Modifying Gravity using Topological Terms

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Keywords: Noether Symmetry, Classical and Quantum Cosmology, Modified Theories of Gravity, Gauss-Bonnet Invariant

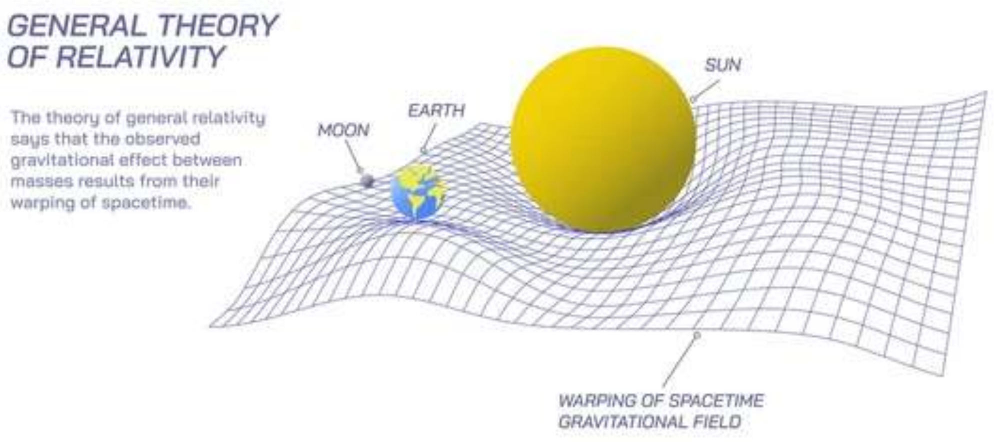
Plan of the presentation:

➢ *Preliminaries*

- *Successes and shortcomings of Einstein's General Relativity*
- *Extended gravity and topological Invariants*

➢ *Gauss-Bonnet theory*

- f(G) cosmology
- f(G) Black Holes



➢ *Chern-Simons theory*

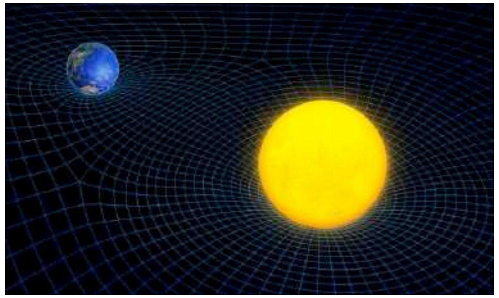
- Chern-Simons cosmology
- Chern-Simons Black Holes
- Application to electromagnetic theory

➢ *Conclusions and perspectives*

1. Preliminaries

Prediction of General Relativity

Describes the gravitational interaction through the space-time curvature



$$S_{GR} \equiv \frac{1}{2}\int d^4x\,\sqrt{-g}\,R$$

First theory to successfully pass the Solar System Tests

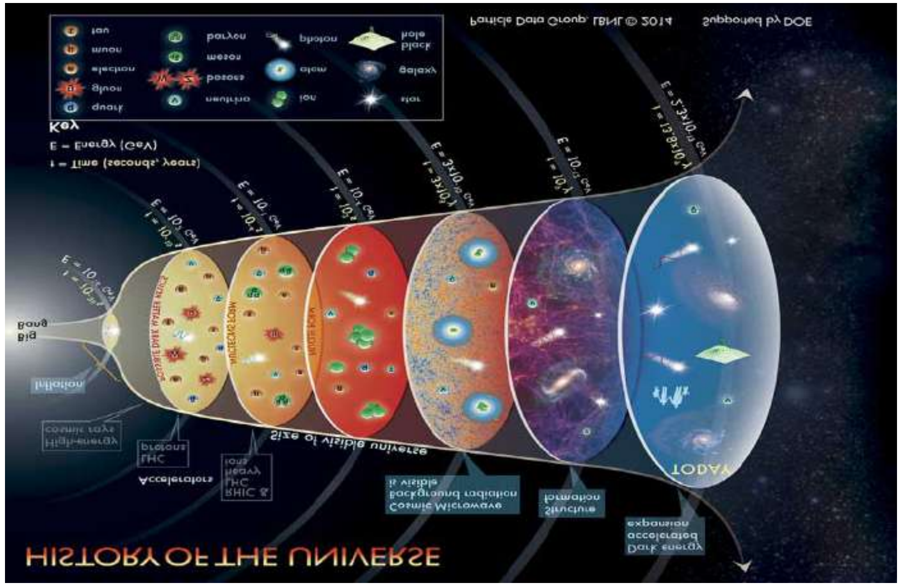
In a static and spherically Symmetric background

Schwarzschild Solution

$$ds^2 = \left(1 - \frac{2GM}{c^2 r}\right)c^2 dt^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \mathrm{sen}^2\theta d\phi^2$$

Predicts the middle ephocs crossed by Universe evolution

$$\frac{a}{a_0} = \left(\frac{t}{t_0}\right)^{\frac{2}{3(\gamma+1)}} \qquad \rho(t) = \left[6\pi G_N(1+\gamma^2)t^2\right]^{-1}$$

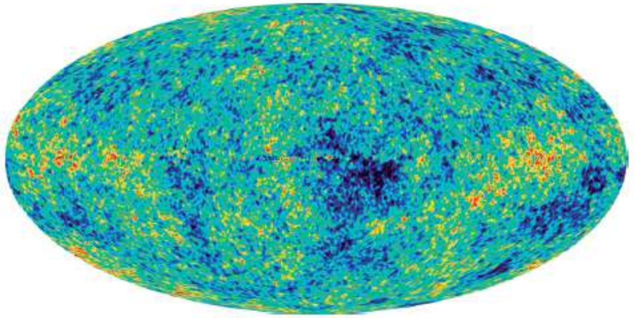


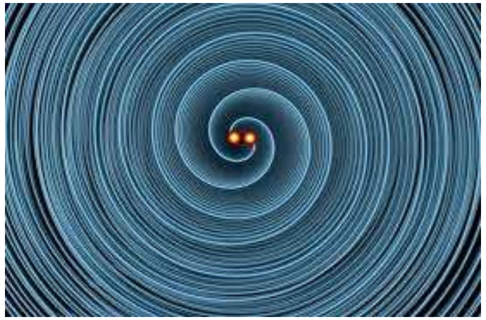
Reatività Generale $\boxed{R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}}$

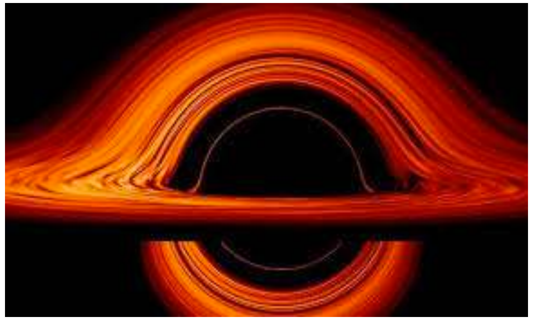
Modern Cosmology

Gravitational Waves

Black Holes







$$ds^2 = dt^2 - a(t)^2 \delta_{ij} dx^i dx^j$$

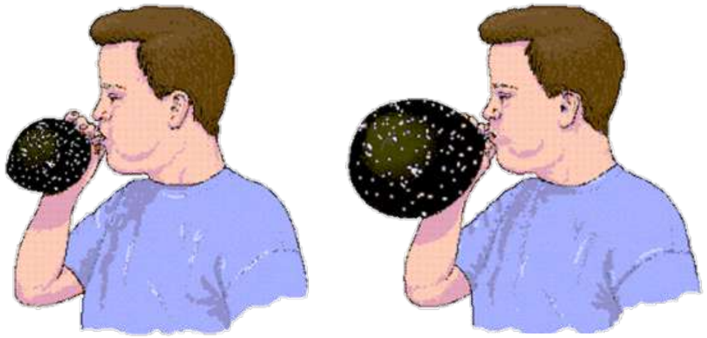
$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

$$ds^2 = e^{\nu(r,t)} c^2 dt^2 - e^{\lambda(r,t)} dr^2 - r^2 (\sin^2\theta d\phi^2 + d\theta^2).$$

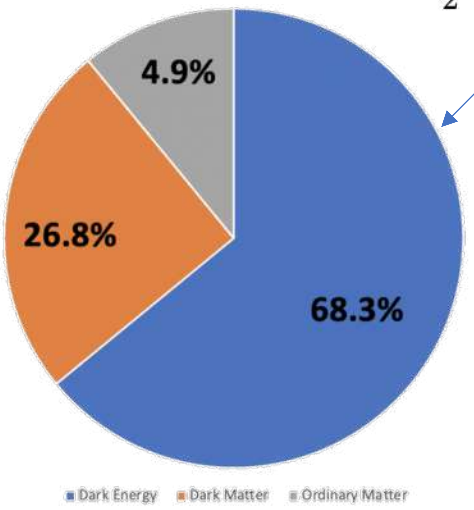
Shortcomings of General Relativity

Large scales

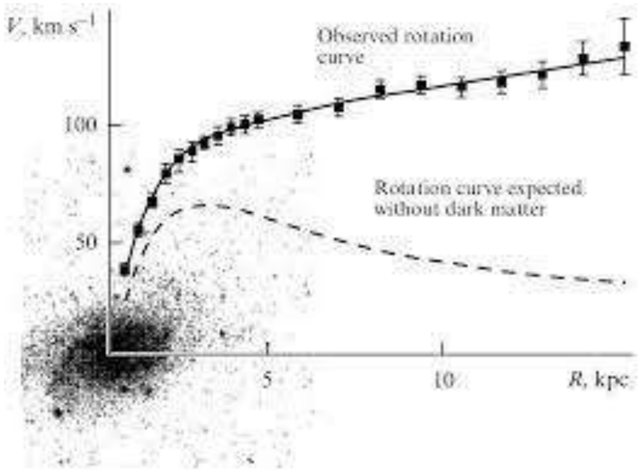
- Accelerated expansion of the Universe
- Inflation

