Finanziato dall'Unione europea NextGenerationEU
Ministero dell'Università e della Ricerca
Italiadomani PIANO NAZIONALE DI RIPRESA E RESILIENZA
ICSC Centro Nazionale di Ricerca in HPC, Big Data and Quantum Computing

Sum rules for Chiral, Conformal and Gravitational Anomaly and the Hadronic Gravitational Form Factors

Dario Melle

University of Salento and INFN

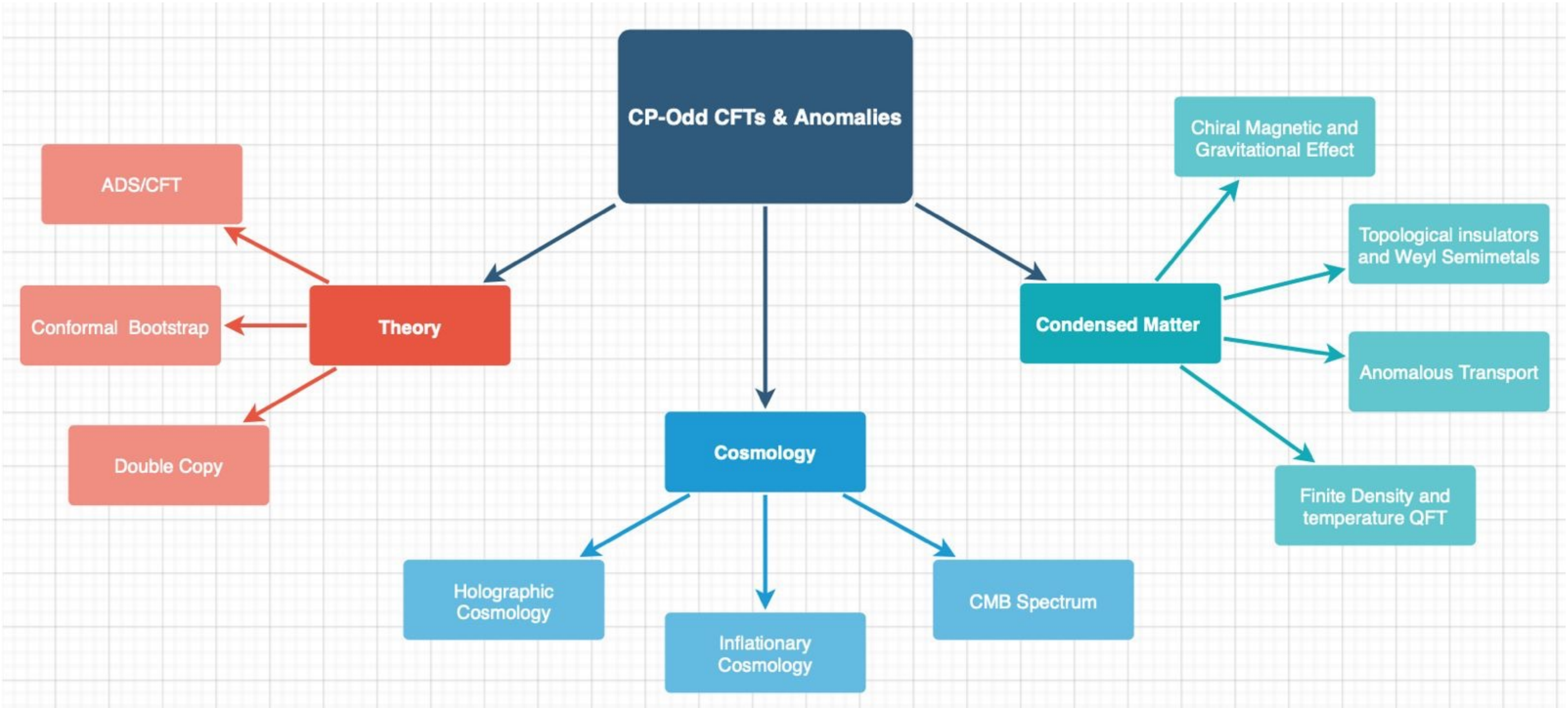
Based on works 2502.03182, 2504.01904 and 2511.-----

UNIVERSITÀ DEL SALENTO L'Ateneo tra i due mari
INFN Istituto Nazionale di Fisica Nucleare SEZIONE DI LECCE
ICSC Centro Nazionale di Ricerca in HPC, Big Data and Quantum Computing

Catania November 13, 2025

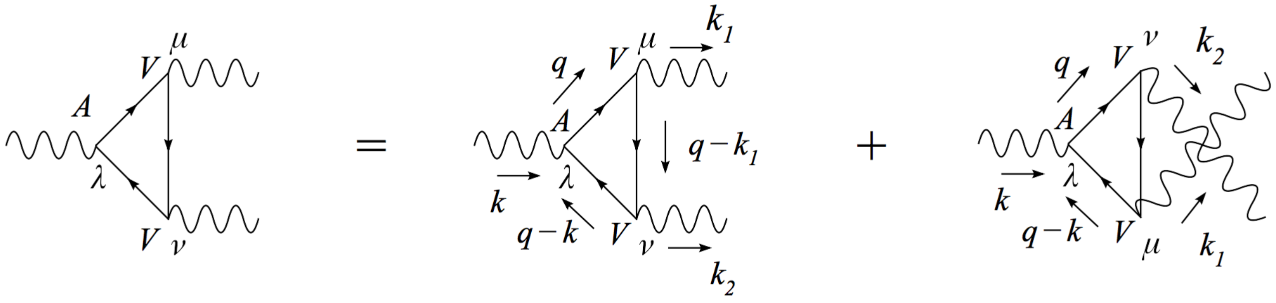
Outline

- Some motivation
- Chiral sum rules and the anomaly pole of the <AVV>
- Sum rules in the <ATT>
- Conformal sum rules and the <TJJ> extra pole
- Future Developments


CP-Odd CFTs & Anomalies
Theory
ADS/CFT
Conformal Bootstrap
Double Copy
Cosmology
Holographic Cosmology
Inflationary Cosmology
CMB Spectrum
Condensed Matter
Chiral Magnetic and Gravitational Effect
Topological insulators and Weyl Semimetals
Anomalous Transport
Finite Density and temperature QFT

AVV diagram

$$\Delta_{\alpha\mu\nu}(p_1,p_2) = \int \frac{d^4k}{(2\pi)^4}\mathrm{Tr}\left(\frac{1}{\not{k}-\not{p}_1-m}\gamma_\mu\frac{1}{\not{k}-m}\gamma_\nu\frac{1}{\not{k}+\not{p}_2-m}\gamma_\alpha\gamma_5\right)+[(p_1,\mu)\leftrightarrow(p_2,\nu)]$$



