SELECTING MODIFIED GRAVITY THROUGH NOETHER SYMMETRY

THE CASE OF *GAUSS-BONNET COSMOLOGY*

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THE STATE-OF-THE-ART

GENERAL RELATIVITY

FOUR BASIC ASSUMPTIONS:

- The Principle of Relativity
- The Equivalence Principle
- The Principle of General Covariance
- The Principle of Causality

$$S_{HE} = \frac{1}{2k} \int \sqrt{-g} R d^4 x$$

SUCCESSES

-------------- WEAK FIELD TESTS---------------------

- Light deflection due to the presence of the Sun
- Saphiro Delay
- The exact estimation of the precession of Mercury's perihelion in its orbit around the Sun

------------- STRONG FIELD TESTS-------------------

- Black Holes
- Gravitational Waves

SHORTCOMINGS

----------------- IR SCALES-------------------

- Galaxy rotation curves problem
- The late-time accelerated expansion of the Universe

----------------- UV SCALES-------------------

- Quantum Gravity problem

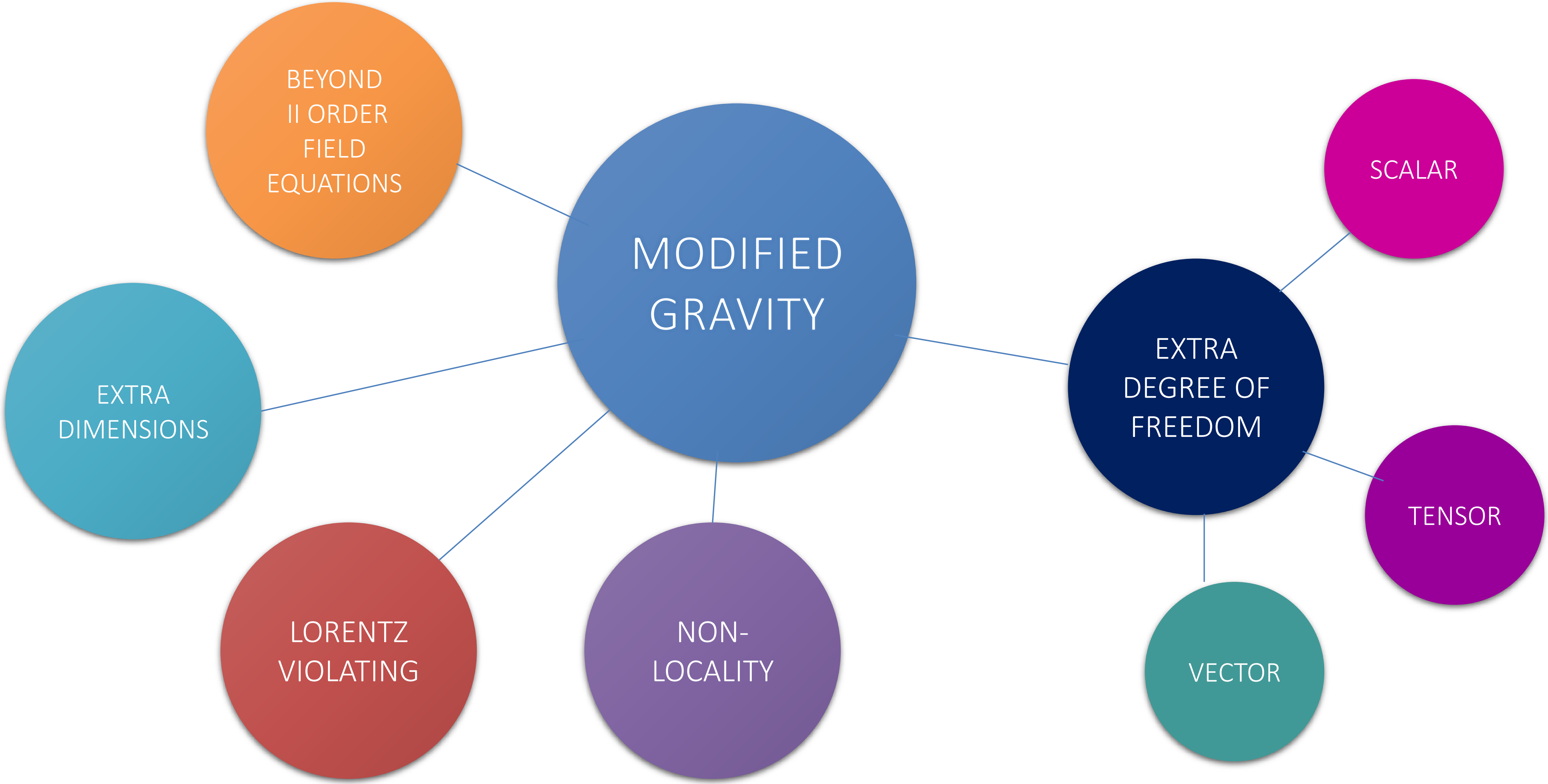
MODIFIED GRAVITY

GEOMETRIC PART

MATTER CONTENT

