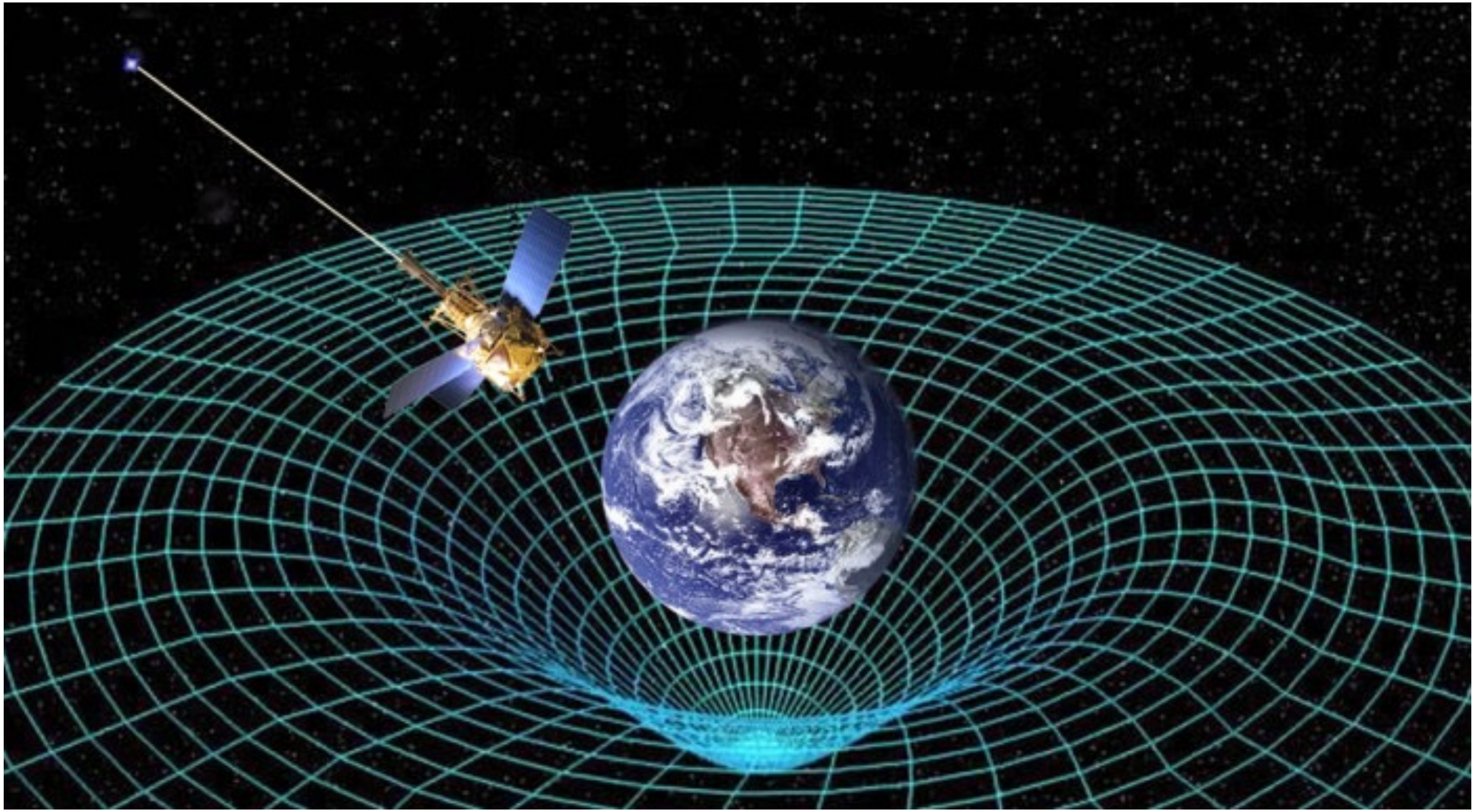
Probing spacetime dynamics by high-precision tests in General Relativity and its modifications/extensions

Salvatore Capozziello



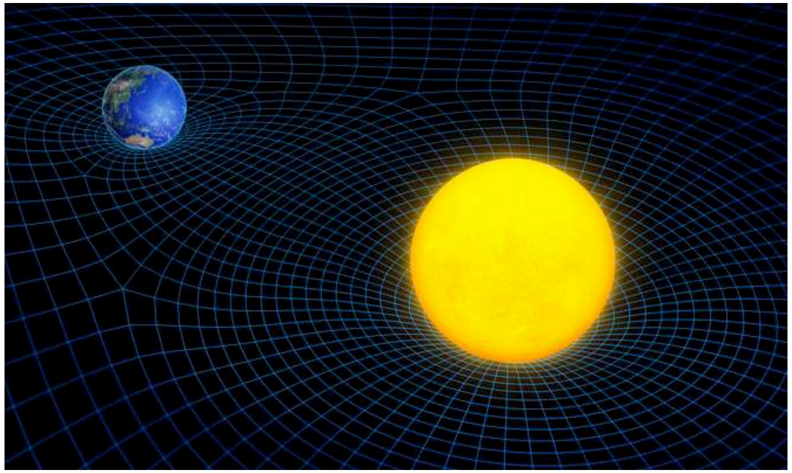
Outline

- *General Relativity, alternatives and extensions*
- *Tests of theories of gravity*
- *Sagnac and Lense-Thirring effects*
- *Inertial guidance gyros*
- *Gyroscope precession*
- *Gravitoelectromagnetism*
- *Case studies: Scalar-tensor and Horava-Lifshitz gravity*
- *The weak-field limit*
- *Experimental constraints from space and ground-based experiments (LARES, GPB, GINGER)*
- *Conclusions*

