

# Radiation-reaction correction to scattering binary dynamics at the Next-to-Leading Post-Newtonian Order

Sara Rufrano Aliberti

Based on **Phys. Rev. D 112, 104005**  
with Donato Bini and Andrea Geralico

13th November 2025

QGSKY Annual Meeting 2025, Catania

# Motivations

- Gravitational interaction between two bodies:
  - capture  $\rightarrow$  elliptic-like orbit;
  - escape  $\rightarrow$  hyperbolic-like or parabolic-like orbits

# Motivations

- Gravitational interaction between two bodies:
  - capture  $\rightarrow$  elliptic-like orbit;
  - escape  $\rightarrow$  hyperbolic-like or parabolic-like orbits  $\rightarrow$  never been detected.

# Motivations

- Gravitational interaction between two bodies:
  - capture  $\rightarrow$  elliptic-like orbit;
  - escape  $\rightarrow$  hyperbolic-like or parabolic-like orbits  $\rightarrow$  never been detected.
- GW observatories are improving ([LVK Collab., Phys. Rev. Lett. 135, 111403])  $\rightarrow$  expected to detect scattering processes;

# Motivations

- Gravitational interaction between two bodies:
  - capture  $\rightarrow$  elliptic-like orbit;
  - escape  $\rightarrow$  hyperbolic-like or parabolic-like orbits  $\rightarrow$  never been detected.
- GW observatories are improving ([LVK Collab., Phys. Rev. Lett. 135, 111403])  $\rightarrow$  expected to detect scattering processes;
- Radiation-reaction effects play important role in dynamics of binary system;
- Presence of radiation-reaction force: modification of the conservative equation of motion (EoM);

[Damour, Deruelle: 1981,1985; Damour, 1982; Blanchet, 1997; ...]

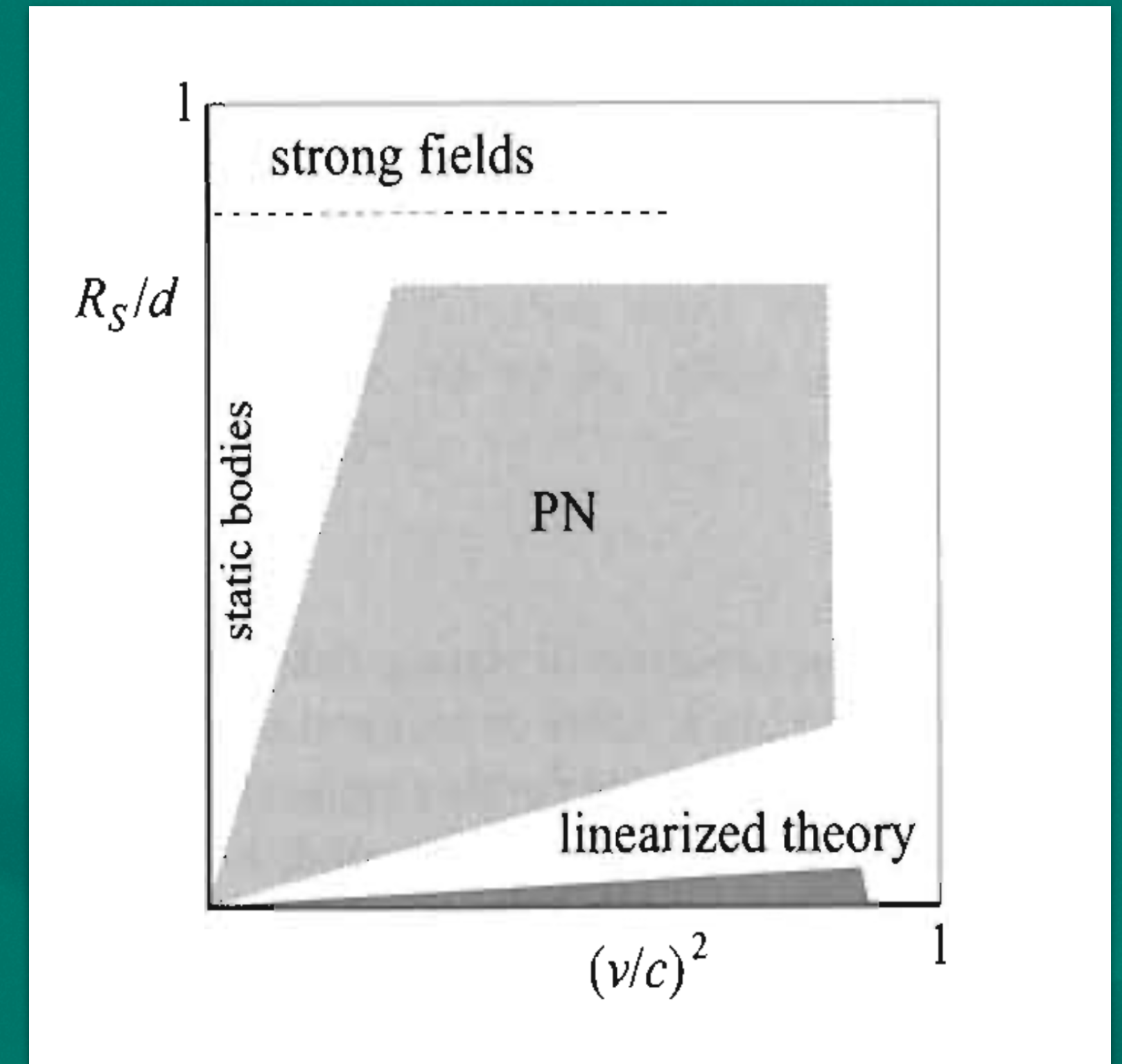
# Motivations

- Gravitational interaction between two bodies:
  - capture  $\rightarrow$  elliptic-like orbit;
  - escape  $\rightarrow$  hyperbolic-like or parabolic-like orbits  $\rightarrow$  never been detected.
- GW observatories are improving ([LVK Collab., Phys. Rev. Lett. 135, 111403])  $\rightarrow$  expected to detect scattering processes;
- Radiation-reaction effects play important role in dynamics of binary system;
- Presence of radiation-reaction force: modification of the conservative equation of motion (EoM);

[Damour, Deruelle: 1981,1985; Damour, 1982; Blanchet, 1997; ...]
- Aim: determine the radiation-reaction correction to the quasi-Keplerian orbital parameters at 3.5PN order.

# Post-Newtonian (PN) Formalism

- Idea: to go beyond the limit of sources moving in flat space-time.
- Self-gravitating system + flat space-time  $\rightarrow$  Newtonian gravity.
- (Moderately) relativistic system + gravitational forces  $\rightarrow$  Post-Newtonian formalism.
- Slowly moving and weakly self-gravitating sources  $\rightarrow$   $(v/c)^2 \sim R_S/d \ll 1 \rightarrow$  PN formalism ✓.