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}$$

MODIFIED GRAVITY



MODIFIED GRAVITY

ALTERNATIVE THEORIES

RELAXING BASIC ASSUMPTIONS

EXTENDED THEORIES

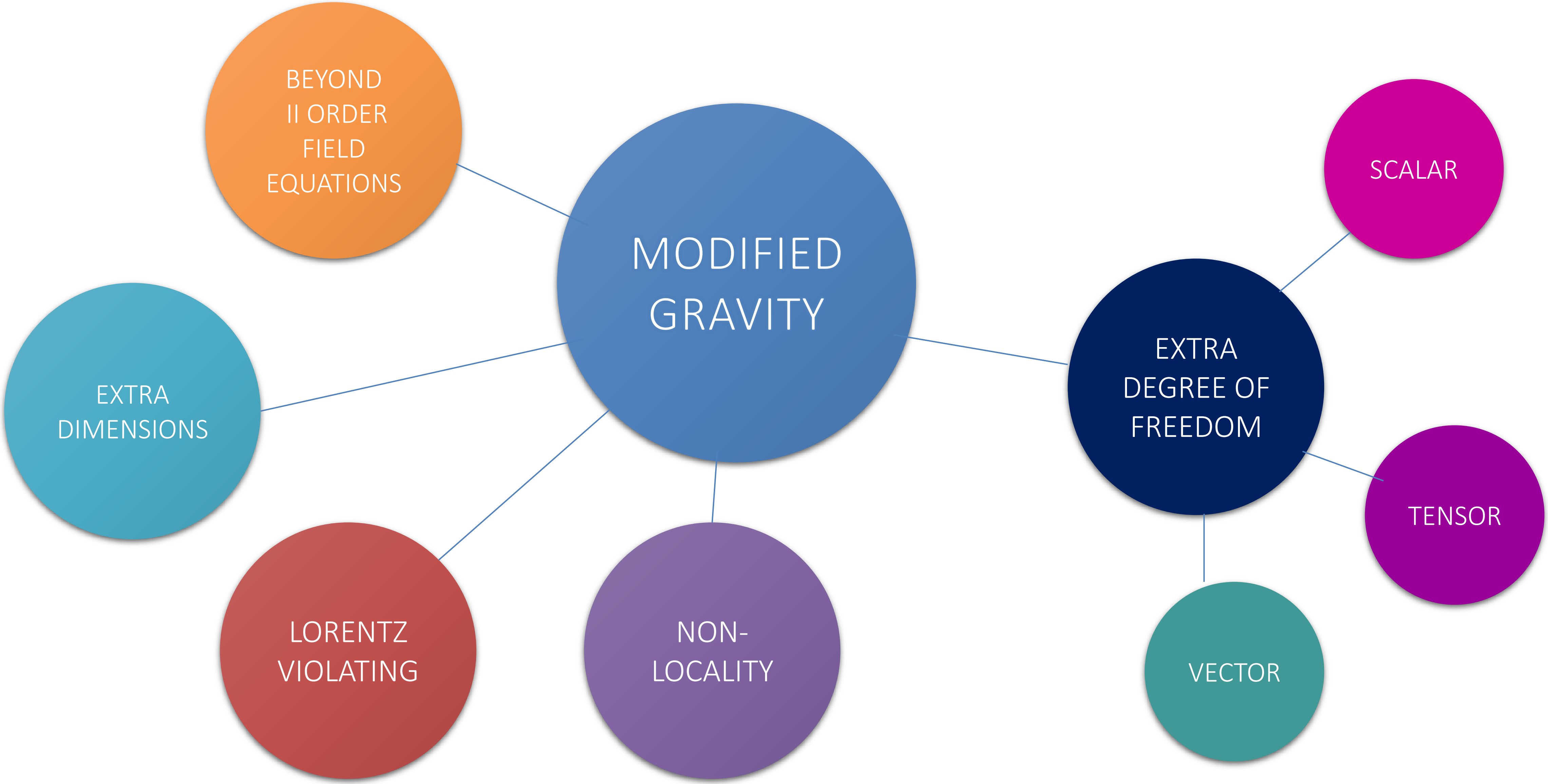
SUBSTITUING THE LINEAR DEPENDENCE OF THE ACTION ON THE RICCI SCALAR

ATTENTION!

A new relativistic theory must satisfies few requirements from a phenomenological point of view:

- To recover Newtonian dynamics in weak-field, slow motion limit
- To pass Classical Solar System tests
- To adress Large-Scale Structure problem
- To reproduce in a self-consistent way the main cosmological parameters

MODIFIED GRAVITY



NOETHER SYMMETRY APPROACH

NOETHER'S FIRST THEOREM

To every differentiable symmetry generated by local actions, there corresponds a conserved current (Noether current)

WHY?

EXTEND/MODIFY GRAVITY → HIGHLY COMPLEX FIELD EQUATIONS

NOETHER SYMMETRY APPROACH → SYMMETRIES → CONSERVED QUANTITIES

ATTENTION!

NOT EVERY MODEL IDENTIFIED BY THE NOETHER SYMMETRY APPROACH WILL BE NECESSARILY VIABLE!