Schoutens used in this parametrization

$$p_2^{\mu_1}\epsilon^{\mu_2\mu_3p_1p_2} = p_2^{\mu_2}\epsilon^{\mu_1\mu_3p_1p_2} - p_2^{\mu_3}\epsilon^{\mu_1\mu_2p_1p_2} - p_2^2\epsilon^{\mu_1\mu_2\mu_3p_1} + (p_1\cdot p_2)\,\epsilon^{\mu_1\mu_2\mu_3p_2}$$
$$p_1^{\mu_2}\epsilon^{\mu_1\mu_3p_1p_2} = p_1^{\mu_1}\epsilon^{\mu_2\mu_3p_1p_2} + p_1^{\mu_3}\epsilon^{\mu_1\mu_2p_1p_2} - p_1^2\epsilon^{\mu_1\mu_2\mu_3p_2} + (p_1\cdot p_2)\,\epsilon^{\mu_1\mu_2\mu_3p_1}$$

$$\begin{aligned}\langle J^{\mu_1}(p_1)J^{\mu_2}(p_2)J_5^{\mu_3}(q)\rangle &= F_1\left(\epsilon^{\mu_1\mu_2p_1p_2}p_1^{\mu_3}+\epsilon^{\mu_1\mu_2p_1p_2}p_2^{\mu_3}\right)\\ +F_2\left(\epsilon^{\mu_2\mu_3p_1p_2}p_1^{\mu_1}-\epsilon^{\mu_1\mu_2\mu_3p_2}s_1\right)&+F_3\left(\epsilon^{\mu_1\mu_3p_1p_2}p_2^{\mu_2}-\epsilon^{\mu_1\mu_2\mu_3p_1}s_2\right)\end{aligned}$$

$$\nabla_\mu\langle J_5^\mu\rangle = a_1\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}+a_2\varepsilon^{\mu\nu\rho\sigma}R^{\alpha\beta}{}_{\mu\nu}R_{\alpha\beta\rho\sigma},$$

$$
\begin{aligned}
F_1 =&\frac{1}{64\pi^2\lambda(s,s_1,s_2)^2}\Big(2\left(s^2-2(s_1+s_2)s+(s_1-s_2)^2\right)(s-s_1-s_2)\\
&+2\left((s_1+s_2)s^2-2\left(s_1^2-4s_2s_1+s_2^2\right)s+(s_1-s_2)^2(s_1+s_2)\right)\mathrm{B}_0\left(s,m^2\right)\\
&-2s_1\left((s-s_1)^2-5s_2^2+4(s+s_1)s_2\right)\mathrm{B}_0\left(s_1,m^2\right)\\
&-2s_2\left(s^2+4s_1s-2s_2s-5s_1^2+s_2^2+4s_1s_2\right)\mathrm{B}_0\left(s_2,m^2\right)\\
&+4\big(m^2(s-s_1-s_2)\left(s^2-2(s_1+s_2)s+(s_1-s_2)^2\right)\\
&-s_1s_2\left(-2s^2+(s_1+s_2)s+(s_1-s_2)^2\right)\big)\mathrm{C}_0\left(s,s_1,s_2,m^2\right)\Big)
\end{aligned}
$$

Off-Shell

$$
\lim_{q^2\to 0} q^2\left\langle J^{\mu_1}(p_1)J^{\mu_2}(p_2)J^{\mu_3}(q)\right\rangle = 0.
$$

$$
\begin{aligned}
F_2 =&\frac{1}{2\pi^2\lambda(s,s_1,s_2)^2}\Big(\left(s^2-2(s_1+s_2)s+(s_1-s_2)^2\right)(s-s_1+s_2)\\
&+s\left((s-s_1)^2-5s_2^2+4(s+s_1)s_2\right)\mathrm{B}_0\left(s,m^2\right)\\
&-\left(s^3-(2s_1+s_2)s^2+\left(s_1^2+8s_2s_1-s_2^2\right)s+(s_1-s_2)^2s_2\right)\mathrm{B}_0\left(s_1,m^2\right)\\
&+\left(-5s^2+4(s_1+s_2)s+(s_1-s_2)^2\right)s_2\mathrm{B}_0\left(s_2,m^2\right)\\
&+2\big((s-s_1+s_2)\left((s-s_1)^2+s_2^2-2(s+s_1)s_2\right)m^2\\
&+ss_2\left(s^2+s_1s-2s_2s-2s_1^2+s_2^2+s_1s_2\right)\big)\mathrm{C}_0\left(s,s_1,s_2,m^2\right)\Big)
\end{aligned}
$$

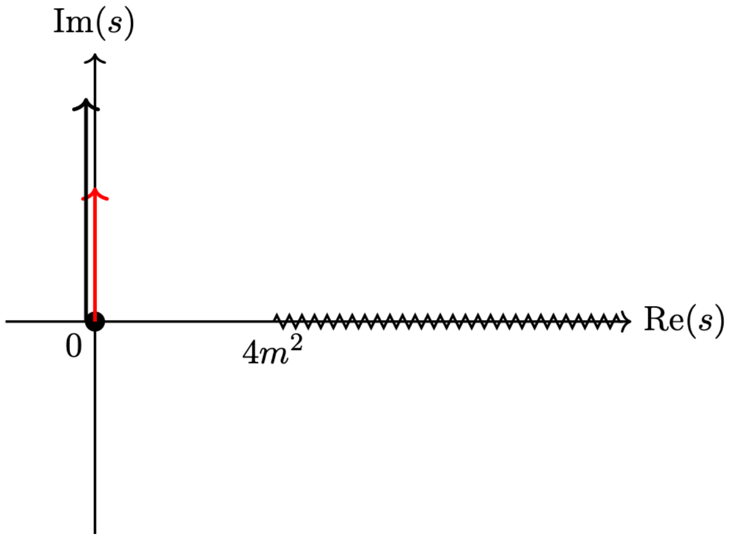
On-Shell m=0

$$
\lim_{q^2\to 0} q^2\left\langle J^{\mu_1}(p_1)J^{\mu_2}(p_2)J^{\mu_3}(q)\right\rangle = \frac{1}{2\pi^2}q^{\mu_3}\epsilon^{\mu_1\mu_2p_1p_2}
$$

$$
\begin{aligned}
F_3 =&-\frac{1}{2\pi^2\lambda(s,s_1,s_2)^2}\Big(\left(s^2-2(s_1+s_2)s+(s_1-s_2)^2\right)(s+s_1-s_2)\\
&+s\left(s^2+4s_1s-2s_2s-5s_1^2+s_2^2+4s_1s_2\right)\mathrm{B}_0\left(s,m^2\right)\\
&+s_1\left(-5s^2+4(s_1+s_2)s+(s_1-s_2)^2\right)\mathrm{B}_0\left(s_1,m^2\right)\\
&-\left((s+s_1)(s-s_1)^2+(s+s_1)s_2^2-2\left(s^2-4s_1s+s_1^2\right)s_2\right)\mathrm{B}_0\left(s_2,m^2\right)\\
&+2\big((s+s_1-s_2)\left((s-s_1)^2+s_2^2-2(s+s_1)s_2\right)m^2\\
&+ss_1\left((s-s_1)^2-2s_2^2+(s+s_1)s_2\right)\big)\mathrm{C}_0\left(s,s_1,s_2,m^2\right)\Big).
\end{aligned}
$$

No particle pole

$$p_1^\mu \Delta_{\alpha\mu\nu}(p_1,p_2) = 0, \quad p_2^\nu \Delta_{\alpha\mu\nu}(p_1,p_2) = 0.$$