# Radiation-reaction correction

- Relative acceleration  $\mathbf{a} = \mathbf{a}_1 - \mathbf{a}_2$  in the center-of-mass (CM) frame:

$$\mathbf{a} = \mathbf{a}_{\text{cons}} + \mathbf{A}^{\text{rr}},$$

$$\eta = \frac{1}{c}, M = m_1 + m_2$$

with

$$\mathbf{A}^{\text{rr}} = \eta^5 \mathbf{A}_{2.5PN}^{\text{rr}} + \eta^7 \mathbf{A}_{3.5PN}^{\text{rr}} + \eta^9 \mathbf{A}_{4.5PN}^{\text{rr}} + \mathcal{O}(\eta^{10}).$$

# Radiation-reaction correction

- Relative acceleration  $\mathbf{a} = \mathbf{a}_1 - \mathbf{a}_2$  in the center-of-mass (CM) frame:

$$\mathbf{a} = \mathbf{a}_{\text{cons}} + \mathbf{A}^{\text{rr}},$$

$$\eta = \frac{1}{c}, M = m_1 + m_2$$

with

$$\mathbf{A}^{\text{rr}} = \eta^5 \mathbf{A}_{2.5PN}^{\text{rr}} + \eta^7 \mathbf{A}_{3.5PN}^{\text{rr}} + \eta^9 \mathbf{A}_{4.5PN}^{\text{rr}} + \mathcal{O}(\eta^{10}).$$

- Leading Order (LO) at 2.5PN:

[B. R. Iyer and C. M. Will, 10.1103/PhysRevD.52.6882]

# Radiation-reaction correction

- Relative acceleration  $\mathbf{a} = \mathbf{a}_1 - \mathbf{a}_2$  in the center-of-mass (CM) frame:

$$\mathbf{a} = \mathbf{a}_{\text{cons}} + \mathbf{A}^{\text{rr}},$$

$$\eta = \frac{1}{c}, M = m_1 + m_2$$

with

$$\mathbf{A}^{\text{rr}} = \eta^5 \mathbf{A}_{2.5PN}^{\text{rr}} + \eta^7 \mathbf{A}_{3.5PN}^{\text{rr}} + \eta^9 \mathbf{A}_{4.5PN}^{\text{rr}} + \mathcal{O}(\eta^{10}).$$

- Leading Order (LO) at 2.5PN:

[B. R. Iyer and C. M. Will, 10.1103/PhysRevD.52.6882]

Energy flux at infinity  $\longrightarrow \dot{E} \sim (\ddot{M}_{ij})^2$

# Radiation-reaction correction

- Relative acceleration  $\mathbf{a} = \mathbf{a}_1 - \mathbf{a}_2$  in the center-of-mass (CM) frame:

$$\mathbf{a} = \mathbf{a}_{\text{cons}} + \mathbf{A}^{\text{rr}},$$

$$\eta = \frac{1}{c}, M = m_1 + m_2$$

with

$$\mathbf{A}^{\text{rr}} = \eta^5 \mathbf{A}_{2.5PN}^{\text{rr}} + \eta^7 \mathbf{A}_{3.5PN}^{\text{rr}} + \eta^9 \mathbf{A}_{4.5PN}^{\text{rr}} + \mathcal{O}(\eta^{10}).$$

- Leading Order (LO) at 2.5PN:

[B. R. Iyer and C. M. Will, 10.1103/PhysRevD.52.6882]

Energy flux at infinity  $\longrightarrow \dot{E} \sim (\ddot{M}_{ij})^2 \rightarrow \dot{E} \sim \dot{M}_{ij} M_{ij}^{(5)}$

# Radiation-reaction correction

- Relative acceleration  $\mathbf{a} = \mathbf{a}_1 - \mathbf{a}_2$  in the center-of-mass (CM) frame:

$$\mathbf{a} = \mathbf{a}_{\text{cons}} + \mathbf{A}^{\text{rr}},$$

$$\eta = \frac{1}{c}, M = m_1 + m_2$$

with

$$\mathbf{A}^{\text{rr}} = \eta^5 \mathbf{A}_{2.5PN}^{\text{rr}} + \eta^7 \mathbf{A}_{3.5PN}^{\text{rr}} + \eta^9 \mathbf{A}_{4.5PN}^{\text{rr}} + \mathcal{O}(\eta^{10}).$$

- Leading Order (LO) at 2.5PN:

[B. R. Iyer and C. M. Will, 10.1103/PhysRevD.52.6882]

Energy flux at infinity  $\longrightarrow \dot{E} \sim (\ddot{M}_{ij})^2 \rightarrow \dot{E} \sim \dot{M}_{ij} M_{ij}^{(5)} \sim M v^i (x^j M_{ij}^{(5)})$

# Radiation-reaction correction

- Relative acceleration  $\mathbf{a} = \mathbf{a}_1 - \mathbf{a}_2$  in the center-of-mass (CM) frame:

$$\mathbf{a} = \mathbf{a}_{\text{cons}} + \mathbf{A}^{\text{rr}},$$

$$\eta = \frac{1}{c}, M = m_1 + m_2$$

with

$$\mathbf{A}^{\text{rr}} = \eta^5 \mathbf{A}_{2.5PN}^{\text{rr}} + \eta^7 \mathbf{A}_{3.5PN}^{\text{rr}} + \eta^9 \mathbf{A}_{4.5PN}^{\text{rr}} + \mathcal{O}(\eta^{10}).$$

- Leading Order (LO) at 2.5PN:

[B. R. Iyer and C. M. Will, 10.1103/PhysRevD.52.6882]

Energy flux at infinity  $\longrightarrow \dot{E} \sim (\ddot{M}_{ij})^2 \rightarrow \dot{E} \sim \dot{M}_{ij} M_{ij}^{(5)} \sim M v^i (x^j M_{ij}^{(5)})$

$$\dot{E} \sim M \mathbf{v} \cdot \mathbf{a}$$

# Radiation-reaction correction

- Relative acceleration  $\mathbf{a} = \mathbf{a}_1 - \mathbf{a}_2$  in the center-of-mass (CM) frame:

$$\mathbf{a} = \mathbf{a}_{\text{cons}} + \mathbf{A}^{\text{rr}},$$

$$\eta = \frac{1}{c}, M = m_1 + m_2$$

with

$$\mathbf{A}^{\text{rr}} = \eta^5 \mathbf{A}_{2.5PN}^{\text{rr}} + \eta^7 \mathbf{A}_{3.5PN}^{\text{rr}} + \eta^9 \mathbf{A}_{4.5PN}^{\text{rr}} + \mathcal{O}(\eta^{10}).$$

- Leading Order (LO) at 2.5PN:

[B. R. Iyer and C. M. Will, 10.1103/PhysRevD.52.6882]

Energy flux at infinity  $\longrightarrow \dot{E} \sim (\ddot{M}_{ij})^2 \rightarrow \dot{E} \sim \dot{M}_{ij} M_{ij}^{(5)} \sim M v^i (x^j M_{ij}^{(5)})$

$$\dot{E} \sim M \mathbf{v} \cdot \mathbf{a}$$

$$\rightarrow a^i \sim x^j M_{ij}^{(5)} = \mathcal{O}(\eta^5)$$

# Radiation-reaction correction

- CM radiation reaction relative force and acceleration:

$$\mathbf{F}^{\text{rr}} = \mu \mathbf{A}^{\text{rr}},$$

$$\rightarrow \mathbf{A}^{\text{rr}} = -\frac{8}{5}\nu \left(\frac{GM}{r^2}\right) \left(\frac{GM}{c^3 r}\right) \left[ -\left(A_{2.5\text{PN}} + \eta^2 A_{3.5\text{PN}} + \dots\right) \dot{r}\mathbf{n} + \left(B_{2.5\text{PN}} + \eta^2 B_{3.5\text{PN}} + \dots\right) \mathbf{v} \right]$$

$$\eta = \frac{1}{c}, M = m_1 + m_2,$$
$$\mu = \frac{m_1 m_2}{M}, \nu = \frac{\mu}{M}, \mathbf{n} = \mathbf{r}/r$$

# Radiation-reaction correction

- CM radiation reaction relative force and acceleration:

$$\mathbf{F}^{\text{rr}} = \mu \mathbf{A}^{\text{rr}},$$

$$\rightarrow \mathbf{A}^{\text{rr}} = -\frac{8}{5}\nu \left(\frac{GM}{r^2}\right) \left(\frac{GM}{c^3 r}\right) \left[ -\left(A_{2.5\text{PN}} + \eta^2 A_{3.5\text{PN}} + \dots\right) \dot{r}\mathbf{n} + \left(B_{2.5\text{PN}} + \eta^2 B_{3.5\text{PN}} + \dots\right) \mathbf{v} \right]$$

- Correction to Newtonian acceleration ( $\propto GM/r^2$ );

$$\eta = \frac{1}{c}, M = m_1 + m_2,$$
$$\mu = \frac{m_1 m_2}{M}, \nu = \frac{\mu}{M}, \mathbf{n} = \mathbf{r}/r$$