- THEORETICAL LIMITATIONS
- RULED OUT BY EXPERIMENTAL EVIDENCE

NOETHER SYMMETRY APPROACH

NOETHER'S FIRST THEOREM

To every differentiable symmetry generated by local actions, there corresponds a conserved current (Noether current)

$$\begin{cases} \mathcal{L}\left(x^a, \phi^i, \partial_a \phi^i\right) \to \mathcal{L}\left(\tilde{x}^a, \tilde{\phi}^i, \partial_a \tilde{\phi}^i\right) \\ \tilde{x}^a = x^a + \delta x^a \\ \tilde{\phi}^i = \phi^i + \delta \phi^i \end{cases}$$

$$\partial_a \tilde{\phi}^i = \frac{d\tilde{\phi}^i}{d\tilde{x}^a} = \frac{d\phi^i + d(\delta\phi^i)}{dx^a + d(\delta x^a)} = \left(\frac{d\phi^i}{dx^a} + \frac{d\left(\delta\phi^i\right)}{dx^a}\right)\left(1 + \frac{d\left(\delta x^b\right)}{dx^b}\right)^{-1} \sim \left(\frac{d\phi^i}{dx^a} + \frac{d\left(\delta\phi^i\right)}{dx^a}\right)\left(1 - \frac{d\left(\delta x^b\right)}{dx^b}\right)$$

$$\begin{aligned} \partial_a \tilde{\phi}^i &= \partial_a \phi^i - \partial_a \phi^i \partial_b\left(\delta x^b\right) + \partial_a\left(\delta\phi^i\right) - \partial_a\left(\delta\phi^i\right)\partial_b\left(\delta x^b\right) \sim 0 \\ &= \partial_a\, \phi^i + \left[\partial_a\left(\delta\phi^i\right) - \partial_a \phi^i \partial_b\left(\delta x^b\right)\right] \\ &= \partial_a \phi^i + \delta\left(\partial_a \phi^i\right) \end{aligned}$$

NOETHER SYMMETRY APPROACH

NOETHER'S FIRST THEOREM

To every differentiable symmetry generated by local actions, there corresponds a conserved current (Noether current)

$$\begin{cases} \mathcal{L}\left(x^a, \phi^i, \partial_a \phi^i\right) \to \mathcal{L}\left(\tilde{x}^a, \tilde{\phi}^i, \partial_a \tilde{\phi}^i\right) \\ \tilde{x}^a = x^a + \delta x^a \\ \tilde{\phi}^i = \phi^i + \delta\phi^i \\ \partial_a \tilde{\phi}^i = \partial_a \phi^i + \delta\left(\partial_a \phi^i\right) = \partial_a\, \phi^i + \left[\partial_a\left(\delta\phi^i\right) - \partial_a \phi^i \partial_b\left(\delta x^b\right)\right] \end{cases}$$

THE FIRST PROLONGATION OF NOETHER VECTOR

$$X^{[1]} = \delta x^a \frac{\partial}{\partial x^a} + \delta\phi^i \frac{\partial}{\partial \phi^i} + \delta\left(\partial_a \phi^i\right) \frac{\partial}{\partial\left(\partial_a \phi^i\right)}$$

THE NOETHER IDENTITY

$$\boxed{X^{[1]}\mathcal{L} + \partial_a(\delta x^a)\mathcal{L} = \partial_a g^a}$$

NOETHER SYMMETRY APPROACH

NOETHER'S FIRST THEOREM

To every differentiable symmetry generated by local actions, there corresponds a conserved current (Noether current)

WHAT ABOUT CONSERVED QUANTITIES?

$$\mathcal{L}\left(x^a, \phi^i, \partial_a \phi^i\right) \rightarrow \mathcal{L}\left(\tilde{x}^a, \tilde{\phi}^i, \partial_a \tilde{\phi}^i\right)$$

$$\left[\delta x^a \frac{\partial}{\partial x^a} + \delta \phi^i \frac{\partial}{\partial \phi^i} + \delta\left(\partial_a \phi^i\right) \frac{\partial}{\partial(\partial_a \phi^i)}\right] \mathcal{L} + \partial_a(\delta x^a)\mathcal{L} = \partial_a g^a$$

PROPERLY REWRITING

$$\frac{d}{dx^a}\left(\mathcal{L}\,\delta x^a - \frac{\partial \mathcal{L}}{\partial(\partial_b \phi^i)} \partial_b \phi^i \delta x^a + \frac{\partial \mathcal{L}}{\partial(\partial_a \phi^i)} \delta \phi^i - g^a\right) = 0$$

$$\rightarrow \boxed{j^a = \mathcal{L}\,\delta x^a - \frac{\partial \mathcal{L}}{\partial(\partial_b \phi^i)} \partial_b \phi^i \delta x^a + \frac{\partial \mathcal{L}}{\partial(\partial_a \phi^i)} \delta \phi^i - g^a}$$

RECIPE FOR A GOOD THEORY

1. Choose a modified theory;
2. Establish a metric;
3. Write the gravitational action;
4. Find canonical point-like Lagrangian;
5. Find the dynamical system composed by the Euler-Lagrange equations and the energy condition

$$\begin{cases} \dfrac{d}{dt}\left(\dfrac{\partial \mathcal{L}}{\partial \dot{q}^i}\right) - \dfrac{\partial \mathcal{L}}{\partial q^i} = 0 & \text{Euler-Lagrange equations} \\ E_{\mathcal{L}} = \dot{q}^i \dfrac{\partial \mathcal{L}}{\partial \dot{q}^i} - \mathcal{L} = 0 & \text{Null energy condition} \end{cases}$$

6. Select viable models applying the Noether symmetry approach;
7. Solve the dynamical system and find the cosmological solutions!