General Relativity:
foundations and predictions

General Relativity

GR describes gravitational interaction by the spacetime curvature



The theory successfully passes the Solar System Tests

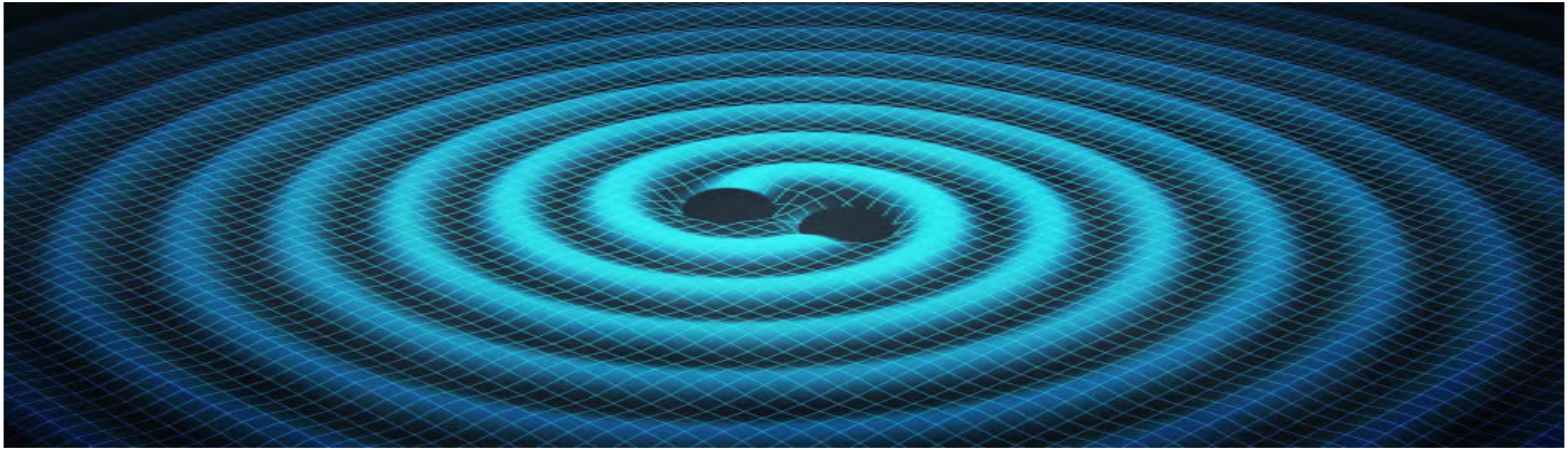
Static and spherically Background: the paradigm

Schwarzschild Solution

$$ds^2 = \left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \mathrm{sen}^2\theta d\phi^2$$

- **Black Holes**

- **Gravitational Waves**



General Relativity: shortcomings

Shortcomings in GR

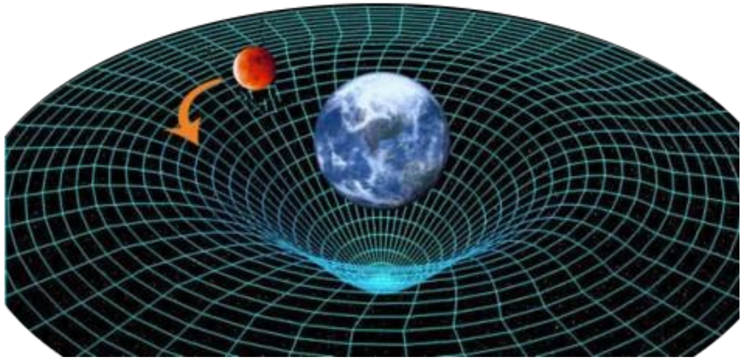
Large Scales

- Universe accelerated expansion
- Inflation
- Galaxy Rotation Curve
- Mass-Radius diagram of Neuton Stars
- Fine-tuning cosmological parameters

Small Scales

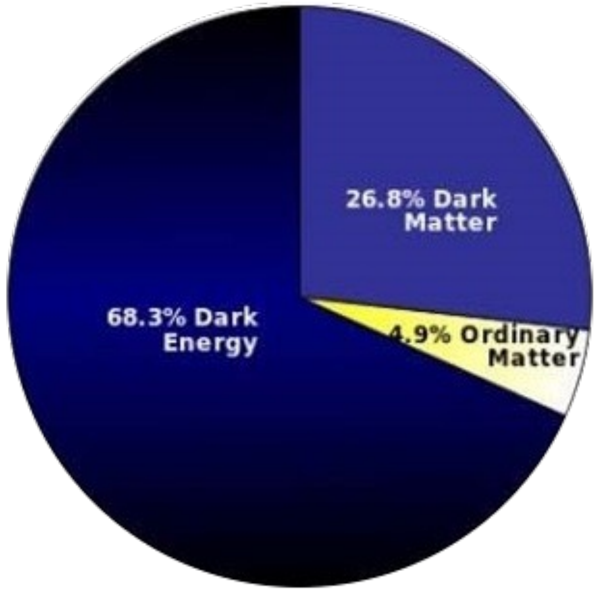
- Renormalizability
- GR cannot be quantized
- GR cannot be treated under the same standard of the other gauge interactions
- Discrepancy between theoretical and experimental value of Λ
- Spacetime singularities