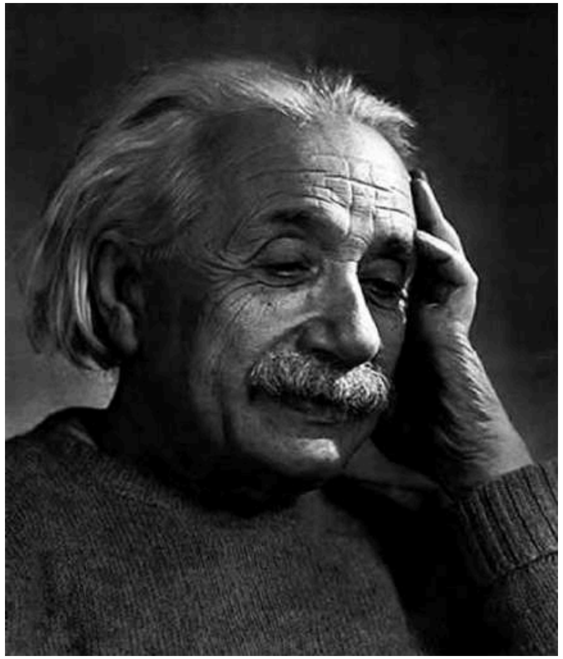


$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}$$

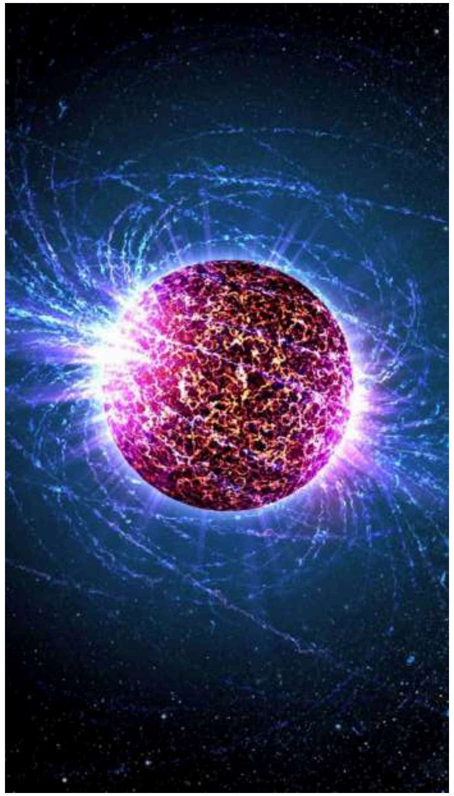


- Galaxy rotation curve





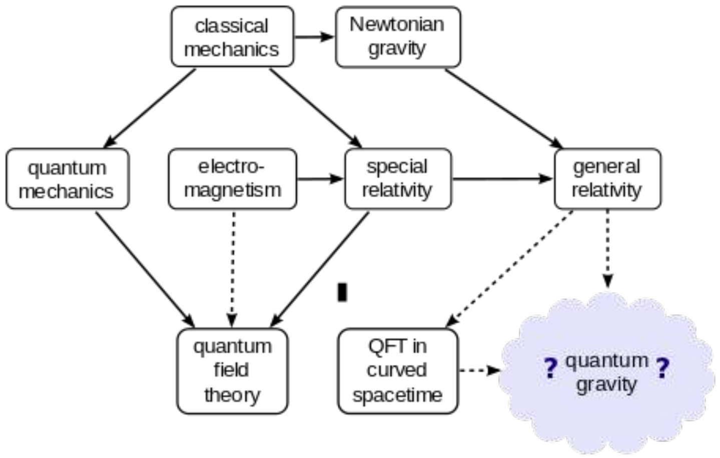
- Mass-Radius diagram of Neutron Stars

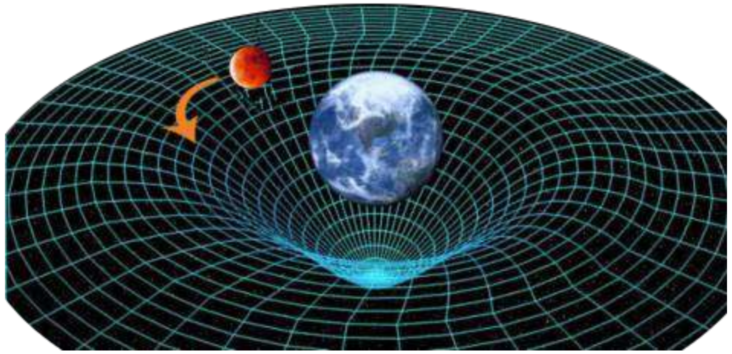


Shortcomings of General Relativity

Small scales

- Renormalizability
- GR cannot be quantized



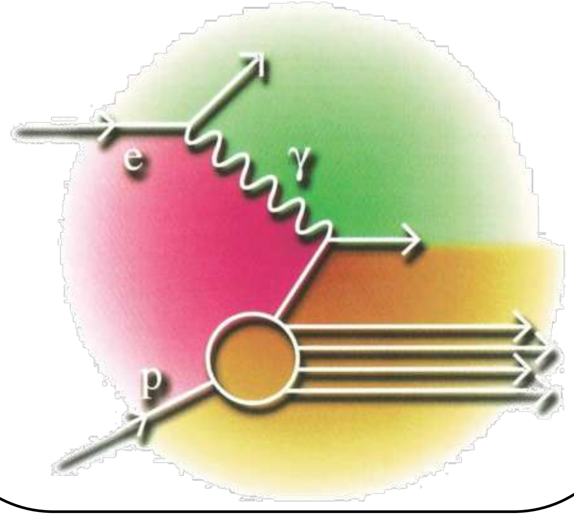


- Discrepancy between theoretical and experimental values of Λ

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}$$

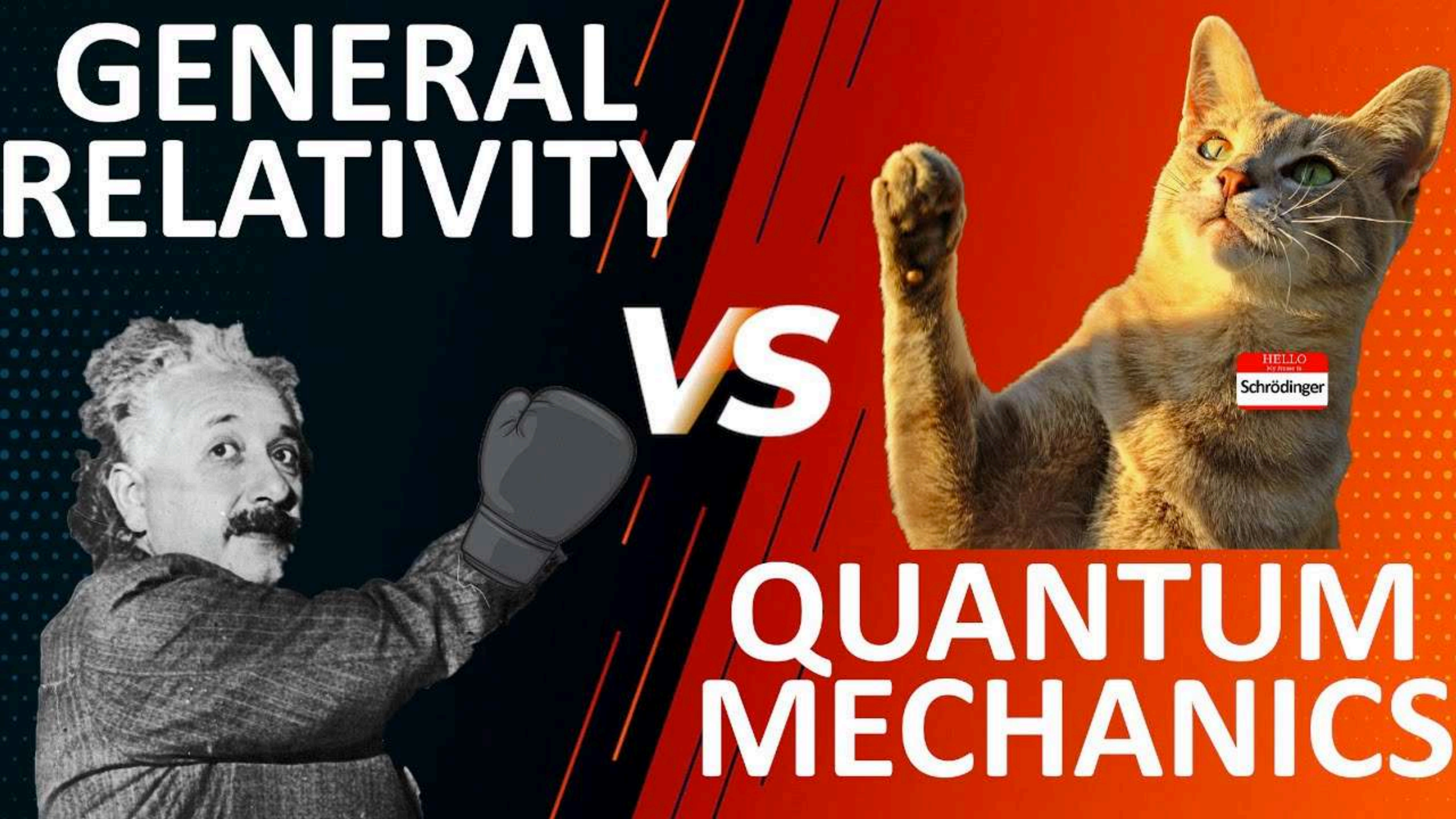
$$\frac{\rho'_\Lambda}{\rho_\Lambda} = \frac{\hbar^2}{96\pi^3 G H_0^2} \sim 10^{121}$$

- Cannot be treated under the same standard as other interactions



- Space-time singularity

Unfortunately, so far, no theory is capable of addressing all these problems at once


GENERAL
RELATIVITY
VS
HELLO
Schrödinger
QUANTUM
MECHANICS

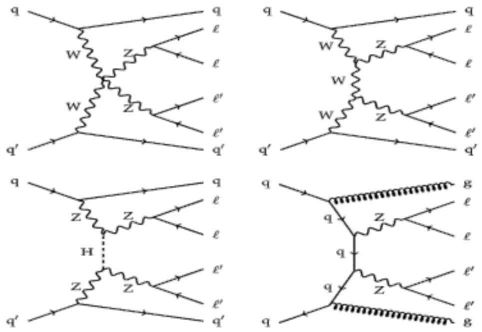
Modified Theories of Gravity

Classification

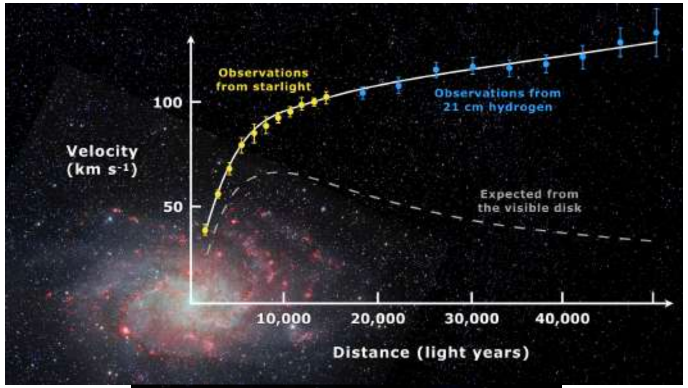
- Extended action → $f(R)$
- Modified Action → $f(T)$
- Coupling To Scalar fields → $\varphi \cdot R$

Motivations:

1. Could account for UV and IR quantum corrections

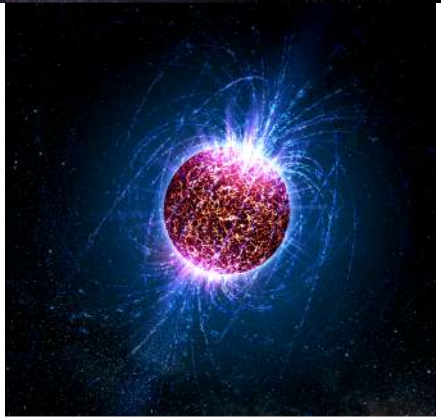


2. Fit the galaxy rotation curve



3. Contains GR as a particular limit
4. Reproduce both late and early time cosmic evolutions

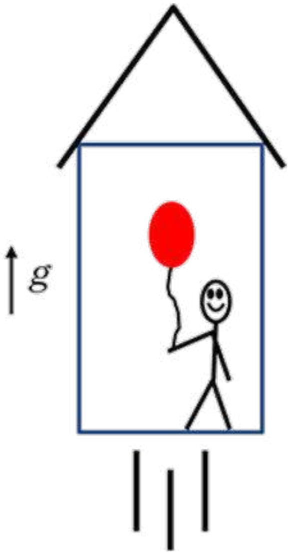
5. Predict the right mass-radius relation of some neutron star without invoking exotic EOS



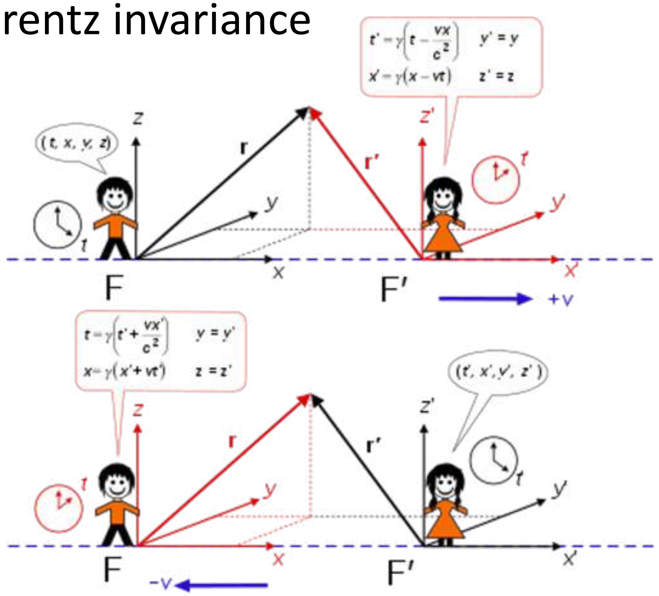
Modified theories of gravity

- Relax some assumptions of GR

➢ Equivalence principle

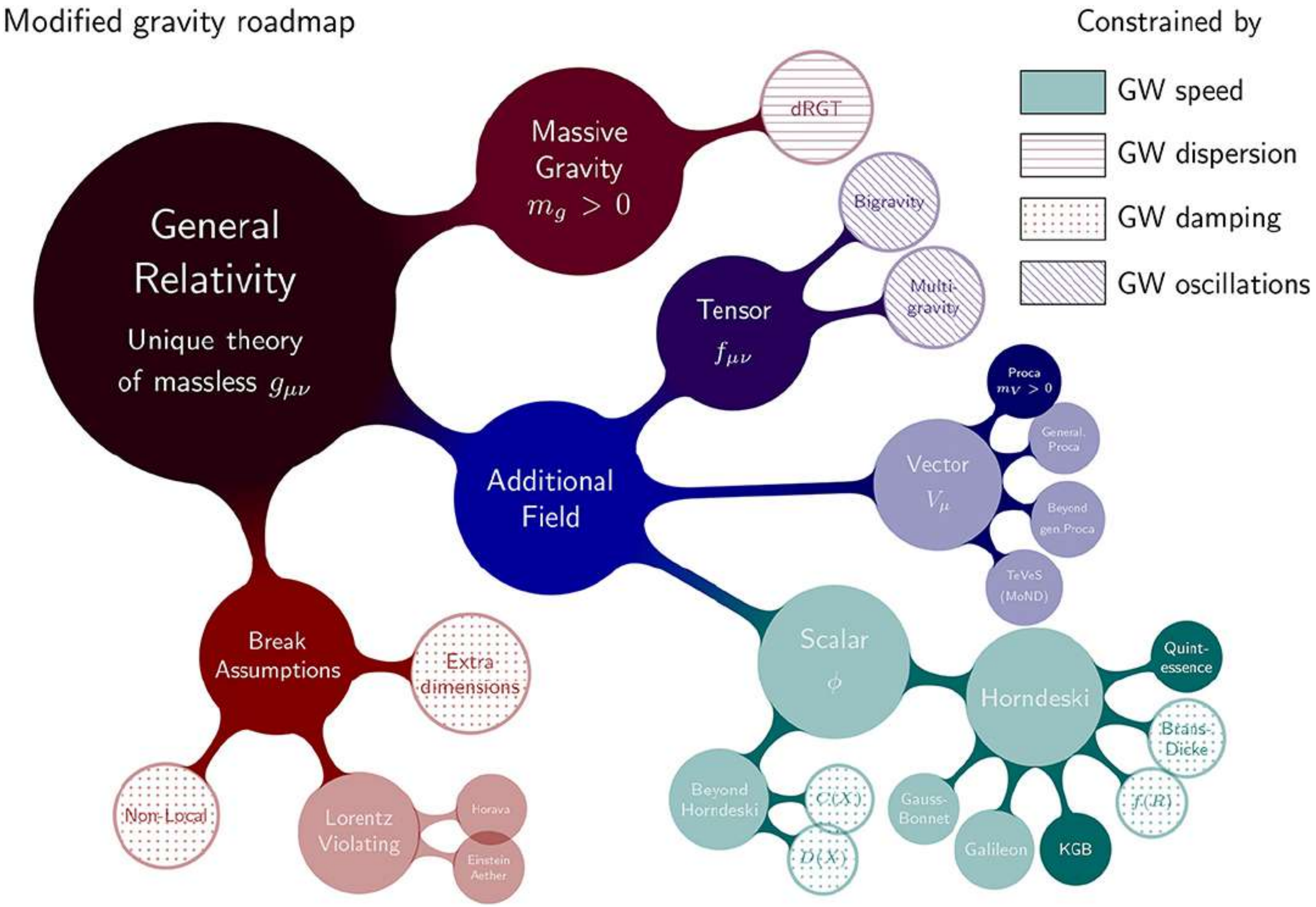


➢ Lorentz invariance



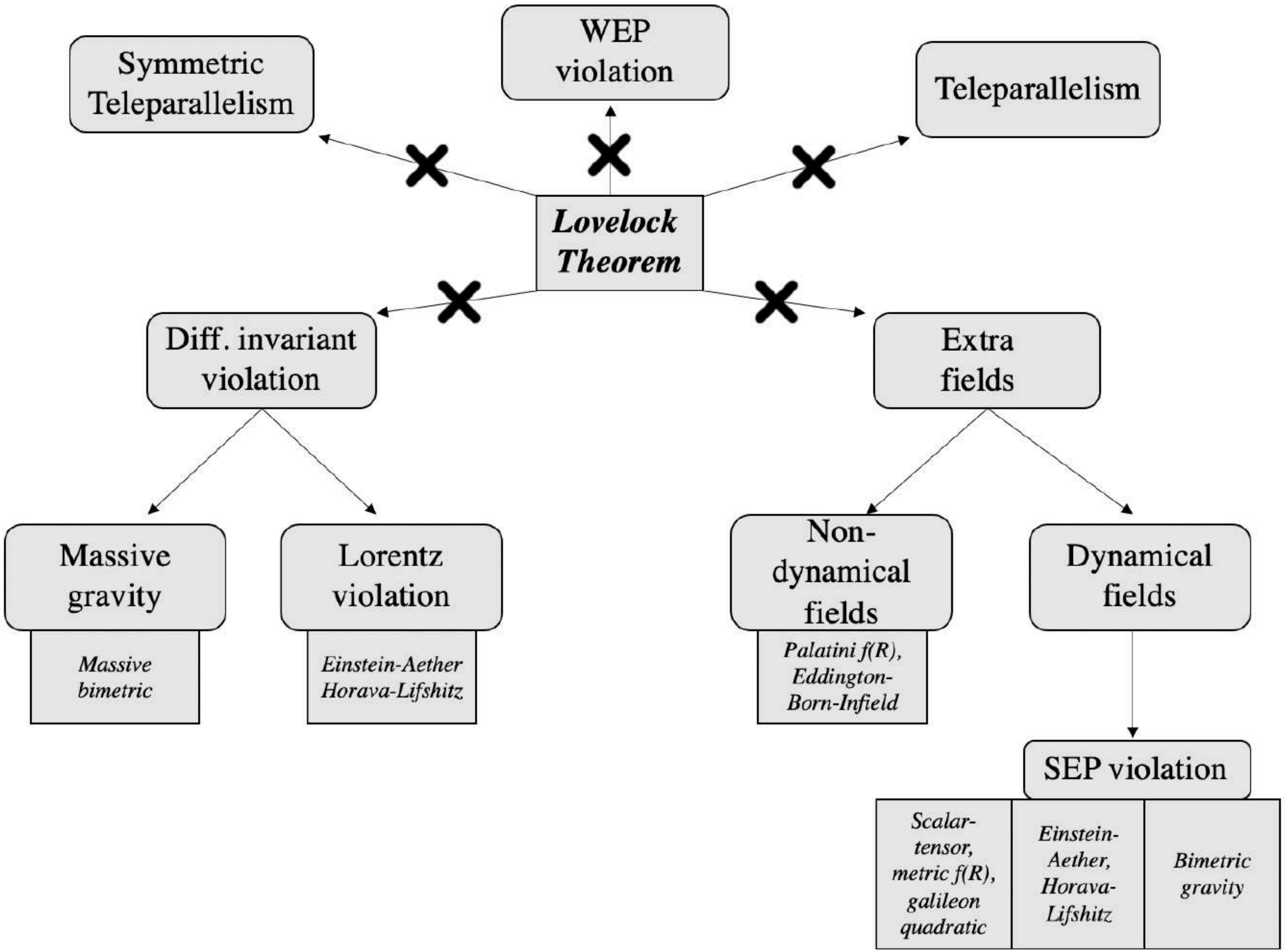
➢ Second-order field equations

$$S = \int \sqrt{-g} F(\phi, R, \Box^z R, R^{\mu\nu} R_{\mu\nu}, R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma}) \qquad z \in \mathbb{Z}$$