GAUSS-BONNET TERM

GENERAL ACTION OF A MODIFIED THEORY OF GRAVITY

$$S = \int \sqrt{-g} F\left(\phi, R, \Box R, \Box^2 R, \dots, \Box^k R, R^{\mu\nu} R_{\mu\nu}, R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma}\right) d^4x$$

Let us define the following two higher-order curvature invariants

$$P \equiv g^{\mu\rho} R^{\nu\sigma} R_{\mu\nu\rho\sigma}, \qquad Q = R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma}$$

and consider the general action

$$S = \int \sqrt{-g} f(R, P, Q) d^4x$$

Among all the possible combinations of R, P, and Q, the only action leading to ghost-free modes is

$$S = \int \sqrt{-g} (R^2 - 4P + Q) d^4x$$

The quantity

$$\boxed{\mathcal{G} \equiv R^2 - 4P + Q}$$

is the GAUSS-BONNET TOPOLOGICAL INVARIANT.

GAUSS-BONNET TERM

WHY IS THE GAUSS-BONNET TERM IMPORTANT?

- In four-dimensions, the Gauss-Bonnet term turns out to be a topological surface term. It can contribute in reducing the dynamics and providing exact solutions.

- A linear contribution of the Gauss Bonnet term does not change the dynamics but a function of it can be physically interesting.

- In cosmological context, both P and Q are similar to R^2. Hence, $f(\mathcal{G}) = \sqrt{\mathcal{G}} \sim R$
  $\rightarrow$ General relativity is restored!

- It naturally appears in the formulation of Quantum Field Theory in curved spacetime.

THE GAUSS-BONNET COSMOLOGY

- Non minimal coupling between the scalar field and the geometric term described by the coupling function $F(\phi, \mathcal{G})$;
- Four dimensions;
- Kinetic and potential terms;
- No matter term;
- Spatially flat Friedman-Lemaitre-Robertson-Walker (FLRW) metric

$$ds^2 = dt^2 - a^2(t)\delta_{ij}dx^i dx^j$$

$$S = \int \sqrt{-g}\left[F(\phi, \mathcal{G}) + \omega(\phi)\partial_\mu\phi\partial^\mu\phi + V(\phi)\right]d^4x$$

THE GAUSS-BONNET TERM IN SPATIALLY FLAT FLRW METRIC

$$\mathcal{G} = 24\frac{\dot{a}^2\ddot{a}}{a^3} = \frac{8}{a^3}(3\dot{a}^2\ddot{a}) = \frac{8}{a^3}\frac{d(\dot{a}^3)}{dt}$$

INTEGRATING OVER THE THREE-DIMENSIONAL HYPERSURFACE

$$\int \sqrt{-g}\, d^4x \rightarrow \int a^3 dt$$

$$S = \int a^3\left[F(\phi, \mathcal{G}) + \omega(\phi)\dot{\phi}^2 + V(\phi)\right]dt$$

THE GAUSS-BONNET COSMOLOGY

CANONICAL POINT-LIKE LAGRANGIAN

LAGRANGIAN MULTIPLIERS METHOD

$$S = \int a^3 \left\{ F(\phi, \mathcal{G}) + \omega(\phi)\dot{\phi}^2 + V(\phi) - \lambda \left[\mathcal{G} - \frac{8}{a^3}\frac{d(\dot{a}^3)}{dt} \right] \right\} dt$$

$$\frac{\delta S}{\delta \mathcal{G}} = a^3 F_{\mathcal{G}} - a^3 \lambda = 0 \rightarrow \lambda = F_{\mathcal{G}}$$

$$S = \int a^3 \left\{ F(\phi, \mathcal{G}) + \omega(\phi)\dot{\phi}^2 + V(\phi) - F_{\mathcal{G}} \left[\mathcal{G} - \frac{8}{a^3}\frac{d(\dot{a}^3)}{dt} \right] \right\} dt$$

$$S = \int \left[a^3 F + \omega(\phi)\dot{\phi}^2 a^3 + V(\phi) a^3 - a^3 F_{\mathcal{G}} \mathcal{G} + 8F_{\mathcal{G}} \frac{d(\dot{a}^3)}{dt} \right] dt$$

