Metric-Affine Theories of Gravity

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Metric-Affine Theories of Gravity

Metric tensor and **affine connection** are two independent geometrical structures.

$$\Gamma^{\lambda}{}_{\mu\nu} = \tilde{\Gamma}^{\lambda}{}_{\mu\nu} + K^{\lambda}{}_{\mu\nu} + L^{\lambda}{}_{\mu\nu}$$

- $\tilde{\Gamma}^{\lambda}{}_{\mu\nu} = \frac{1}{2} g^{\alpha\lambda}\left(g_{\mu\alpha,\nu} + g_{\alpha\nu,\mu} - g_{\mu\nu,\alpha}\right)$ **Levi-Civita connection**;
- $K^{\lambda}{}_{\mu\nu} = \frac{1}{2} g^{\alpha\lambda}\left(T_{\mu\alpha\nu} + T_{\nu\alpha\mu} - T_{\alpha\mu\nu}\right)$ **Contortion tensor**;
- $L^{\lambda}{}_{\mu\nu} = \frac{1}{2} g^{\alpha\lambda}\left(Q_{\alpha\mu\nu} - Q_{\mu\alpha\nu} - Q_{\nu\alpha\mu}\right)$ **Disformation tensor**.

Metric-Affine Theories of Gravity

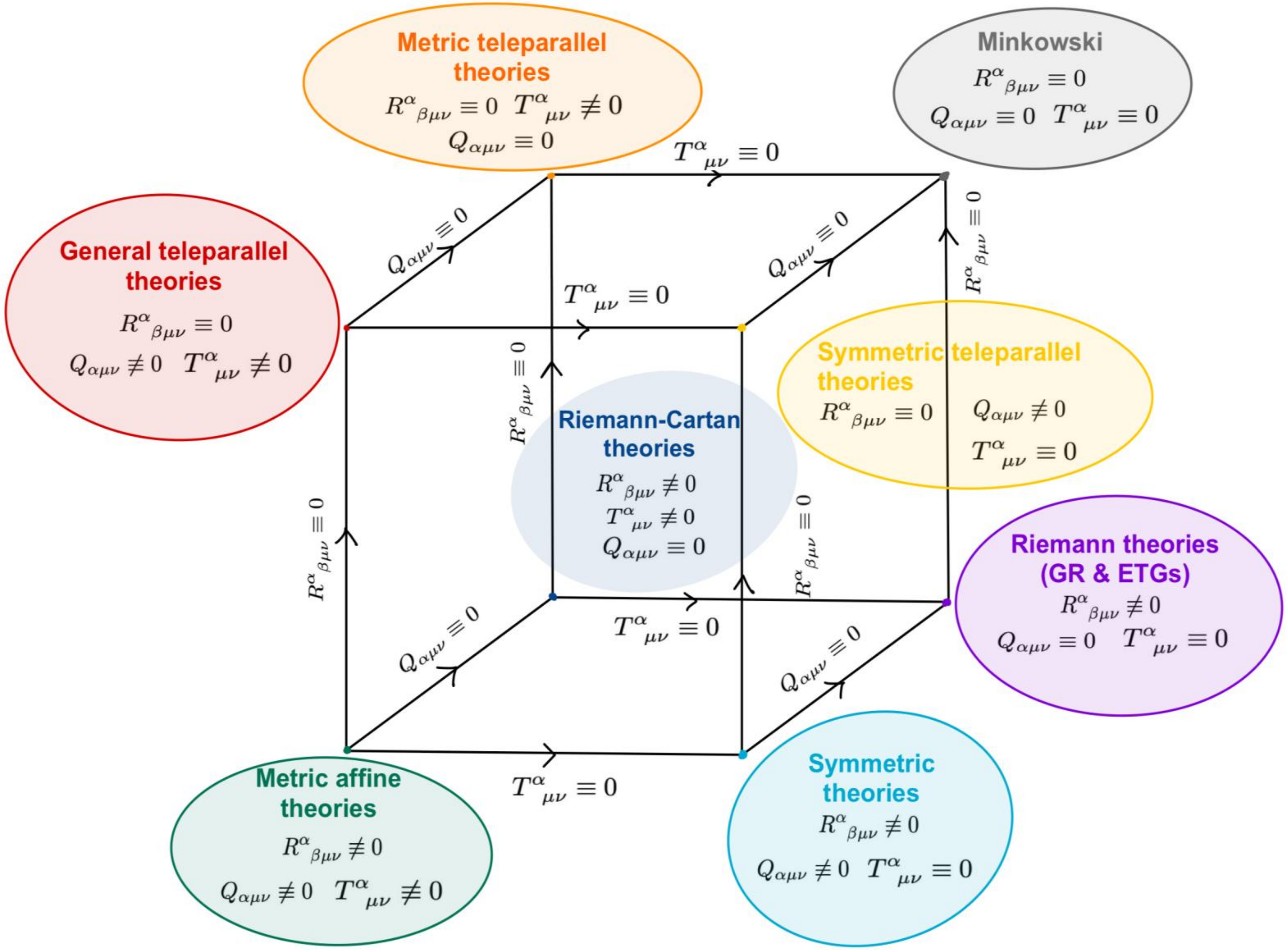
Through the affine connection, it is possible to define:

- the **curvature tensor** $R^{\alpha}_{\beta\mu\nu} = \partial_{\mu}\Gamma^{\alpha}_{\beta\nu} - \partial_{\nu}\Gamma^{\alpha}_{\beta\mu} + \Gamma^{\alpha}_{\sigma\mu}\Gamma^{\sigma}_{\beta\nu} - \Gamma^{\alpha}_{\sigma\nu}\Gamma^{\sigma}_{\beta\mu}$;

- the **torsion tensor** $T^{\alpha}_{\mu\nu} = \Gamma^{\alpha}_{\nu\mu} - \Gamma^{\alpha}_{\mu\nu}$;

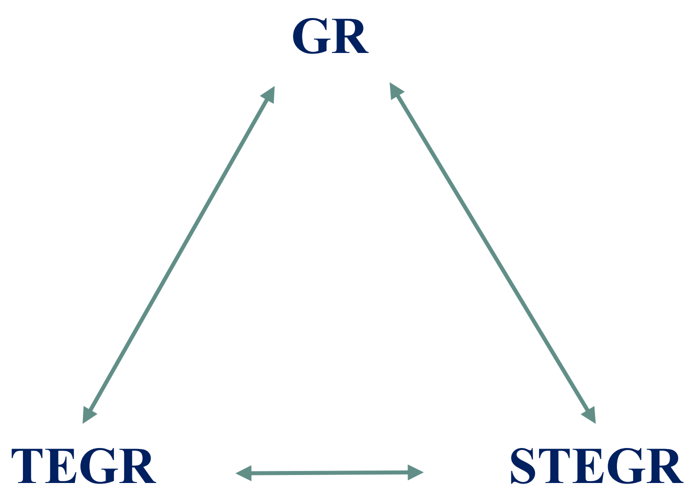
- the **nonmetricity tensor** $Q_{\alpha\mu\nu} = \nabla_{\alpha} g_{\mu\nu} = \partial_{\alpha} g_{\mu\nu} - \Gamma^{\sigma}_{\mu\alpha}\, g_{\sigma\nu} - \Gamma^{\sigma}_{\nu\alpha} g_{\mu\sigma}$.