# Radiation-reaction correction

- CM radiation reaction relative force and acceleration:

$$\mathbf{F}^{\text{rr}} = \mu \mathbf{A}^{\text{rr}},$$

$$\eta = \frac{1}{c}, M = m_1 + m_2, \\ \mu = \frac{m_1 m_2}{M}, \nu = \frac{\mu}{M}, \mathbf{n} = \mathbf{r}/r$$

$$\rightarrow \mathbf{A}^{\text{rr}} = -\frac{8}{5}\nu \left(\frac{GM}{r^2}\right) \left(\frac{GM}{c^3 r}\right) \left[ -\left(A_{2.5\text{PN}} + \eta^2 A_{3.5\text{PN}} + \dots\right) \dot{r}\mathbf{n} + \left(B_{2.5\text{PN}} + \eta^2 B_{3.5\text{PN}} + \dots\right) \mathbf{v} \right]$$

- Correction to Newtonian acceleration ( $\propto GM/r^2$ );
- $\mathbf{A}^{\text{rr}} \rightarrow 0$  for test body limit (gravitational radiation  $\rightarrow 0$ );

# Radiation-reaction correction

- CM radiation reaction relative force and acceleration:

$$\mathbf{F}^{\text{rr}} = \mu \mathbf{A}^{\text{rr}},$$

$$\eta = \frac{1}{c}, M = m_1 + m_2, \\ \mu = \frac{m_1 m_2}{M}, \nu = \frac{\mu}{M}, \mathbf{n} = \mathbf{r}/r$$

$$\rightarrow \mathbf{A}^{\text{rr}} = -\frac{8}{5}\nu \left(\frac{GM}{r^2}\right) \left(\frac{GM}{c^3 r}\right) \left[ -\left(A_{2.5\text{PN}} + \eta^2 A_{3.5\text{PN}} + \dots\right) \dot{r}\mathbf{n} + \left(B_{2.5\text{PN}} + \eta^2 B_{3.5\text{PN}} + \dots\right) \mathbf{v} \right]$$

- Correction to Newtonian acceleration ( $\propto GM/r^2$ );
- $\mathbf{A}^{\text{rr}} \rightarrow 0$  for test body limit (gravitational radiation  $\rightarrow 0$ );
- $\mathbf{A}^{\text{rr}}$  related to the emission of GWs (nonlinear in  $G$ ) and dissipative (odd in velocities);

# Radiation-reaction correction

- CM radiation reaction relative force and acceleration:

$$\mathbf{F}^{\text{rr}} = \mu \mathbf{A}^{\text{rr}},$$

$$\eta = \frac{1}{c}, M = m_1 + m_2, \\ \mu = \frac{m_1 m_2}{M}, \nu = \frac{\mu}{M}, \mathbf{n} = \mathbf{r}/r$$

$$\rightarrow \mathbf{A}^{\text{rr}} = -\frac{8}{5}\nu \left(\frac{GM}{r^2}\right) \left(\frac{GM}{c^3 r}\right) \left[ -\left(A_{2.5\text{PN}} + \eta^2 A_{3.5\text{PN}} + \dots\right) \dot{r}\mathbf{n} + \left(B_{2.5\text{PN}} + \eta^2 B_{3.5\text{PN}} + \dots\right) \mathbf{v} \right]$$

- Correction to Newtonian acceleration ( $\propto GM/r^2$ );
- $\mathbf{A}^{\text{rr}} \rightarrow 0$  for test body limit (gravitational radiation  $\rightarrow 0$ );
- $\mathbf{A}^{\text{rr}}$  related to the emission of GWs (nonlinear in  $G$ ) and dissipative (odd in velocities);
- Leading order  $\mathcal{O}(\eta^5) \rightarrow A_{2.5\text{PN}}$  and  $B_{2.5\text{PN}}$  functions of  $\mathcal{O}(\eta^2) \rightarrow 6$  parameters;

# Radiation-reaction correction

- CM radiation reaction relative force and acceleration:

$$\mathbf{F}^{\text{rr}} = \mu \mathbf{A}^{\text{rr}},$$

$$\eta = \frac{1}{c}, M = m_1 + m_2, \\ \mu = \frac{m_1 m_2}{M}, \nu = \frac{\mu}{M}, \mathbf{n} = \mathbf{r}/r$$

$$\rightarrow \mathbf{A}^{\text{rr}} = -\frac{8}{5}\nu \left(\frac{GM}{r^2}\right) \left(\frac{GM}{c^3 r}\right) \left[ -\left(A_{2.5\text{PN}} + \eta^2 A_{3.5\text{PN}} + \dots\right) \dot{r} \mathbf{n} + \left(B_{2.5\text{PN}} + \eta^2 B_{3.5\text{PN}} + \dots\right) \mathbf{v} \right]$$

- Correction to Newtonian acceleration ( $\propto GM/r^2$ );
- $\mathbf{A}^{\text{rr}} \rightarrow 0$  for test body limit (gravitational radiation  $\rightarrow 0$ );
- $\mathbf{A}^{\text{rr}}$  related to the emission of GWs (nonlinear in  $G$ ) and dissipative (odd in velocities);
- Leading order  $\mathcal{O}(\eta^5) \rightarrow A_{2.5\text{PN}}$  and  $B_{2.5\text{PN}}$  functions of  $\mathcal{O}(\eta^2) \rightarrow 6$  parameters;
- Next-to-Leading order  $\mathcal{O}(\eta^{7/2}) \rightarrow A_{3.5\text{PN}}$  and  $B_{3.5\text{PN}}$  functions of  $\mathcal{O}(\eta^4) \rightarrow 12$  parameters.

# Radiation-reaction correction

$$A_i \text{ and } B_i \rightarrow \text{energy and angular momentum balance} \rightarrow \begin{cases} \frac{dE}{dt} \propto -\Phi_E \\ \frac{dJ}{dt} \propto -\Phi_J \end{cases}$$

# Radiation-reaction correction

$$A_i \text{ and } B_i \rightarrow \text{energy and angular momentum balance} \rightarrow \begin{cases} \frac{dE}{dt} \propto -\Phi_E \\ \frac{dJ}{dt} \propto -\Phi_J \end{cases}$$

- PN expression of  $E$  and  $J$  constant in time

# Radiation-reaction correction

$$A_i \text{ and } B_i \rightarrow \text{energy and angular momentum balance} \rightarrow \begin{cases} \frac{dE}{dt} \propto -\Phi_E \\ \frac{dJ}{dt} \propto -\Phi_J \end{cases}$$

- PN expression of  $E$  and  $J$  constant in time  $\rightarrow$  contribution from radiation-reaction parts;

# Radiation-reaction correction

$$A_i \text{ and } B_i \rightarrow \text{energy and angular momentum balance} \rightarrow \begin{cases} \frac{dE}{dt} \propto -\Phi_E \\ \frac{dJ}{dt} \propto -\Phi_J \end{cases}$$

- PN expression of  $E$  and  $J$  constant in time  $\rightarrow$  contribution from radiation-reaction parts;
- Energy and angular momentum can be ambiguous when they are not conserved:
  - (6) + 6 parameters at LO;
  - (12) + 12 parameters at NLO.

[G. A. Schott, Philos. Mag. 29,49 (1915)]

# Radiation-reaction correction

$$A_i \text{ and } B_i \rightarrow \text{energy and angular momentum balance} \rightarrow \begin{cases} \frac{dE}{dt} \propto -\Phi_E \\ \frac{dJ}{dt} \propto -\Phi_J \end{cases}$$

- PN expression of  $E$  and  $J$  constant in time  $\rightarrow$  contribution from radiation-reaction parts;

- Energy and angular momentum can be ambiguous when they are not conserved:

[G. A. Schott, Philos. Mag. 29,49 (1915)]

- (6) + 6 parameters at LO;

- (12) + 12 parameters at NLO.