The Longitudinal part is:

$$\Phi_0(p_1^2,p_2^2,p_3^2,m^2) \;=\; \frac{g^2 m^2}{2\pi^2}\,\frac{1}{q^2} \mathrm{C}_0\left(q^2,p_1^2,p_2^2,m^2\right) + \frac{g^2}{4\pi^2}\,\frac{1}{q^2}.$$

$$\Phi_0(q^2,s_1,s_2,m^2) \;=\; \frac{1}{2\pi i}\oint_C \frac{\Phi_0(s,s_1,s_2,m^2)}{s-q^2}\,ds,$$

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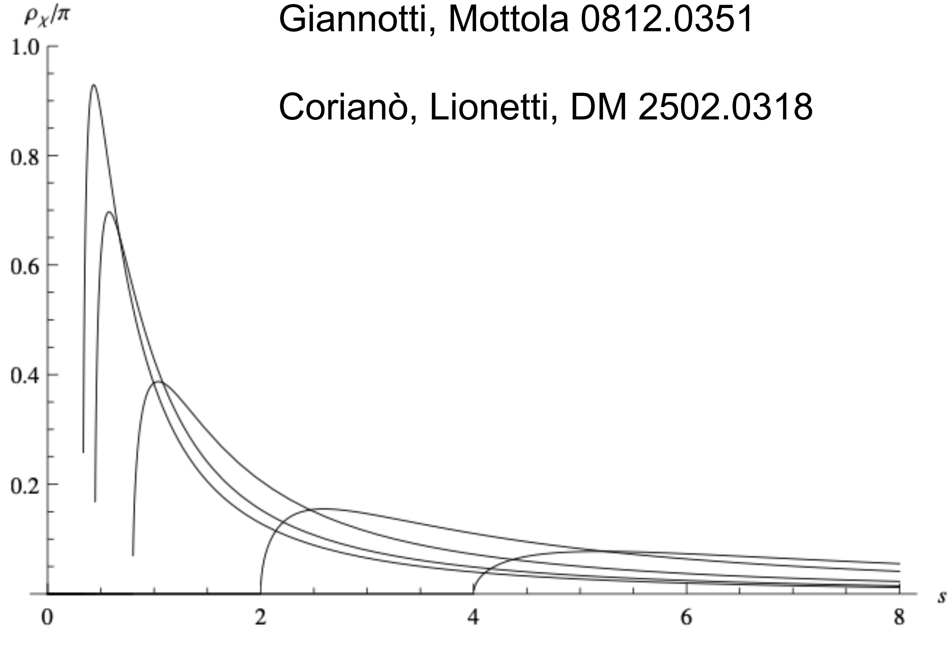
Corianò, Lionetti, DM 2502.0318

$$\mathrm{disc}\,\Phi_0(q^2,p_1^2,p_2^2,m^2) = -4i\pi m^2\, C_0(q^2,p_1^2,p_2^2,m^2)\,\delta(q^2) - 2\pi i\,\delta(q^2) + \frac{2m^2}{q^2}\mathrm{disc}\,C_0(q^2,p_1^2,p_2^2,m^2).$$



(a)

$$\begin{aligned}\rho(s,s_1,s_2,m^2) \;&\equiv\; \frac{2m^2}{q^2}\mathrm{disc}\,C_0(q^2,p_1^2,p_2^2,m^2)\theta(q^2-4m^2) \\ &=\; \frac{2m^2\pi\log\left(\frac{m^2-s(w-\bar z)(\bar w - z)}{m^2-s(w-z)(\bar w-\bar z)}\right)}{s^2(z-\bar z)},\end{aligned}$$

$$\frac{1}{\pi}\int_{4m^2}^{\infty}\Delta\Phi_0(s,p_1^2,p_2^2,m^2)ds = \frac{g^2}{2\pi^2},$$

$$\Phi_0(q^2) = \frac{1}{\pi} \int_0^\infty \frac{\Delta\Phi_0(s)}{s - q^2}\, ds$$

$$\mathcal{L}^{-1}\Big\{\int_0^\infty \frac{\Delta\Phi_0}{s-q^2}\, ds\Big\}(t) = \frac{1}{\pi}\int_0^\infty \Delta\Phi_0(s) e^{q^2 t}\, ds \qquad f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} F(s) e^{st}\, ds,$$

$$\lim_{t\to 0} \mathcal{L}^{-1}\{\Phi_0\}(t) = \frac{1}{\pi}\int_0^\infty \Delta\Phi_0\, ds$$

$$\mathcal{L}^{-1}\{\Phi_0\}(t) = -\int_0^1 dx \int_0^{1-x} dy\, \frac{g^2\left(m^2 \exp\left(\frac{t\left(-m^2+p_1^2xy-p_2^2x^2-p_2^2xy+p_2^2x\right)}{xy+y^2-y}\right) - 2m^2 + p_1^2xy - p_2^2x^2 - p_2^2xy + p_2^2x\right)}{2\pi^2\left(m^2 - p_1^2xy + p_2^2x^2 + p_2^2xy - p_2^2x\right)}$$

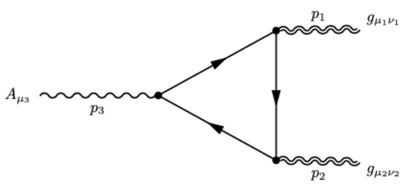
$$\frac{1}{\pi}\int_{4m^2}^\infty \Delta\Phi_0(s, p_1^2, p_2^2, m^2) ds = \frac{g^2}{2\pi^2},$$

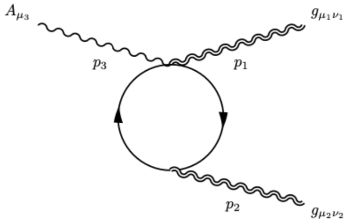
ATT correlator

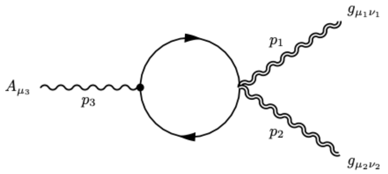
$$\langle T^{\mu_1\nu_1}T^{\mu_2\nu_2}J_A^{\mu_3}\rangle = \langle t^{\mu_1\nu_1}t^{\mu_2\nu_2}j_A^{\mu_3}\rangle + \langle t^{\mu_1\nu_1}t^{\mu_2\nu_2}j_{A\;loc}^{\mu_3}\rangle + \langle t^{\mu_1\nu_1}_{loc}t^{\mu_2\nu_2}j_A^{\mu_3}\rangle + \langle t^{\mu_1\nu_1}t^{\mu_2\nu_2}_{loc}j_A^{\mu_3}\rangle$$

$$\langle t^{\mu_1\nu_1}t^{\mu_2\nu_2}j_{A\;loc}^{\mu_3}\rangle = \frac{p_3^{\mu_3}}{p_3^2}\Pi^{\mu_1\nu_1}_{\alpha_1\beta_1}(p_1)\Pi^{\mu_2\nu_2}_{\alpha_2\beta_2}(p_2)\epsilon^{\alpha_1\alpha_2p_1p_2}\left(\bar{F}_1 g^{\beta_1\beta_2}(p_1\cdot p_2) + \bar{F}_2 p_1^{\beta_2}p_2^{\beta_1}\right)$$

$$\begin{aligned}\bar{F}_1 &= \frac{1}{24\pi^2} + \frac{m^2}{2\pi^2\lambda\,(s-s_1-s_2)}\Big\{2\left[\lambda m^2 + s s_1 s_2\right]C_0\left(s,s_1,s_2,m^2\right) + Q(s)\left[s(s_1+s_2)-(s_1-s_2)^2\right] \\ &\quad - s_2Q(s_2)\left[s+s_1-s_2\right] - s_1Q(s_1)\left[s-s_1+s_2\right] + \lambda\Big\},\\ \bar{F}_2 &= -\frac{1}{24\pi^2} + \frac{m^2}{2\pi^2\lambda^2}\Big\{2\left[\lambda\left(m^2(-s+s_1+s_2)-2s_1s_2\right) + 3s_1s_2\left((s_1-s_2)^2 - s(s_1+s_2)\right)\right]C_0\left(s,s_1,s_2,m^2\right) \\ &\quad + s_1Q(s_1)\left[\lambda + 6s_2(s+s_1-s_2)\right] + s_2Q(s_2)\left[\lambda+6s_1(s-s_1+s_2)\right] - Q(s)\left[12ss_1s_2 + \lambda(s_1+s_2)\right] \\ &\quad + \lambda(-s+s_1+s_2)\Big\}.\end{aligned}$$