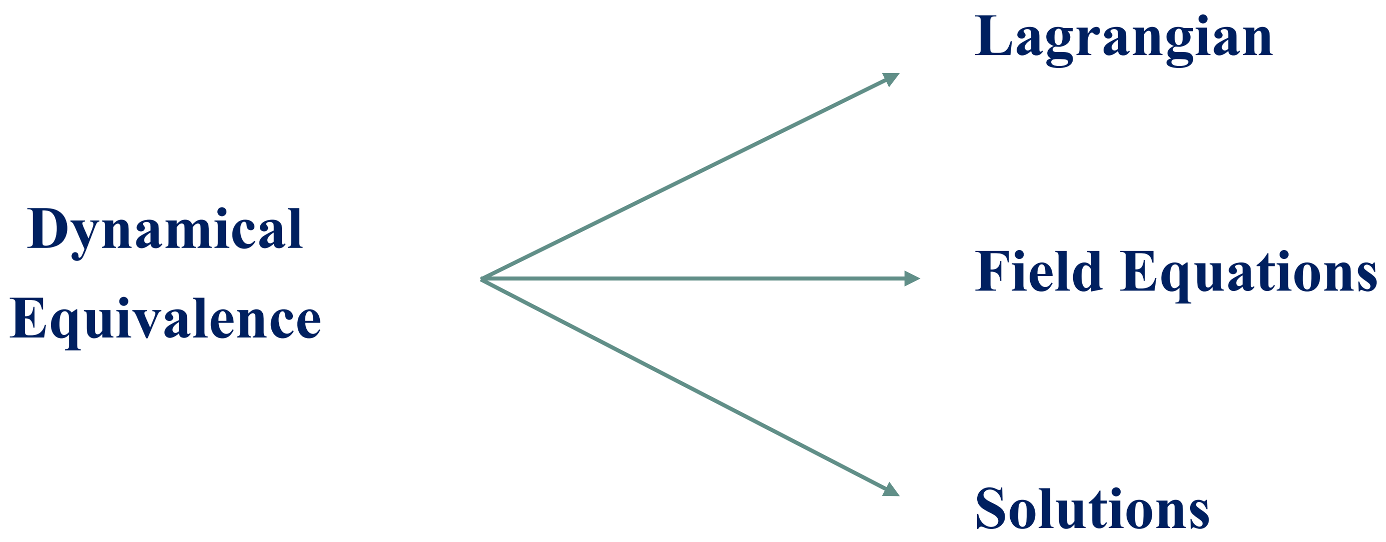
Metric-Affine Theories of Gravity



The Geometric Trinity of Gravity

General Relativity (GR) can be formulated also through torsion or non-metricity in Teleparallel Equivalent of GR (TEGR) and Symmetric TEGR (STEGR), respectively. They form the so-called **Geometric Trinity of Gravity**.





The Geometric Trinity of Gravity

The equivalence of the three representations is first shown via the Lagrangian approach:

$$R = \tilde{R} \rightarrow \mathcal{L}_{GR} = \tilde{R}$$

$$R = \tilde{R} - T + \hat{B} = 0 \rightarrow \mathcal{L}_{TEGR} = T - \hat{B}$$

$$R = \tilde{R} - Q - B = 0 \rightarrow \mathcal{L}_{STEGR} = Q + B$$

TEGR and STEGR Lagrangian are equivalent to the GR Hilbert-Einstein action up to a boundary terms B.

The Extended Geometric Trinity of Gravity

It is possible to consider generalizations of the Geometric Trinity Gravity, replacing the scalars $\tilde{R}$, T, and Q with arbitrary smooth functions in the respective actions; this led to three dynamical equivalent extended theories:

- Extended Metric Gravity, $f(\tilde{R})$;
- Extended Teleparallel Gravity, $f(T, \hat{B})$;
- Extended Symmetric Teleparallel Gravity, $f(Q, B)$.

The equivalence is based on the choice of the smooth function f.

The Extended Geometric Trinity of Gravity

In fact, the equivalence of the Ricci scalar with the torsion and the non-metricity scalar up to the respective boundary terms leds to:

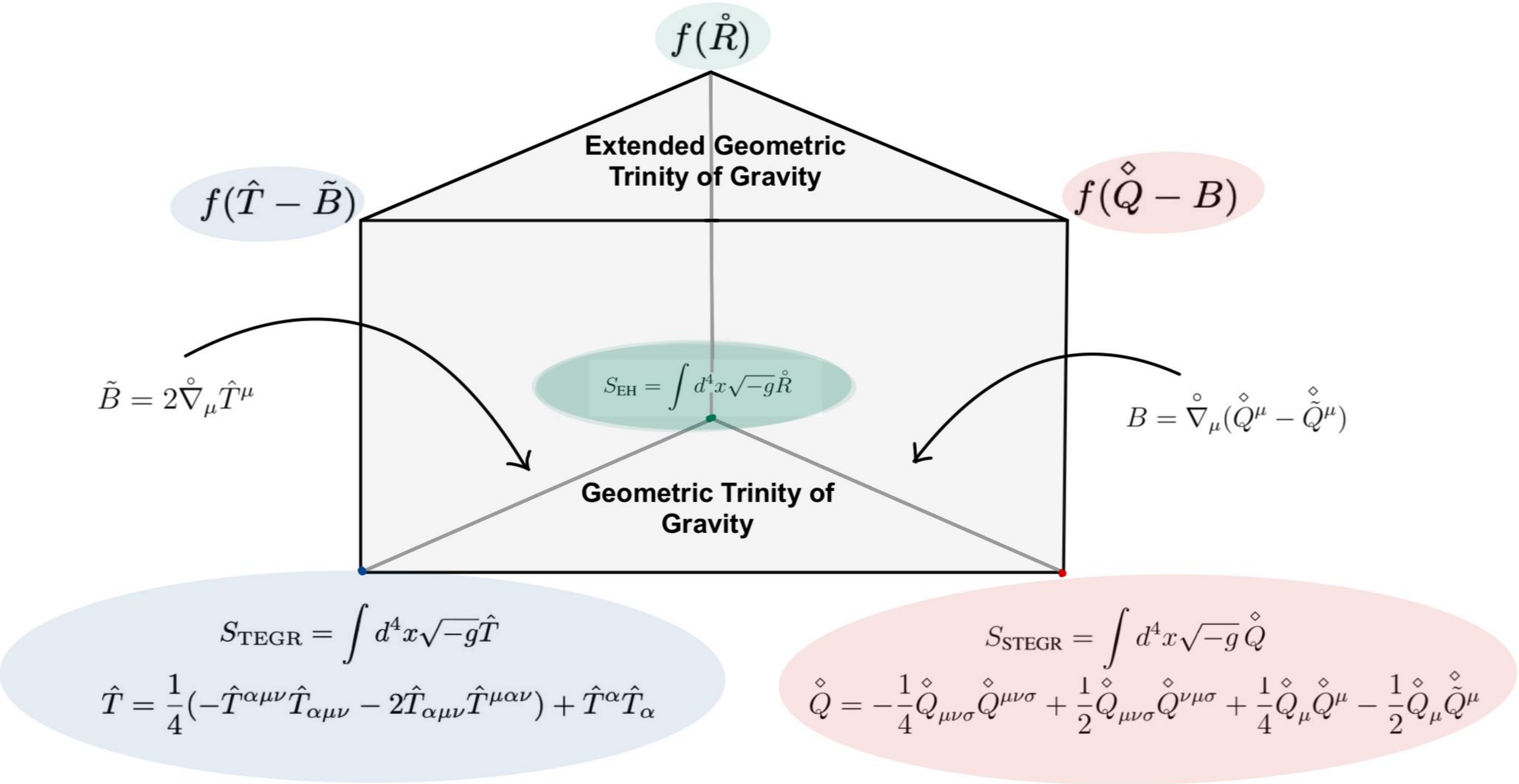
$$R = \tilde{R} \longrightarrow f(\tilde{R})$$

$$R = 0 \rightarrow \tilde{R} = T - \hat{B} \longrightarrow f(\tilde{R}) = f(T, \hat{B}) = f(T - \hat{B})$$

$$R = 0 \rightarrow \tilde{R} = Q + B \longrightarrow f(\tilde{R}) = f(Q, B) = f(Q + B)$$

These conditions define the equivalence between $f(\tilde{R})$ Lagrangian and field equations with the $f(T, \hat{B})$, and $f(Q, B)$ theories.
In this perspective, $f(\tilde{R})$, $f(T, \hat{B})$, and $f(Q, B)$ form the so-called **Extended Geometric Trinity of Gravity**.

The Extended Geometric Trinity of Gravity



The Extended Geometric Trinity of Gravity

The alternative approach to discuss extensions of gravity theories is constructing effective stress-energy tensors containing the further geometric DoFs. In this picture, the more **geometry** appear as additional **matter fields**.