Examples of Modified Gravity Potentials

Modified Gravity Model	Corrected potential	Yukawa parameters
$f(R)$	$\Phi(r) = -\frac{G_N M}{r}\left[1 + \alpha\, e^{-m_R r}\right]$ $\mathbf{A} = \frac{G_N}{r^3}\,\mathbf{r}\times\mathbf{J}$	${m_R}^2 = -\frac{f_R(0)}{6 f_{RR}(0)}$
$f(R, \Box R) = R + a_0 R^2 + a_1 R \Box R$	$\Phi(r) = -\frac{G_N M}{r}\left(1 + c_0 e^{(-r/l_0)} + c_1 e^{(-r/l_1)}\right)$ $\mathbf{A} = \frac{G_N}{r^3}\,\mathbf{r}\times\mathbf{J}$	$c_{0,1} = \frac{1}{6} \mp \frac{a_0}{2\sqrt{9a_0^2 + 6a_1}}$ $l_{0,1} = \sqrt{-3a_0 \pm \sqrt{9a_0^2 + 6a_1}}$
$f(R, R_{\alpha\beta}R^{\alpha\beta}, \phi) + \omega(\phi)\phi_{;\alpha}\phi^{;\alpha}$	$\Phi(r) = -\frac{G_N M}{r}\Big[1 + g(\xi,\eta)\, e^{-m_R \tilde{k}_R r} +$ $+ [1/3 - g(\xi,\eta)]\, e^{-m_R \tilde{k}_\phi r} - \frac{4}{3}\, e^{-m_Y r}\Big]$ $\mathbf{A} = \frac{G_N}{r^3}\left[1 - (1 + m_Y r)\, e^{-m_Y r}\right]\mathbf{r}\times\mathbf{J}$	${m_R}^2 = -\frac{1}{3 f_{RR}(0,0,\phi^{(0)}) + 2 f_Y(0,0,\phi^{(0)})}$ ${m_Y}^2 = \frac{1}{f_Y(0,0,\phi^{(0)})}$ ${m_\phi}^2 = -\frac{f_{\phi\phi}(0,0,\phi^{(0)})}{2\omega(\phi^{(0)})}$ $\xi = \frac{3 f_{R\phi}(0,0,\phi^{(0)})^2}{2\omega(\phi^{(0)})}$ $\eta = \frac{m_\phi}{m_R}$ $g(\xi,\eta) = \frac{1 - \eta^2 + \xi + \sqrt{\eta^4 + (\xi-1)^2 - 2\eta^2(\xi+1)}}{6\sqrt{\eta^4 + (\xi-1)^2 - 2\eta^2(\xi+1)}}$ $\tilde{k}^2_{R,\phi} = \frac{1 - \xi + \eta^2 \pm \sqrt{(1 - \xi + \eta^2)^2 - 4\eta^2}}{2}$



Symmetric Teleparallelism
WEP violation
Teleparallelism
Lovelock Theorem
Diff. invariant violation
Extra fields
Massive gravity
Lorentz violation
Non-dynamical fields
Dynamical fields
Massive bimetric
Einstein-Aether Horava-Lifshitz
Palatini f(R), Eddington-Born-Infield
SEP violation
Scalar-tensor, metric f(R), galileon quadratic
Einstein-Aether, Horava-Lifshitz
Bimetric gravity

Example: f(R) Gravity $S = \int \sqrt{-g} f(R)\, d^4x.$

$$G_{\mu\nu} = \frac{1}{f_R(R)} \left\{ \frac{1}{2} g_{\mu\nu} \left[f(R) - R f_R(R) \right] + f_R(R)_{;\mu;\nu} - g_{\mu\nu} \Box f_R(R) \right\}$$

Effective Energy-Momentum tensor

$$\overline{T}^{GF}_{\mu\nu} = \frac{1}{f_R(R)} \left\{ \frac{1}{2} g_{\mu\nu} \left[f(R) - R f_R(R) \right] + f_R(R)_{;\mu;\nu} - g_{\mu\nu} \Box f_R(R) \right\} \longrightarrow G_{\mu\nu} = \overline{T}^{GF}_{\mu\nu}$$

Example: Scalar-Tensor Gravity $S = \int \sqrt{-g} \left\{ f(\phi)\, R + \frac{\omega(\phi)}{2} g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} - V(\phi) \right\} d^4x$

Second-order field equations

Can be related to f(R) gravity through conformal transformations

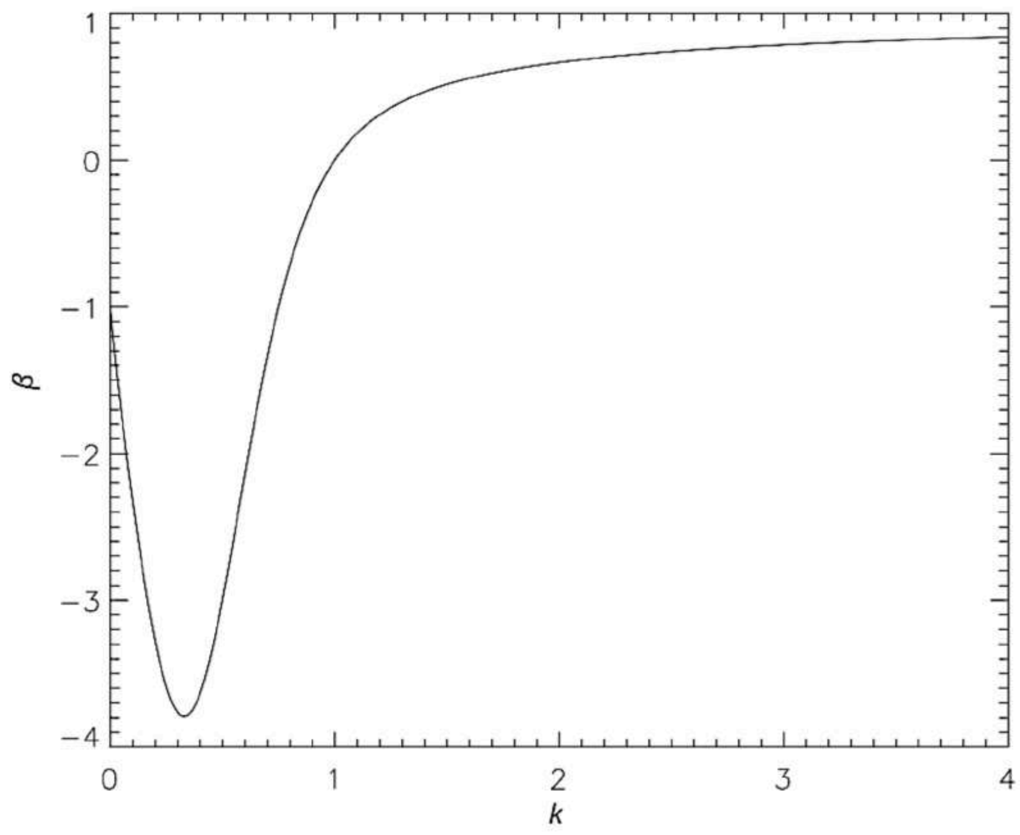
T.P. Sotiriou and V. Faraoni. Rev. Mod Phys. ***82*** *(2010), 451-497*

Newtonian potential in the weak-field limit

Selected theory
$f(R) = R^k$

Static potential
$\Phi(r) = -\frac{GM}{2r}\left[1 + \left(\frac{r}{r_c}\right)^{\beta}\right]$

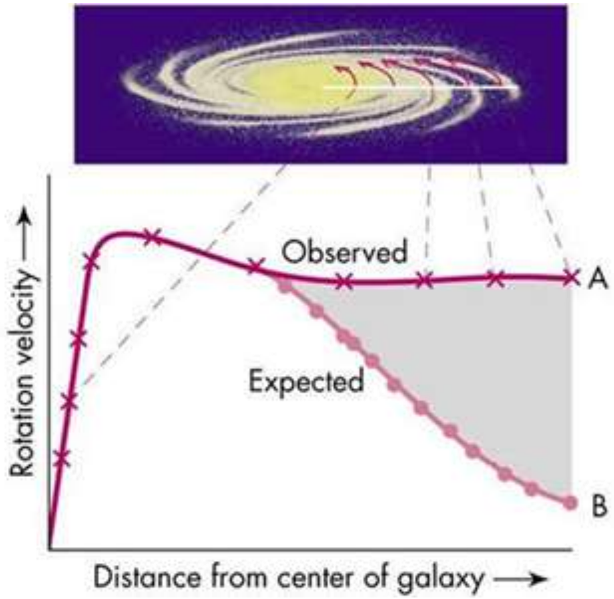
$$\beta = \frac{12k^2 - 7k - 1 - \sqrt{36k^4 + 12k^3 - 83k^2 + 50k + 1}}{6k^2 - 4k + 2}$$



Galaxy rotation curve

$$\boxed{f(R) = R^k}$$

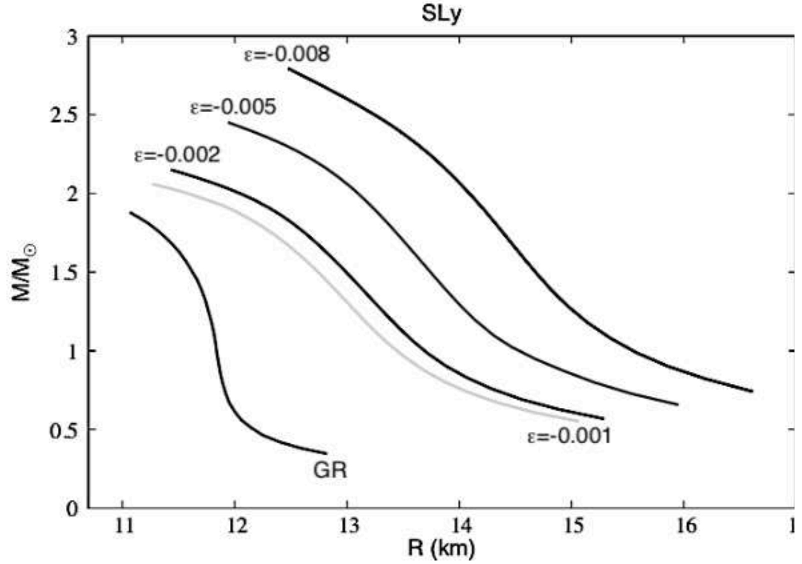
$$\boxed{k = 0.817}$$

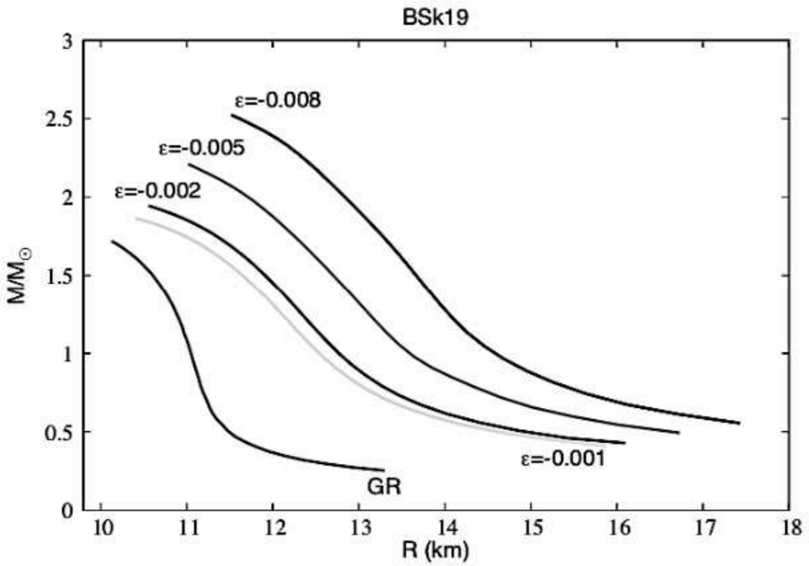


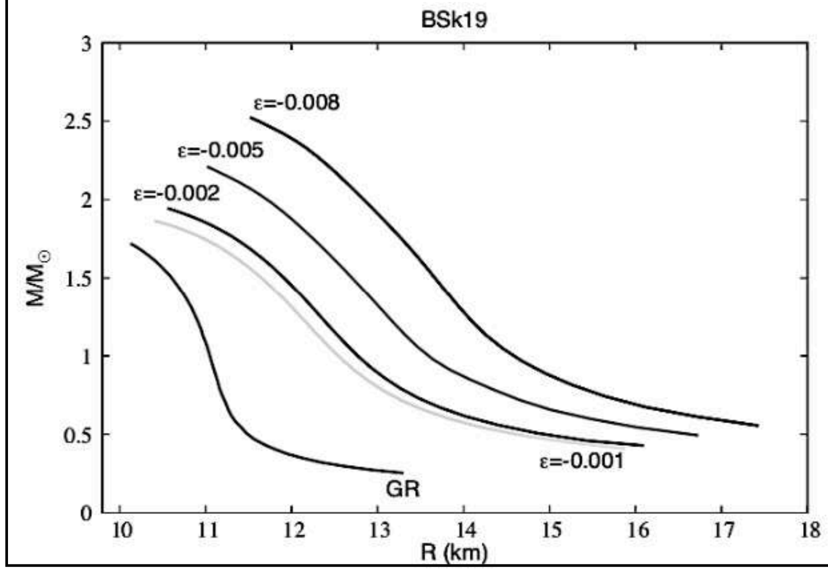
No need for Dark Matter?

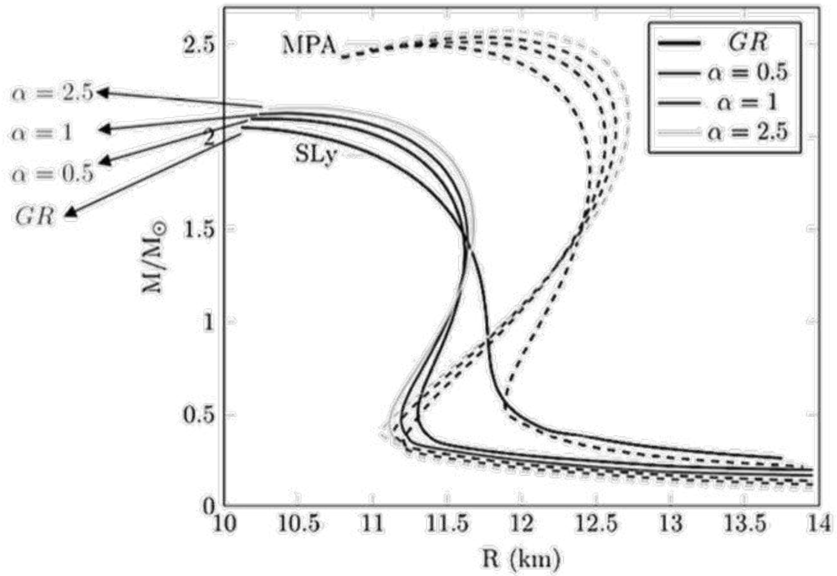
f(R) theory $\longrightarrow \begin{cases} f(R) = R + \alpha R^2 \\ f(R) = f_0 R^k \quad (k = 1 + \boldsymbol{\varepsilon}) \end{cases}$

