Covariant approach to the Dirac field in LRS space-times

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  - Velocity and spin vector fields as generators of time-like and space-like congruences.
  - Apply to LRS space-times types I, II, III for perfect and non-perfect spinor fluids.

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- Projection operators:

$$U_{ab} = u_a u_b, \quad h_{ab} = -n_a n_b, \quad N_{ab} = g_{ab} - u_a u_b + n_a n_b.$$

LRS space-times classes

In LRS space-times, n^i is a preferred spatial direction: 2-spatial components vanish.

- The covariant derivatives of the time-like and space-like congruence are

$$\nabla_i u_j = \Sigma\left(n_i n_j + \frac{1}{2}N_{ij}\right) + \frac{1}{3}\Theta\left(N_{ij} - n_i n_j\right) - A u_i n_j + \Omega\varepsilon_{ij},$$

$$\nabla_i n_j = \frac{1}{2}\phi N_{ij} + \xi\varepsilon_{ij} - A u_i u_j + \left(\Sigma - \frac{1}{3}\Theta\right) n_i u_j.$$

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  with

$$\Theta := \nabla_i u^i, \quad A := (u^a \nabla_a u_i) n^i, \quad \Sigma := \nabla_{(a} u_{b)} n^a n^b,$$

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- For any covariant scalar f, we define:

$$\dot{f} = u^i \nabla_i f, \qquad \hat{f} = n^i \nabla_i f$$

- Energy–momentum tensor:

$$T_{ab} = \mu u_a u_b - p(N_{ab} - n_a n_b) - Q(n_a u_b + n_b u_a) + \frac{1}{2}\Pi(N_{ab} + 2n_a n_b).$$

with μ the energy density, p the isotropic pressure, Q the momentum density, Π the anisotropic pressure.

[0] Clarkson C. 2007, Phys.Rev. D 76 104034

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- The variables that covariantly describe LRS space-times are

$$\{A, \Theta, \Sigma, \Omega, \phi, \xi, E, H, \mu, p, Q, \Pi\}$$

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- Scalars fully describe the kinematics, Weyl curvature, and matter content.
- Covariant equations come from the decomposition of the Ricci identities, Bianchi identities and conservation laws:

$$(\nabla_c \nabla_d - \nabla_d \nabla_c)\, u_a(n_a) = R_{abcd}\, u^b(n^b), \quad \nabla_b T^{ab} = 0, \quad \nabla_{[a} R_{bc]de} = 0.$$

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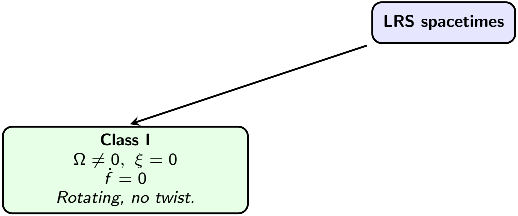

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Class I
$\Omega \neq 0,\ \xi = 0$
$\dot{f} = 0$
Rotating, no twist.

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$\Omega = 0,\ \xi = 0$
Non-rotating, twist-free.

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Class III
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$\hat{f} = 0$
Twisting, non-rotating.

[0] van Elst H. and Ellis G. F. R. 1996,Class. Quantum Grav. 13 1099

Polar Formalism

- A regular spinor field ψ can always be written in **polar form**:

$$\psi = \sqrt{\frac{\rho}{2}}\, e^{-\frac{i}{2}\beta\gamma^5}\, \boldsymbol{L}^{-1} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

where ρ is the **density**, β the **chiral angle** and $\boldsymbol{L}$ has the structure of a spinor transformation.

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- Associated bilinears:

$$\bar{\psi}\psi = \rho\cos\beta, \quad i\bar{\psi}\gamma^5\psi = \rho\sin\beta$$

$$U^a = \bar{\psi}\boldsymbol{\gamma}^a\psi = \rho u^a, \quad S^a = \bar{\psi}\boldsymbol{\gamma}^a\gamma^5\psi = \rho s^a$$

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- The orthogonal unit vectors u^a and s^a define the time-like and space-like directions of the $(1+1+2)$ splitting:

$$u_a u^a = -s_a s^a = 1 \quad \text{and} \quad u_a s^a = 0$$

- Covariant derivative of ψ:

$$\boldsymbol{\nabla}_k\psi = \left(\frac{1}{2}\nabla_k \ln\rho - \frac{i}{2}\nabla_k\beta\gamma^5 - iP_k - \frac{1}{2}R_{abk}\boldsymbol{s}^{ab}\right)\psi$$

where P_k and $R_{abk} = -R_{bak}$ are called the *momentum* and *tensorial connection*.

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$$\begin{cases} \nabla_i U^i = 0 \\ (\nabla_i\beta + B_i)U^i + 2P_i S^i = 0 \\ \nabla^{[a}U^{b]} + \varepsilon^{abpq}\nabla_p\beta\, U_q - \frac{1}{2}R^{ij}{}_p\,\varepsilon_{ijqk}U^k\varepsilon^{abpq} + 2\varepsilon^{abpq}P_pS_q - 2mM^{ab} = 0 \end{cases}$$

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$$\Pi = -\tfrac{\rho}{6}(\hat{\beta} - \Omega), \quad Q = -\tfrac{\rho}{4}(\dot{\beta} + \xi),$$

These express the **energy density, pressure, momentum density, and anisotropic stress** of the spinor field in geometric terms.

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This framework allows us to analyze the effective spinorial fluid in LRS space-times $\Longrightarrow$ coupling of Dirac and covariant equations.

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\hat{\Pi} + \hat{p} = -\frac{3}{2}\Pi\phi + \Pi A + A(\mu + p) \\
\hat{\Omega} = -\Omega(A + \phi) \\
\hat{\phi} = -\frac{1}{2}\phi^2 - A\phi + 2\Omega^2 - (\mu + p) - \Pi \\
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  - No solutions in which all variables are free.

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- $\phi = 0 \Rightarrow E = (\Sigma - \frac{1}{3}\Theta)(\Sigma + \frac{2}{3}\Theta) + \frac{2}{3}\mu$: Homogeneous: Bianchi-I, spinorial dust

Spinorial fluid in LRSIII space–times

LRSIII: $\xi \neq 0$, $\Omega = 0$,$\hat{f} = 0 \;\; \forall f$ covariant scalar.

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$$2m \sin\beta = 0 \implies \beta = k\pi$$

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- No LRSIII solutions exist both in the case of perfect and non-perfect spinorial fluid.

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Main results

- **LRS III** $\rightarrow$ ruled out.
- **LRS I** $\rightarrow$ allowed for non-perfect fluids; perfect fluids lead to stringent constraints.
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Future Perspectives

- Analysis of the resulting systems of equations.
- Generalization of the formalism by changing the choice of congruences.

Thank you for your attention!