No theory is capable of solving these problems at once so far



Cosmological issues

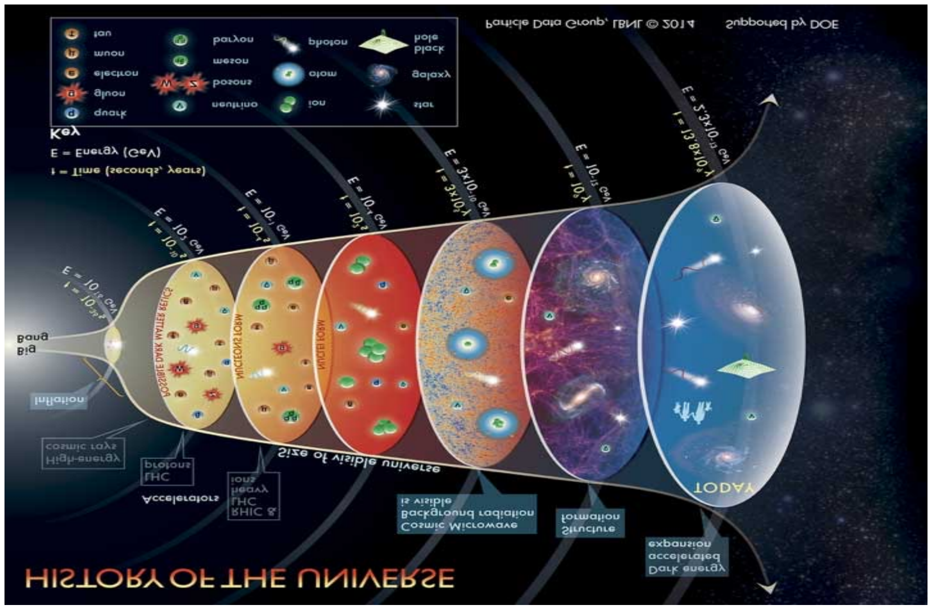
$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$