Mass-Radius Diagram









Gauss-Bonnet and Chern-Simons

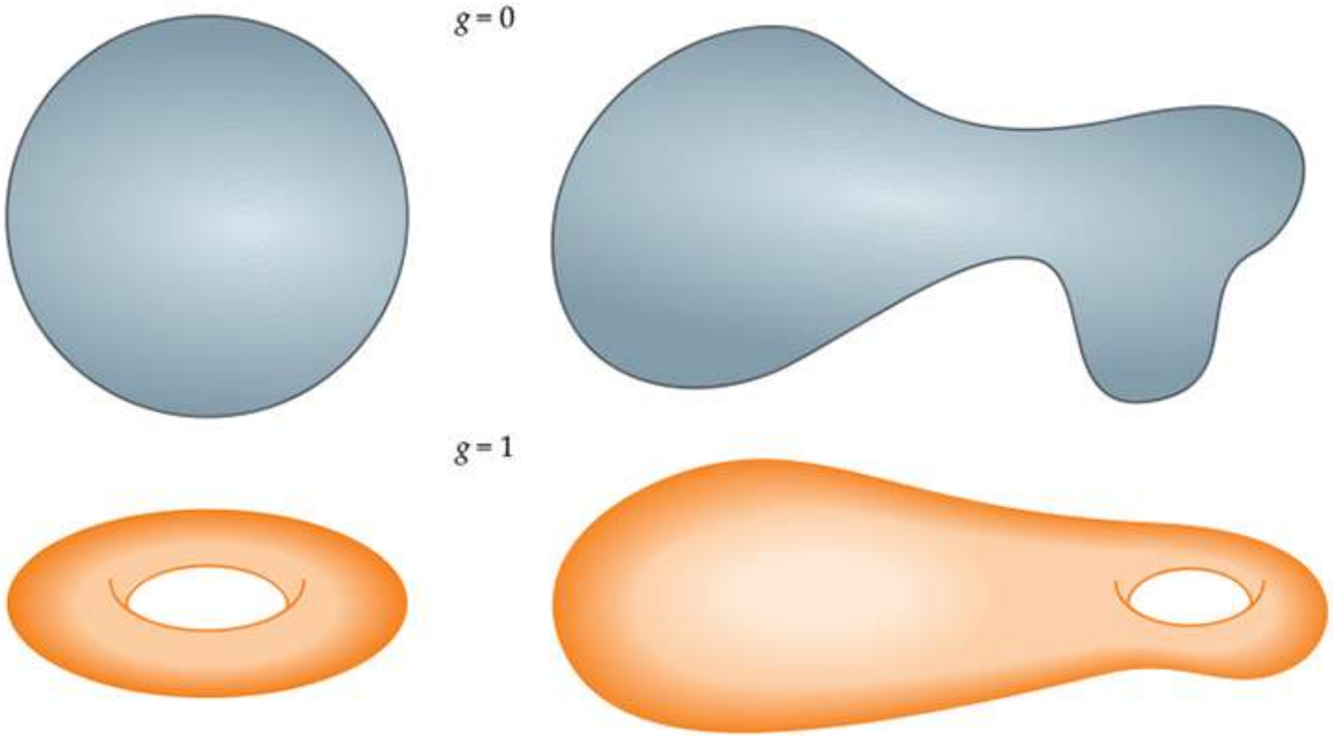
In this framework, modified theories can also include topological invariants:

Gauss—Bonnet theory

Chern—Simons theory

Topological Invariants

Depend on the topology, independently of the space-time geometry



$$\frac{1}{2\pi}\int K\, dA = 2(g-1)$$

Torus cannot be obtained by deforming the sphere

They do not depend on the local form of the spacetime, but only relies on its global structure

2. *Gauss-Bonnet Theory*

Gauss-Bonnet Theory

$$Q \equiv R^{\mu\nu p\sigma} R_{\mu\nu p\sigma}$$

General extended action with second-order curvature invariants: $$S = \int \sqrt{-g}\, f(R, P, Q)\, d^4x$$

$$P \equiv g^{\mu p} R^{\nu\sigma} R_{\mu\nu p\sigma}$$

There is only a particular combination…. $\mathcal{G} \equiv R^2 - 4P + Q$

Euler characteristic

…. Which provides a topological surface term: $$\int_{\mathcal{M}} \sqrt{-g}\, \mathcal{G}\, d^4x = \chi(\mathcal{M})$$

Euler density

Usually the following action is considered:

$$S = \int \sqrt{-g} \left(\frac{1}{2} R + f(\mathcal{G}) \right) d^4x$$

So that, when $f(\mathcal{G}) \to 0$, GR is safely recovered

Plays the role of cosmological constant

S. Nojiri, S.D. Odintsov. Phys. Lett. B ***631*** *(2005),1-6*

Gauss-Bonnet Theory

Here, due to

$$\sqrt{\mathcal{G}} \sim \sqrt{R^2}$$

We consider

$$S = \int \sqrt{|g|} f(\mathcal{G})\, d^{d+1}x \longrightarrow$$

$$\boxed{\begin{aligned} &-\left(2RR_{\mu\nu} - 4R_{\mu p}R^{p}{}_{\nu} + 2R_{\mu}{}^{p\sigma\tau}R_{\nu p\sigma\tau} - 4R^{\alpha\beta}R_{\mu\alpha\nu\beta}\right) f_{\mathcal{G}}(\mathcal{G}) \\ &+\left(2RD_{\mu}D_{\nu} + 4G_{\mu\nu}\Box - 4R^{p}_{\{\nu}D_{\mu\}}D_{p} + 4g_{\mu\nu}R^{p\sigma}D_{p}D_{\sigma} \right. \\ &\left. -4R_{\mu\alpha\nu\beta}D^{\alpha}D^{\beta}\right) f_{\mathcal{G}}(\mathcal{G}) + \frac{1}{2}g_{\mu\nu}f(\mathcal{G}) = 0\,, \end{aligned}}$$

<u>This theory can reproduce GR results even without adding the scalar curvature</u>

Why considering the Gauss—Bonnet term?

1. ***The Gauss-Bonnet term is a topological surface term and reduces the dynamics***

2. ***The Gauss-Bonnet term naturally emerges in gauge theories of gravity (e.g. Lovelock gravity)***

3. ***In homogeneous cosmology, it turns out that*** $f(\mathcal{G}) = \sqrt{\mathcal{G}} \longrightarrow R$

How to select the form of the action?

Noether Point Symmetries

$$\bar{t} = \bar{t}(t, q; \varepsilon) \simeq t + \varepsilon\xi(t, q)$$
$$\bar{q}^i = \bar{q}^i(t, q; \varepsilon) \simeq q^i + \varepsilon\eta^i(t, q)$$

→ *1-parameter (ε) group of point transformations*

$$\boldsymbol{X} = \xi(t, q)\frac{\partial}{\partial t} + \eta^i(t, q)\frac{\partial}{\partial q^i}$$

→ *infinitesimal group generator*

$$\boldsymbol{X}^{[1]} = \boldsymbol{X} + \eta^{[1]i}\frac{\partial}{\partial \dot{q}^i} = \boldsymbol{X} + \left(\dot{\eta}^i - \dot{\xi}\dot{q}^i\right)\frac{\partial}{\partial \dot{q}^i}$$

→ *"first prolongation" of the infinitesimal generator*

Noether Theorem. *If and only if it exists a function $g(t, q(t))$ such that*

$$\boldsymbol{X}^{[1]}L + \dot{\xi}L = \dot{g},$$

*then the one-parameter group of point transformations generated by **$\boldsymbol{X}$** is a one-parameter group of Noether point symmetries for the dynamical system described by L.*

The associated first integral of motion is:

$$I(t, q, \dot{q}) = \xi\left(\dot{q}\frac{\partial L}{\partial \dot{q}^i} - L\right) - \eta^i\frac{\partial L}{\partial \dot{q}^i} + g$$

The system contains the unknown function $\boldsymbol{f(G)}$, so that it can provide, in principle, an explicit form for $\boldsymbol{f(G)}$ related to the existence of symmetries

*k.F. Dialektopoulos and S. Capozziello. IJGMMP **15** (2018)*

Noether Symmetry Approach

The recipe:

1. We consider a point-like Lagrangian

2. We write the ansatz for $\boldsymbol{X}$ ed $\boldsymbol{X}^{[1]}$

3. We expand the Noether point symmetries existence condition

$$\boldsymbol{X}^{[1]}L + \dot{\xi}L = \dot{g}$$

to obtain a polynomial depending on $\xi(t,q)$, $\eta^i(t,q)$, $\dot{g}(t,q)$ and products of the Lagrangian velocities $(e.g.\ \dot{\eta}^i\ \dot{\eta}^j\ \dot{\xi}\ \ldots)$

3. We obtain a system of PDEs for $\xi, \eta^i, \dot{g}$

The system contains the unknown function $\boldsymbol{f(G,\phi)}$, so that it can provide, in principle, an explicit form for $\boldsymbol{f(G,\phi)}$ related to the existence of symmetries

Noether symmetry approach to modified Gauss—Bonnet theory

$$ds^2 = dt^2 - a(t)^2\delta_{ij}dx^i dx^j \longrightarrow \mathcal{G} = \frac{p(d)\left[(d-3)\dot{a}^4 + 4a\dot{a}^2\ddot{a}\right]}{a^4}$$

with $p(d) = d(d-1)(d-2)$

$$S = \int \sqrt{-g} f(\mathcal{G})\, d^{d+1}x \longrightarrow S = 2\pi^2 \int \left[a^d f(\mathcal{G}) - \lambda \left\{\mathcal{G} - \frac{p(d)\left[(d-3)\dot{a}^4 + 4a\dot{a}^2\ddot{a}\right]}{a^4}\right\} + \mathcal{L}_m\right] d^{d+1}x$$

Point-like Lagrangian

$$\mathcal{L} = \frac{1}{3}a^{d-4}\left[(3-d)p(d)\dot{a}^4 f_{\mathcal{G}}(\mathcal{G}) + 3a^4[f(\mathcal{G}) - \mathcal{G}f_{\mathcal{G}}(\mathcal{G})] - 4ap(d)\dot{a}^3\dot{\mathcal{G}}f_{\mathcal{G}\mathcal{G}}(\mathcal{G})\right] + \mathcal{L}_m$$

Symmetry generator

$$\mathcal{X} = \xi(a,\mathcal{G},t)\partial_t + \alpha(a,\mathcal{G},t)\partial_a + \beta(a,\mathcal{G},t)\partial_{\mathcal{G}}$$

One set of solutions

$$f(\mathcal{G}) = f_0\mathcal{G}^k \quad \begin{cases} a(t) = a_0 e^{qt}\,, \quad \mathcal{G}(t) = (d+1)p(d)\,q^4\,, \quad k = \dfrac{d+1}{4} \\[2ex] a(t) = a_0 t^{-4\frac{(k-1)(4k-1)}{4k-d-1}}\,, \quad \mathcal{G}(t) = \dfrac{256\left[(k-1)(4k-1)\right]^3\left[4+(d+1)(4k-5)\right]p(d)}{((d+1)t-4kt)^4}\,, \end{cases}$$

*F. Bajardi and S. Capozziello. EPJC **80** (2020), 704*

Cosmological applications

In four dimensions the Lagrangian becomes

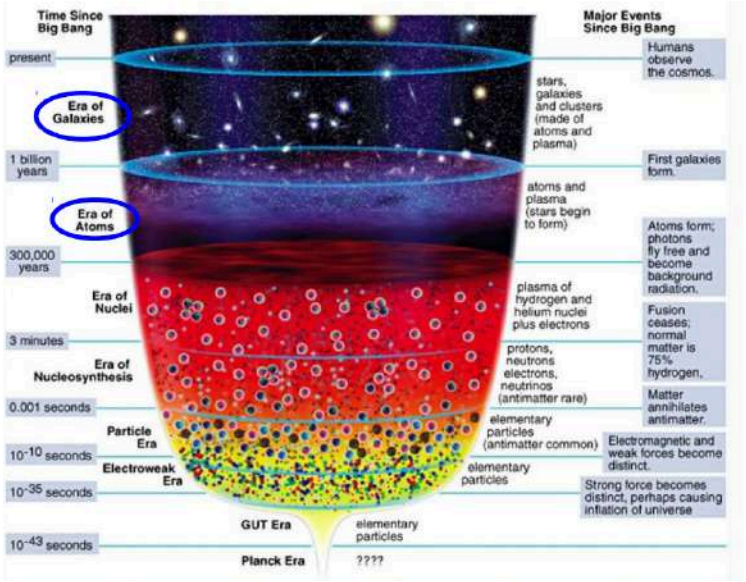
$$\mathcal{L}^{(4)} = a^3[f(\mathcal{G}) - \mathcal{G} f_{\mathcal{G}}(\mathcal{G})] - 8\dot{a}^3 f_{\mathcal{G}\mathcal{G}}(\mathcal{G})\dot{\mathcal{G}} + \rho_0 a^{-3w}$$

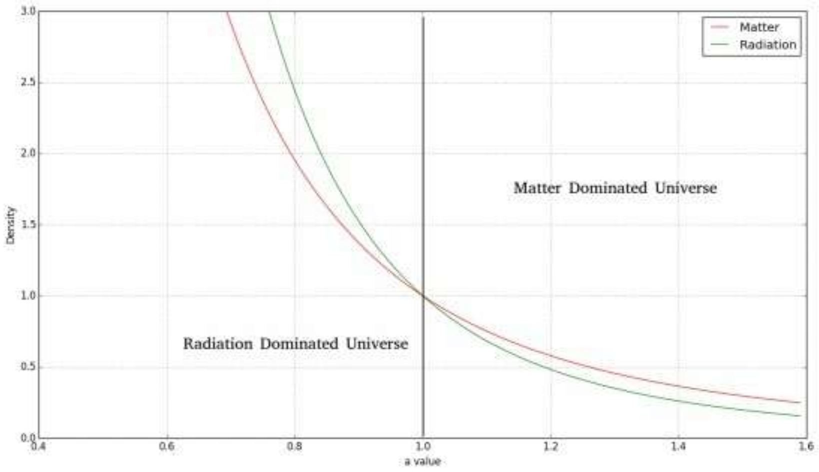
and yields

$$\begin{cases} a(t) = a_0 t^{1-4k} \qquad \mathcal{G}(t) = -96k(1-4k)^3 t^{-4} \equiv \mathcal{G}_0 t^{-4}\,, \qquad w = 0 \\ \\ a(t) = e^{nt} \quad \mathcal{G}(t) = 24m^4 \quad w = -1. \end{cases}$$

from which it is possible to distinguish the epochs crossed by the Universe

$$\boxed{\begin{aligned} k &= \frac{1}{8} \rightarrow a(t) \sim t^{\frac{1}{2}} \quad \mathcal{G} = -\frac{3}{2}t^{-4} \rightarrow \text{Radiation} \\ k &= \frac{1}{6} \rightarrow a(t) \sim t^{\frac{1}{3}} \quad \mathcal{G} = -\frac{16}{27}t^{-4} \rightarrow \text{Stiff matter} \\ k &= \frac{1}{12} \rightarrow a(t) \sim t^{\frac{2}{3}} \quad \mathcal{G} = -\frac{64}{27}t^{-4} \rightarrow \text{Dust matter} \end{aligned}}$$





<u>To describe the early time evolution we need a quantum formalism</u>

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Applications to energy conditions

Null Energy Condition (NEC) $\rightarrow \rho + p \geq 0$

Weak Energy Condition (WEC) $\rightarrow \rho \geq 0\,;\ \rho + p \geq 0$

Dominant Energy Condition (DEC) $\rightarrow \rho - |p| \geq 0$

Strong Energy Condition (SEC) $\rightarrow \rho + p \geq 0\,;\ \rho + 3p \geq 0$

Modified gravity field equations

$$FG_{\mu\nu} = T_{\mu\nu}^{(Grav\ Field)} + T_{\mu\nu}^{(Matter)}$$

Modified Gauss-Bonnet field equations

$$G_{\mu\nu}(2R - 4\Box) f_{\mathcal{G}}(\mathcal{G}) = T_{\mu\nu} - \Big[\Big(R^2 - 4R_{\mu p}R^{p}{}_{\nu} + 2R_{\mu}{}^{p\sigma\tau}R_{\nu p\sigma\tau} - 4R^{\alpha\beta}R_{\mu\alpha\nu\beta} \Big) f_{\mathcal{G}}(\mathcal{G}) - \Big(2RD_\mu D_\nu - 4R^{p}_{\{\nu}D_{\mu\}}D_p + 4g_{\mu\nu}R^{p\sigma}D_pD_\sigma - 4R_{\mu\alpha\nu\beta}D^\alpha D^\beta \Big) f_{\mathcal{G}}(\mathcal{G}) - \frac{1}{2}g_{\mu\nu}f(\mathcal{G}) \Big] .$$

$$\rho^{(GF)} = \frac{f_0}{2}\mathcal{G}^{k-3} \Big[-24k(k-1)H^2\left(\mathcal{G}\ddot{\mathcal{G}} + (k-2)\dot{\mathcal{G}}^2\right) -72k\mathcal{G}^2H^4 - 48k\mathcal{G}H^2\left(2\mathcal{G}(\dot{H}+H^2) + (k-1)H\dot{\mathcal{G}}\right) + \mathcal{G}^3 \Big]$$