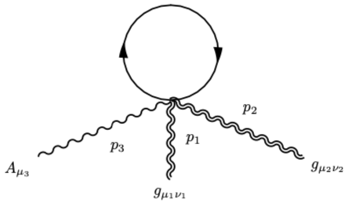




$$\langle t^{\mu_1\nu_1}t^{\mu_2\nu_2}j^{\mu_3}_{A\,loc}\rangle = (\bar{F}_1 - \bar{F}_2)\frac{p_3^{\mu_3}}{4\,p_3^2}\left\{\left[\varepsilon^{\nu_1\nu_2p_1p_2}\left(p_1\cdot p_2\, g^{\mu_1\mu_2} - p_1^{\mu_2}p_2^{\mu_1}\right) + (\mu_1\leftrightarrow\nu_1)\right] + (\mu_2\leftrightarrow\nu_2)\right\}$$

$$+(\bar{F}_1+\bar{F}_2)\,\frac{p_3^{\mu_3}}{4\,p_3^2}\left\{\left[\varepsilon^{\nu_1\nu_2p_1p_2}\left(p_1\cdot p_2\, g^{\mu_1\mu_2} + p_1^{\mu_2}p_2^{\mu_1}\right) + (\mu_1\leftrightarrow\nu_1)\right] + (\mu_2\leftrightarrow\nu_2)\right\}$$

$$+(\bar{F}_1+\bar{F}_2)\frac{(p_1\cdot p_2)}{p_1^2p_2^2}\frac{p_3^{\mu_3}}{4\,p_3^2}\left\{\left[\varepsilon^{\nu_1\nu_2p_1p_2}\left((p_1\cdot p_2)\,p_1^{\mu_1}p_2^{\mu_2} - p_2^2\,p_1^{\mu_1}p_1^{\mu_2} - p_1^2\,p_2^{\mu_1}p_2^{\mu_2}\right) + (\mu_1\leftrightarrow\nu_1)\right] + (\mu_2\leftrightarrow\nu_2)\right\}$$



ATT sum rule

Corianò, Lionetti, DM 2502.0318

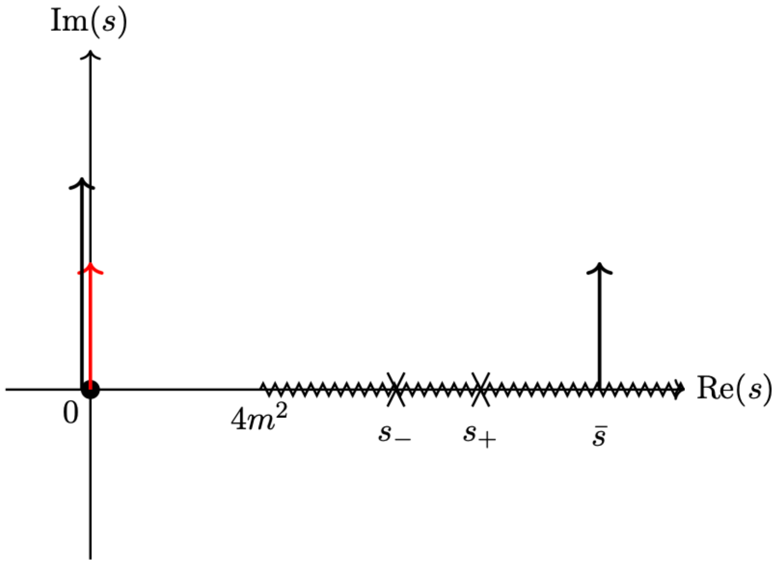
$$s_\pm = (\sqrt{s_1} \pm \sqrt{s_2})^2$$

$$\text{disc}(\phi_{ATT})_{pole} = \text{disc}(\phi_{ATT})_0 + \text{disc}(\phi_{ATT})_{s_1+s_2} + \text{disc}(\phi_{ATT})_{s_-} + \text{disc}(\phi_{ATT})_{s_+}.$$

$$\text{disc}(\phi_{ATT})_0 = 2\pi i \frac{\delta(s)}{12\pi^2} - \frac{2\pi i \delta(s)}{12\pi^2 (s_1 - s_2)^3 (s_1 + s_2)} \Psi(s, s_1, s_2, m^2)$$

$$\begin{aligned}\Psi(s,s_1,s_2,m^2) \equiv \Big(& 12m^2\big(-\left((s_1-s_2)\left(s_1^2+s_2^2\right)\mathrm{B}_0\left(s,m^2\right)\right) \\ & + s_1\left(s_1^2+2s_1s_2+3s_2^2\right)\mathrm{B}_0\left(s_1,m^2\right) - s_2\left(3s_1^2+2s_1s_2+s_2^2\right)\mathrm{B}_0\left(s_2,m^2\right) \\ & + (s_1-s_2)\left(2m^2\left(s_1^2+s_2^2\right)+s_1s_2(s_1+s_2)\right)\mathrm{C}_0\left(s,s_1,s_2,m^2\right)\big) \\ & + 12m^2\left(s_1^2+s_2^2\right)(s_1-s_2)\Big).\end{aligned}$$

$$\begin{aligned}\text{disc}(\phi_{ATT})_{s_1+s_2} = & \frac{2\pi i\, m^2 \delta(s-s_1-s_2)}{4\pi^2(s_1+s_2)} \Big(-2\mathrm{B}_0\left(s_1+s_2,m^2\right) + \mathrm{B}_0\left(s_1,m^2\right) \\ & + \mathrm{B}_0\left(s_2,m^2\right) + \left(4m^2-s_1-s_2\right)\mathrm{C}_0\left(s_1,s_2,s_1+s_2,m^2\right) + 2\Big).\end{aligned}$$



$$\frac{1}{\pi}\int_0^\infty \Delta\phi_{ATT}\, ds = \frac{1}{12\pi^2}$$

TJJ diagram

Quark sector

$$\langle T_{\mu_1\nu_1}(\boldsymbol{p}_1) J^{\mu_2 a_2}(\boldsymbol{p}_2) J^{\mu_3 a_3}(\boldsymbol{p}_3)\rangle_q = \langle t_{\mu_1\nu_1}(\boldsymbol{p}_1) j^{\mu_2 a_2}(\boldsymbol{p}_2) j^{\mu_3 a_3}(\boldsymbol{p}_3)\rangle_q$$
$$+ 2\mathscr{T}_{\mu_1\nu_1}{}^{\alpha}(\boldsymbol{p}_1)\left[\delta^{\mu_3}_{[\alpha}p_{3\beta]}\langle J^{\mu_2 a_2}(\boldsymbol{p}_2)J^{\beta a_3}(-\boldsymbol{p}_2)\rangle_q + \delta^{\mu_2}_{[\alpha}p_{2\beta]}\langle J^{\mu_3 a_3}(\boldsymbol{p}_3)J^{\beta a_2}(-\boldsymbol{p}_3)\rangle_q\right]$$
$$+ \frac{1}{d-1}\pi_{\mu_1\nu_1}(\boldsymbol{p}_1)\mathcal{A}_q^{\mu_2\mu_3 a_2 a_3},$$