Extra scalar fields and new particles for Dark Side

Extra scalar fields for inflationary expansion

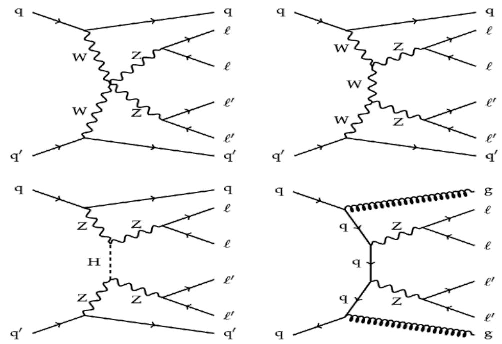
Cosmological constant puzzle



Alternative Theories of Gravity

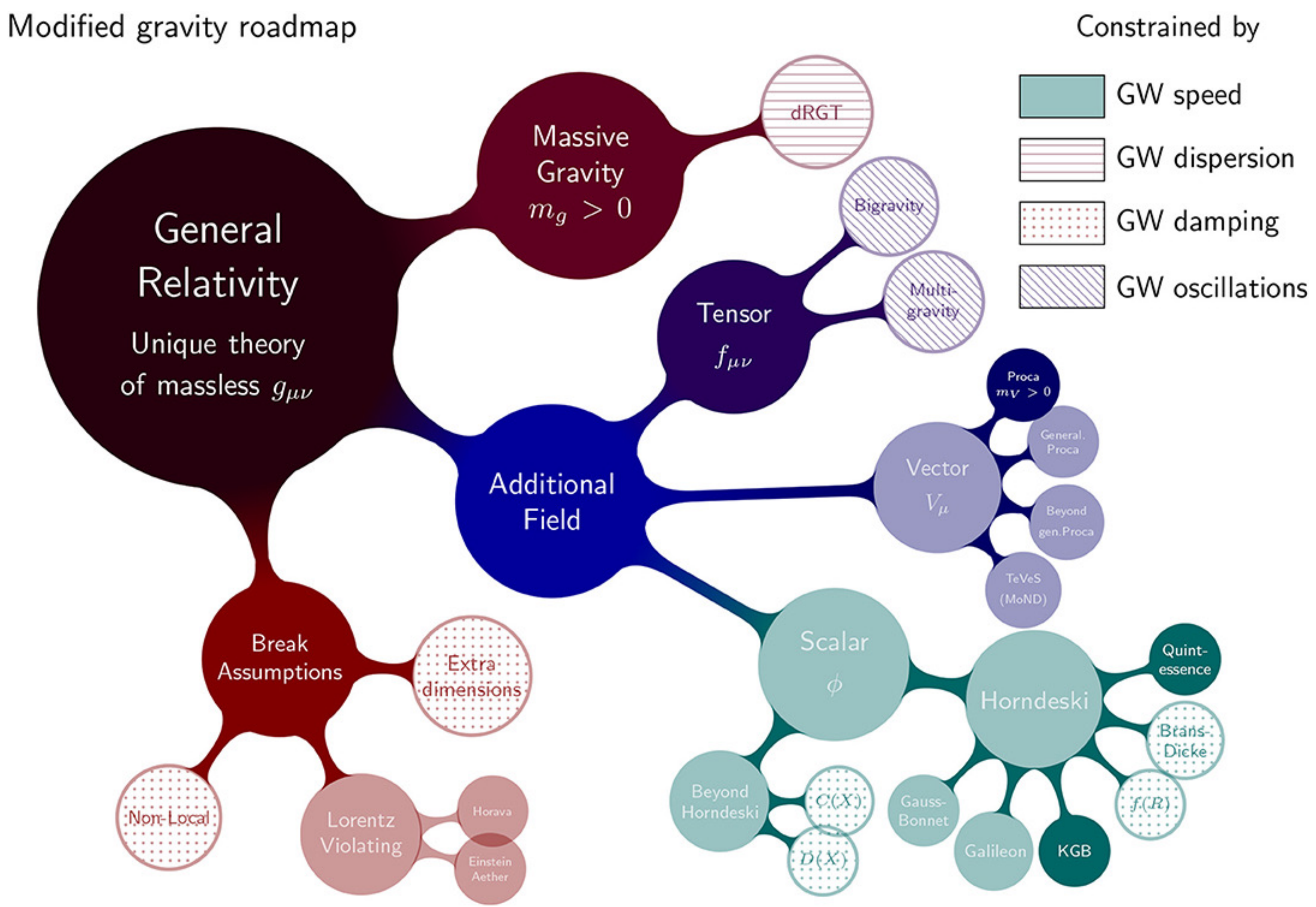
Classification

- **Extended Theories**: $f(R)$ gravity. GR is recovered for $f(R) = R$
- **Modified Theories**: $f(T)$ teleparallel gravity. Torsion T instead of curvature R
- Non-minimaly coupled scalar fields φR. Effective gravitational couplings



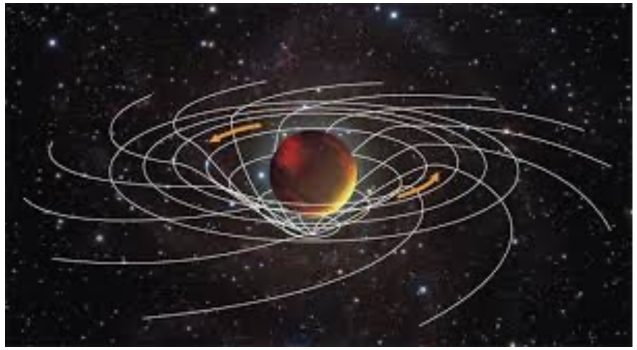
Motivations

- Could account for UV quantum corrections?
- Could reproduce IR cosmic evolution?
- Could account for the Mach Principle?
- Could address Dark Matter and Dark Energy?
- Could probe Equivalence Principle at quantum scales?
- Key roles of Multimessenger Astronomy and gravitational waves


Modified gravity roadmap
Constrained by
GW speed
GW dispersion
GW damping
GW oscillations
General Relativity
Unique theory of massless $g_{\mu\nu}$
Massive Gravity
$m_g > 0$
dRGT
Tensor
$f_{\mu\nu}$
Bigravity
Multi-gravity
Additional Field
Proca
$m_V > 0$
General. Proca
Vector
V_μ
Beyond gen.Proca
TeVeS (MoND)
Scalar
ϕ
Horndeski
Quint-essence
Brans-Dicke
$f(R)$
KGB
Galileon
Gauss-Bonnet
Beyond Horndeski
$C(X)$
$D(X)$
Break Assumptions
Extra dimensions
Non-Local
Lorentz Violating
Horava
Einstein Aether

Is it possible to find out probes and test-beds for other theories of gravity?

- *Geodesic motions around compact objects e.g. SgrA**
- **Sagnac and Lense-Thirring effects**
- *Exact torsion-balance experiments*
- *Microgravity experiments from atomic physics*
- *Violation of the Equivalence Principle*
- *Effective masses related to further gravitational degrees of freedom*
- ***Extra gravitational modes over GR***



Sagnac effect

Consider an observer in motion in a general spacetime background.
It is always possible to define the so-called **proper reference frame** of the observer in which the spacetime metric reduces to the following form:

$$\mathrm{d}s^2 = -(\mathrm{d}x^{\hat{0}})^2\left[1+\frac{2}{c^2}\vec{a}\cdot\vec{x}+\frac{1}{c^4}\left(\vec{a}\cdot\vec{x}\right)^2-\frac{1}{c^2}\left(\vec{\Omega}\times\vec{x}\right)^2+R_{\hat{0}\hat{i}\hat{0}\hat{j}}x^{\hat{i}}x^{\hat{j}}\right]$$
$$+2\mathrm{d}x^{\hat{0}}\mathrm{d}x^{\hat{i}}\left[\frac{1}{c}\epsilon_{\hat{i}\hat{j}\hat{k}}\Omega^{\hat{j}}x^{\hat{k}}-\frac{2}{3}R_{\hat{0}\hat{j}\hat{i}\hat{k}}x^{\hat{j}}x^{\hat{k}}\right]+\mathrm{d}x^{\hat{i}}\mathrm{d}x^{\hat{j}}\left[\delta_{\hat{i}\hat{j}}-\frac{1}{3}R_{\hat{i}\hat{k}\hat{j}\hat{l}}x^{\hat{k}}x^{\hat{l}}\right]+o(r^2)$$