Energy density of the gravitational field for

$$f(\mathcal{G}) = f_0\mathcal{G}^k$$

$$p^{(GF)} = \frac{f_0}{2}\mathcal{G}^{k-3} \Big[16k(k-1)(\dot{H}+H^2)\left(\mathcal{G}\ddot{\mathcal{G}} + (k-2)\dot{\mathcal{G}}^2\right) + 24k\mathcal{G}^2H^4 +16k\mathcal{G}(\dot{H}+H^2)\left(3\mathcal{G}(\dot{H}+H^2) + 2(k-1)H\dot{\mathcal{G}}\right) +24k\mathcal{G}H^2\left(4\mathcal{G}(\dot{H}+H^2) + (k-1)H\dot{\mathcal{G}}\right) - \mathcal{G}^3 \Big] ,$$

Pressure of the gravitational field for

$$f(\mathcal{G}) = f_0\mathcal{G}^k$$

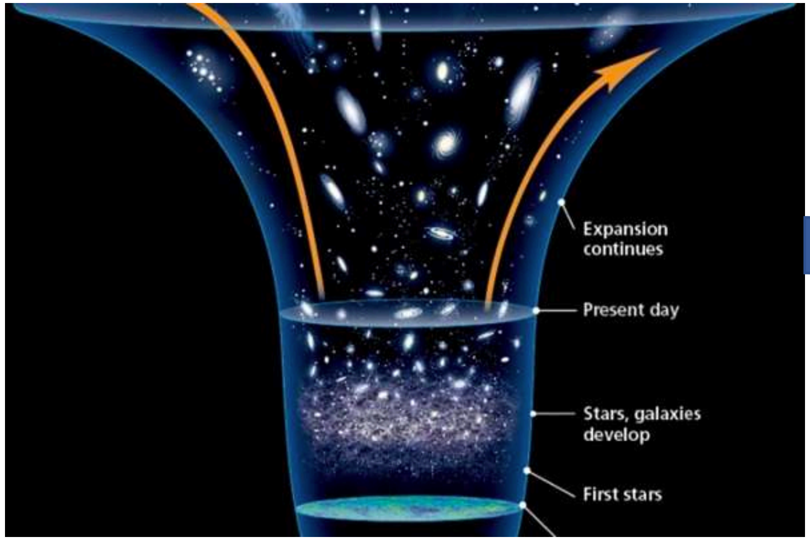
F. Bajardi. Eur.Phys. J. C ***84*** *(2024) 12, 1298*

With the help of cosmographic parameters

$$q = -1 - \frac{\dot{H}}{H^2} \qquad j = 1 + \frac{\ddot{H} + 3\dot{H}H}{H^3}$$

$$s = 1 + \frac{\dddot{H} + 3\ddot{H}H + 3\dot{H}^2 + 6H^2\dot{H} + H\ddot{H}}{H^4}$$

$$q = -0.81,\ j = 2.16,\ s = -0.22$$



*The **EC** are satisfied for*

$$\mathrm{NEC} \to 0 \leq k \leq 1.457 \vee k \geq 2.977$$

$$\mathrm{WEC} \to 0.093 \leq k \leq 1.457 \vee k \geq 2.977$$

$$\mathrm{DEC} \to 0.113 \leq k \leq 1.457 \vee k \geq 2.977$$

$$\mathrm{SEC} \to 0 \leq k \leq 0.189,$$

$$0.113 < k < 0.189.$$

<u>f(G) gravity can mimick dark energy</u>

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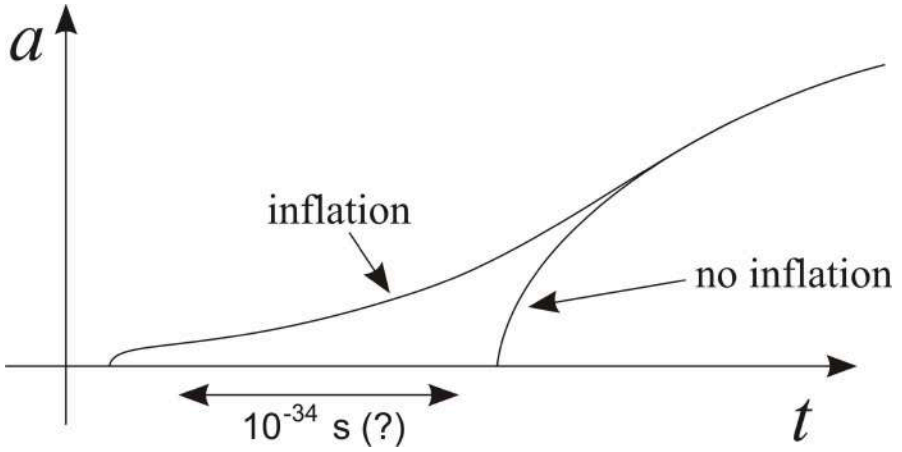
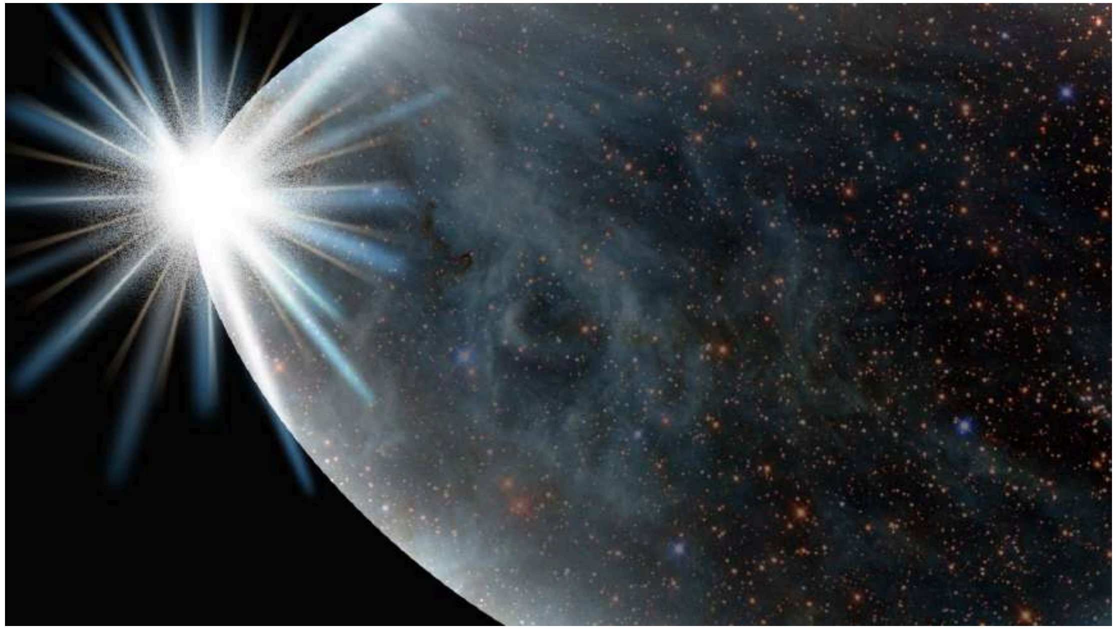
Slow-roll inflation: *inflation is driven by a scalar field rolling down a potential energy hill. Inflation occurs as soon as the scalar field rolling is slow with respect to the Universe expansion.*

Slow-Roll approximation

$$|\varepsilon| \equiv \left| -\frac{\dot{H}}{H^2} \right| \ll 1 \quad |\eta| \equiv \left| -\frac{\ddot{H}}{2H\dot{H}} \right| \ll 1$$





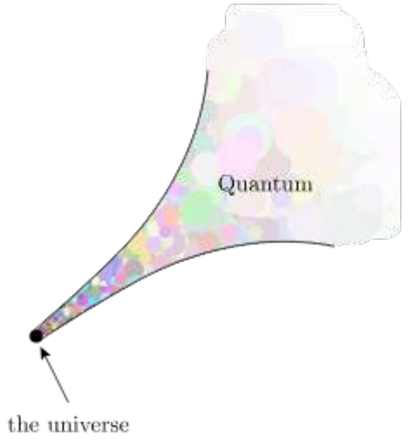
$$f(\mathcal{G}) = f_0 \mathcal{G}^k$$

Thanks to the EC, it turns out that inflation in f(G) gravity is realized when $k \ll 0 \vee k \gg \frac{1}{2}$

Application to Quantum Cosmology

Using canonical quantization rules

$$\begin{cases} [\hat{h}_{ij}(x), \hat{\pi}^{kl}(x')] = i\, \delta_{ij}^{kl}\, \delta^3(x-x') \\ \delta_{ij}^{kl} = \frac{1}{2}(\delta_i^k \delta_j^l + \delta_i^l \delta_j^k) \\ [\hat{h}_{ij}, \hat{h}_{kl}] = 0 \\ [\hat{\pi}^{ij}, \hat{\pi}^{kl}] = 0. \end{cases}$$



One gets a Schroedinger-like equation

$$\left(2\, \nabla^2 - \frac{1}{2}\sqrt{h}\, {}^3R\right) |\psi>= 0$$

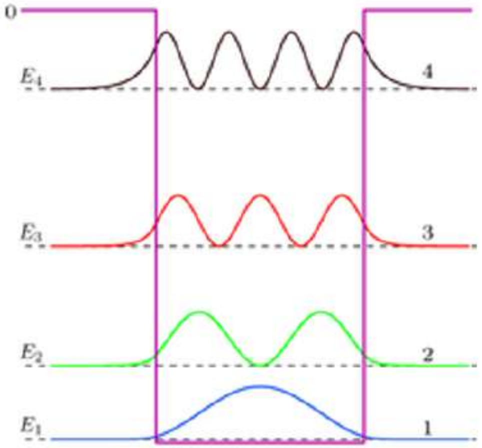
If the solution is oscillating

$$|\psi>\sim e^{ij_k Q^k}$$

Hartle Criterium

But…

$$\cancel{<\psi|\psi>\equiv \int \psi^* \psi\, dx^3}$$



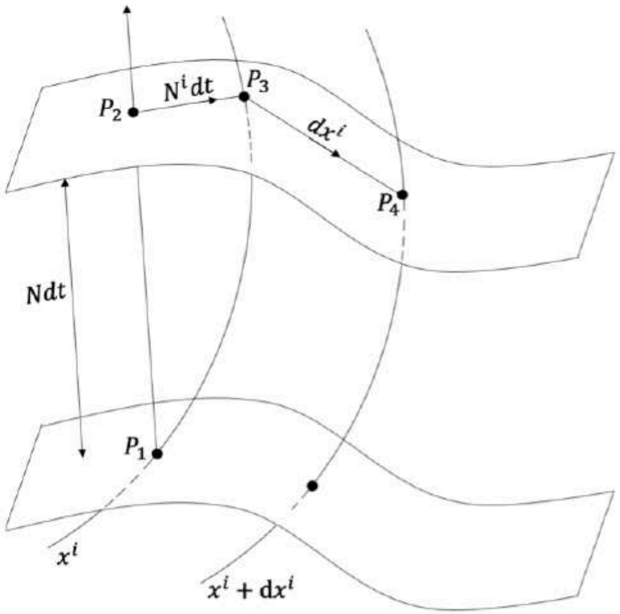
Many World

Enucleation from nothing

Probability of the quantum Universe to evolve towards our classical Universe

Quantum cosmology

$$ds^2 = g_{\mu\nu}dX^\mu dX^\nu = -(N^2 - N_iN^i)dt^2 + 2N^i dx_i dt + h_{ij}dx^i dx^j$$



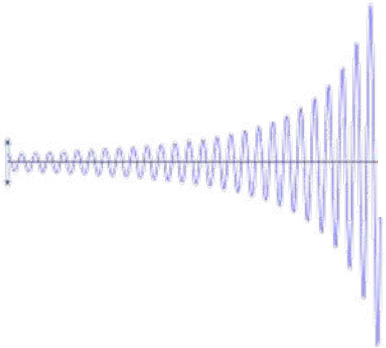
$$\boxed{\mathcal{H} = \int (\pi^{ij}\dot{h}_{ij} - \mathscr{L})d^3x}$$

$$\begin{cases} h_{ij} \to \hat{h}_{ij} \\ \pi \to \hat{\pi} = -i\dfrac{\delta}{\delta N} \\ \pi^i \to \hat{\pi}^i = -i\dfrac{\delta}{\delta N_i} \\ \pi^{ij} \to \hat{\pi}^{ij} = -i\dfrac{\delta}{\delta h_{ij}} \end{cases} \xrightarrow{\text{Noether symmetries}} \begin{cases} \hat{\mathscr{H}}|\psi>= 0 \\ -i\partial_1|\psi>= j_1|\psi> \\ -i\partial_2|\psi>= j_2|\psi> \\ \vdots \\ -i\partial_m|\psi>= j_m|\psi> \end{cases}$$

$\hat{\mathcal{H}}_0|\psi>= 0$	$|\psi>\sim e^{ij_kQ^k}$

Hartle's Criterion

But it is not the probability amplitude



Oscillating wave function of the universe

Applications of f(G) gravity to quantum cosmology

By a Legendre transformation of the Lagrangian one gets:

$$\mathcal{H}=\frac{f_0}{k}\mathcal{G}^k a^3+\pi_a\left(-\frac{\pi_{\mathcal{G}}}{8f_0}\mathcal{G}^{2-k}\right)^{\frac{1}{3}}$$

Imposing

$$\begin{cases}\pi_{\mathcal{G}}=-i\dfrac{\partial}{\partial\mathcal{G}}\\[2ex] \pi_a=-i\dfrac{\partial}{\partial a}\\[2ex] \boxed{\mathcal{H}\psi=0\,.}\end{cases}$$

$$i\frac{\partial}{\partial\mathcal{G}}\psi(a,\mathcal{G})=8f_0\Sigma_0\mathcal{G}^{4k-2}\psi(a,\mathcal{G})$$

$$\psi(a,\mathcal{G})=\psi_0\exp\left\{i\left[-\frac{f_0}{4k}\left(\Sigma_0\right)^{-\frac{1}{3}}a^4+\frac{8f_0\Sigma_0\mathcal{G}^{4k-1}}{1-4k}\right]\right\}$$

$$\frac{\partial S}{\partial a}=\pi_a\quad\rightarrow\quad\mathcal{G}^k a^3=24\mathcal{G}^{k-2}\dot{a}^3\dot{\mathcal{G}}$$

The wave function can be recast as

$$\psi(a,\mathcal{G})\sim e^{iS}$$

with

$$S=-\frac{f_0}{4k}\left(\Sigma_0\right)^{-\frac{1}{3}}a^4+\frac{8f_0\Sigma_0}{1-4k}\mathcal{G}^{4k-1}$$

Hartle's criterion is recovered and classical trajectories are provided by Hamilton-Jacobi equations

$$\frac{\partial S}{\partial a}=\pi_a\quad\rightarrow\quad\mathcal{G}^k a^3=24\mathcal{G}^{k-2}\dot{a}^3\dot{\mathcal{G}}$$