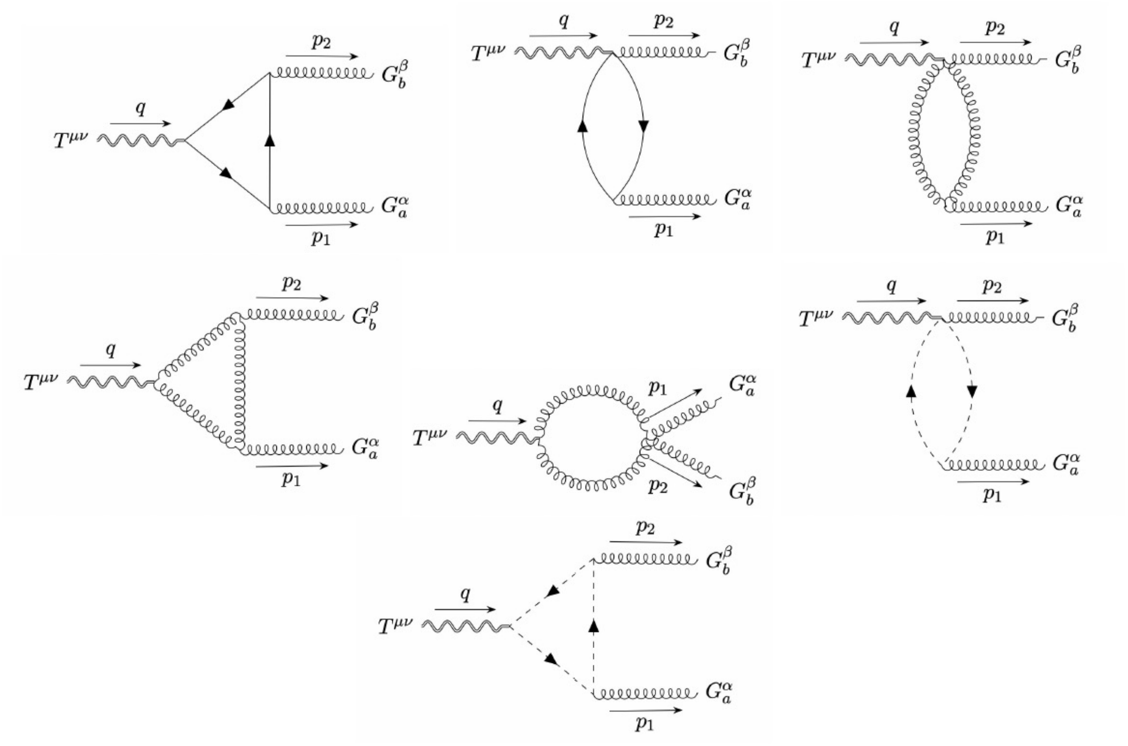
$$\mathscr{T}_{\mu\nu\alpha}(\boldsymbol{p}) = \frac{1}{p^2}\left[2p_{(\mu}\delta_{\nu)\alpha} - \frac{p_\alpha}{d-1}\left(\delta_{\mu\nu} + (d-2)\frac{p_\mu p_\nu}{p^2}\right)\right]$$

WIs

$$p_1^{\nu_1}\langle T_{\mu_1\nu_1}(p_1)J^{\mu_2 a_2}(p_2)J^{\mu_3 a_3}(p_3)\rangle_q = 2\delta^{\mu_3}_{[\mu_1}p_{3\alpha]}\langle J^{\mu_2 a_2}(p_2)J^{\alpha a_3}(-p_2)\rangle_q + 2\delta^{\mu_2}_{[\mu_1}p_{2\alpha]}\langle J^{\alpha a_2}(p_3)J^{\mu_3 a_3}(-p_3)\rangle_q,$$
$$p_{2\mu_2}\langle T_{\mu_1\nu_1}(p_1)J^{\mu_2 a_2}(p_2)J^{\mu_3 a_3}(p_3)\rangle_q = 0,$$
$$\langle T(p_1)J^{\mu_2 a_2}(p_2)J^{\mu_3 a_3}(p_3)\rangle_q = \mathcal{A}_q^{\mu_2\mu_3 a_2 a_3},$$

Gluon sector

$$\langle T^{\mu_1\nu_1}(p_1)J^{a_2\mu_2}(p_2)J^{a_3\mu_3}(p_3)\rangle_g = \langle t^{\mu_1\nu_1}(p_1)j^{a_2\mu_2}(p_2)j^{a_3\mu_3}(p_3)\rangle_g + \langle t^{\mu_1\nu_1}(p_1)j^{a\mu_2}_{loc}(p_2)j^{a_3\mu_3}(p_3)\rangle_g + \langle t^{\mu_1\nu_1}(p_1)j^{a_2\mu_2}(p_2)j^{a_3\mu_3}_{loc}(p_3)\rangle_g$$
$$+ 2\mathcal{I}^{\mu_1\nu_1\rho}(p_1)\left[\delta^{\mu_3}_{[\rho}p_{3\sigma]}\langle J^{a_2\mu_2}(p_2)J^{a_3\sigma}(-p_2)\rangle_g + \delta^{\mu_2}_{[\rho}p_{2\sigma]}\langle J^{a_3\mu_3}(p_3)J^{a_2\sigma}(-p_3)\rangle_g\right] + \frac{1}{d-1}\pi^{\mu_1\nu_1}(p_1)\left[\mathcal{A}_g^{\mu_2\mu_3 a_2 a_3} + \mathcal{B}_g^{\mu_2\mu_3 a_2 a_3}\right]$$



$$g_{\mu\nu}\langle T^{\mu\nu}\rangle = b_1 E_4 + b_2 C^{\mu\nu\rho\sigma}C_{\mu\nu\rho\sigma} + b_3 \nabla^2 R + b_4 F^{\mu\nu}F_{\mu\nu},$$