It is based on:

- The second Bianchi Identity: $\nabla_\lambda R^\alpha_{\beta\mu\nu} + \nabla_\mu R^\alpha_{\beta\lambda\nu} + \nabla_\nu R^\alpha_{\beta\mu\lambda} = (T^\sigma_{\mu\lambda} R^\alpha_{\beta\sigma\nu} + T^\sigma_{\nu\lambda} R^\alpha_{\beta\mu\sigma} + T^\sigma_{\nu\mu} R^\alpha_{\beta\sigma\lambda})$;

- the covariant conservation of the stress-energy tensor: $\nabla_\mu T^{\mu\nu} = 0$.

Thus, starting from the second Bianchi Identity for a general linear affine connection, it is possible to infer the field equations in TEGR and STEGR framework, which are equivalent to Einstein field equations.

The Extended Geometric Trinity of Gravity

Curvature, torsion, and non-metricity are simply alternative ways of representing the same gravitational field, accounting for the same DoFs.

The conservation laws can also be applied to EGTG. In fact, the $f(\tilde{R})$ field equations can be rewritten w.r.t. the Einstein tensor:

$$G_{\mu\nu} = \chi(T^{m}_{\mu\nu} + T^{eff}_{\mu\nu}),$$

separating the usual matter energy-momentum tensor and the effective energy-momentum tensor, which is given by the geometrical contribution:

$$T^{eff}_{\mu\nu} = \frac{1}{f(\tilde{R})}\left(\nabla_\mu\nabla_\nu f(\tilde{R})_{\tilde{R}} - f\left(\tilde{R}\right)_{\tilde{R}} g_{\mu\nu} + g_{\mu\nu}\frac{f\left(\tilde{R}\right) - \tilde{R} f(\tilde{R})_{\tilde{R}}}{2}\right).$$

The Einstein tensor as usual is $G_{\mu\nu} = \tilde{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu}\tilde{R}$.

The Extended Geometric Trinity of Gravity

Since for the second Bianchi Identity $\nabla_\mu G^{\mu\nu} = 0$, as well as the energy-momentum tensor is divergence free, we get that:

$$\nabla_\mu T^{\mu\nu}_{eff} = 0$$

Hence, the continuity equation holds also for the $T^{\mu\nu}_{eff}$. As a consequence, generalized conservation will be also valid for the field equations of $f(T - \hat{B})$ and $f(Q + B)$, according to their dynamical equivalence to $f(\tilde{R})$.

Projective Symmetries

Projective transformations preserve the so-called **projective structure**, defined as the set of all solutions of the autoparallel equation of a manifold. Thus, in GR test particles paths are determined by the geodesic equation:

$$\frac{d^2x^\lambda}{d\tau^2} + \tilde{\Gamma}^\lambda_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = f(\lambda)\frac{dx^\mu}{d\lambda},$$

that throughout an affine parametrization $\tau(\lambda)$ turns out to be:

$$\frac{d^2x^\lambda}{d\tau^2} + \tilde{\Gamma}^\lambda_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0,$$

and are interpreted as the ***shortest*** line after minimizing the $\int ds$.

Projective Symmetries

On the other hand, the auto-parallels equation:

$$u^{\lambda}\nabla_{\lambda}u^{\mu} = f(\lambda)\frac{dx^{\mu}}{d\lambda}$$

should be considered as the ***straightest*** lines, after having considered the minimization of $\int ds$, and its solution depends only by the connection ∇. Two affine connections on a manifold are called **projectively equivalent** if their geodesics coincide (the affine parametrizations may, of course, differ) and belong to the same lightcone projective structure if they admit the same lightcone geodesics: $u^{\mu}u_{\mu} = 0$.

Projective Symmetries

A transformation of the connection that leaves the projective structure invariant is known as projective invariance and it is such that:

$$\delta\Gamma^{\alpha}{}_{\mu\nu}u^{\mu}u^{\nu} = hu^{\alpha}.$$

In the context of MAGs, the most general projective transformation is:

$$\Gamma^{\alpha}{}_{\mu\nu} \to \Gamma^{\alpha}{}_{\mu\nu} + \delta^{\alpha}_{\mu}p_{\nu} + \delta^{\alpha}_{\nu}q_{\mu} + g_{\mu\nu}r^{\alpha} + \varepsilon^{\sigma\alpha}{}_{\mu\nu}s_{\sigma},$$

and torsion and non-metricity tensor, respectively transform as:

$$\delta T^{\rho}_{\mu\nu} = \delta^{\rho}_{\nu}(p-q)_{\mu} - \delta^{\rho}_{\mu}(p-q)_{\nu} + 2\varepsilon^{\sigma\rho}{}_{\mu\nu}s_{\sigma}$$
$$\delta Q_{\rho\mu\nu} = -2g_{\mu\nu}p_{\rho} - g_{\mu\rho}(q+r)_{\nu} - g_{\nu\rho}(q+r)_{\mu}$$

