-Shenker-Stanford Conjecture in General Relativity and beyond

INFN

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Contents

Some introductory remarks on the MSS Conjecture

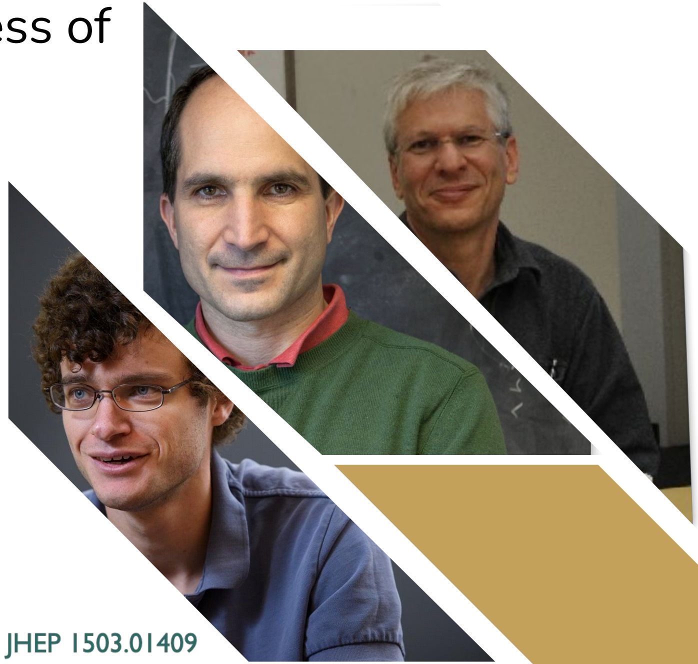
The Maldacena-Shenker-Stanford Conjecture

In thermal quantum systems with a large number of degrees of freedom it is conjectured that chaos cannot develop faster than exponentially and that there exists an universal upper bound to the Lyapunov exponent that parametrically controls the increasing chaoticness of the system.

$$\boxed{\lambda \leq \frac{2\pi}{\beta} = 2\pi T}$$

Motivated by the outcome of the holographic calculations of the out-of-time order correlation function (OTOC)

$$C(t) = -\langle [\hat{A}(t), \hat{B}(0)]^2 \rangle$$



J. Maldacena, S. H. Shenker and D. Stanford (2016) JHEP 1503.01409

The OTOC function as an indicator of quantum chaos

$$\langle \hat{A}(t)\hat{B}(0)\hat{B}(0)\hat{A}(t)\rangle,\ \langle \hat{B}(0)\hat{A}(t)\hat{A}(t)\hat{B}(0)\rangle$$

$$\langle \hat{A}(t)\hat{B}(0)\hat{A}(t)\hat{B}(0)\rangle,\ \langle \hat{B}(0)\hat{A}(t)\hat{B}(0)\hat{A}(t)\rangle$$

A first definition of quantum chaos

$$\hat{A}(t) \to \hat{x}(t)$$

$$\hat{B}(0) \to \hat{p}(0)$$

$$\boxed{C(t) = -\langle [\hat{A}(t), \hat{B}(0)]^2\rangle}$$

$$C(t) = -\langle [\hat{x}(t), \hat{p}(0)]^2\rangle \sim \hbar^2 e^{2\lambda t}$$

Semi-classical limit

$$[\hat{U}, \hat{V}] \to i\hbar\{U, V\}$$

$$[\hat{x}(t), \hat{p}(0)] \to i\hbar\{x(t), p(0)\} = i\hbar \frac{\partial x(t)}{\partial x(0)} \sim e^{\lambda t}$$