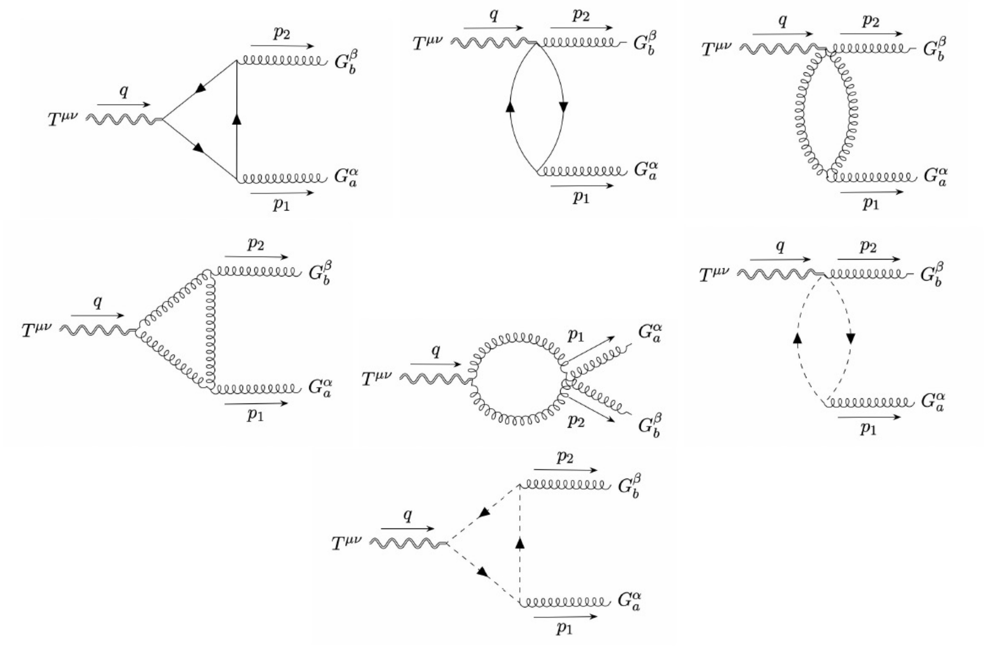
$$S_{pole} = \beta(g)\int d^4x\, d^4y\, R^{(1)}(x)\,\square^{-1}(x,y)\,F^a_{\alpha\beta}F^{a\alpha\beta}$$

$$\beta(g) = \frac{dg}{d\ln(\mu^2)} = -\beta_0\frac{g^3}{16\pi^2}, \qquad \beta_0 = \frac{11}{3}C_A - \frac{2}{3}n_f$$

Corianò, Lionetti, DM, Tommasi 2409.05609

TJJ trace of the quark sector

$$g_{\mu\nu}\left\langle T^{\mu\nu}J^{\alpha a}J^{\beta b}\right\rangle_q = \left(\phi_1^{\alpha\beta}(p_1,p_2,q,m)+\phi_2^{\alpha\beta}(p_1,p_2,q,m)\right)\delta^{ab}$$



$$\phi_1^{\alpha\beta ab} = \left(-\frac{2}{3}n_f\frac{g_s^2}{16\pi^2}+\chi_0(p_1,p_2,q,m)\right)\delta^{ab}u^{\alpha\beta}(p_1,p_2)$$

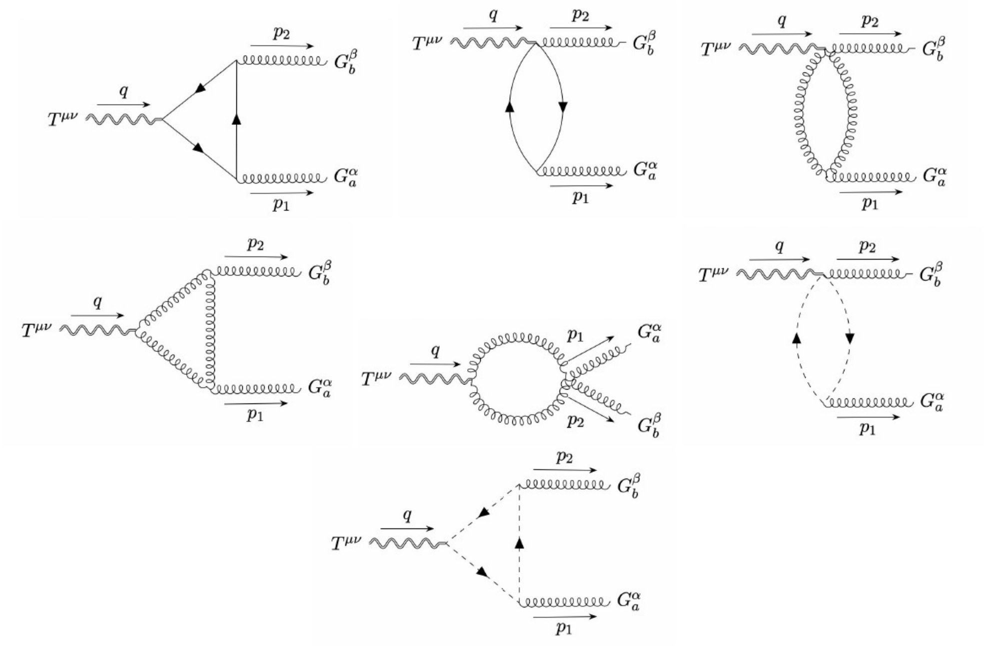
$$\begin{aligned}\chi_0 =& g_s^2 m^4 H_1 C_0\left(p_1^2,p_2^2,q^2,m^2\right)+g_s^2 m^2\Big(H_2 C_0\left(p_1^2,p_2^2,q^2,m^2\right)\\ &+H_3\bar{B}_0(q^2,m^2)+H_4\bar{B}_0(p_1^2,m^2)+H_5\bar{B}_0(p_2^2,m^2)+H_6\Big)\end{aligned}$$

$$\phi_2^{\alpha\beta}(p_1,p_2,q,m)=\chi_1(p_1,p_2,q,m)v^{\alpha\beta}+\left(B_2 p_1^\alpha p_1^\beta + B_2(p_1\leftrightarrow p_2)p_2^\alpha p_2^\beta + B_3 p_1^\alpha p_2^\beta\right)$$

These are proportional to the equation of motion

TJJ trace of the gluon sector

$$g_{\mu\nu}\left\langle T^{\mu\nu} J^{\alpha a} J^{\beta b}\right\rangle_g = \left(\frac{g_s^2}{16\pi^2}\ \frac{11}{3}C_A\right) u^{\alpha\beta}\delta^{ab} + \mathcal{B}_g^{\alpha\beta}\delta^{ab}$$



$$\mathcal{B}_g^{\alpha\beta ab} = \chi_g(p_1,p_2,q)u^{\alpha\beta}\delta^{ab} + \phi_g^{\alpha\beta ab}$$

All the for factors are proportional to the equation of motion

$$\phi_g^{\alpha\beta ab} = \chi_g'(p_1,p_2,q)v^{\alpha\beta}\delta^{ab} + C_1\, p_1^\alpha\, p_1^\beta\delta^{ab} + C_2\, p_1^\alpha\, p_2^\beta\delta^{ab} + C_1(p_1 \leftrightarrow p_2)\, p_2^\alpha\, p_2^\beta\delta^{ab}$$