INTEGRATING BY PARTS

$$8 \int F_{\mathcal{G}} \frac{d(\dot{a}^3)}{dt} dt = 8F_{\mathcal{G}}\dot{a}^3 - 8 \int \left[F_{\phi\mathcal{G}}\dot{\phi}\dot{a}^3 + F_{\mathcal{G}\mathcal{G}}\dot{\mathcal{G}}\dot{a}^3 \right] dt$$

$$S = \int \left[a^3 F + \omega(\phi)\dot{\phi}^2 a^3 + V(\phi) a^3 - a^3 F_{\mathcal{G}} \mathcal{G} + 8\dot{a}^3 \left[F_{\phi\mathcal{G}}\dot{\phi} + F_{\mathcal{G}\mathcal{G}}\dot{\mathcal{G}} \right] \right] dt$$

$$\rightarrow \boxed{\mathcal{L} = a^3 F + \omega(\phi)\dot{\phi}^2 a^3 + V(\phi) a^3 - a^3 F_{\mathcal{G}} \mathcal{G} + 8\dot{a}^3 \left[F_{\phi\mathcal{G}}\dot{\phi} + F_{\mathcal{G}\mathcal{G}}\dot{\mathcal{G}} \right]}$$

THE GAUSS-BONNET COSMOLOGY

DYNAMICAL SYSTEM

$$
\begin{cases}
\dfrac{d}{dt}\left(\dfrac{\partial \mathcal{L}}{\partial \dot{a}}\right)-\dfrac{\partial \mathcal{L}}{\partial a}=0 \\[2ex]
\dfrac{d}{dt}\left(\dfrac{\partial \mathcal{L}}{\partial \dot{\phi}}\right)-\dfrac{\partial \mathcal{L}}{\partial \phi}=0 \\[2ex]
\dfrac{d}{dt}\left(\dfrac{\partial \mathcal{L}}{\partial \dot{\mathcal{G}}}\right)-\dfrac{\partial \mathcal{L}}{\partial \mathcal{G}}=0 \\[2ex]
\dot{a}\dfrac{\partial \mathcal{L}}{\partial \dot{a}}+\dot{\mathcal{G}}\dfrac{\partial \mathcal{L}}{\partial \dot{\mathcal{G}}}+\dot{\phi}\dfrac{\partial \mathcal{L}}{\partial \dot{\phi}}-\mathcal{L}=0
\end{cases}
$$

$$
8\dot{a}^2\left(\ddot{\mathcal{G}}\frac{\partial^2 F}{\partial \mathcal{G}^2}+2\dot{\mathcal{G}}\dot{\phi}\frac{\partial^3 F}{\partial \phi \partial \mathcal{G}^2}+\dot{\mathcal{G}}^2\frac{\partial^3 F}{\partial \mathcal{G}^3}+\ddot{\phi}\frac{\partial^2 F}{\partial \phi \partial \mathcal{G}}+\dot{\phi}^2\frac{\partial^3 F}{\partial \phi^2 \partial \mathcal{G}}\right)+16\dot{a}\ddot{a}\left(\dot{\mathcal{G}}\frac{\partial^2 F}{\partial \mathcal{G}^2}+\dot{\phi}\frac{\partial^2 F}{\partial \phi \partial \mathcal{G}}\right)+
$$
$$
-a^2\left(F-\mathcal{G}\frac{\partial F}{\partial \mathcal{G}}+V(\phi)+\omega(\phi)\dot{\phi}^2\right)=0
$$

$$
24\dot{a}^2\ddot{a}\frac{\partial^2 F}{\partial \phi \partial G}+a^3\left(-\frac{\partial F}{\partial \phi}+\mathcal{G}\frac{\partial^2 F}{\partial \phi \partial \mathcal{G}}-\frac{\partial V(\phi)}{\partial \phi}+2\omega(\phi)\ddot{\phi}+\dot{\phi}^2\frac{\partial \omega(\phi)}{\partial \phi}\right)+6a^2\dot{a}\omega(\phi)\dot{\phi}=0
$$

$$
\mathcal{G}=24\,\frac{\dot{a}^2\ddot{a}}{a^3}
$$

$$
24\dot{a}^3\left(\dot{\mathcal{G}}\frac{\partial^2 F}{\partial \mathcal{G}^2}+\dot{\phi}\frac{\partial^2 F}{\partial \phi \partial \mathcal{G}}\right)-a^3\left(F-\mathcal{G}\frac{\partial F}{\partial \mathcal{G}}+V(\phi)-\omega(\phi)\dot{\phi}^2\right)=0
$$

NOETHER SYMMETRY APPROACH

APPLICATION

MINISUPERSPACE:

$$\mathcal{S} = \{a(t), \mathcal{G}(t), \phi(t)\} = q^i$$

SET OF INFINITESIMAL TRANSFORMATIONS:

$$\begin{cases} \mathcal{L}(t, q^i, \dot{q}^i) \rightarrow \mathcal{L}(\tilde{t}, \tilde{q}^i, \tilde{\dot{q}}^i) \\ \tilde{t} = t + \epsilon\xi + \mathcal{O}(\epsilon^2) \\ \tilde{q}^i = q^i + \epsilon\eta^i + \mathcal{O}(\epsilon^2) \\ \tilde{\dot{q}}^i = \dot{q}^i + \epsilon\eta^{[1]} = \dot{q}^i + \epsilon(\dot{\eta}^i - \dot{q}^i\dot{\xi}) \end{cases}$$