[B. R. Iyer and C. M. Will, 10.1103/PhysRevD.52.6882]

$$\dot{r} F_r^{\text{rr}} + \dot{\phi} F_\phi^{\text{rr}} + \frac{dE_{\text{Schott}}}{dt} = -\Phi_E,$$

$$F_\phi^{\text{rr}} + \frac{dJ_{\text{Schott}}}{dt} = -\Phi_J.$$

- LO: 10 constraints over 12 parameters  $\rightarrow$  2 residual gauge parameters;
- NLO: 18 constraints over 24 parameters  $\rightarrow$  6 residual gauge parameters.

# Radiation-reaction correction

- Leading-order 2.5PN (harmonic coordinates): [B. R. Iyer and C. M. Will, 10.1103/PhysRevD.52.6882]

$$A_{2.5\text{PN}, \text{h}} = 3v^2 + \frac{17}{3} \frac{GM}{r},$$
$$B_{2.5\text{PN}, \text{h}} = v^2 + 3 \frac{GM}{r}.$$

Harmonic condition:

$$\partial_\nu(\sqrt{-g} g^{\mu\nu}) = 0$$

- Next-to-Leading order 3.5PN (harmonic coordinates): [S. Nissanke and L. Blanchet, 2005]

$$A_{3.5\text{PN}, \text{h}} = \left(-\frac{183}{28} - \frac{15}{2}\nu\right) v^4 + \left(-\frac{173}{14} + \frac{181}{6}\nu\right) v^2 \frac{GM}{r} + \left(\frac{285}{4} + \frac{15}{2}\nu\right) v^2 \dot{r}^2 + \left(-\frac{147}{4} - 47\nu\right) \dot{r}^2 \frac{GM}{r}$$
$$- 70\dot{r}^4 + \left(-\frac{989}{14} - 23\nu\right) \frac{G^2 M^2}{r^2},$$
$$B_{3.5\text{PN}, \text{h}} = \left(-\frac{313}{28} - \frac{3}{2}\nu\right) v^4 + \left(\frac{205}{42} + \frac{37}{2}\nu\right) v^2 \frac{GM}{r} + \left(\frac{339}{4} + \frac{3}{2}\nu\right) v^2 \dot{r}^2 + \left(-\frac{205}{12} - \frac{106}{3}\nu\right) \dot{r}^2 \frac{GM}{r}$$
$$- 75\dot{r}^4 + \left(-\frac{1325}{42} - 13\nu\right) \frac{G^2 M^2}{r^2}$$

# Variation of constants method at the 3.5PN order

- Radiation reaction corrections to dynamics → Lagrange's method of variation of constants;

- Relative acceleration in CM frame  $\mathbf{a}$ :

[Damour, Gopakumar, Iyer, gr-qc/0404128]

[Bini, Damour, Geralico, 2210.07165]

$$\mathbf{a} = \mathbf{a}_{\text{cons}} + \mathbf{A}^{\text{rr}},$$

- $\mathbf{a}_{\text{cons}} \rightarrow$  unperturbed integrable part;
- $\mathbf{A}^{\text{rr}} \rightarrow$  perturbation.
- Motion confined in a plane → polar coordinates;
- Functional form of the unperturbed equations of motion:

$$\begin{cases} r = S(l, c_1, c_2), \\ \dot{r} = \bar{n}(c_1, c_2) \frac{\partial S(l, c_1, c_2)}{\partial l}, \\ \phi = c_\phi + \bar{W}(l, c_1, c_2), \\ \dot{\phi} = \bar{n}(c_1, c_2) \frac{\partial \bar{W}(l, c_1, c_2)}{\partial l}, \end{cases}$$

$c_1, c_2, c_l, c_\phi$   
Integration  
constants

Average anomaly



with

$$l = \bar{n}(c_1, c_2)(t - t_0) + c_l$$



Mean motion

# Variation of constants method at the 3.5PN order

- From 1PN quasi-Keplerian parametrization of the hyperbolic motion:

[Cho,Dandapat,Gopakumar, 2111.00818]

[Bini, Damour, Geralico, 2007.11239]

$$\begin{cases} r = S(l, c_1, c_2), \\ \phi = c_\phi + \bar{W}(l, c_1, c_2), \end{cases} \rightarrow \begin{cases} S = \bar{a}_r(e_r \cosh u - 1), \\ \bar{W} = 2K \arctan \left[ \sqrt{\frac{e_\phi + 1}{e_\phi - 1}} \tanh \frac{u}{2} \right], \end{cases}$$

with  $u = u(l, c_1, c_2)$  through  $l = e_t \sinh u - u$ ;

# Variation of constants method at the 3.5PN order

- From 1PN quasi-Keplerian parametrization of the hyperbolic motion:

[Cho,Dandapat,Gopakumar, 2111.00818]

[Bini, Damour, Geralico, 2007.11239]

$$\begin{cases} r = S(l, c_1, c_2), \\ \phi = c_\phi + \bar{W}(l, c_1, c_2), \end{cases} \rightarrow \begin{cases} S = \bar{a}_r(e_r \cosh u - 1), \\ \bar{W} = 2K \arctan \left[ \sqrt{\frac{e_\phi + 1}{e_\phi - 1}} \tanh \frac{u}{2} \right], \end{cases}$$

with  $u = u(l, c_1, c_2)$  through  $l = e_t \sinh u - u$ ;

- Perturbed solution:  $c_1, c_2, c_l, c_\phi \rightarrow c_1(t), c_2(t), c_l(t), c_\phi(t)$ , satisfying

$$\begin{cases} \frac{dc_1}{dt} = \frac{\partial c_1}{\partial \mathbf{v}} \cdot \mathbf{A}_{rr}, \\ \frac{dc_2}{dt} = \frac{\partial c_2}{\partial \mathbf{v}} \cdot \mathbf{A}_{rr}, \\ \frac{dc_l}{dt} = - \left( \frac{\partial S}{\partial l} \right)^{-1} \left[ \frac{\partial S}{\partial c_1} \frac{dc_1}{dt} + \frac{\partial S}{\partial c_2} \frac{dc_2}{dt} \right], \\ \frac{dc_\phi}{dt} = - \frac{\partial \bar{W}}{\partial l} \frac{dc_l}{dt} - \frac{\partial \bar{W}}{\partial c_1} \frac{dc_1}{dt} - \frac{\partial \bar{W}}{\partial c_2} \frac{dc_2}{dt}. \end{cases}$$

# Variation of constants method at the 3.5PN order

- $c_1 = \bar{a}_r, c_2 = e_r$ :

[Cho, Gopakumar, Haney, Lee, 1807.02380]

[Bini, Damour, Geralico, 2007.11239]

[Arun, Blanchet, Iyer, Qusailah, 2007.11239]

$$\bar{a}_r = \frac{1}{2\bar{E}} \left[ 1 - \frac{\bar{E}}{2}(-7 + \nu)\eta^2 + \mathcal{O}(\eta^4) \right],$$

$$e_r = \sqrt{1 + 2\bar{E}j^2} \left[ 1 + \frac{\bar{E}}{2} \frac{5\bar{E}j^2(\nu - 3) + 2(\nu - 6)}{1 + 2\bar{E}j^2} \eta^2 + \mathcal{O}(\eta^4) \right],$$

with

$$\bar{E} = \frac{1}{2}v^2 - \frac{GM}{r} + \eta^2 \left[ \frac{3}{8}(1 - 3\nu)v^4 + \frac{1}{2}(3 + \nu)v^2 \frac{GM}{r} + \frac{1}{2}\nu \frac{GM}{r} \dot{r}^2 + \frac{1}{2} \frac{G^2 M^2}{r^2} \right] + \mathcal{O}(\eta^4),$$

$$j = (xv_y - yv_x) \left[ 1 + \eta^2 \left( \frac{1}{2}(1 - 3\nu)v^2 + (3 + \nu) \frac{GM}{r} \right) + \mathcal{O}(\eta^4) \right].$$

$$\eta = \frac{1}{c}, M = m_1 + m_2, \\ \mu = \frac{m_1 m_2}{M}, \nu = \frac{\mu}{M}, \mathbf{n} = \mathbf{r}/r$$