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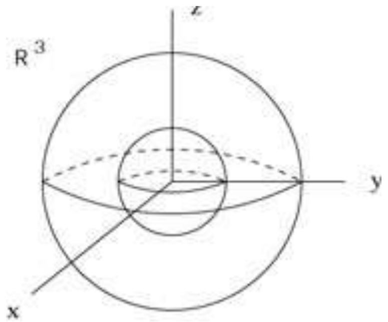
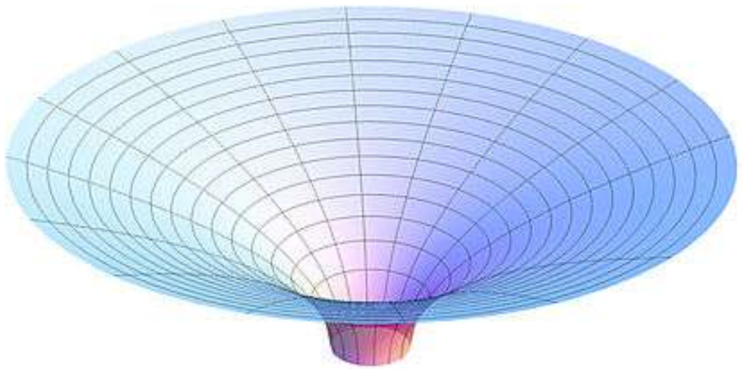
Spherical symmetry

Number of spatial dimensions

$$ds^2 = P(r)^2 dt^2 - Q(r)^2 dr^2 - r^2 \underbrace{d\Omega^2_{d-1}}_{\text{d-1 sphere}}$$





$$d\Omega^2_{d-1} = \Sigma_{j=1}^{d-1} d\theta_j^2 + \sin^2\theta_j d\phi^2$$

In this spacetime, the **GB** term can be written as:

$$\mathcal{G}^{(d+1)} = \frac{(d-2)(d-1)}{r^4 P Q^5} \Big\{(d-3)P\left(Q^2-1\right)\left[(d-4)Q^3-(d-4)Q+4rQ'\right]$$
$$-4r\left[(d-3)Q^3P'+rQ^3P''-(d-3)QP'-rQP''\right.$$
$$\left.-rQ^2P'Q'+3rP'Q'\right]\Big\}\,,$$

Note that for d=3 it turns into a topological surface term, while for d<3 it vanishes

We assume the Birkhoff theorem to hold for this model

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We start from

d+1 dimensional representation of the GB term in spherical symmetry

$$\mathcal{S} = \int d^{d+1}x\, r^{d-1} PQ \left[f(\mathcal{G}) - \lambda\left(\mathcal{G} - \tilde{\mathcal{G}}\right)\right]$$

The action can be varied with respect to G, in order to find out the Lagrange multiplier

$$\begin{aligned}\mathscr{L}(r,P,Q,\mathcal{G}) =& r^{d-1}PQ\left[f - \mathcal{G}f_{\mathcal{G}}\right] \\ &+ \frac{(d-1)(d-2)r^{d-5}(Q^2-1)}{Q^4}\left\{(d-3)Pf_{\mathcal{G}}\left[(d-4)Q(Q^2-1) + 4rQ'\right]\right. \\ &\left.+4r^2QP'\mathcal{G}'f_{\mathcal{G}\mathcal{G}}\right\}.\end{aligned}$$

Imposing the symmetry existence condition

$$X^{[1]}\mathscr{L} + \partial_\mu \xi^\mu \mathscr{L} = \partial_\mu g^\mu$$

with generator

$$\mathcal{X} = \xi(r,\mathcal{G},P,Q)\partial_r + \eta^{\mathcal{G}}(r,\mathcal{G},P,Q)\partial_{\mathcal{G}} + \eta^P(r,\mathcal{G},P,Q)\partial_P + \eta^Q(r,\mathcal{G},P,Q)\partial_Q$$

The function and the symmetry generator can be selected

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Results provided by the approach

$$\left[\begin{array}{l} \mathcal{X} = c_1 r\partial_r - 4c_1\mathcal{G}\partial_\mathcal{G} + (4k-d)c_1 P\partial_P \\ \mathcal{X} = c_1 r\partial_r + c_2\partial_P \end{array}\right\}$$

Only non-trivial function containing symmetries

$$f = f_0\mathcal{G}^k$$

Once replaced into the field equations leads to

$\mathbf{P(r)^2}$	$\mathbf{Q(r)^2}$	$\mathbf{d}$	$\mathbf{k}$
$1+e^{-2c_2\sqrt{c_1-4r}r^{\frac{3}{2}-\frac{d}{2}}}$	$1/P(r)^2$	$d\geq 3$	$k>0,\neq 1$
$P_0^2\left(1-\frac{k_3}{r^{\frac{d}{2}-2}}\right)$	$1/P(r)^2$	$d>3$	$k=1$
$1\pm r^{2-\frac{d}{2}}\sqrt{\frac{4k_1 d}{120\binom{d+1}{d-4}}}\pm r^2\sqrt{\frac{\mathcal{G}_0(d-3)}{120\binom{d+1}{d-4}}}$	$1/P(r)^2$	$d>3$	$k=\frac{d+1}{4}$
$\forall P(r)$	$\frac{1}{3}\left(A(r)-e^{q_0}P'(r)+\frac{e^{2q_0}}{A(r)}P'(r)^2\right)$	$d=3$	$k>0$
$-\frac{1}{2}\exp\left[tanh^{-1}\left(\sqrt{\frac{\mathcal{G}_0}{30}}\frac{r^2}{2}\right)\right]\sqrt{4-\frac{\mathcal{G}_0 r^4}{30}}$	$1+\frac{\sqrt{\mathcal{G}_0}r^2}{2\sqrt{30}}$	$d=4$	$k=5/4$
1	1	$d=4$	$\forall k$

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Four-dimensional limit (d = 3)

Imposing d=3 and $Q(r) = P(r)^n$

The field equations folds into a single equation of the form

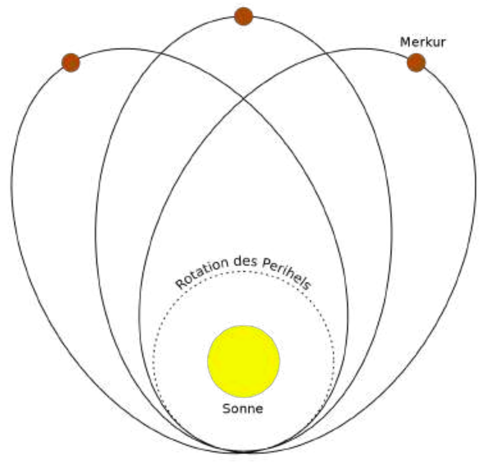
$$k(P^{2n} - 3)P'^2 - P(P^{2n} - 1)P'' = 0$$

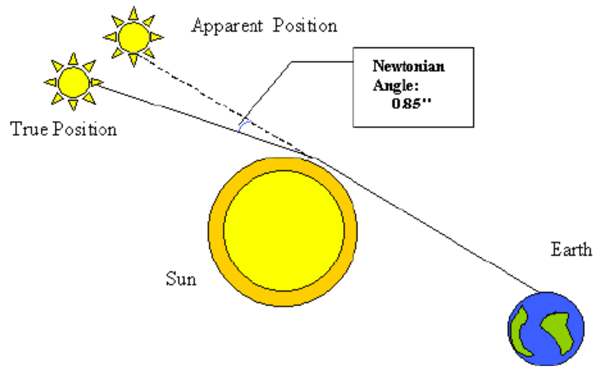
Whose solution is

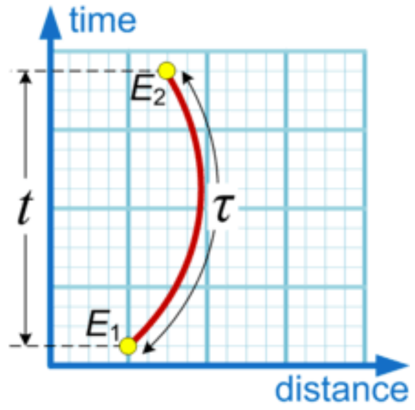
$$P(r)^2 = -2c_1\left[(r + c_2)\left(\frac{6r}{M(r)} + 1\right)\right] + \frac{3}{8}\left[\frac{M(r)^2 + 9}{M(r)} + 3\right]$$

with

$$M(r) = \sqrt[3]{128c_1^2r^2 + 16\left(16c_1c_2 - 9\right)c_1r + 64\sqrt{c_1^3\left(c_2 + r\right)^3\left(4c_1r + 4c_2c_1 - 1\right)} + 128c_2^2c_1^2 - 144c_2c_1 + 27}.$$







Further analysis is needed to check whether the theory passes at least all the solar system tests

3. Chern-Simons Theory

Basic foundations of Lovelock and Chern-Simons theories

Lovelock action

$SO(1, D-1)$

$$S = \kappa \int_{\mathcal{M}} \sum_{i=0}^{\frac{D}{2}} \alpha_i \mathscr{L}^{(D,i)}$$

$$\mathscr{L}^{(D,i)} = \epsilon_{a_1,a_2\ldots a_D} R^{a_1 a_2} \wedge R^{a_3 a_4} \wedge \ldots \wedge R^{a_{2i-1} a_{2i}} \wedge e^{a_{2i+1}} \wedge e^{a_{2i+2}} \wedge \ldots \wedge e^{a_D}$$

It is the most general action ***without torsion*** which leads to ***second-order field equations***

*The **GB** term naturally emerges*

$$\mathcal{G} = \epsilon_{a_1,a_2,a_3\ldots a_n} R^{a_1,a_2} \wedge R^{a_3,a_4} \wedge e^{a_5} \wedge \ldots \wedge e^{a_n}$$

Four-dimensional limit

Topological surface

It follows that

Lovelock Theorem:

The H-E action is the most general four-dimensional action leading to second-order field equations

$$S = \kappa \int \sqrt{-g}\,(\alpha_0 + \alpha_1 R + \alpha_2 \mathcal{G})\, d^4x$$

$$\mathscr{L}^{(4)} = \epsilon_{abcd}\left[\alpha_2 R^{ab} \wedge R^{cd} + \alpha_1 R^{ab} \wedge e^c \wedge e^d + \alpha_0 e^a \wedge e^a \wedge e^b \wedge e^c \wedge e^d\right]$$

J. Zanelli. arXiv:hep-th/0502193

Chern-Simons theory

Starting from Lovelock action, it is possible to select the coefficients such that the resulting theory is invariant with respect to some gauge group

$$S = \kappa \int_{\mathcal{M}} \sum_{i=0}^{\frac{D}{2}} \alpha_i \mathscr{L}^{(D,i)}$$

Proceed by trial and error

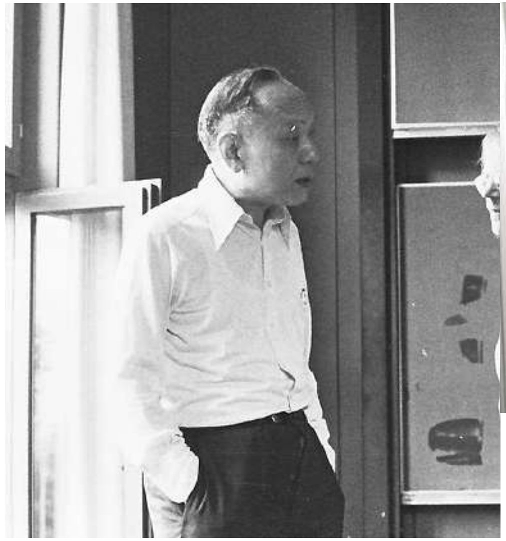
Find a methodical procedure

Why Chern-Simons?

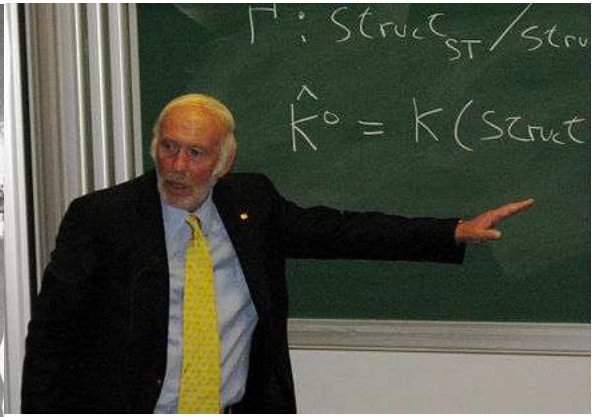
- It fits the formalism of QFT
- It can be quantized
- It can be applied to SUGRA
- It can be renormalized
- AdS/CFT correspondence

All D-dimensional Lagrangians whose exterior derivative provides a topological surface term, are *quasi* gauge-invariant $E = d\mathcal{L}.$

Chern-Simons Lagrangians



Shiing-Shen Chern



James Harris Simons

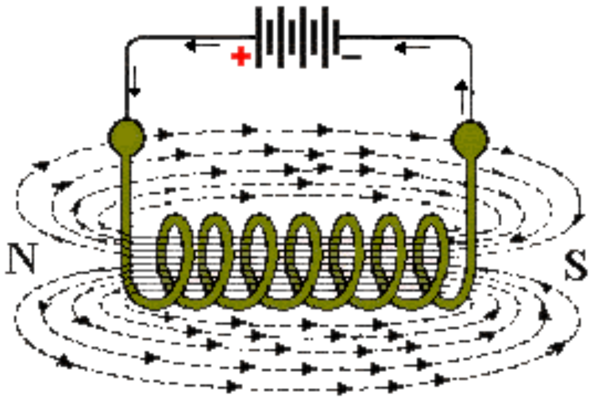
$D=3$ Chern-Simons Lagrangian	Top. Invariant($\mathcal{P}$)	Group
$L_3^{(A)dS} = \epsilon_{abc}(R^{ab} \pm \frac{e^a e^b}{3l^2})e^c$	$E_4 = \epsilon_{abc}(R^{ab} \pm \frac{e^a e^b}{l^2})T^c$	$SO(4)\dagger$
$L_3^{Lorentz} = \omega^a{}_b d\omega^b{}_a + \frac{2}{3}\omega^a{}_b\omega^b{}_c\omega^c{}_a$	$P_4^{Lorentz} = R^a{}_b R^b{}_a$	$SO(2,1)$
$L_3^{Torsion} = e^a T_a$	$N_4 = T^a T_a - e^a e^b R_{ab}$	$SO(2,1)$
$L_3^{U(1)} = AdA$	$P_4^{U(N)} = FF$	$U(1)$
$L_3^{SU(N)} = tr[\mathbf{AdA}+\frac{2}{3}\mathbf{AAA}]$	$P_4^{SU(4)} = tr[\mathbf{FF}]$	$SU(N)$

*Examples of three-dimensional **CS** Lagrangians*

Chern-Simons theory: three other examples

$\mathscr{L} = AdA$ → U(1) invariant

Related topological surface



Electromagnetism ← $P_4^{U(N)} = dAdA$

$\mathscr{L} = \mathrm{Tr}\left(\mathbf{A}\wedge d\mathbf{A} + \frac{2}{3}\mathbf{A}\wedge\mathbf{A}\wedge\mathbf{A}\right)$ → SU(N) invariant

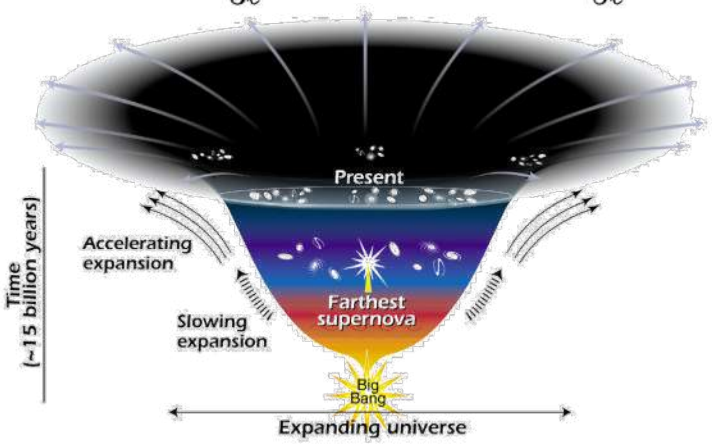
Related topological surface

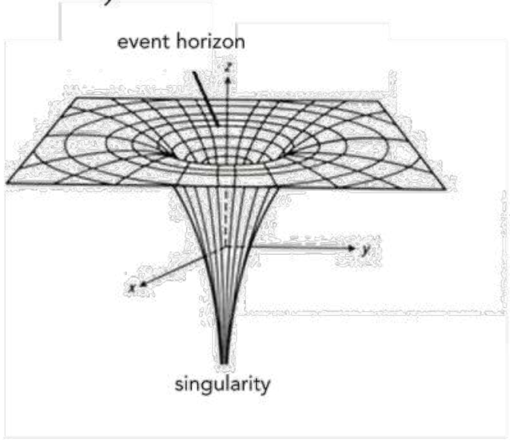
Biological systems ← $P_4^{SU(4)} = Tr[dAdA]$