TJJ sum rule

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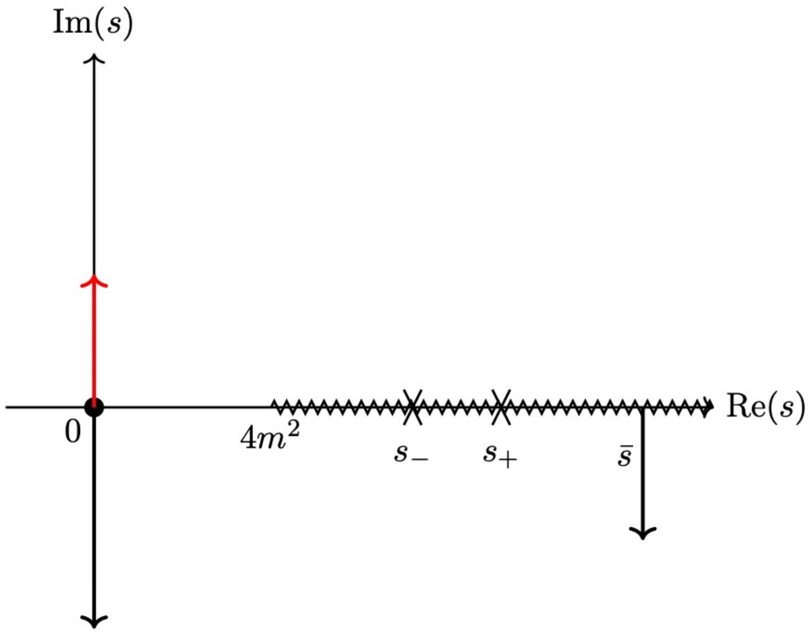
$$\langle T^{\mu\nu}(q)J^{a\alpha}(p_1)J^{b\beta}(p_2)\rangle_{tr} = \frac{1}{3\,q^2}\hat{\pi}^{\mu\nu}(q)\Bigg[\ \left(\frac{g_s^2}{16\pi^2}\left(\frac{11}{3}C_A - \frac{2}{3}n_f\right) + \chi_0(p_1,p_2,q,m) + \chi_g(p_1,p_2,q)\right)\delta^{ab}u^{\alpha\beta}(p_1,p_2) + \left(\phi_2^{\alpha\beta ab} + \phi_g^{\alpha\beta ab}\right)\Bigg]$$

$$\Phi_{TJJ}(p_1^2,p_2^2,q^2,m^2) \equiv \frac{1}{3q^2}\mathcal{A} = \frac{1}{3q^2}\left(\frac{g_s^2}{48\pi^2}(11C_A - 2n_f) + \chi_0(p_1,p_2,q,m) + \chi_g(p_1,p_2,q)\right)$$

$$\mathrm{disc}(\Phi_{TJJ})_{pole} = \mathrm{disc}(\Phi_{TJJ})_0 + \mathrm{disc}(\Phi_{TJJ})_{s_1+s_2} + \mathrm{disc}(\Phi_{TJJ})_{s_-} + \mathrm{disc}(\Phi_{TJJ})_{s_+}$$

$$\mathrm{disc}(\Phi_{TJJ})_{cont} = \frac{n_F\, g_s^2\, m^2 K_1(s,s_1,s_2)\,\mathrm{disc}\,B_0\left(s,m^2\right)}{12\pi^2 s(s-s_1-s_2)\lambda^2} + \frac{n_F\, g_s^2\, m^2 K_2(s,s_1,s_2)\,\mathrm{disc}\,C_0\left(s,s_1,s_2,m^2\right)}{24\pi^2 s(s-s_1-s_2)\lambda^2}$$
$$\frac{C_A g_s^2(s_1+s_2)\,\mathrm{disc}\,B_0(s)}{48\pi^2\lambda} + \frac{C_A g_s^2\, K_3\,\mathrm{disc}\,C_0\left(s,s_1,s_2\right)}{96\pi^2 s(s-s_1-s_2)\lambda}$$

$$\int_{4m^2}^{\infty}\Delta\Phi_{TJJ}(s,p_1^2,p_2^2,m^2)\,ds = \frac{g_s^2}{144\pi^2}\left(11C_A - 2n_f\right)$$



TJJ and the dilaton pole

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Off-Shell

$$\lim_{s\to 0} q^2 \langle T^{\mu\nu}(q) J^{\alpha a}(p_1) J^{\beta b}(p_2)\rangle = 0.$$

On-Shell

$$\lim_{s\to 0} q^2 \langle T^{\mu\nu}(q) J^{\alpha a}(p_1) J^{\beta b}(p_2)\rangle = -\frac{g_s^2}{48\pi^2}\left(\frac{2}{3}n_f - \frac{11}{3}C_A\right)\tilde{\phi}_1^{\mu\nu\alpha\beta} - \frac{g_s^2}{288\pi^2}(n_f - C_A)\,\tilde{\phi}_2^{\mu\nu\alpha\beta}.$$

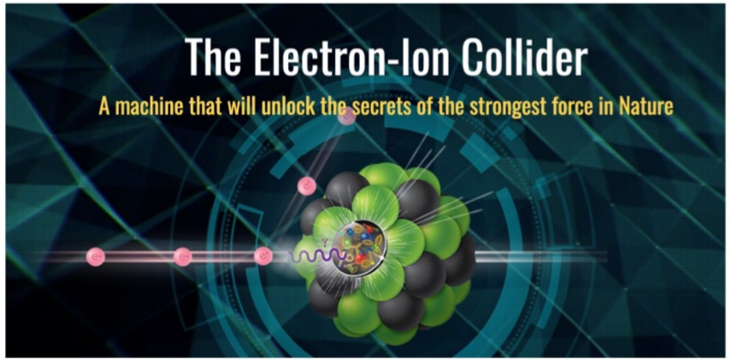
$$\tilde{\phi}_1^{\mu\nu\alpha\beta}(p_1,p_2) = (s\, g^{\mu\nu} - q^\mu q^\nu)\, u^{\alpha\beta}(p_1,p_2)$$

$$\tilde{\phi}_2^{\mu\nu\alpha\beta}(p_1,p_2) = -2\, u^{\alpha\beta}(p_1,p_2)\left[s\, g^{\mu\nu} + 2(p_1^\mu p_1^\nu + p_2^\mu p_2^\nu) - 4\,(p_1^\mu p_2^\nu + p_2^\mu p_1^\nu)\right]$$