$\vec{a}$	proper acceleration ($\vec{a}\approx-\vec{g}$ on Earth's surface)	$\vec{\Omega}$	**Angular velocity vector** *
$r=\lvert\vec{x}\rvert$	spatial distance from observer's worldline	$R_{\hat{\mu}\hat{\nu}\hat{\alpha}\hat{\beta}}$	Riemann tensor components

Independent of background geometry then it is a possible test for any theory of gravity

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$$
\begin{aligned}
\mathrm{d}s^2 = \; & -(\mathrm{d}x^{\hat{0}})^2\left[1 + \frac{2}{c^2}\vec{a}\cdot\vec{x} + \cancel{\frac{1}{c^4}(\vec{a}\cdot\vec{x})^2} - \cancel{\frac{1}{c^2}\left(\vec{\Omega}\times\vec{x}\right)^2} + \cancel{R_{\hat{0}\hat{i}\hat{0}\hat{j}}x^{\hat{i}}x^{\hat{j}}}\right] \\
& + 2\mathrm{d}x^{\hat{0}}\mathrm{d}x^{\hat{i}}\left[\boxed{\frac{1}{c}\epsilon_{\hat{i}\hat{j}\hat{k}}\Omega^{\hat{j}}x^{\hat{k}}} - \cancel{\frac{2}{3}R_{\hat{0}\hat{j}\hat{i}\hat{k}}x^{\hat{j}}x^{\hat{k}}}\right] + \mathrm{d}x^{\hat{i}}\mathrm{d}x^{\hat{j}}\left[\delta_{\hat{i}\hat{j}} - \cancel{\frac{1}{3}R_{\hat{i}\hat{k}\hat{j}\hat{l}}x^{\hat{k}}x^{\hat{l}}}\right] + \cancel{o(r^2)}
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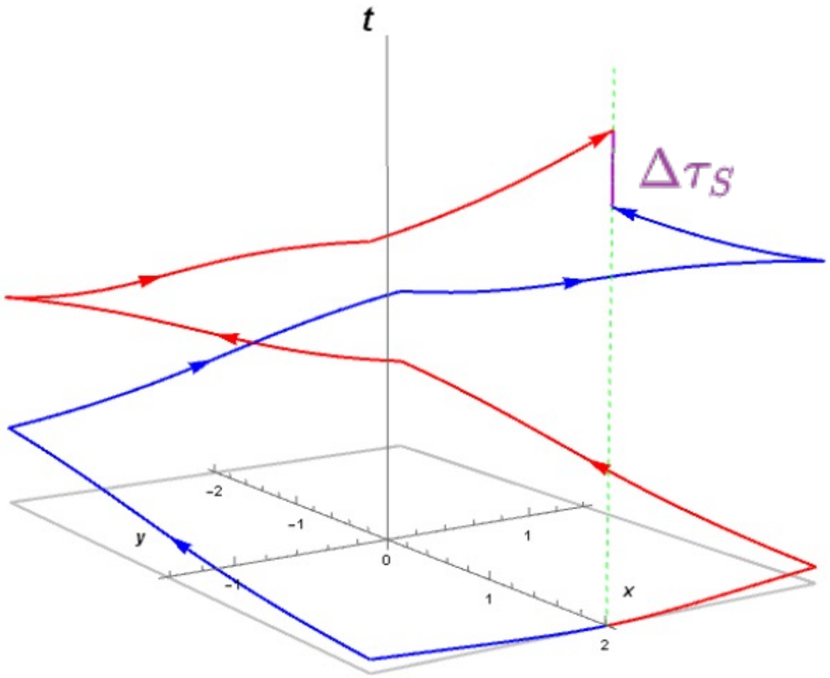
Sagnac effect



The figure on the left depicts two **counterpropagating light rays**: they follow different null-geodesics with a common spatial projection.

A stationary observer ($\vec{a}$, $\vec{\Omega}$ time indep.) on the optical path measures the following difference in the roundtrip time intervals:

$$\Delta\tau_S = -\frac{2}{c}\sqrt{-g_{00}(q)}\int_S \frac{g_{0i}}{g_{00}}\mathrm{d}l^i$$