Quadratic Non-metric Gravity

Here, we consider a non-metric quadratic Lagrangian:

$$\mathcal{L} = \mathrm{a}_0 R + \mathrm{a}_1 \mathrm{Q}_{\alpha\mu\nu}\mathrm{Q}^{\alpha\mu\nu} + \mathrm{a}_2 \mathrm{Q}_{\nu\mu\rho}\mathrm{Q}^{\rho\mu\nu} + \mathrm{a}_3 \mathrm{Q}^{\mu}\mathrm{Q}_{\mu} + \mathrm{a}_4 \widehat{\mathrm{Q}}^{\mu}\widehat{\mathrm{Q}}_{\mu} + \mathrm{a}_5 \widehat{\mathrm{Q}}^{\mu}\mathrm{Q}_{\mu} + \lambda_{\rho}^{\mu\nu}\left(\Gamma^{\rho}_{\mu\nu} - \Gamma^{\rho}_{\nu\mu}\right)$$

We work in the so-called Palatini formulation, and impose that torsion is zero by adding a Lagrange multiplier $\lambda_{\rho}^{\mu\nu}\left(\Gamma^{\rho}_{\nu\mu} - \Gamma^{\rho}_{\mu\nu}\right)$.

We apply the general projective transformation to this Lagrangian, without the direction $\varepsilon^{\sigma\alpha}{}_{\mu\nu}s_{\sigma}$, obtaining the following:

$$\begin{gathered}
p^{\rho}Q_{\rho}(4a_1 + 16a_3 + 2a_5) + \\
p^{\rho}\widehat{\mathrm{Q}}_{\rho}(4a_2 + 4a_4 + 8a_5) + \\
q^{\rho}Q_{\rho}(2a_2 + 4a_3 + 5a_5) + \\
q^{\rho}\widehat{\mathrm{Q}}_{\rho}(4a_1 + 2a_2 + 10a_4 + 2a_5) + \\
\mathrm{r}^{\rho}Q_{\rho}\left(\frac{1}{2}\mathrm{a}_0 + 2a_2 + 4a_3 + 5a_5\right) + \\
\mathrm{r}^{\rho}\widehat{\mathrm{Q}}_{\rho}(\mathrm{a}_0 + 4a_1 + 2a_2 + 10a_4 + 2a_5)
\end{gathered}$$

Quadratic Non-metric Gravity

Collecting the terms together, $\mathrm{p}^{\rho} = \mathrm{q}^{\rho} = \mathrm{r}^{\rho}$, expressing the solution with respect to p^{ρ}, we obtain:

$$p^{\rho} Q_{\rho} \left(\frac{1}{2} \mathrm{a}_0 + 4a_1 + 2a_2 + 20a_3 + 7a_5 \right)$$
$$p^{\rho} \widehat{\mathrm{Q}}_{\rho} (\mathrm{a}_0 + 4a_1 + 6a_2 + 14a_4 + 10a_5)$$

Then we applied to the Lagrangian $\mathcal{L}$, the usual projective transformation:

$$\Gamma^{\alpha}{}_{\mu\nu} \to \Gamma^{\alpha}{}_{\mu\nu} + \delta^{\alpha}_{\mu} p_{\nu} + \delta^{\alpha}_{\nu} q_{\mu},$$

and applying the same collecting method, we obtain:

$$p^{\rho} Q_{\rho} (4a_1 + 2a_2 + 20a_3 + 7a_5)$$
$$p^{\rho} \widehat{\mathrm{Q}}_{\rho} (4a_1 + 6a_2 + 14a_4 + 10a_5).$$