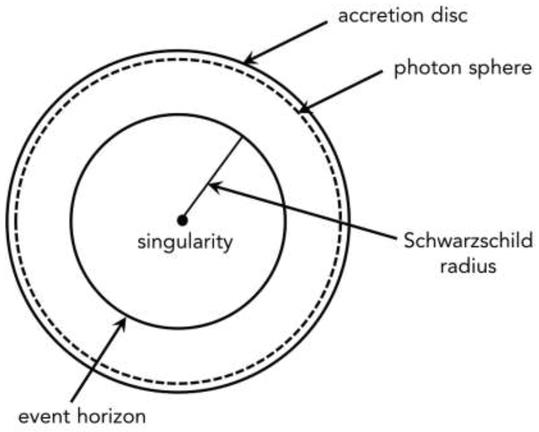
$$\mathscr{L}_5^{AdS} = \frac{1}{\ell}\epsilon_{abcde}\left(R^{ab}\wedge R^{cd}\wedge e^e + \frac{2}{3\ell^2}R^{ab}\wedge e^c\wedge e^d\wedge e^e + \frac{1}{5\ell^4}e^a\wedge e^b\wedge e^c\wedge e^d\wedge e^e\right)$$

→ AdS invariant

Related topological surface







Cosmology and Black holes ←

$$E_6 = 2R^{abcd}R_{cdef}R^{ef}_{ab} + 8R^{ab}_{cd}R^{ce}_{bf}R^{df}_{ae} + 24R^{abcd}R_{cdbe}R^{e}_{a} + 3RR^{abcd}R_{abcd} + 24R^{abcd}R_{ac}R_{bd} + 16R^{ab}R_{bc}R^{c}_{a} - 12RR^{ab}R_{ab} + R^3\,.$$

Cosmological applications

$$ds^2 = dt^2 - \frac{a(t)^2}{1-kr^2}dr^2 - r^2 d\Omega^2_{d-1}$$

$$S = \kappa \int |e| \left[\alpha_0 + \alpha_1 \mathcal{R}^{(d+1)} + \alpha_2 \mathcal{G}^{(d+1)}\right] d^{d+1}x$$

Starting line element and starting action

d+1 dimensional expression of scalar curvature and GB term

$$\mathcal{R}^{(d+1)} = -d\left[2\frac{\ddot{a}}{a} + (d-1)\left(\frac{\dot{a}^2+k}{a^2}\right)\right]$$

$$\mathcal{G}^{(d+1)} = d(d-1)(d-2)\left[(d-3)\left(\frac{\dot{a}^4}{a^4} + 2k\frac{\dot{a}^2}{a^4} + \frac{k^2}{a^4}\right) + \frac{4}{3a^3}\frac{d}{dt}\left(\dot{a}^3\right) + 4k\frac{\ddot{a}}{a^3}\right]$$

$$\mathcal{L} = \frac{r^{d-2}a^{d-4}}{3\sqrt{1-kr^2}} \Big\{3a^2[a^2\alpha_0 + \alpha_1 d(d-1)(\dot{a}^2-k)] - \alpha_2 d(d-1)(d-2)(d-3)(\dot{a}^4 + 6k\dot{a}^2 - 3k^2)\Big\}$$

Cosmological Lagrangian

Case	α_0	α_1	α_2	k	Scale Factor	Dimension
Einstein–de Sitter	$\neq 0$	$\neq 0$	0	$\neq 0$	$a(t) = \pm\sqrt{\frac{\alpha_1 k\, d(d-1)}{\alpha_0 - \alpha_0 \coth^2\left[\sqrt{\alpha_0}\left(c_1 + \frac{t}{\sqrt{\alpha_1(d-1)d}}\right)\right]}}$	Any
				0	$a(t) = a_0 e^{\pm\sqrt{\frac{\alpha_0}{\alpha_1 d(d-1)}}\,t}$	
					$a(t) = a_0 e^{\pm\sqrt{\frac{\alpha_0}{12\alpha_1}}\,t}$	5
Pure Gauss–Bonnet	$\neq 0$	0	$\neq 0$	0	$a(t) = a_0 \exp\left\{\pm\sqrt[4]{\frac{-\alpha_0}{d(d-1)(d-2)(d-3)\alpha_2}}\,t\right\}$ $a(t) = b(t)$	Any
					$a(t) = a_0 \exp\left\{\pm\sqrt[4]{\frac{-\alpha_0}{24\alpha_2}}\,t\right\}$	5
	0	0	$\neq 0$	$\neq 0$	$a(t) = \sqrt{-k}\,t$	Any
				0	$a(t) \sim \text{Const.}$	
					$a(t) \sim \text{Const.}$	5
Lovelock	$\neq 0$	$\neq 0$	$\neq 0$	0	$a(t) = a_0 \exp\left\{\pm\sqrt{\frac{2\alpha_0}{\pm\sqrt{(d-1)d\,[\alpha_1^2(d-1)d - 4\alpha_0\alpha_2(d-3)(d-2)]} + \alpha_1 d(d-1)}}\,t\right\}$	Any
					$a(t) = a_0 \exp\left\{\pm\sqrt{\frac{\alpha_0}{\pm 2\sqrt{9\alpha_1^2 - 6\alpha_0\alpha_2} + 6\alpha_1}}\,t\right\}$	5
	0	$\neq 0$	$\neq 0$	$\neq 0$	$a(t) = \pm\sqrt{\frac{-\alpha_2 k(d-3)(d-2)}{\alpha_1}} \sinh\left[\sqrt{\alpha_1}\left(\frac{t}{\sqrt{\alpha_2(d-3)(d-2)}} + c_1\right)\right]$	Any
				0	$a(t) = a_0 \exp\left\{\pm\sqrt{\frac{\alpha_1}{\alpha_2\,(d-2)\,(d-3)}}\,t\right\}$	
					$a(t) = a_0 \exp\left\{\pm\sqrt{\frac{\alpha_1}{2\alpha_2}}\,t\right\}$	5
Chern–Simons	$\frac{1}{5l^4}$	$\frac{2}{3l^2}$	1	0	$a(t) = a_0 \exp\left\{\pm\frac{1}{l}\sqrt{\frac{1}{6}\left(1 \pm \sqrt{\frac{7}{10}}\right)}\,t\right\}$	

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Spherical symmetry

$$ds^2 = P(r)^2 dt^2 - Q(r)^2 dr^2 - r^2 d\Omega^2_{d-1}$$

$$S = \kappa \int |e| \left[\alpha_0 + \alpha_1 \mathcal{R}^{(d+1)} + \alpha_2 \mathcal{G}^{(d+1)}\right] d^{d+1}x$$

Starting line element and starting action

d+1 dimensional expression of scalar curvature and GB term

$$\mathcal{R}^{(d+1)} = \frac{2r\left\{Q\left[(d-1)P' + rP''\right] - rP'Q'\right\} + (1-d)P\left[(d-2)Q^3 + (2-d)Q + 2rQ'\right]}{r^2 P Q^3}$$

$$\mathcal{G}^{(d+1)} = \frac{(d-2)(d-1)}{r^4 P Q^5} \Big\{(d-3)P\left(Q^2-1\right)\left[(d-4)Q^3 - (d-4)Q + 4rQ'\right]$$
$$-4r\Big[(d-3)Q^3P' + rQ^3P'' - (d-3)QP' - rQP''$$
$$-rQ^2P'Q' + 3rP'Q'\Big]\Big\} \,,$$

$$\mathcal{L}^{(d+1)} = \frac{r^{d-5}P}{Q^4} \Big\{\alpha_0 r^4 Q^5 - \alpha_1 (d-1) r^2 Q^2 [(d-2)Q(Q^2-1) + 2rQ']$$
$$+\alpha_2 (d-3)(d-2)(d-1)(Q^2-1)[(d-4)Q(Q^2-1) + 4rQ']\Big\}$$

Spherically symmetric Lagrangian

Case	α_0 α_1 α_2	$P(r)^2$, $Q^2(r)$	Dimension
Einstein–de Sitter	$\neq 0$ $\neq 0$ 0	$P(r)^2 = 1/Q(r)^2 = 1 + \frac{c_1}{r^{d-2}} - \frac{\alpha_0}{\alpha_1 d(d-1)} r^2$	Any
		$P(r)^2 = 1/Q(r)^2 = 1 + \frac{c_1}{r^2} - \frac{\alpha_0}{12\alpha_1} r^2$	5
Pure Gauss–Bonnet	$\neq 0$ 0 $\neq 0$	$P(r)^2 = 1/Q(r)^2 = 1 \pm \frac{1}{r^{d/2-2}} \sqrt{\frac{c_1}{6\alpha_2 \binom{d-1}{d-4}} - r^d \frac{\alpha_0}{24\alpha_2 \binom{d}{d-4}}}$	Any
		$P(r)^2 = 1/Q(r)^2 = 1 \pm \sqrt{1 + c_1 - \frac{\alpha_0}{24\alpha_2} r^4}$	5
		$P^2(r) = P_0^2 \sqrt{48\alpha_2 (2c_1 + 1) \mp 4\sqrt{6}\sqrt{\alpha_2 (24\alpha_2 + 96\alpha_2 c_1 - \alpha_0 r^4)} - \alpha_0 r^4}$ $\quad Q^2(r) = \frac{2\left(12\alpha_2 \pm \sqrt{6}\sqrt{\alpha_2 (24\alpha_2 + 96\alpha_2 c_1 - \alpha_0 r^4)}\right)}{\alpha_0 r^4 - 96\alpha_2 c_1}$	
	0 0 $\neq 0$	$P(r)^2 = 1/Q(r)^2 = 1 + \frac{c_1}{r^{\frac{d}{2}-2}}$	Any
		Const.	5
Lovelock	$\neq 0$ $\neq 0$ $\neq 0$	$P(r)^2 = 1/Q(r)^2 = 1 \pm \frac{1}{r^{d/2-2}} \sqrt{\frac{c_1}{6\alpha_2 \binom{d-1}{d-4}} + r^d \left(\frac{\alpha_1^2}{16\alpha_2^2 \binom{d-2}{d-4}^2} - \frac{\alpha_0}{24\alpha_2 \binom{d}{d-4}}\right)} - \frac{\alpha_1}{4\alpha_2 \binom{d-2}{d-4}} r^2$	Any
		$P(r)^2 = 1/Q(r)^2 = 1 - \frac{\alpha_1 r^2}{4\alpha_2} \pm \frac{\sqrt{3r^4 (3\alpha_1^2 - 2\alpha_0\alpha_2) + 6\alpha_2 c_1}}{12\alpha_2}$	5
		$P(r)^2 = P_0^2 \sqrt{-4c_1 + \alpha_0 r^4 - 12\alpha_1 r^2} \left[\frac{3\sqrt{r^4 uw + 2ux} + \sqrt{3}\left(\alpha_0 x - r^2 w(-3\alpha_1 + z)\right)}{-6\alpha_1 + \alpha_0 r^2 + 2z}\right]^{y/2} \left[\frac{6\alpha_1 - \alpha_0 r^2 + 2z}{3\sqrt{r^4 vw + 2vx} + \sqrt{3}\left(r^2 w(z + 3\alpha_1) + \alpha_0 x\right)}\right]^{s/2}$ $\quad Q(r)^2 = \frac{-3\alpha_1 r^2 + 12\alpha_2 \pm \sqrt{3}\sqrt{8c_1\alpha_2 - 2\alpha_0\alpha_2 r^4 + 3\alpha_1^2 r^4 + 48\alpha_2^2}}{\frac{\alpha_0}{2} r^4 - 6\alpha_1 r^2 - 2c_1}$	
	0 $\neq 0$ $\neq 0$	$P(r)^2 = 1/Q(r)^2 = 1 \pm \frac{1}{r^{d/2-2}} \sqrt{\frac{c_1}{6\alpha_2 \binom{d-1}{d-4}} + \frac{\alpha_1^2}{16\alpha_2^2 \binom{d-2}{d-4}^2} r^d} - \frac{\alpha_1}{4\alpha_2 \binom{d-2}{d-4}} r^2$	Any
		$P(r)^2 = 1/Q(r)^2 = 1 - \frac{\alpha_1 r^2}{4\alpha_2} \pm \frac{\sqrt{9\alpha_1^2 r^4 + 6\alpha_2 c_1}}{12\alpha_2}$	5
		$P(r)^2 = P_0^2 \left(2c_1 + \alpha_1 r^2\right) \sqrt{\frac{\sqrt{16\alpha_2(\alpha_2 + c_1) + \alpha_1^2 r^4} + \alpha_1 r^2}{8\alpha_2^2 + 2\alpha_2\left(\sqrt{16\alpha_2(\alpha_2 + c_1) + \alpha_1^2 r^4} + 4c_1\right) + c_1\left(\sqrt{16\alpha_2(\alpha_2 + c_1) + \alpha_1^2 r^4} - \alpha_1 r^2\right)}}$, $\quad Q(r)^2 = \frac{-4\alpha_2 - \sqrt{16\alpha_2(\alpha_2 + c_1) + \alpha_1^2 r^4} + \alpha_1 r^2}{4c_1 + 2\alpha_1 r^2}$	
		$P(r)^2 = P_0^2 \sqrt{\frac{8\alpha_2^2 + 2\alpha_2\left(\sqrt{16\alpha_2(\alpha_2 + c_1) + \alpha_1^2 r^4} + 4c_1\right) + c_1\left(\sqrt{16\alpha_2(\alpha_2 + c_1) + \alpha_1^2 r^4} - \alpha_1 r^2\right)}{\sqrt{16\alpha_2(\alpha_2 + c_1) + \alpha_1^2 r^4} + \alpha_1 r^2}}$ $\quad Q(r)^2 = \frac{-4\alpha_2 + \sqrt{16\alpha_2(\alpha_2 + c_1) + \alpha_1^2 r^4} + \alpha_1 r^2}{4c_1 + 2\alpha_1 r^2}$	
Chern–Simons	$\frac{1}{5l^4}$ $\frac{2}{3l^2}$ 1	$P(r)^2 = 1/Q(r)^2 = 1 - \frac{r^2}{6l^2} \pm \sqrt{\frac{7r^4}{360l^4} + \frac{c_1}{24}}$	
		$P(r)^2 = P_0^2 \sqrt{-4c_1 + \frac{r^4}{5l^4} - \frac{8r^2}{l^2}} \left[\frac{\sqrt{3}\left(\frac{x}{5l^4} - r^2 w\left(-\frac{2}{l^2} + z\right)\right) + 3\sqrt{u(r^4 w + 2x)}}{\frac{r^2}{5l^4} - \frac{4}{l^2} + 2z}\right]^{y/2} \left[\frac{-\frac{r^2}{5l^4} + \frac{4}{l^2} + 2z}{\sqrt{3}\left(\frac{x}{5l^4} + r^2 w\left(z + \frac{2}{l^2}\right)\right) + 3\sqrt{v(r^4 w + 2x)}}\right]^{s/2}$ $\quad Q(r)^2 = \frac{\sqrt{24c_1 + \frac{14r^4}{5l^4} + 144} - \frac{2r^2}{l^2} + 12}{-2c_1 + \frac{r^4}{10l^4} - \frac{4r^2}{l^2}}$	