NOETHER IDENTITY:

$$X^{[1]}\mathcal{L} + \dot{\xi}\mathcal{L} = \dot{g}$$

where

FIRST PROLONGATION OF NOETHER VECTOR

$$\begin{aligned} X^{[1]} &= \xi\frac{\partial}{\partial t} + \eta^i\frac{\partial}{\partial q^i} + (\dot{\eta}^i - \dot{q}^i\dot{\xi})\frac{\partial}{\partial(\dot{q}^i)} \\ &= \xi\frac{\partial}{\partial t} + \alpha\frac{\partial}{\partial a} + \beta\frac{\partial}{\partial \mathcal{G}} + \gamma\frac{\partial}{\partial \phi} + (\dot{\alpha} - \dot{a}\dot{\xi})\frac{\partial}{\partial \dot{a}} + (\dot{\beta} - \dot{\mathcal{G}}\dot{\xi})\frac{\partial}{\partial \dot{\mathcal{G}}} + (\dot{\gamma} - \dot{\phi}\dot{\xi})\frac{\partial}{\partial \dot{\phi}} \end{aligned}$$

GENERATORS OF INFINITESIMAL TRANSFORMATIONS

$$\eta^i = \{\alpha(t, a, \mathcal{G}, \phi), \beta(t, a, \mathcal{G}, \phi), \gamma(t, a, \mathcal{G}, \phi)\}$$

NOETHER SYMMETRY APPROACH

$$\omega(\phi) = \omega_0$$

$$V(\phi) = 0$$

$$V(\phi) = V_0\phi^p$$

$$V(\phi) = V_0 \exp(p\phi)$$

$$F(\phi, \mathcal{G}) = f_0\phi^w\mathcal{G}^k$$

$$F(\phi, \mathcal{G}) = f_o \exp(w\phi)\mathcal{G}^k$$

$$\omega(\phi) = \omega_0\phi^q$$

$$V(\phi) = 0$$

$$V(\phi) = V_0\phi^p$$

$$V(\phi) = V_0 \exp(p\phi)$$

$$F(\phi, \mathcal{G}) = f_0\phi^w\mathcal{G}^k$$

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$$\omega(\phi) = \omega_0 \exp(q\phi)$$

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$$F(\phi, \mathcal{G}) = f_0\phi^w\mathcal{G}^k$$

$$F(\phi, \mathcal{G}) = f_o \exp(w\phi)\mathcal{G}^k$$

NOETHER SYMMETRY APPROACH

1

$$\boxed{\omega(\phi) = \omega_0}$$

$$\boxed{V(\phi) = 0}$$

$$V(\phi) = V_0 \phi^p$$

$$V(\phi) = V_0 \exp(p\phi)$$

$$\boxed{F(\phi, \mathcal{G}) = f_0 \phi^w \mathcal{G}^k}$$

$$F(\phi, \mathcal{G}) = f_o \exp(w\phi) \mathcal{G}^k$$

2

3

$$\boxed{\boxed{\omega(\phi) = \omega_0 \phi^q}}$$

$$\boxed{V(\phi) = 0}$$

$$\boxed{V(\phi) = V_0 \phi^p}$$

$$V(\phi) = V_0 \exp(p\phi)$$

$$\boxed{\boxed{F(\phi, \mathcal{G}) = f_0 \phi^w \mathcal{G}^k}}$$

$$F(\phi, \mathcal{G}) = f_o \exp(w\phi) \mathcal{G}^k$$

4

$$\boxed{\omega(\phi) = \omega_0 \exp(q\phi)}$$

$$\boxed{V(\phi) = 0}$$

$$V(\phi) = V_0 \phi^p$$

$$V(\phi) = V_0 \exp(p\phi)$$

$$F(\phi, \mathcal{G}) = f_0 \phi^w \mathcal{G}^k$$

$$\boxed{F(\phi, \mathcal{G}) = f_o \exp(w\phi) \mathcal{G}^k}$$

FIRST CASE

$$\omega(\phi) = \omega_0$$
$$V(\phi) = 0$$
$$F(\phi, \mathcal{G}) = f_0 \phi^w \mathcal{G}^k$$

CANONICAL POINT-LIKE LAGRANGIAN:

$$\mathcal{L} = a^3 \omega_0 \dot{\phi}^2 - f_0 (k-1) \mathcal{G}^k \phi^w a^3 - 8 f_0 k \dot{a}^3 \mathcal{G}^{k-2} \phi^{w-1} [(k-1)\phi \dot{\mathcal{G}} + w \mathcal{G} \dot{\phi}]$$

INFINITESIMAL GENERATORS OF THE COORDINATE TRANSFORMATIONS

$$\alpha(a) = \alpha_0 a, \qquad \beta(\mathcal{G}) = -4\xi_0 \mathcal{G}, \qquad \gamma(\phi) = \frac{6(1-2k)\alpha_0}{-2+8k-w}\phi, \qquad \xi(t) = -\frac{3(w-2)\alpha_0}{-2+8k-w} t$$

In order to find *analytic solutions* to the equations of motion, we set

$$w = 2 \;\rightarrow\; F(\phi, \mathcal{G}) = f_0 \phi^2 \mathcal{G}^k$$

$$S = \int \sqrt{-g} [f_0 \phi^2 \mathcal{G}^k + \omega_0 \partial^\mu \phi \partial_\mu \phi] d^4x$$

$$a(t) = a_0 \exp(mt), \qquad \phi(t) = \phi_0 \exp(nt), \qquad \mathcal{G} = 24 m^4$$

where a_0, ϕ_0, m, n are real constants.