Lyapunov exponent

Sensitive dependence on initial conditions

$$F(t) = Tr\left[wB(0)wA(t)wB(0)wA(t)\right]$$

$$C'(t) = -Tr\left[w^2[A(t), B(0)]w^2[A(t), B(0)]\right], \quad w^4 = \frac{1}{Z}e^{-\beta H}$$

$$C'(t) = Tr[w^2 A(t)B(0)w^2 B(0)A(t)] + Tr[w^2 B(0)A(t)w^2 A(t)B(0)] - F\left(t + i\frac{\beta}{4}\right) - F\left(t - i\frac{\beta}{4}\right)$$

A second definition of quantum chaos

The holographic computation of the OTOC

$$C_{QC} = \langle A(t_1)B(t_2)A(t_3)B(t_4)\rangle$$

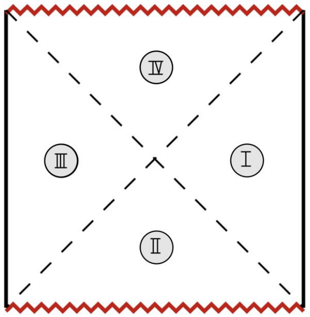
$$|\psi\rangle = A(t_3)B(t_4)|TFD\rangle$$

$$|\psi'\rangle = B(t_2)A(t_1)|TFD\rangle$$

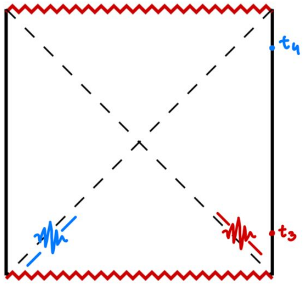
Thermofield double state

$$|TFD\rangle = \frac{1}{Z}\sum_n e^{-\beta E_n/2}|n\rangle_1 \times |n\rangle_2$$

Eternal black hole



$$ds^2 = -\frac{4l^2}{(1+uv)^2}dudv + R^2\frac{(1-uv)^2}{(1+uv)^2}d\phi^2$$



Bulk representation of $|\psi\rangle = A(t_3)B(t_4)|TFD\rangle$

in-state of a gravitational scattering process

$$\langle A(t)B(0)A(t)B(0)\rangle = \langle A(t)^2\rangle\langle B(0)^2\rangle\left(1 + \#iG_N e^{\frac{2\pi}{\beta}t} + \ldots\right)$$

Black holes as the fastest scramblers in nature

$$\boxed{\lambda = \frac{2\pi}{\beta}}$$

J. Maldacena (2003) JHEP 0106112
S. H. Shenker and D. Stanford (2013) JHEP 1306.0622, 1312.3296

The EGTG and the violation of the MSS Conjecture

General Relativity as a purely metric theory

Levi-Civita connection:

(i) Metric-compatibility

$$\nabla_\alpha g_{\mu\nu} = 0$$

(ii) Symmetry under the interchange of lower indices

$$\Gamma^\lambda{}_{\mu\nu} = \Gamma^\lambda{}_{\nu\mu}$$

$$T^\lambda{}_{\mu\nu} = \Gamma^\lambda{}_{\mu\nu} - \Gamma^\lambda{}_{\nu\mu} = 0$$

$$\downarrow$$

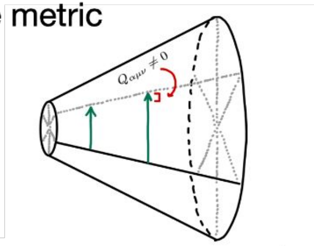
$$\boxed{\mathring{\Gamma}^\lambda{}_{\mu\nu} \equiv \left\{ \begin{array}{c} \lambda \\ \mu\nu \end{array} \right\} = \frac{1}{2} g^{\lambda\rho}(g_{\mu\rho,\nu} + g_{\nu\rho,\mu} - g_{\mu\nu,\rho})}$$

Metric-affine theories and the separate roles of the metric and the affine connection

The connection is not a-priori chosen to be the Levi-Civita connection:

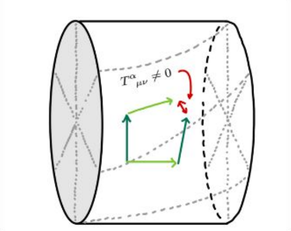
(i') The connection fails to covariantly conserve the metric

$$Q_{\alpha\mu\nu} = \nabla_\alpha g_{\mu\nu}$$
$$= \partial_\alpha g_{\mu\nu} - \Gamma^\lambda{}_{\alpha\mu} g_{\lambda\nu} - \Gamma^\lambda{}_{\alpha\nu} g_{\mu\lambda}$$



(ii') The torsion tensor is not identically null

$$T^\lambda{}_{\mu\nu} = \Gamma^\lambda{}_{\mu\nu} - \Gamma^\lambda{}_{\nu\mu}$$



$$\downarrow$$

$$\boxed{\Gamma^\alpha{}_{\mu\nu} = \mathring{\Gamma}^\alpha{}_{\mu\nu} + K^\alpha{}_{\mu\nu} + L^\alpha{}_{\mu\nu}}$$

$$K^\alpha{}_{\mu\nu} = \frac{1}{2}(T^\alpha{}_{\mu\nu} + T_\mu{}^\alpha{}_\nu + T_\nu{}^\alpha{}_\mu)$$ **Contorsion tensor**

$$L^\alpha{}_{\mu\nu} = \frac{1}{2}(Q^\alpha{}_{\mu\nu} - Q_\mu{}^\alpha{}_\nu - Q_\nu{}^\alpha{}_\mu)$$ **Disformation tensor**

J. B. Jiménez, L. Heisenberg, and T. S. Koivisto (2019) Universe 5 1903.06830

Metric-affine theories

$$R^{\alpha}{}_{\beta\mu\nu} = \Gamma^{\alpha}{}_{\nu\beta,\mu} - \Gamma^{\alpha}{}_{\mu\beta,\nu} + \Gamma^{\alpha}{}_{\mu\sigma}\Gamma^{\sigma}{}_{\nu\beta} - \Gamma^{\alpha}{}_{\nu\sigma}\Gamma^{\sigma}{}_{\mu\beta} =$$
$$= \mathring{R}^{\alpha}{}_{\beta\mu\nu} + \mathring{\nabla}_{\mu}D^{\alpha}{}_{\nu\beta} - \mathring{\nabla}_{\nu}D^{\alpha}{}_{\mu\beta} + D^{\sigma}{}_{\nu\beta}D^{\alpha}{}_{\mu\sigma} - D^{\sigma}{}_{\mu\beta}D^{\alpha}{}_{\nu\sigma}$$
$$T^{\lambda}{}_{\mu\nu} = \Gamma^{\lambda}{}_{\mu\nu} - \Gamma^{\lambda}{}_{\nu\mu}$$
$$Q_{\alpha\mu\nu} = \nabla_{\alpha}g_{\mu\nu}$$

- Correspondingly to the newly introduced torsion and non-metricity tensors, one defines:

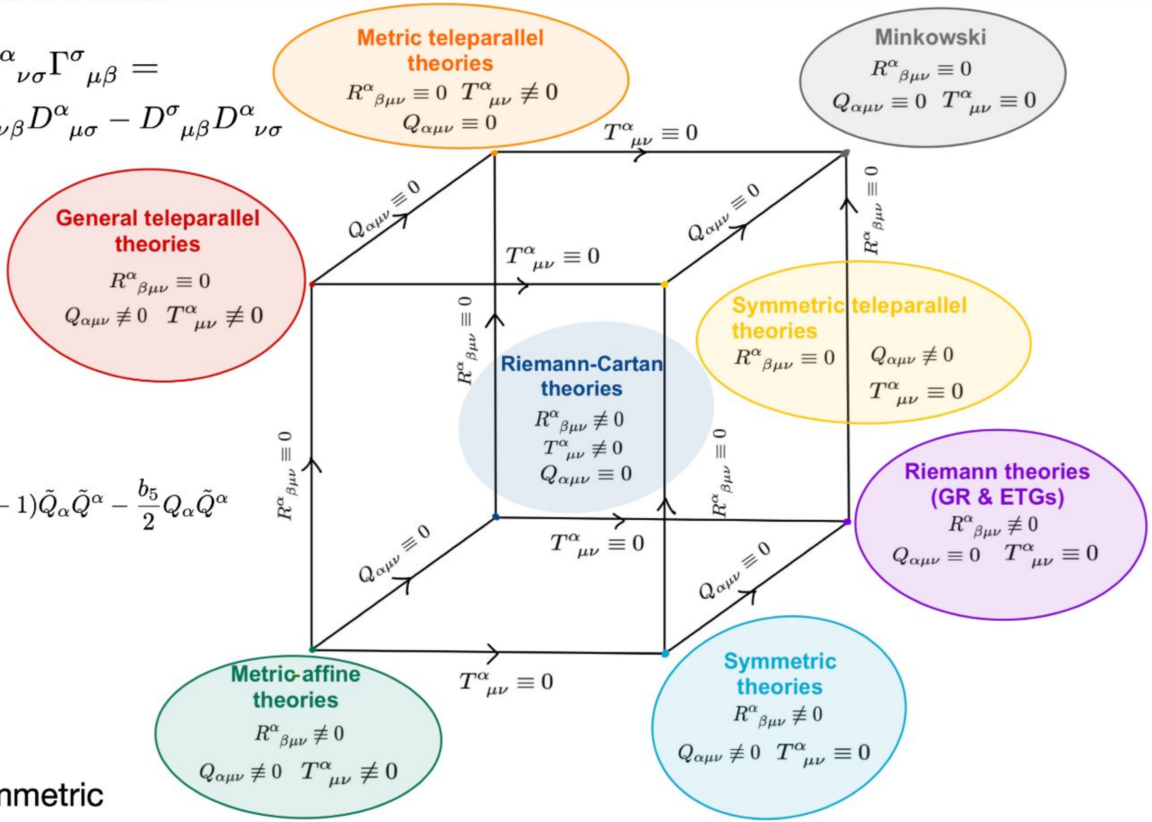
$$\mathbb{T} \equiv \frac{a_1}{4}T_{\alpha\mu\nu}T^{\alpha\mu\nu} + \frac{a_2}{2}T_{\alpha\mu\nu}T^{\mu\alpha\nu} - a_3T_{\alpha}T^{\alpha}$$

$$\mathbb{Q} \equiv -\frac{b_1}{4}Q_{\alpha\mu\nu}Q^{\alpha\mu\nu} + \frac{b_2}{2}Q_{\alpha\mu\nu}Q^{\mu\alpha\nu} + \frac{b_3}{4}Q_{\alpha}Q^{\alpha} - (b_4-1)\tilde{Q}_{\alpha}\tilde{Q}^{\alpha} - \frac{b_5}{2}Q_{\alpha}\tilde{Q}^{\alpha}$$

- In the framework of teleparallel theories the following convenient notation will be employed:

$\hat{\Gamma}^{\alpha}{}_{\mu\nu} \longleftarrow$ Over-hats in the metric teleparallel framework.

$\overset{\diamond}{\Gamma}{}^{\alpha}{}_{\mu\nu} \longleftarrow$ Over-diamonds in the symmetric teleparallel framework.



S. Capozziello, S. Cesare, C. Ferrara (2025) Eur.Phys.J.C. 2503.08167
S. Bahamonde, et al. (2023) Rep. Prog. Phys. 86 026901

The Geometric Trinity of Gravity and the Extended Geometric Trinity of Gravity

- TEGR, STEGR and GR define a set of three dynamically equivalent theories: the Geometric Trinity of Gravity (GTG).