# Variation of constants method at the 3.5PN order

- Leading order:

[Bini, Damour, Geralico, 2210.07165]

$$\frac{d\bar{a}_r}{dt} = -2\bar{a}_r^2 \mathbf{v} \cdot \mathbf{A}^{\text{rr}},$$

$$\frac{de_r}{dt} = \frac{e_r^2 - 1}{e_r} \bar{a}_r \mathbf{v} \cdot \mathbf{A}^{\text{rr}} + \frac{\sqrt{e_r^2 - 1}}{e_r \sqrt{a_r}} [\mathbf{x} \times \mathbf{A}^{\text{rr}}]_z.$$

- Next-to-Leading order:

$$\bar{a}_r(\mathbf{x}, \mathbf{v}) = \bar{a}_r^{(0)}(\mathbf{x}, \mathbf{v}) + \eta^2 \bar{a}_r^{(2)}(\mathbf{x}, \mathbf{v}),$$

$$e_r(\mathbf{x}, \mathbf{v}) = e_r^{(0)}(\mathbf{x}, \mathbf{v}) + \eta^2 e_r^{(2)}(\mathbf{x}, \mathbf{v}),$$

$$\mathbf{A}^{\text{rr}}(\mathbf{x}, \mathbf{v}) = \mathbf{A}^{(0)\text{rr}}(\mathbf{x}, \mathbf{v}) + \eta^2 \mathbf{A}^{(2)\text{rr}}(\mathbf{x}, \mathbf{v}),$$

where  $\mathbf{A}^{(0)\text{rr}} \equiv \mathbf{A}_{2.5\text{PN}}^{\text{rr}}$ ,  $\mathbf{A}^{(2)\text{rr}} \equiv \mathbf{A}_{3.5\text{PN}}^{\text{rr}}$ . Therefore:

$$\begin{aligned} \bar{a}_r(\mathbf{x}, \mathbf{v}) &= \bar{a}_r^{(0)}(\mathbf{x}, \mathbf{v}) + \eta^2 \bar{a}_r^{(2)}(\mathbf{x}, \mathbf{v}) \\ \mathbf{A}^{\text{rr}}(\mathbf{x}, \mathbf{v}) &= \mathbf{A}^{(0)\text{rr}}(\mathbf{x}, \mathbf{v}) + \eta^2 \mathbf{A}^{(2)\text{rr}}(\mathbf{x}, \mathbf{v}) \end{aligned} \rightarrow \frac{d\bar{a}_r}{dt} = \frac{\partial \bar{a}_r}{\partial \mathbf{v}} \cdot \mathbf{A}^{\text{rr}} \rightarrow \begin{aligned} \frac{d\bar{a}_r^{(0)}}{dt} &= \frac{\partial \bar{a}_r^{(0)}}{\partial v^i} A_i^{(0)\text{rr}}, \\ \frac{d\bar{a}_r^{(2)}}{dt} &= \frac{\partial \bar{a}_r^{(0)}}{\partial v^i} A_i^{(2)\text{rr}} + \frac{\partial \bar{a}_r^{(2)}}{\partial v^i} A_i^{(0)\text{rr}}, \end{aligned}$$

# Variation of constants method at the 3.5PN order

- Evolution of the radiative-reaction part  $\delta^{\text{rr}}c_\alpha(t)$  from:

$$\left\{ \begin{array}{l} \frac{dc_1}{dt} = \frac{\partial c_1}{\partial \mathbf{v}} \cdot \mathbf{A}^{\text{rr}} \\ \frac{dc_2}{dt} = \frac{\partial c_2}{\partial \mathbf{v}} \cdot \mathbf{A}^{\text{rr}} \\ \frac{dc_l}{dt} = - \left( \frac{\partial S}{\partial l} \right)^{-1} \left[ \frac{\partial S}{\partial c_1} \frac{dc_1}{dt} + \frac{\partial S}{\partial c_2} \frac{dc_2}{dt} \right] \\ \frac{dc_\phi}{dt} = - \frac{\partial \bar{W}}{\partial l} \frac{dc_l}{dt} - \frac{\partial \bar{W}}{\partial c_1} \frac{dc_1}{dt} - \frac{\partial \bar{W}}{\partial c_2} \frac{dc_2}{dt} \end{array} \right. \longrightarrow \delta^{\text{rr}}c_\alpha(u) = \int_{-\infty}^u \frac{dc_\alpha}{dt} \frac{dt}{du^*} du^*$$

# Results

- Radiation-reaction contribution to the relative scattering angle:

$$[\delta^{rr}\phi]^{(0)} = \frac{\nu}{\bar{a}_r^{5/2}} \left[ \frac{2(121e_r^2 + 304)}{15(e_r^2 - 1)^3} \arccos\left(-\frac{1}{e_r}\right) + \frac{2(72e_r^4 + 1069e_r^2 + 134)}{45e_r^2(e_r^2 - 1)^{5/2}} \right],$$
$$[\delta^{rr}\phi]^{(2)} = \frac{\nu}{\bar{a}_r^{7/2}} \left[ \frac{48(7e_r^2 + 8)}{5(e_r^2 - 1)^{7/2}} \arccos^2\left(-\frac{1}{e_r}\right) + \frac{[(1316\nu + 2783)e_r^4 - (36400\nu + 90420)e_r^2 - 51296\nu - 58376]}{420(e_r^2 - 1)^4} \arccos\left(-\frac{1}{e_r}\right) + \frac{[(10080\nu + 51840)e_r^6 - (175420\nu + 1396049)e_r^4 - (1049720\nu + 1003562)e_r^2 - 80640\nu + 157576]}{6300e_r^2(e_r^2 - 1)^{7/2}} \right],$$
$$\rightarrow \chi_{\text{rr}}^{2.5\text{PN}+3.5\text{PN}} = [\delta^{\text{rr}}\phi]^{(0)} + \eta^2[\delta^{\text{rr}}\phi]^{(2)}$$

# What's next?

- Radiative-reaction corrections to quasi-Keplerian parameters at 4.5PN order;
- Application to losses of: [Arun, Blanchet, Iyer, Qusailah, 2007.11239]
  - Energy;
  - Angular momentum;
  - Linear momentum.

# What's next?

- Radiative-reaction corrections to quasi-Keplerian parameters at 4.5PN order;
- Application to losses of: [Arun, Blanchet, Iyer, Qusailah, 2007.11239]
  - Energy;
  - Angular momentum;
  - Linear momentum.

Thank you for the attention!