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Classical electromagnetism

In coordinates representation

$$S = \int \mathbf{A}d\mathbf{A} = \int (\partial_\mu A_\nu - \partial_\nu A_\mu) A_p \, dx^\mu \wedge dx^\nu \wedge dx^p = \int \epsilon^{\mu\nu p} F_{\mu\nu} A_p \, d^3x$$

Can be added to the free EM Lagrangian

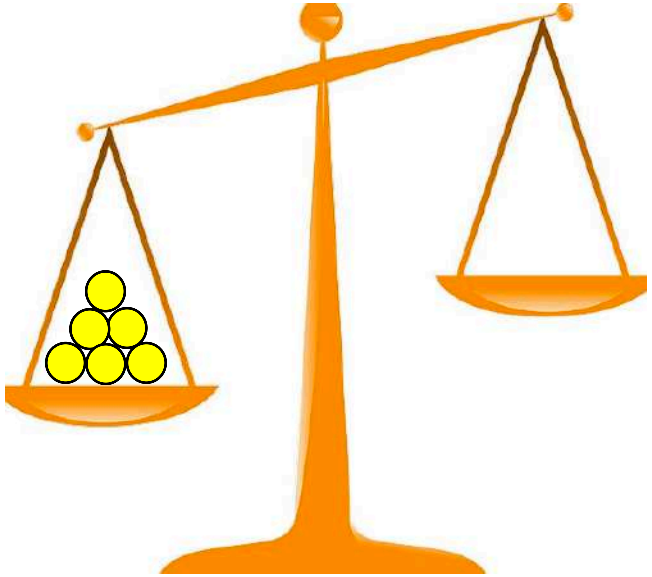
$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{1}{2} m \, \epsilon^{\mu\nu p} F_{\mu\nu} A_p$$

By varying with respect to the gauge connection

$$(\Box + m^2)\left(\epsilon_{\alpha\beta\tau} F^{\alpha\beta}\right) = 0$$

→ **Massive wave equations** → **Massive photons**

Klein-Gordon equation for the vector field $\epsilon_{\alpha\beta\tau} F^{\alpha\beta}$



Proca wave equation: $(\Box + m^2) A^\beta = 0$ → Breaks U(1) invariance	
Chern-Simons wave equation: $(\Box + m^2)\left(\epsilon_{\alpha\beta\tau} F^{\alpha\beta}\right) = 0$ → U(1) invariant, but not conformally invariant	
Special relativity is preserved	

4. *Gauss-Bonnet Invariant emerging from symmetry*

Starting from a general **fourth-order Lagrangian**

$$\mathcal{L} = \sqrt{-g}\left(\alpha_1 R^2 + \alpha_2 R^{\mu\nu}R_{\mu\nu} + \alpha_3 R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}\right)$$

The goal is to select **coefficients** by symmetry arguments

To this end, we focus on a general **spherically-symmetric-like background**

$$ds^2 = P(r)^2 dt^2 - Q(r)^2 dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

And use Lagrange multiplier to find the **point-like Lagrangian**

$$\begin{aligned}\mathcal{L} = \frac{1}{r^2 P Q^5} \quad & \Big(\alpha_2 \Big\{2\left[rQP' - P\left(rQ' + Q^3 - Q\right)\right]^2 + r^2\left(-rQP'' + rP'Q' + 2PQ'\right)^2 + r^2\left[rP'Q' - Q\left(rP'' + 2P'\right)\right]^2\Big\} \\ & + 4\alpha_3\left\{r^4 P'^2 Q'^2 - 2r^4 QP'P''Q' + Q^2\left(r^4 P''^2 + 2r^2 P'^2\right) + P^2\left(2r^2 Q'^2 + Q^6 - 2Q^4 + Q^2\right)\right\} \\ & + 4\alpha_1\left\{r\left[rP'Q' - Q\left(rP'' + 2P'\right)\right] + P\left(2rQ' + Q^3 - Q\right)\right\}^2\Big),\end{aligned}$$

We then apply **the Noether symmetry existence condition** in the **minisuperspace**

$$X^{[1]}\mathscr{L} + \partial_\mu \xi^\mu \mathscr{L} = \partial_\mu g^\mu \qquad \eta^i = \{\alpha(P,Q,r), \beta(P,Q,r)\}$$

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$$
\begin{aligned}
a: \quad & 4r(4\alpha_1+\alpha_2)\left(Q^2-1\right)\left(Q\frac{\partial\alpha}{\partial r}-P\frac{\partial\beta}{\partial r}\right)-2(2\alpha_1+\alpha_2+2\alpha_3)\left[Q\left(Q^2-1\right)^2\left(\alpha+P\frac{\partial\xi}{\partial r}\right)+\beta P\left(Q^4+2Q^2-3\right)\right] \\
& +r^2Q^4\frac{\partial g}{\partial r}=0 \\
b: \quad & (2\alpha_1+\alpha_2+2\alpha_3)\left[3\beta P+Q\left(\alpha-P\frac{\partial\xi}{\partial r}\right)\right]=0 \\
c: \quad & r(4\alpha_1+\alpha_2)\left(Q\frac{\partial\alpha}{\partial r}-P\frac{\partial\beta}{\partial r}\right)+2\alpha_1P\left[\beta(Q^2-3)-Q(Q^2-1)\frac{\partial\xi}{\partial r}\right]=0 \\
d: \quad & (2\alpha_1+\alpha_2+2\alpha_3)\frac{\partial\alpha}{\partial Q}=0 \\
e: \quad & 2r(4\alpha_1+\alpha_2)Q\frac{\partial\alpha}{\partial r}-2Q^2\left(10\alpha_1+\alpha_2-2\alpha_1Q^2\right)\frac{\partial\alpha}{\partial Q}-(8\alpha_1+3\alpha_2+4\alpha_3)\left[5\beta P-Q\left(\alpha-P\frac{\partial\xi}{\partial r}+2P\frac{\partial\beta}{\partial Q}\right)\right]=0 \\
f: \quad & r(2\alpha_1+\alpha_2+2\alpha_3)\frac{\partial\alpha}{\partial Q}-2(4\alpha_1+\alpha_2)P\frac{\partial\xi}{\partial Q}=0 \\
g: \quad & (2\alpha_1+\alpha_2+2\alpha_3)\frac{\partial\xi}{\partial Q}=0 \\
h: \quad & 2r(4\alpha_1+\alpha_2)\frac{\partial\alpha}{\partial Q}-(8\alpha_1+3\alpha_2+4\alpha_3)P\frac{\partial\xi}{\partial Q}=0 \\
i: \quad & r(2\alpha_1+\alpha_2+2\alpha_3)Q\frac{\partial\alpha}{\partial r}-(4\alpha_1+\alpha_2)\left[4\beta P+Q\left(Q\frac{\partial\alpha}{\partial Q}-P\frac{\partial\beta}{\partial Q}\right)\right]+2\alpha_1PQ^2(-1+Q^2)\frac{\partial\xi}{\partial Q}=0 \\
j: \quad & 4r^2(8\alpha_1+3\alpha_2+4\alpha_3)P\frac{\partial\beta}{\partial r}-r^2Q^5\frac{\partial g}{\partial Q}+2(2\alpha_1+\alpha_2+2\alpha_3)PQ^2(-1+Q^2)^2\frac{\partial\xi}{\partial Q} \\
& +4r(4\alpha_1+\alpha_2)\left(-2\beta P(-2+Q^2)+Q(-1+Q^2)\left(\alpha-Q\frac{\partial\alpha}{\partial Q}+P\frac{\partial\beta}{\partial Q}\right)\right)-4r^2Q(10\alpha_1+\alpha_2-2\alpha_1Q^2)\frac{\partial\alpha}{\partial r}=0 \\
k: \quad & 4\beta P(10\alpha_1+\alpha_2-\alpha_1Q^2)+2(4\alpha_1+\alpha_2)r\left(-Q\frac{\partial\alpha}{\partial r}+P\frac{\partial\beta}{\partial r}\right)+PQ(10\alpha_1+\alpha_2-2\alpha_1Q^2)\left(\frac{\partial\xi}{\partial r}-\frac{\partial\beta}{\partial Q}-\frac{\partial\alpha}{\partial P}\right) \\
& +(8\alpha_1+3\alpha_2+4\alpha_3)\left(Q^2\frac{\partial\alpha}{\partial Q}+P^2\frac{\partial\beta}{\partial P}\right)=0 \\
l: \quad & 2r(4\alpha_1+\alpha_2)P\frac{\partial\beta}{\partial r}+(8\alpha_1+3\alpha_2+4\alpha_3)\left[3\beta P+Q\left(\alpha+P\frac{\partial\xi}{\partial r}-2P\frac{\partial\alpha}{\partial P}\right)\right]+2P^2(10\alpha_1+\alpha_2-2\alpha_1Q^2)\frac{\partial\beta}{\partial P}=0 \\
m: \quad & r(2\alpha_1+\alpha_2+2\alpha_3)\left\{5\beta P+Q\left[\alpha+P\left(3\frac{\partial\xi}{\partial r}-2\frac{\partial\beta}{\partial Q}-2\frac{\partial\alpha}{\partial P}\right)\right]\right\}+4(4\alpha_1+\alpha_2)PQ\left(-Q\frac{\partial\xi}{\partial Q}+P\frac{\partial\xi}{\partial P}\right)=0 \\
n: \quad & (2\alpha_1+\alpha_2+2\alpha_3)\frac{\partial\xi}{\partial P}=0 \\
o: \quad & 2r^2\left(2\alpha_1+\alpha_2+2\alpha_3\right)Q\frac{\partial\alpha}{\partial r}+2PQ^2\left(10\alpha_1+\alpha_2-2\alpha_1Q^2\right)\frac{\partial\xi}{\partial Q} \\
& +2r(4\alpha_1+\alpha_2)\left[-5\beta P-2PQ\frac{\partial\xi}{\partial r}-2Q^2\frac{\partial\alpha}{\partial Q}+2PQ\frac{\partial\beta}{\partial Q}+PQ\frac{\partial\alpha}{\partial P}\right]-(8\alpha_1+3\alpha_2+4\alpha_3)P^2Q\frac{\partial\xi}{\partial P}=0 \\
p: \quad & r(2\alpha_1+\alpha_2+2\alpha_3)\frac{\partial\beta}{\partial P}+2(4\alpha_1+\alpha_2)Q\frac{\partial\xi}{\partial P}=0 \\
q: \quad & 2r(4\alpha_1+\alpha_2)\frac{\partial\beta}{\partial P}+(8\alpha_1+3\alpha_2+4\alpha_3)Q\frac{\partial\xi}{\partial P}=0 \\
r: \quad & 2r^2(2\alpha_1+\alpha_2+2\alpha_3)P\frac{\partial\beta}{\partial r}-(8\alpha_1+3\alpha_2+4\alpha_3)PQ^2\frac{\partial\xi}{\partial Q} \\
& +2r(4\alpha_1+\alpha_2)\left[4\beta P+\alpha Q+2PQ\frac{\partial\xi}{\partial r}-PQ\frac{\partial\beta}{\partial Q}-2PQ\frac{\partial\alpha}{\partial P}+2P^2\frac{\partial\beta}{\partial P}\right]+2P^2Q(10\alpha_1+\alpha_2-2\alpha_1Q^2)\frac{\partial\xi}{\partial P}=0 \\
s: \quad & (2\alpha_1+\alpha_2+2\alpha_3)\left[4\beta P+\alpha Q+PQ\left(\frac{\partial\xi}{\partial r}-\frac{\partial\beta}{\partial Q}-\frac{\partial\alpha}{\partial P}\right)\right] \\
t: \quad & r^2(2\alpha_1+\alpha_2+2\alpha_3)P\frac{\partial\beta}{\partial r}+r(4\alpha_1+\alpha_2)\left(3\beta P+\alpha Q-PQ\frac{\partial\alpha}{\partial P}+P^2\frac{\partial\beta}{\partial P}\right)+2\alpha_1P^2Q(-1+Q^2)\frac{\partial\xi}{\partial P}=0 \\
u: \quad & 4r^2(8\alpha_1+3\alpha_2+4\alpha_3)Q\frac{\partial\alpha}{\partial r}-r^2PQ^4\frac{\partial g}{\partial P}+2(2\alpha_1+\alpha_2+2\alpha_3)P^2Q(-1+Q^2)^2\frac{\partial\xi}{\partial P} \\
& -4r^2P(10\alpha_1+\alpha_2-2\alpha_1Q^2)\frac{\partial\beta}{\partial r}+4r(4\alpha_1+\alpha_2)P\left[\beta(Q^2-3)-(Q^2-1)\left(Q\frac{\partial\alpha}{\partial P}-P\frac{\partial\beta}{\partial P}\right)\right]=0 \\
v: \quad & (2\alpha_1+\alpha_2+2\alpha_3)\frac{\partial\beta}{\partial P}=0
\end{aligned}
$$

Whose **only** solution is

$$\xi = \xi_0 + \xi_1 r, \quad \alpha = 2\xi_1 P + k_1 r + k_2, \quad \beta = \frac{\xi_1 Q(Q^2-1)}{(Q^2-3)}, \quad g = \frac{8k_1\alpha_3(1-Q^2)}{Q^3} + g_2$$

with

$$2\alpha_1 + \alpha_2 + 2\alpha_3 = 0$$
$$\alpha_1 - \alpha_3 = 0.$$

$$\mathcal{L} = \sqrt{-g}\,\alpha_3\left(R^2 - 4R^{\mu\nu}R_{\mu\nu} + R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}\right) = \sqrt{-g}\,\alpha_3\mathcal{G}$$

The Gauss-Bonnet term is the only quadratic term containing symmetries

This is interesting in view of investigating the power of symmetries in gravity models

Example:

$$S = \kappa\int\sqrt{-g}\,(\alpha_0 + \alpha_1 R + \alpha_2\mathcal{G})\,d^4x$$

$$\mathscr{L}^{(4)} = \epsilon_{abcd}\left[\alpha_2 R^{ab}\wedge R^{cd} + \alpha_1 R^{ab}\wedge e^c\wedge e^d + \alpha_0 e^a\wedge e^a\wedge e^b\wedge e^c\wedge e^d\right]$$

5. Conclusions and Perspectives

Conclusions and perspectives: Gauss-Bonnet

We considered:

- *Cosmology and applications (EC, exact solutions)*
- *Spherical symmetry (Black hole solutions)*

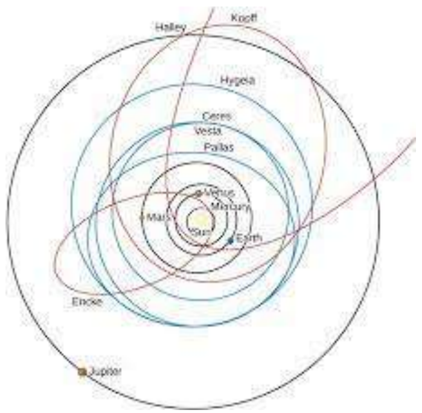
$$S = \int \sqrt{-g} f(\mathcal{G}) \; d^{d+1}x$$

Future perspectives:

- *Consider different backgrounds*
- *Constraining the free parameters by experiments*
- *Study the four-dimensional limit of the spherically symmetric solution via PPN formalism*
- *Consider other modifications including other topological surfaces (Pontrjagin scalar, Kretschmann scalar)*

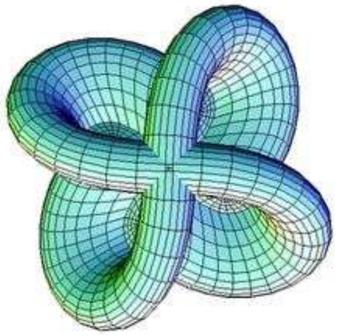
Conclusions and perspectives: Chern-Simons

Application of **Chern-Simons Gravity** to:



➢ ***Spherical symmetry***

Perspectives → *Study the orbits and the Solar System Test*



➢ ***Cosmology***

Perspectives → *Extend Lovelock gravity*

Perspectives → *Study other topological theories*

Thank you

for your attention