- The extensions:

$$\mathring{R} \to f(\mathring{R}), \;\; \hat{T} \to f(\hat{T}),$$
$$\overset{\diamond}{Q} \to f(\overset{\diamond}{Q})$$

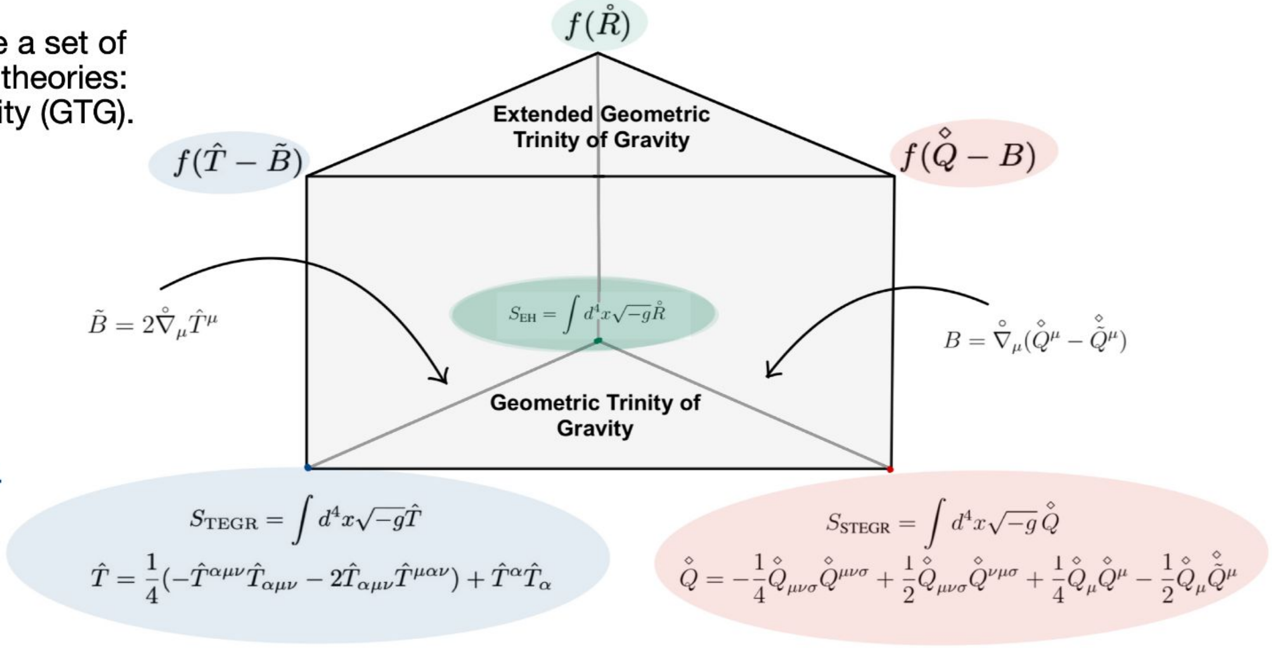
  are clearly not equivalent.

- Including boundary terms in the functional dependence + specializing to the cases:

$$f(\hat{T}, \tilde{B}) = f(\hat{T} - \tilde{B})$$
$$f(\overset{\diamond}{Q}, B) = f(\overset{\diamond}{Q} - B)$$

$$\downarrow \oplus \; f(\mathring{R})$$

Extended Geometric Trinity of Gravity (EGTG)



S. Capozziello, S. Cesare, C. Ferrara (2025) Eur.Phys.J.C. 2503.08167
S. Capozziello, V. De Falco and C. Ferrara (2022) Eur. Phys. J. C. 2208.03011
S. Capozziello, V. De Falco and C. Ferrara (2023) Eur. Phys. J. C. 2307.13280

The violation of the MSS Conjecture

$$ds^2 = \frac{1}{\Lambda \cos^2 \tau}(-d\tau^2 + dx^2) + \frac{1}{\Lambda}(d\theta^2 + \sin^2\theta d\varphi^2) =$$

$$= e^{2\rho}(-d\tau^2 + dx^2) + e^{-2\phi}(d\theta^2 + \sin^2\theta d\varphi^2)$$ The Nariai solution