# Results

- $\delta^{\text{rr}} c_\alpha = \delta^{\text{rr}} c_\alpha^{(0)} + \eta^2 \delta^{\text{rr}} c_\alpha^{(2)}$ :

$$AT(u) \equiv \arctan \left[ \sqrt{\frac{e_r + 1}{e_r - 1}} \tanh \left( \frac{u}{2} \right) \right] + \arctan \left[ \sqrt{\frac{e_r + 1}{e_r - 1}} \right], \quad \chi \equiv e_r \cosh(u) - 1$$

$$\begin{aligned} \delta^{\text{rr}} \bar{a}_r^{(2)} = & \frac{\nu}{\bar{a}_r^{5/2}} \left\{ \left[ -\frac{32(e_r^2 - 1)^2}{\chi^7} + \frac{(\frac{56}{3}\nu + \frac{274}{15})(e_r^2 - 1)}{\chi^6} + \frac{((-\frac{24}{5}\nu + \frac{12}{5})e_r^2 - 24\nu - \frac{9166}{75})}{\chi^5} + \frac{((-\frac{256}{15}\nu - \frac{157}{2})e_r^2 + \frac{1223}{25} + \frac{104}{15}\nu)}{\chi^4(e_r^2 - 1)} \right. \right. \\ & + \frac{((-\frac{16}{5}\nu - \frac{272}{35})e_r^4 + (\frac{112}{45}\nu - \frac{28277}{1050})e_r^2 - \frac{18017}{525} - \frac{344}{15}\nu)}{\chi^3(e_r^2 - 1)^2} + \frac{(-\frac{3823}{420}e_r^4 + (-\frac{23161}{175} - \frac{1124}{45}\nu)e_r^2 - \frac{16327}{525} - \frac{512}{15}\nu)}{\chi^2(e_r^2 - 1)^3} \\ & \left. + \frac{((-\frac{235733}{2100} - \frac{352}{45}\nu)e_r^4 + (-\frac{191726}{525} - \frac{4796}{45}\nu)e_r^2 - \frac{21107}{525} - \frac{944}{15}\nu)}{\chi(e_r^2 - 1)^4} \right] e_r \sinh u + \frac{((-\frac{235733}{2100} - \frac{352}{45}\nu)e_r^4 + (-\frac{191726}{525} - \frac{4796}{45}\nu)e_r^2 - \frac{21107}{525} - \frac{944}{15}\nu)}{(e_r^2 - 1)^4} \\ & \left. + \frac{(-\frac{3823}{210}e_r^6 + (-\frac{328}{5}\nu - 471)e_r^4 + (-\frac{3472}{15}\nu - \frac{18476}{35})e_r^2 - \frac{1912}{105} - \frac{288}{5}\nu)}{(e_r^2 - 1)^{9/2}} AT(u) \right\} \end{aligned}$$

# Results

- $\delta^{\text{rr}} c_\alpha = \delta^{\text{rr}} c_\alpha^{(0)} + \eta^2 \delta^{\text{rr}} c_\alpha^{(2)}$ :

$$AT(u) \equiv \arctan \left[ \sqrt{\frac{e_r + 1}{e_r - 1}} \tanh \left( \frac{u}{2} \right) \right] + \arctan \left[ \sqrt{\frac{e_r + 1}{e_r - 1}} \right], \quad \chi \equiv e_r \cosh(u) - 1$$

$$\begin{aligned} \delta^{\text{rr}} e_r^{(2)} = \frac{\nu}{\bar{a}_r^{7/2}} & \left\{ \left[ \frac{16(e_r^2 - 1)^3}{\chi^7} + \frac{\left( -\frac{137}{15} - \frac{28}{3}\nu \right) (e_r^2 - 1)^2}{\chi^6} + \frac{\left( \left( \frac{114}{5} + \frac{12}{5}\nu \right) e_r^2 + \frac{2783}{75} + 12\nu \right) (e_r^2 - 1)}{\chi^5} \right. \right. \\ & + \frac{\left( \left( \frac{2333}{60} + \frac{68}{15}\nu \right) e_r^2 - \frac{1807}{75} + \frac{8}{15}\nu \right)}{\chi^4} + \frac{\left( \left( \frac{94}{35} + \frac{28}{5}\nu \right) e_r^4 + \left( \frac{223361}{6300} - \frac{296}{45}\nu \right) e_r^2 - \frac{5717}{1575} + \frac{64}{5}\nu \right)}{\chi^3(e_r^2 - 1)} \\ & + \frac{\left( \frac{8737}{840} e_r^4 + \left( \frac{1042}{45}\nu + \frac{500513}{6300} \right) e_r^2 + \frac{32}{5}\nu - \frac{5627}{1575} \right)}{\chi^2(e_r^2 - 1)^2} + \frac{\left( -\frac{144}{35} e_r^6 + \left( \frac{488}{45}\nu + \frac{1456529}{12600} \right) e_r^4 + \left( \frac{3214}{45}\nu + \frac{135823}{900} \right) e_r^2 + \frac{32}{5}\nu - \frac{5627}{1575} \right)}{\chi(e_r^2 - 1)^3} \Bigg] \sinh(u) \\ & + \frac{\left( -\frac{144}{35} e_r^6 + \left( \frac{488}{45}\nu + \frac{1456529}{12600} \right) e_r^4 + \left( \frac{3214}{45}\nu + \frac{135823}{900} \right) e_r^2 + \frac{32}{5}\nu - \frac{5627}{1575} \right)}{e_r(e_r^2 - 1)^3} + \frac{\left( \frac{5281}{420} e_r^4 + \left( \frac{2585}{7} + 68\nu \right) e_r^2 + \frac{14258}{105} + \frac{328}{3}\nu \right) e_r}{(e_r^2 - 1)^{7/2}} AT(u) \Bigg\} \end{aligned}$$

# Results

$$\chi \equiv e_r \cosh(u) - 1$$

- $\delta^{\text{rr}} c_\alpha = \delta^{\text{rr}} c_\alpha^{(0)} + \eta^2 \delta^{\text{rr}} c_\alpha^{(2)}$ :

$$\begin{aligned} \delta^{\text{rr}} c_l^{(2)} = & \frac{\nu \left( -25200 e_r^8 + 100800 e_r^6 - 151200 e_r^4 + 100800 e_r^2 - 25200 \right)}{1575 \chi^7 \bar{a}_r^{7/2} e_r^2} \\ & + \frac{\nu \left[ 105(140\nu + 377) e_r^6 - 315(140\nu + 377) e_r^4 + 315(140\nu + 377) e_r^2 - 105(140\nu + 377) \right]}{1575 \chi^6 \bar{a}_r^{7/2} e_r^2} \\ & + \frac{\nu \left[ -630(6\nu - 47) e_r^6 - 168(155\nu + 786) e_r^4 + 42(1510\nu + 4173) e_r^2 - 84(400\nu + 867) \right]}{1575 \chi^5 \bar{a}_r^{7/2} e_r^2} \\ & + \frac{\nu \left[ -630(122\nu - 37) e_r^4 + 840(113\nu + 87) e_r^2 - 210(86\nu + 459) \right]}{1575 \chi^4 \bar{a}_r^{7/2} e_r^2} \\ & + \frac{\nu \left[ -60(147\nu - 269) e_r^4 + 10(13594\nu + 13369) e_r^2 - 10(1932\nu + 3223) \right]}{1575 \chi^3 \bar{a}_r^{7/2} e_r^2} \\ & + \frac{\nu \left[ 90(1386\nu - 113) e_r^2 - 90(112\nu - 1) \right]}{1575 \chi^2 \bar{a}_r^{7/2} e_r^2} + \frac{32\nu (28\nu - 39)}{35 \chi \bar{a}_r^{7/2}} \end{aligned}$$