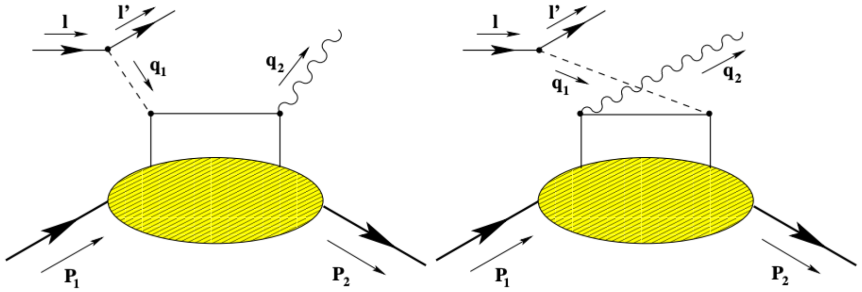
Brookhaven National Laboratory

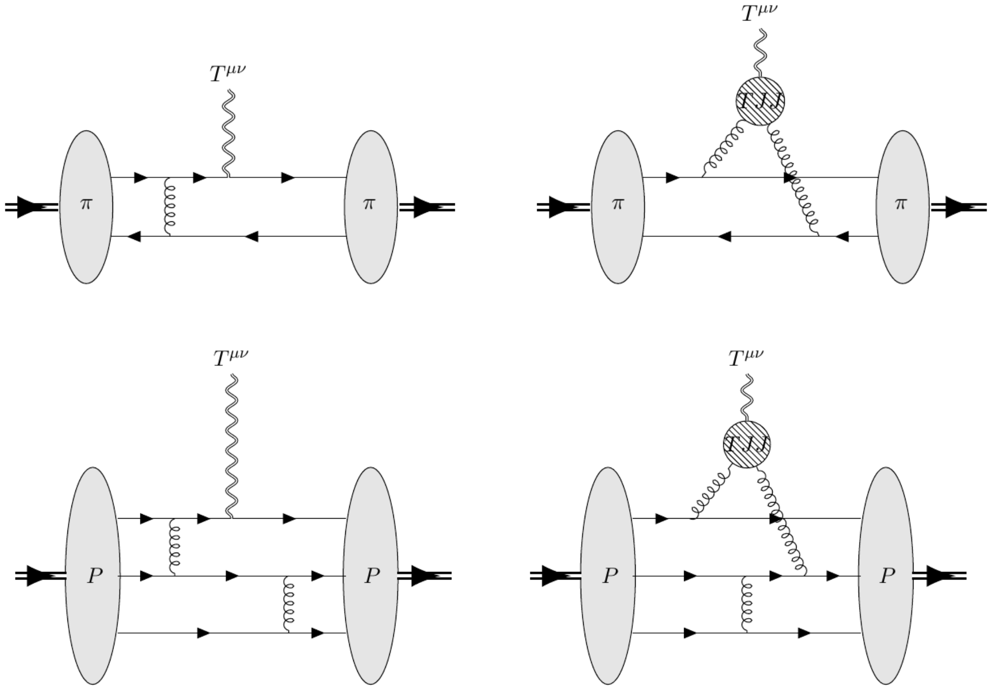
Jefferson Lab

The Electron-Ion Collider (EIC) at Brookhaven National Laboratory is designed to have a highly flexible energy range, with the capability to collide electrons with protons and nuclei at center-of-mass energies ranging from approximately **20 GeV** to **140 GeV (e-p and e-Ions)**



Invariant amplitudes of DVCS
Related to form factors of
GFF of the proton





$$\langle \pi(p_2)|\hat{T}^{\mu\nu}|\pi(p_1)\rangle = 2P^\mu P^\nu A^\pi(q^2) + \tfrac{1}{2}\left(q^\mu q^\nu - g^{\mu\nu} q^2\right) D^\pi(q^2),$$

$$\langle p', s'|T_{\mu\nu}(0)|p, s\rangle = \bar{u}'\left[A(t)\,\frac{\gamma_{\{\mu}P_{\nu\}}}{2} + B(t)\,\frac{i\,P_{\{\mu}\sigma_{\nu\}\rho}\Delta^\rho}{4M} + D(t)\,\frac{\Delta_\mu\Delta_\nu - g_{\mu\nu}\Delta^2}{4M} + M\sum_{\hat{a}}\bar{c}^{\hat{a}}(t)\,g_{\mu\nu}\right]u$$

Ji's Sum rule

$$J_q + J_g = \frac{1}{2}[A_q(0) + B_q(0)] + \frac{1}{2}[A_g(0) + B_g(0)] = \frac{1}{2}$$

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$$\langle \pi(p_2)|T^{\mu\nu}|\pi(p_1)\rangle = \frac{i\, f_\pi^2 C_F g_s^2}{N_C^2} \int_0^1 d\alpha_1 \int_0^1 d\alpha_2\, \tilde{\phi}_\pi(\alpha_1)\, K_\pi^{\mu\nu}(\alpha_1,\alpha_2;q)\, \tilde{\phi}_\pi(\alpha_2).$$

$$T_1^{\mu\nu} = p_1^\mu p_2^\nu, \quad T_2^{\mu\nu} = p_2^\mu p_1^\nu, \quad T_3^{\mu\nu} = p_1^\mu p_1^\nu, \quad T_4^{\mu\nu} = p_2^\mu p_2^\nu, \quad T_5^{\mu\nu} = g^{\mu\nu}.$$

$$K_\pi^{\mu\nu}(\alpha_1,\alpha_2;q) = \sum_{i=1}^{5} F_i^{TJJ}(\alpha_1,\alpha_2;q)\, T_i^{\mu\nu}(p_1,p_2),$$

$$\langle \pi(p_2)|\hat{T}^{\mu\nu}|\pi(p_1)\rangle = 2P^\mu P^\nu A^\pi(q^2) + \tfrac{1}{2}\left(q^\mu q^\nu - g^{\mu\nu} q^2\right) D^\pi(q^2),$$

$$F_1^{TJJ} = F_2^{TJJ} = \frac{4\left(q^2\left(4(\alpha_1-1)(\alpha_2-1)A_2 + (\alpha_1-\alpha_2)A_3\right) + 4A_4\right)}{(\alpha_1-1)\alpha_1(\alpha_2-1)\alpha_2 q^4},$$

$$A_\pi(q^2) = \int_0^1 d\alpha_1 \int_0^1 d\alpha_2\, \tilde{\phi}_\pi(\alpha_1)\left(F_1^{\text{tree}} + F_1^{TJJ} + F_3^{\text{tree}} + F_3^{TJJ}\right)\tilde{\phi}_\pi(\alpha_2),$$

$$D_\pi(q^2) = -\int_0^1 d\alpha_1 \int_0^1 d\alpha_2\, \tilde{\phi}_\pi(\alpha_1)\, \frac{2\left(F_5^{\text{tree}} + F_5^{TJJ}\right)}{q^2}\, \tilde{\phi}_\pi(\alpha_2).$$

$$F_3^{TJJ} = \frac{8\left(A_3 - 2(\alpha_1-1)A_2\right)}{\alpha_1(\alpha_2-1)\alpha_2 q^2},$$

$$F_4^{TJJ} = -\frac{8\left(2(\alpha_2-1)A_2 + A_3\right)}{(\alpha_1-1)\alpha_1\alpha_2 q^2},$$

$$\begin{aligned}
F_5^{TJJ} = -\frac{4}{3(\alpha_1-1)\alpha_1(\alpha_2-1)\alpha_2 q^2}\Bigg[& q^2\Big(\left((\alpha_2-2)^2 + \alpha_1^2 + (6\alpha_2-4)\alpha_1\right)A_2 \\
& + 2\left((2\alpha_2-1)\alpha_1 - \alpha_2 + 2\right)\mathcal{A}\Big) \\
& + 2\Big(2(\alpha_1+\alpha_2-2)\,\mathcal{J}\mathcal{J}[(\alpha_2-1)p_2 - (\alpha_1-1)p_1] \\
& - 2(\alpha_1+\alpha_2)\,\mathcal{J}\mathcal{J}[\alpha_1 p_1 - \alpha_2 p_2] + A_4 + 2C_5\Big)\Bigg].
\end{aligned}$$

$$\mathcal{A} = \frac{1}{3}\,\frac{g_s^2}{16\pi^2}\left(11C_A - 2n_f\right),$$

Future developments

- Sudakov?
- Anomaly role in the pion and proton GFFs
- Clarify the Anomaly interplay in the Ji’s sum rule of the proton
- Full calculation of the NLO for the pion and proton GFF

Thank you for your attention