Derivation of the perturbed field equations

$$\frac{f_R(R_0) - 2\Lambda f_{RR}(R_0)}{2\Lambda\cos^2\tau}\delta R + \frac{f(R)}{\Lambda\cos^2\tau}\delta\rho + (-\delta\ddot{\rho} + 2\delta\ddot{\phi} + \delta\rho'' - 2\tan\tau\delta\dot{\phi})f_R(R_0) + \tan\tau f_{RR}(R_0)\delta\dot{R} - f_{RR}\delta R'' = 0$$

$$\frac{2\Lambda f_{RR}(R_0) - f_R(R_0)}{2\Lambda\cos^2\tau}\delta R + f_R(R_0)[\delta\ddot{\rho} - \delta\rho'' + 2\delta\phi'' - 2\delta\dot{\phi}\tan\tau] - \frac{f(R_0)}{\Lambda\cos^2\tau}\delta\rho + \tan\tau f_{RR}(R_0)\delta\dot{R} - f_{RR}(R_0)\delta\ddot{R} = 0$$

$$\frac{2\Lambda f_{RR}(R_0) - f_R(R_0)}{2\Lambda}\delta R + \cos^2\tau f_R(R_0)[-\delta\ddot{\phi} + \delta\phi''] + \frac{\delta\phi}{\Lambda}f(R_0) + \cos^2\tau f_{RR}(R_0)(\delta R'' - \delta\ddot{R}) = 0$$

$$2f_R(R_0)(\delta\dot{\phi}' - \delta\phi'\tan\tau) + f_{RR}(R_0)(\tan\tau\delta R' - \delta\dot{R}') = 0$$

where $\delta R = 4\Lambda(\delta\phi - \delta\rho) + 2\Lambda\cos^2\tau[\delta\ddot{\rho} - \delta\rho'' - 2\delta\ddot{\phi} + 2\delta\phi'']$

$$\frac{d^2\delta\phi}{dt^2} + \frac{d\delta\phi}{dt} - \frac{2(2\alpha - 1)}{3\alpha}\delta\phi = 0$$

S. Nojiri and S. D. Odintsov (2013) Class. Quant. Grav. 1301.2775
A. Addazi, S. Capozziello and S. D. Odintsov (2021) Phys. Lett. B 2103.16856

The violation of the MSS Conjecture

$$\frac{d^2\delta\phi}{dt^2} + \frac{d\delta\phi}{dt} - \frac{2(2\alpha-1)}{3\alpha}\delta\phi = 0$$

$$\delta\phi(t) = \delta\phi(0)\, e^{at}$$

$$\frac{2\Lambda f_{RR}(R_0)}{f_R(R_0)}$$

Stability parameter

(*) Model dependence

$$a \equiv a_{\pm} = \frac{-1 \pm \sqrt{1+4m^2}}{2}, \quad m^2 = \frac{2(2\alpha-1)}{3\alpha}$$

$m^2 > 0$

$$\boxed{\lambda_g = a_+ = \frac{-1+\sqrt{1+4m^2}}{2}}$$

Gravitational Lyapunov exponent

(*) Violation in the EGTG framework

$$g_{\mu\nu}[-f(\hat{T},\tilde{B}) + (f_{\tilde{B}} + f_{\hat{T}})\tilde{B}] + 2\mathring{R}_{\mu\nu} f_{\hat{T}} - 4[(\partial^\rho f_{\hat{T}}) + (\partial^\rho f_{\tilde{B}})]\hat{S}_{\mu\rho\nu} + 2\mathring{\nabla}_\mu \mathring{\nabla}_\nu f_{\tilde{B}} - 2g_{\mu\nu}\mathring{\Box} f_{\tilde{B}} = 0$$

$$f_{\overset{\diamond}{Q}}(\mathring{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathring{R}) + 2\overset{\diamond}{P}{}^{\lambda}{}_{\mu\nu}\mathring{\nabla}_\lambda(f_{\overset{\diamond}{Q}} + f_B) + \mathring{\nabla}_\mu\mathring{\nabla}_\nu f_B - g_{\mu\nu}\mathring{\Box} f_B - \frac{1}{2}g_{\mu\nu}(f - f_B B - f_{\overset{\diamond}{Q}}\overset{\diamond}{Q}) = 0$$