# Results

$$\chi \equiv e_r \cosh(u) - 1$$

- $\delta^{\text{rr}} c_\alpha = \delta^{\text{rr}} c_\alpha^{(0)} + \eta^2 \delta^{\text{rr}} c_\alpha^{(2)}$ :

$$\begin{aligned} \delta^{\text{rr}} c_\phi^{(2)} = & \frac{\nu}{1575 \bar{a}_r^{7/2}} \left\{ -\frac{25200 (e_r^2 - 1)^{7/2}}{\chi^7 e_r^2} + \frac{105 (140\nu + 377) (e_r^2 - 1)^{5/2}}{\chi^6 e_r^2} - \frac{42 (e_r^2 - 1)^{3/2} [15(6\nu - 55)e_r^2 + 800\nu + 1734]}{\chi^5 e_r^2} \right. \\ & - \frac{105 \sqrt{e_r^2 - 1} [(117\nu + 158)e_r^2 - 4(43\nu + 282)]}{\chi^4 e_r^2} + \frac{10 [6(42\nu - 53)e_r^4 + (1750\nu + 6817)e_r^2 - 1932\nu - 4399]}{\chi^3 e_r^2 \sqrt{e_r^2 - 1}} \\ & + \frac{90 [41 + 112\nu - (334 + 231\nu)e_r^2 + (83 + 119\nu)e_r^4]}{\chi^2 e_r^2 (e_r^2 - 1)^{3/2}} - \frac{7560 (2e_r^2 + 13)}{\chi (e_r^2 - 1)^{5/2}} - \frac{75600 e_r \sinh(u)}{\chi^2 (e_r^2 - 1)^2} \arctan \left[ \sqrt{\frac{e_r + 1}{e_r - 1}} \tanh \left( \frac{u}{2} \right) \right] \\ & - \frac{15120 e_r (2e_r^2 + 13) \sinh(u)}{\chi (e_r^2 - 1)^3} \arctan \left[ \sqrt{\frac{e_r + 1}{e_r - 1}} \tanh \left( \frac{u}{2} \right) \right] - \frac{15120 (7e_r^2 + 8)}{(e_r^2 - 1)^{7/2}} \arctan \left[ \sqrt{\frac{e_r + 1}{e_r - 1}} \tanh \left( \frac{u}{2} \right) \right]^2 \\ & \left. + \frac{15120 (7e_r^2 + 8)}{(e_r^2 - 1)^{7/2}} \arctan \left[ \sqrt{\frac{e_r + 1}{e_r - 1}} \right]^2 + \frac{15120 (2e_r^2 + 13)}{(e_r^2 - 1)^3} \arctan \left[ \sqrt{\frac{e_r + 1}{e_r - 1}} \right] \right\} \end{aligned}$$

# Results

$$AT(u) \equiv \arctan \left[ \sqrt{\frac{e_r + 1}{e_r - 1}} \tanh \left( \frac{u}{2} \right) \right] + \arctan \left[ \sqrt{\frac{e_r + 1}{e_r - 1}} \right]$$

$$\bullet \quad \delta^{\text{rr}} n = \frac{\partial n}{\partial \bar{a}_r} \delta^{\text{rr}} \bar{a}_r(t):$$

$$\delta^{\text{rr}} \bar{n}^{(0)} = \frac{\nu}{\bar{a}_r^4} \left\{ -\frac{2(37e_r^4 + 292e_r^2 + 96)}{5(e_r^2 - 1)^{7/2}} AT(u) - \frac{(673e_r^2 + 602)}{15(e_r^2 - 1)^3} - e_r \sinh(u) \left( \frac{(673e_r^2 + 602)}{15(e_r^2 - 1)^3 \chi} + \frac{(111e_r^2 + 314)}{15(e_r^2 - 1)^2 \chi^2} + \frac{2(36e_r^2 + 49)}{15(e_r^2 - 1) \chi^3} + \frac{14}{\chi^4} \right) \right\}$$

$$\delta^{\text{rr}} \bar{n}^{(2)} = \frac{\nu}{\bar{a}_r^5} \left\{ e_r \sinh(u) \left[ \frac{48(e_r^2 - 1)^2}{\chi^7} + \frac{\left(-28\nu - \frac{137}{5}\right)(e_r^2 - 1)}{\chi^6} + \frac{\left(\left(\frac{36}{5}\nu - \frac{18}{5}\right)e_r^2 + 36\nu + \frac{4583}{25}\right)}{\chi^5} + \frac{\left(\left(\frac{51}{4} + \frac{559}{15}\nu\right)e_r^2 - \frac{331}{15}\nu + \frac{1581}{50}\right)}{\chi^4(e_r^2 - 1)} \right. \right. \\ \left. \left. + \frac{\left(\left(-\frac{852}{35} + \frac{44}{5}\nu\right)e_r^4 + \left(-\frac{103}{45}\nu + \frac{19177}{700}\right)e_r^2 + \frac{1303}{45}\nu + \frac{35167}{350}\right)}{\chi^3(e_r^2 - 1)^2} + \frac{\left(\left(\frac{37}{6}\nu - \frac{11717}{280}\right)e_r^4 + \left(\frac{16979}{175} + \frac{4387}{90}\nu\right)e_r^2 + \frac{1519}{45}\nu + \frac{71277}{350}\right)}{\chi^2(e_r^2 - 1)^3} \right. \right. \\ \left. \left. + \frac{\left(\left(-\frac{235367}{1400} + \frac{4421}{90}\nu\right)e_r^4 + \left(\frac{14033}{90}\nu + \frac{204151}{350}\right)e_r^2 + \frac{126457}{350} + \frac{2743}{45}\nu\right)}{\chi(e_r^2 - 1)^4} \right] + \frac{\left(\left(-\frac{235367}{1400} + \frac{4421}{90}\nu\right)e_r^4 + \left(\frac{14033}{90}\nu + \frac{204151}{350}\right)e_r^2 + \frac{126457}{350} + \frac{2743}{45}\nu\right)}{(e_r^2 - 1)^4} \right. \\ \left. + \frac{\left(\left(-\frac{11717}{140} + \frac{37}{3}\nu\right)e_r^6 + \left(\frac{917}{5}\nu - \frac{117}{2}\right)e_r^4 + \left(\frac{4228}{15}\nu + \frac{48294}{35}\right)e_r^2 + \frac{272}{5}\nu + \frac{11036}{35}\right)}{(e_r^2 - 1)^{9/2}} AT(u) \right\}$$

# Results

$$\bullet \quad \delta^{\text{rr}} K = \frac{\partial K}{\partial \bar{a}_r} \delta^{\text{rr}} \bar{a}_r(t) + \frac{\partial K}{\partial e_r} \delta^{\text{rr}} e_r(t):$$

$$\delta^{\text{rr}} K^{(0)} = 0$$

$$\delta^{\text{rr}} K^{(2)} = \frac{\nu}{\bar{a}_r^{7/2}} \left[ \frac{48(7e_r^2 + 8)}{5(e_r^2 - 1)^{7/2}} AT(u) + \frac{24(2e_r^2 + 13)}{5(e_r^2 - 1)^3} + \frac{24e_r \sinh(u)}{(e_r^2 - 1)^2 \chi^2} + \frac{24e_r(2e_r^2 + 13) \sinh(u)}{5\chi(e_r^2 - 1)^3} \right]$$

$$AT(u) \equiv \arctan \left[ \sqrt{\frac{e_r + 1}{e_r - 1}} \tanh \left( \frac{u}{2} \right) \right] + \arctan \left[ \sqrt{\frac{e_r + 1}{e_r - 1}} \right]$$

# Results

$$\bullet \quad \delta^{\text{rr}} l = \int_{-\infty}^t \delta^{\text{rr}} \bar{n}(t') dt' + \delta^{\text{rr}} c_l(t) \equiv \delta^{\text{rr}} \tilde{l}(t) + \delta^{\text{rr}} c_l(t):$$

$$AT(u) \equiv \arctan \left[ \sqrt{\frac{e_r + 1}{e_r - 1}} \tanh \left( \frac{u}{2} \right) \right] + \arctan \left[ \sqrt{\frac{e_r + 1}{e_r - 1}} \right]$$

$$\delta^{\text{rr}} \tilde{l}^{(0)} = \frac{\nu}{\bar{a}_r^{5/2}} \left\{ \left[ \ln(\chi) - \chi - 1 - e_r \sinh(u) + u + \ln(2) - \ln(e_r) \right] B + \frac{2(49 + 36e_r^2)}{15\chi(e_r^2 - 1)} + \frac{7}{\chi^2} \right. \\ \left. + \left[ -e_r \sinh(u) AT(u) + \frac{i}{2} (\text{Li}_2(z) - \text{Li}_2(\bar{z})) - \frac{\sqrt{e_r^2 - 1}}{2} \right] C \right\}$$

with

$$z = e^\nu \frac{e_r}{(1 + i\sqrt{e_r^2 - 1})} = e^{\nu - i \arctan(\sqrt{e_r^2 - 1})} = -e^{\nu + 2i \arctan(\alpha)} \quad \text{due to } \arctan(\sqrt{e_r^2 - 1}) = -2 \arctan \left( \sqrt{\frac{e_r + 1}{e_r - 1}} \right) + \pi,$$

$$B = \frac{602 + 673 e_r^2}{15(e_r^2 - 1)^3}, \quad C = \frac{2(96 + 292 e_r^2 + 37 e_r^4)}{5(e_r^2 - 1)^{7/2}}$$