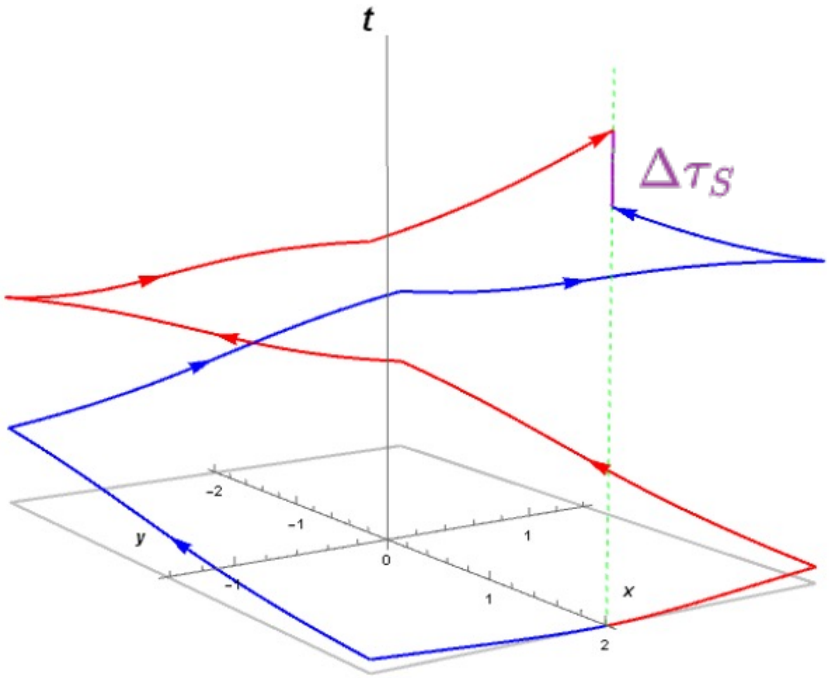
Sagnac time delay

$$\Delta\tau_S \simeq \frac{4}{c^2}\vec{\Omega}\cdot\vec{\mathcal{A}}$$

Area vector of the enclosed path

Angular velocity vector *

Sagnac effect



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Area vector
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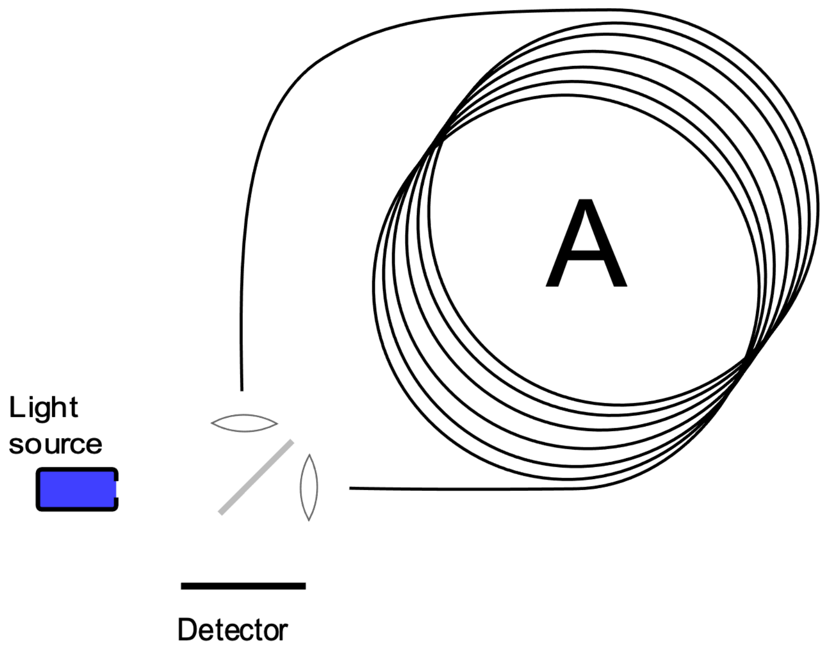
Sagnac effect in a passive interferometer

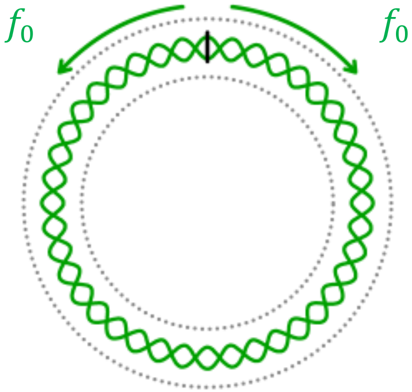


In a passive Sagnac interferometer (e.g. Fiber Optic Gyro) the light generated from an external source is injected into two counter-revolving paths. At the exit port of the interferometer the beams (or photons) acquire a relative optical phase difference:

$$\phi = \frac{8\pi}{\lambda c}\vec{\Omega}\cdot\vec{A}$$

wavelength of light

Sagnac effect in ring resonators



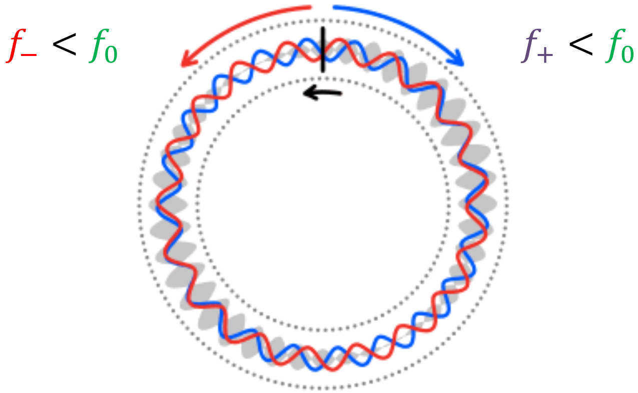
In ring resonator (i.e. a closed optical cavity) resonance conditions are modified by rotation;
the frequency spacing of the lowest energy counterpropagating modes sustained in the cavity is proportional to the rotation rate:

$$f_s = \Delta f = \frac{4\,\vec{\Omega}\cdot\vec{A}}{\lambda P}$$

Sagnac frequency

perimeter of the cavity

Sagnac effect in ring resonators



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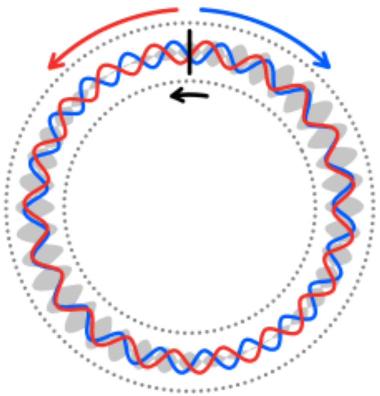
Sagnac frequency

perimeter of the cavity

Sagnac effect in ring resonators



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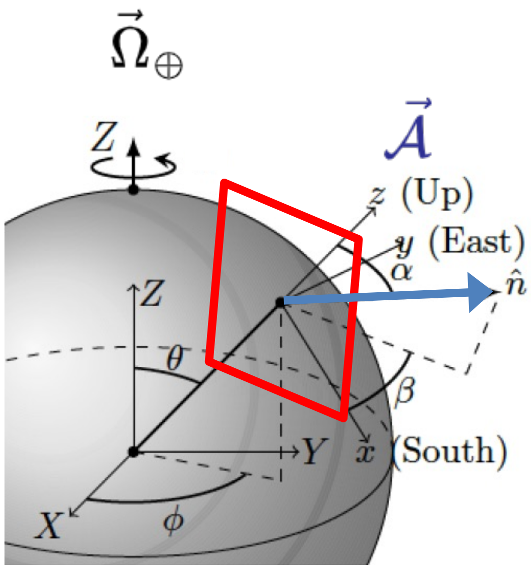


θ	colatitude
α	angle between area vector and local normal
β	angle between area vector and local South direction

$$f_s = \Delta f = \frac{4\,\vec{\Omega}\cdot\vec{A}}{\lambda P}$$

Sagnac frequency

perimeter of the cavity

Lense-Thirring Effect

This effect predicted by GR can be obtained starting from a Kerr-like metric.
It can be adopted to test ***any theory of gravity*** *described by a metric*

$$ds^2 = \mathcal{A}(t,r,\theta)dt^2 + \mathcal{B}(t,r,\theta)dr^2 + \mathcal{C}(t,r,\theta)d\theta^2 + \mathcal{D}(t,r,\theta)\sin^2\theta d\phi^2 + \mathcal{E}(t,r,\theta)dt\,d\phi$$

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4}T_{\mu\nu}$$

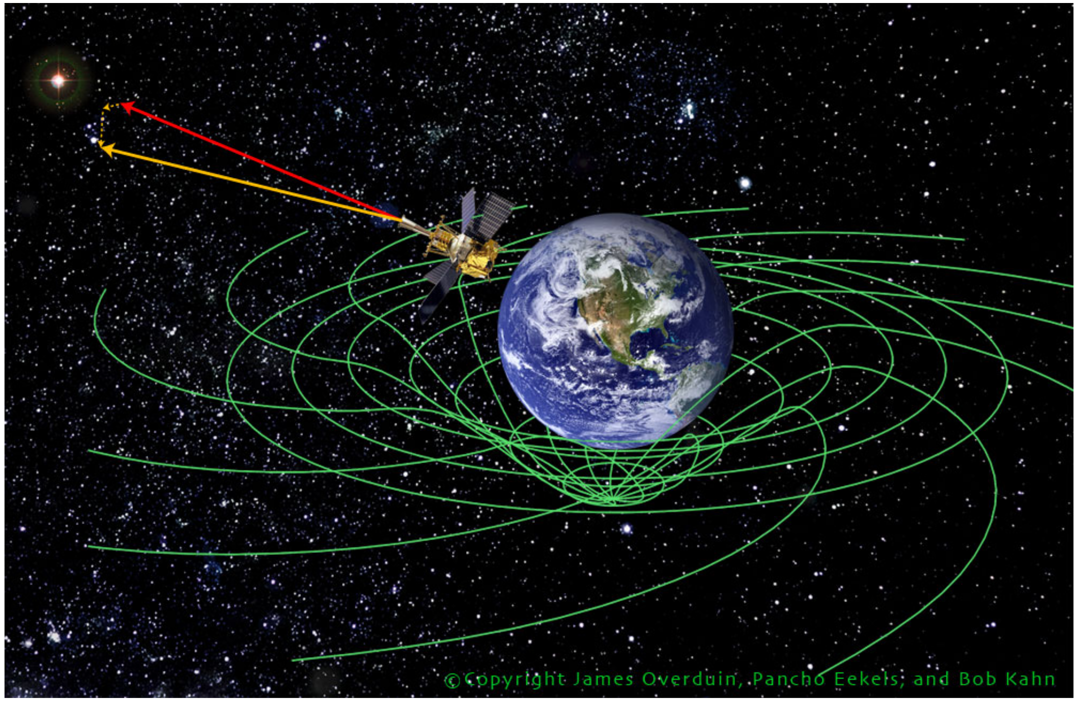
$$ds^2 = \left(1 - \frac{r_S}{r}\right)dt^2 - \frac{1}{1 - \frac{r_S}{r}\frac{J^2}{M^2 r^2}}dr^2 - r^2 d\theta^2 - \left(r^2 + \frac{J^2}{M^2} + \frac{r_S J^2}{M^2 r}\right)d\phi^2 \boxed{- \frac{2 r_S J}{M r}dt\,d\phi}$$