$f(\mathring{R}) = f(\hat{T} - \tilde{B}) = f(\overset{\diamond}{Q} - B)$

$$f_{\mathring{R}}(\mathring{R})\mathring{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}f(\mathring{R}) - \mathring{\nabla}_\mu\mathring{\nabla}_\nu f_{\mathring{R}}(\mathring{R}) + \mathring{\Box} f_{\mathring{R}}(\mathring{R})g_{\mu\nu} = 0$$

What’s next?

On a theoretical level...

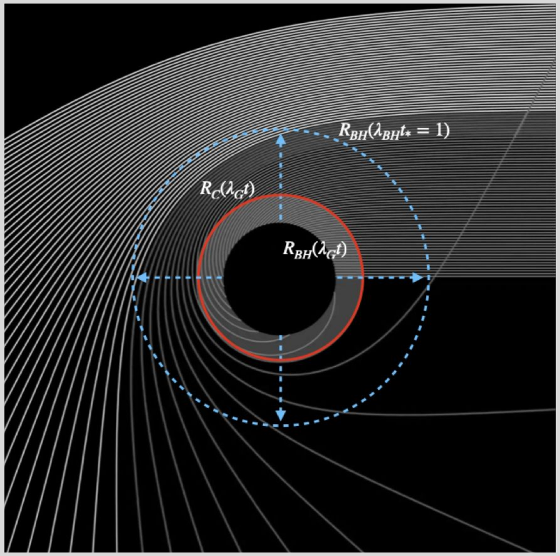
(i) Identifying the MSS Conjecture and its eventual violation as a possible discerning criterion between different theories of gravity.

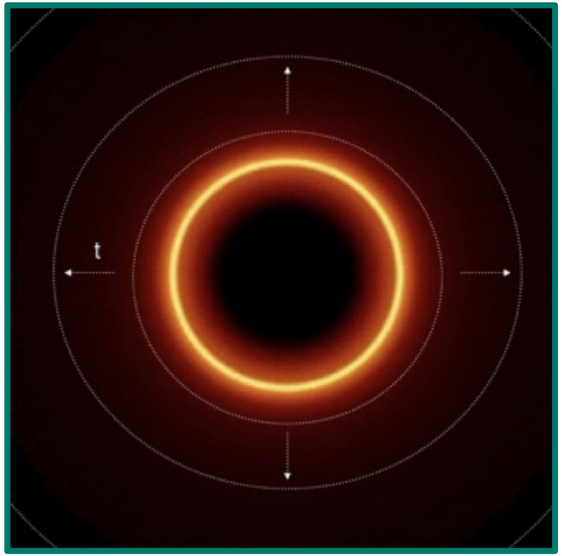
(ii) Delving into the holographic approach to quantum gravity through the analysis of quantum chaos.

(iii) Scrutinizing the entropic content of the MSS Conjecture by employing the Iyer-Wald approach of derivation of the black hole entropy in a general theory of gravity.

On an experimental level...

(iv) Linking the eventual new parameters determining the violation of the MSS bound with experimentally measurable variables (e.g. the photon sphere critical radius).





A. Addazi and S. Capozziello (2023) Phys. Lett. B 2303.01956

A. Addazi, S. Capozziello and S. D. Odintsov (2021) Phys. Lett. B 2103.16856

Thank you for your attention!

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