$$\alpha(a) = \alpha_0 a, \qquad \beta(\mathcal{G}) = 0, \qquad \gamma(\phi) = -\frac{3}{2}\alpha_0 \phi, \qquad \xi(t) = 0$$

CONSTRAINTS ON THE FREE PARAMETERS

$$\begin{cases} f_0 (24m^4)^k [3(k-1)m^2 - 4kmn - 4kn^2] - 3m^2 n^2 \omega_0 = 0 \\ f_0 (24m^4)^k - n\omega_0 (3m+n) = 0 \\ f_0 (24m^4)^k [(k-1)m - 2kn] + mn^2 \omega_0 = 0 \end{cases}$$

$$k = \frac{1}{2}, \qquad n = -\frac{3}{2} m, \qquad f_0 = -\frac{3\sqrt{6}}{16} \omega_0$$

SECOND CASE

$$\omega(\phi) = \omega_0 \phi^q$$
$$V(\phi) = 0$$
$$F(\phi, \mathcal{G}) = f_0 \phi^w \mathcal{G}^k$$

CANONICAL POINT-LIKE LAGRANGIAN:

$$\mathcal{L} = a^3\left[\omega_0 \phi^q \dot{\phi}^2 - f_0(k-1)\mathcal{G}^k \phi^w\right] - 8 f_0 k \dot{a}^3 \mathcal{G}^{k-2} \phi^{w-1}\left[(k-1)\phi\dot{\mathcal{G}} + w\mathcal{G}\dot{\phi}\right]$$

INFINITESIMAL GENERATORS OF THE COORDINATE TRANSFORMATIONS

$$\alpha(a) = \alpha_0 a, \qquad \beta(\mathcal{G}) = -4\xi_0 \mathcal{G}, \qquad \gamma(\phi) = \frac{6(1-2k)\alpha_0}{(2+q)(4k-1)-w}\phi, \qquad \xi(t) = -\frac{3(2+q-w)\alpha_0}{(2+q)(4k-1)-w}t$$

In order to find *analytic solutions* to the equations of motion, we set

$$w = 2 + q \rightarrow \; F(\phi, \mathcal{G}) = f_0 \phi^{2+q} \mathcal{G}^k$$

$$S = \int \sqrt{-g}\left[f_0 \phi^{q+2} \mathcal{G}^k + \omega_0 \phi^q \partial^\mu \phi \partial_\mu \phi\right] d^4x$$

$$\boxed{a(t) = a_0 \exp(mt)\,, \qquad \phi(t) = \phi_0 \exp(nt)\;, \qquad \mathcal{G} = 24 m^4}$$

where a_0, ϕ_0, m, n are real constants.

$$\alpha(a) = \alpha_0 a, \qquad \beta(\mathcal{G}) = 0, \qquad \gamma(\phi) = -\frac{3\alpha_0}{(2+q)}\phi, \qquad \xi(t) = 0$$

CONSTRAINTS ON THE FREE PARAMETERS

$$\begin{cases} f_0(24m^4)^k\left[3(k-1)m^2 - 2kmn(q+2) - kn^2(q+2)^2\right] - 3m^2n^2\omega_0 = 0 \\ n\omega_0\left[6m + n(q+2)\right] - f_0(q+2)(24m^4)^k = 0 \\ f_0(24m^4)^k\left[(k-1)m - kn(q+2)\right] + mn^2\omega_0 = 0 \end{cases}$$

$$\boxed{k = \frac{1}{2}, \qquad n = -\frac{3}{q+2}m, \qquad f_0 = -\frac{3\sqrt{6}}{4(q+2)^2}\omega_0}$$

THIRD CASE

$$\omega(\phi) = \omega_0\phi^q$$
$$V(\phi) = V_0\phi^p$$
$$F(\phi,\mathcal{G}) = f_0\phi^w\mathcal{G}^k$$

CANONICAL POINT-LIKE LAGRANGIAN:

$$\mathcal{L} = a^3\left[\omega_0\phi^q\dot{\phi}^2 - f_0(k-1)\mathcal{G}^k\phi^w\right] - 8f_0k\dot{a}^3\mathcal{G}^{k-2}\phi^{w-1}\left[(k-1)\phi\dot{\mathcal{G}} + w\mathcal{G}\dot{\phi}\right] + a^3V_0\phi^p$$

INFINITESIMAL GENERATORS OF THE COORDINATE TRANSFORMATIONS

$$\alpha(a) = -\frac{1}{3}[\xi_0 + kp\gamma_0]a, \qquad \beta(\mathcal{G}) = -4\xi_0\mathcal{G}, \qquad \gamma(\phi) = \gamma_0\phi, \qquad \xi(t) = -\frac{(p-w)\gamma_0}{4k}t$$