Quadratic Non-metric Gravity

$$\mathcal{M}_{\mu\nu}$$
$$= a_0\left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R\right) - \frac{1}{2} g_{\mu\nu} Q + a_1\left[\nabla_\alpha\left(\sqrt{g} Q^\alpha_{\mu\nu}\right) - \sqrt{g} Q_\nu^{\alpha\beta} Q_{\alpha\beta\mu} + \sqrt{g}\frac{1}{2} Q_{\mu\alpha\beta} Q_\nu^{\alpha\beta}\right]$$
$$+ \frac{1}{2} a_2\left[\nabla_\alpha\left(\sqrt{g}\left(Q^\alpha_{\mu\nu} + Q^\alpha_{\nu\mu}\right)\right) + \sqrt{g}\frac{1}{2}\left(Q_\mu^{\alpha\beta} Q_{\beta\alpha\nu} + Q_\nu^{\alpha\beta} Q_{\beta\alpha\mu}\right)\right]$$
$$+ \frac{1}{2} a_3\left[2\nabla_\alpha\left(\sqrt{g} g_{\mu\nu} Q^\alpha\right) + \sqrt{g} Q_\mu Q_\nu - 2\sqrt{g} Q^\alpha Q_{\alpha\mu\nu}\right] + a_4\left[-\sqrt{g}\widehat{\mathrm{Q}}_\mu \widehat{\mathrm{Q}}_\nu + \nabla_\nu\left(\sqrt{g}\widehat{\mathrm{Q}}_\mu\right) + \nabla_\mu\left(\sqrt{g}\widehat{\mathrm{Q}}_\nu\right)\right]$$
$$+ a_5\left[-\sqrt{g}\widehat{\mathrm{Q}^\alpha} Q_{\alpha\mu\nu} + \nabla_\alpha\left(\sqrt{g} g_{\mu\nu}\widehat{\mathrm{Q}^\alpha}\right) + \frac{1}{2}\left(\nabla_\mu(\sqrt{g} Q_\nu) + \nabla_\nu(\sqrt{g} Q_\mu)\right)\right],$$

$$\mathcal{P}_\alpha^{\mu\nu}$$
$$= a_0\left(Q_\alpha^{\mu\nu} + \delta_\alpha^\nu \widehat{\mathrm{Q}^\mu}\right) + a_1 4 Q_\alpha^{\nu\mu} + 2a_2\left(Q_\alpha^{\mu\nu} + Q_\alpha^{\mu\nu}\right) + a_3 4 Q^\nu \delta_\alpha^\mu + 2a_4\left(\widehat{\mathrm{Q}}_\alpha g^{\mu\nu} + \widehat{\mathrm{Q}^\mu}\delta_\alpha^\nu\right)$$
$$+ a_5\left[2\widehat{\mathrm{Q}^\nu}\delta_\alpha^\mu + Q_\alpha g^{\mu\nu} + Q^\mu \delta_\alpha^\nu\right]$$

Conclusions

- Curvature, torsion, and non-metricity represents the same gravitational field, accounting for the same degrees of freedom.
- The equivalence among GR, TEGR, and STEGR is proved at three levels, and it is extended also for $f(\tilde{R}), f(T-\hat{B}), f(Q+B)$.
- The projective transformations allow to find a families of connections.
- The quadratic non-metric gravity, under a suitable choice of parameter is invariant under a projective transformation.
- The choice of the projective transformation changes the general curvature scalar, and, up to a choice of the coefficient $\mathrm{a_i}$, the Lagrangian $\mathcal{L}$ is invariant under a general projective transformation.
- The conditions on the parameters are the same when applying the projective transformation on the connection and the metric field equations.

Perspectives

- Investigation of metric field equations and connection field equations for quadratic torsion gravity under projective transformations;
- Constrained Hypermomentum under projective invariance:
  $\Delta^{\lambda\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta\Gamma_{\lambda\mu\nu}(\xi^b)}$;
- Investigation on how a projective transformation could eventually allow to connect MAGs and Teleparallel and Symmetric Teleparallel Theories;
- Autoparallel as equations of motion for MAGs.


Thank you!

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