# Results

$$AT(u) \equiv \arctan \left[ \sqrt{\frac{e_r + 1}{e_r - 1}} \tanh \left( \frac{u}{2} \right) \right] + \arctan \left[ \sqrt{\frac{e_r + 1}{e_r - 1}} \right]$$

- $\delta^{\text{rr}} l = \int_{-\infty}^t \delta^{\text{rr}} \bar{n}(t') dt' + \delta^{\text{rr}} c_l(t) \equiv \delta^{\text{rr}} \tilde{l}(t) + \delta^{\text{rr}} c_l(t):$

$$\delta^{\text{rr}} \tilde{l}^{(2)} = \frac{\nu}{\bar{a}_r^{7/2}} \left\{ -\frac{i}{2} \left( \text{Li}_2(z) - \text{Li}_2(\bar{z}) \right) D + \left[ e_r \sinh(u) AT(u) + \frac{\sqrt{e_r^2 - 1}}{2} \right] E + (\chi + 1 + e_r \sinh(u)) F - (u + \ln(\chi) + \ln(2) - \ln(e_r)) G \right. \\ \left. - \frac{e_r^4 \left( \frac{68\nu}{5} - \frac{768}{35} \right) + e_r^2 \left( \frac{949\nu}{90} - \frac{2809}{2100} \right) + \frac{2422\nu}{45} + \frac{4717}{350}}{(e_r^2 - 1)^2 \chi} - \frac{e_r^2 \left( \frac{877\nu}{30} + \frac{11}{40} \right) - \frac{197\nu}{15} - \frac{227}{300}}{(e_r^2 - 1) \chi^2} - \frac{e_r^2 \left( \frac{12\nu}{5} - \frac{6}{5} \right) + 19\nu + \frac{1061}{25}}{\chi^3} \right. \\ \left. + \frac{(e_r^2 - 1)(137 + 140\nu)}{20\chi^4} - \frac{48(e_r^2 - 1)^2}{5\chi^5} \right\}$$

with

$$z = e^\nu \frac{e_r}{(1 + i\sqrt{e_r^2 - 1})} = e^{\nu - i \arctan(\sqrt{e_r^2 - 1})} = -e^{\nu + 2i \arctan(\alpha)} \quad \text{due to } \arctan(\sqrt{e_r^2 - 1}) = -2 \arctan \left( \sqrt{\frac{e_r + 1}{e_r - 1}} \right) + \pi,$$

And  $D, E, F, G$  functions of  $\nu, e_r$  (further details in the draft).

# Fourier space analysis

- Fourier transform of  $f(t)$ :

$$\hat{f}(\omega) = \int_{-\infty}^{+\infty} dt e^{i\omega t} f(t) \longrightarrow \hat{f}(\omega) = \int_{-\infty}^{+\infty} du \frac{dt}{du} e^{i\omega t(u)} f(t(u)).$$

- For  $t \rightarrow \infty$ ,  $f(t(u)) \rightarrow [f]$  (constant)  $\sim$  memory.
- Easier to perform FT taking the large- $e_r$  (eq. large- $j$ ) expansion of integrand, choosing:

$$T = \sinh(u) \rightarrow du = \frac{dT}{\sqrt{1+T^2}}$$

$$\rightarrow \hat{f}(\omega) = \int_{-\infty}^{+\infty} \frac{dT}{\sqrt{1+T^2}} \frac{dt}{du} e^{i\omega t(u)} f(t(u)) \bigg|_{u=\text{arcsinh} T} \quad \text{with} \quad \tilde{\omega} = \frac{\omega j}{p_{\infty}^2}$$

# Fourier space analysis

- Fourier transforms:

$$\begin{aligned}\hat{\delta}^{\text{rr}} \bar{a}_r(\tilde{\omega}) &= \nu \left[ \frac{32}{5} \tilde{\omega} K_0(\tilde{\omega}) \frac{i}{p_\infty j} + \left( \frac{74}{15 \tilde{\omega}} + \frac{118}{15} \tilde{\omega} + \frac{74}{15} \right) \frac{i \pi e^{-\tilde{\omega}}}{p_\infty^2 j^2} \right. \\ &\quad \left. + \left( \frac{196}{9} i \tilde{\omega} K_0(\tilde{\omega}) + \frac{376}{15} i \tilde{\omega}^2 K_1(\tilde{\omega}) - \frac{74}{15} I_1^{\text{at}}(\tilde{\omega}) + \frac{3136}{45} i K_1(\tilde{\omega}) - \frac{74}{15} K_0(\tilde{\omega}) \pi \right) \frac{1}{p_\infty^3 j^3} \right] \\ \hat{\delta}^{\text{rr}} e_r(\tilde{\omega}) &= \nu \left[ \left( -\frac{16}{5} K_1(\tilde{\omega}) - \frac{16}{5} \tilde{\omega} K_0(\tilde{\omega}) \right) i p_\infty^2 \right. \\ &\quad \left. + \left[ \frac{16}{5} K_0(\tilde{\omega}) + \left( -\frac{121}{15 \tilde{\omega}} - \frac{59}{15} \tilde{\omega} - \frac{97}{15} \right) i \pi e^{-\tilde{\omega}} \right] \frac{p_\infty}{j} \right. \\ &\quad \left. + \left( \left( -\frac{778}{45} i \tilde{\omega} + \frac{121}{15} \pi \right) K_0(\tilde{\omega}) + \frac{121}{15} I_1^{\text{at}}(\tilde{\omega}) + \left( -\frac{3152}{45} - \frac{188}{15} \tilde{\omega}^2 \right) i K_1(\tilde{\omega}) \right) \frac{1}{j^2} \right]\end{aligned}$$

# Fourier space analysis

- Fourier transforms:

$$\begin{aligned}\hat{\delta}^{\text{rr}}\bar{n}(\tilde{\omega}) &= \nu \left[ -\frac{48}{5}i\tilde{\omega}K_0(\tilde{\omega})\frac{p_\infty^4}{j} + \left( -\frac{37}{5} - \frac{59}{5}\tilde{\omega} - \frac{37}{5\tilde{\omega}} \right) \frac{i\pi e^{-\tilde{\omega}}p_\infty^3}{j^2} \right. \\ &\quad \left. + \left( -\frac{98}{3}i\tilde{\omega}K_0(\tilde{\omega}) - \frac{188}{5}i\tilde{\omega}^2K_1(\tilde{\omega}) + \frac{37}{5}I_1^{\text{at}}(\tilde{\omega}) - \frac{1568}{15}iK_1(\tilde{\omega}) + \frac{37}{5}K_0(\tilde{\omega})\pi \right) \frac{p_\infty^2}{j^3} \right] \\ \hat{\delta}^{\text{rr}}K(\tilde{\omega}) &= \nu\eta^2 \left[ \frac{96i}{5j^3}p_\infty K_1(\tilde{\omega}) + \frac{\frac{168}{5\tilde{\omega}}i\pi e^{-\tilde{\omega}} + 24i\pi e^{-\tilde{\omega}} - \frac{96}{5}K_0(\tilde{\omega})}{j^4} \right. \\ &\quad \left. + \frac{\left( -\frac{168}{5}\pi - \frac{336}{5}i\tilde{\omega} \right) K_0(\tilde{\omega}) - \frac{168}{5}I_1^{\text{at}}(\tilde{\omega}) + \frac{48}{5}iK_1(\tilde{\omega})}{p_\infty j^5} \right]\end{aligned}$$