In order to find *analytic solutions* to the equations of motion, we set

$$q = \frac{w - 4k + p(2k-1)}{2k}, \qquad w = 2 + \mathrm{q}, \qquad p = w \rightarrow q = q$$

$$F(\phi,\mathcal{G}) = f_0\phi^{2+q}\mathcal{G}^k$$

$$S = \int\sqrt{-g}\left[f_0\phi^{2+q}\mathcal{G}^k + \omega_0\phi^q\partial^\mu\phi\partial_\mu\phi + V_0\phi^{2+q}\right]d^4x$$

$$a(t) = a_0\exp(mt)\,, \qquad \phi(t) = \phi_0\exp(nt)\,, \qquad \mathcal{G} = 24m^4$$

where a_0, ϕ_0, m, n are real constants.

$$\alpha(a) = -\frac{1}{3}k(2+q)\gamma_0 a, \qquad \beta(\mathcal{G}) = 0, \qquad \gamma(\phi) = \gamma_0\phi, \qquad \xi(t) = 0$$

CONSTRAINTS ON THE FREE PARAMETERS

$$\begin{cases} f_0(24m^4)^k[-kn^2(2+q)^2 - 2kmn(q+2) + 3m^2(k-1)] \\ \quad - 3m^2n^2\omega_0 - 3m^2V_0 = 0 \\ -f_0(q+2)(24m^4)^k + n\omega_0[6m + n(q+2)] - (2+q)V_0 = 0 \\ f_0(24m^4)^k[-kn(q+2) + m(k-1)] + mn^2\omega_0 - mV_0 = 0 \end{cases}$$

$$k = \frac{1}{2}, \qquad f_0 = -\frac{\sqrt{6}}{2},$$
$$n = -\frac{3(2+q)}{2\omega_0}m + \frac{\sqrt{m}}{2\omega_0}\frac{(2+q)[3(2+q) - 4\omega_0]}{\sqrt{[(2+q)^4 + 4\omega_0^2]^2 - 4(2+q)^2\omega_0}}$$

FOURTH CASE

$$\omega(\phi) = \omega_0 \exp(q\phi)$$
$$V(\phi) = 0$$
$$F(\phi, \mathcal{G}) = f_0 \exp(w\phi)\, \mathcal{G}^{\frac{1}{2}}$$

CANONICAL POINT-LIKE LAGRANGIAN:

$$\mathcal{L} = 2 f_0 \mathcal{G}^{-\frac{3}{2}} \dot{a}^3 \exp(w\phi)\left(\dot{\mathcal{G}} - 2w\mathcal{G}\dot{\phi}\right) + a^3 \left(\frac{1}{2} f_0 \mathcal{G}^{\frac{1}{2}} \exp(w\phi) + \omega_0 \exp(q\phi)\dot{\phi}^2\right)$$

INFINITESIMAL GENERATORS OF THE COORDINATE TRANSFORMATIONS

$$\alpha(a) = -\frac{\beta_0}{12} a, \qquad \beta(\mathcal{G}) = \beta_0 \mathcal{G}, \qquad \gamma(\phi) = 0, \qquad \xi(t) = -\frac{\beta_0}{4} t$$

In order to find *analytic solutions* to the equations of motion, we set

$$w = \mathrm{q} \rightarrow F(\phi, \mathcal{G}) = f_0 \exp(q\phi)\, \mathcal{G}^{\frac{1}{2}}$$

$$S = \int \sqrt{-g} \left[f_0 \exp(q\phi)\, \mathcal{G}^{\frac{1}{2}} + \omega_0 \exp(q\phi)\, \partial^\mu \phi \partial_\mu \phi \right] d^4x$$

$$a(t) = a_0 \exp(mt)\,, \qquad \phi(t) = \phi_0 \exp(nt)\,, \qquad \mathcal{G} = 24 m^4$$

where a_0, ϕ_0, m, n are real constants.

CONSTRAINTS ON THE FREE PARAMETERS

$$\begin{cases} 2 f_0 m^2 (3m^2 + 2mw\phi_0 + w^2\phi_0^2) + \sqrt{6} m^2 \omega_0 \phi_0^2 = 0 \\ 2\sqrt{6} f_0 m^2 w - \omega_0 \phi_0 (6m + w\phi_0) = 0 \\ \sqrt{6} m^2 \omega_0 \phi_0^2 - 6 f_0 m^3 (m + w\phi_0) = 0 \end{cases}$$

$$\phi_0 = -\frac{3m}{q}, \qquad f_0 = -\left(\frac{3}{2}\right)^{\frac{3}{2}} \frac{\omega_0}{q^2}$$

CONCLUSIONS

1. Despite its successes, General Relativity is not the final theory of gravity.

2. There is no a priori reason to consider a gravitational action linear in the Ricci scalar.

3. The basic idea of modified theories of gravity is changing the gravitational side of the Einstein's field equations rather than the energy-matter content.

4. Too many models can fit observations and experiments because of the large number of free functions and parameters that they contain, so a physical criterion to select viable models is necessary.

5. Any relativistic theory of gravity must satisfy certain minimal requirements from phenomenolog al point of view.

6. Currently, there is no a modified theory able to adress all the shortcomings and, at the same time, to recover the already valid results we know. We can think of them like toy models. Maybe the solution is a combination of them.

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