Angular Momentum

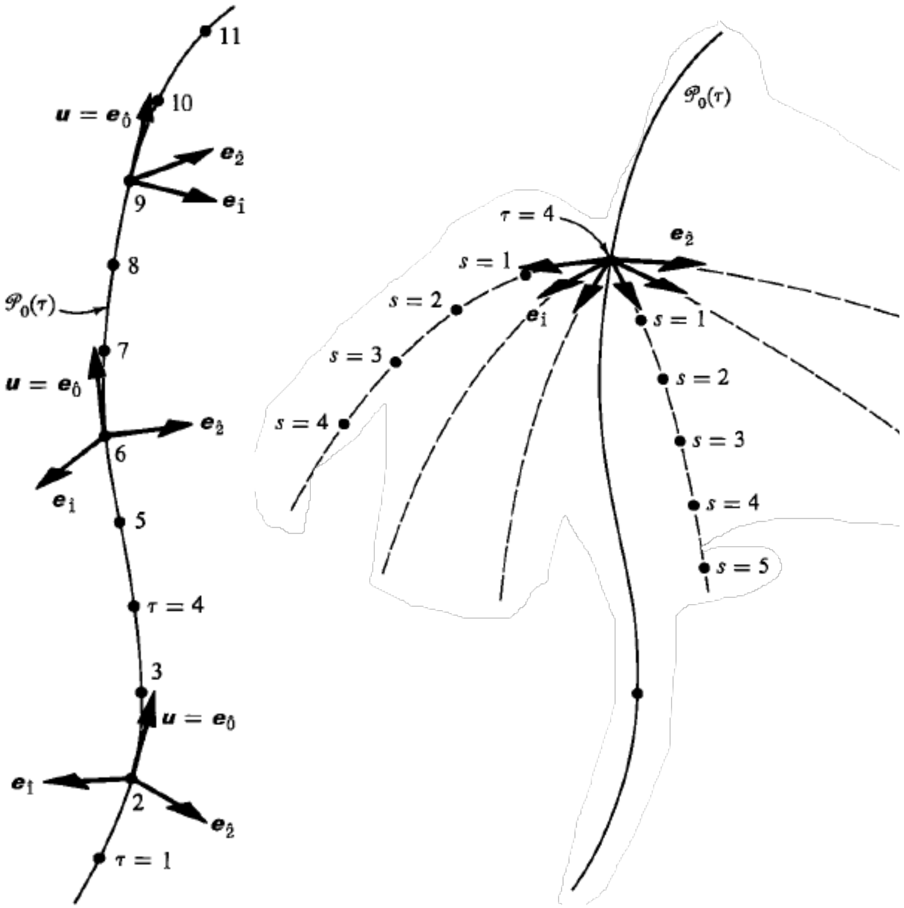
Correction to the precession of a gyroscope near a large rotating mass, due to the dragging of the spacetime!

$$\boxed{\Omega_{LT}^{(GR)} = \frac{r_S}{4 M r^3} J}$$

Gravitomagnetic effects in theories of gravity



Proper reference frame



Misner, C.W., Thorne, K.S. and Wheeler, J.A. (1973)
Gravitation

The proper reference frame of a general observer is defined according to the following procedure:

1) Let τ be the proper time as measured by a clock carried by the observer; call $\gamma(\tau)$ the observer's worldline parametrized by τ;
2) assign at each point of γ a set of three arbitrary normalized vectors $\{\mathbf{e}_{\hat 1}, \mathbf{e}_{\hat 2}, \mathbf{e}_{\hat 3}\}$ that are:
   a) mutually orthogonal,
   b) ortoghonal to the four-velocity $\mathbf{u} = \mathbf{e}_{\hat 0}$;
3) $\{\mathbf{e}_{\hat 0}, \mathbf{e}_{\hat 1}, \mathbf{e}_{\hat 2}, \mathbf{e}_{\hat 3}\}$ define an orthonormal tetrad field along γ that we identify with the proper reference frame of the observer;
4) near the observer, proper reference frame coordinates are uniquely defined accordingly:
   a) the timelike coordinate line is the worldline itself
   b) the spacelike coordinate lines are the spatial geodesics stemming from and tangent to $\{\mathbf{e}_{\hat 1}, \mathbf{e}_{\hat 2}, \mathbf{e}_{\hat 3}\}$

Proper reference frame

$$\nabla_{\mathbf{u}}\mathbf{e}_{\hat{\alpha}} = -\boldsymbol{\Omega}\cdot\mathbf{e}_{\hat{\alpha}} \quad \longleftrightarrow \quad \Omega_{\mu\nu} = a^{\mu}u^{\nu} - u^{\mu}a^{\nu} + u_{\alpha}\Omega_{\beta}\epsilon^{\alpha\beta\mu\nu}$$

proper acceleration $\longleftrightarrow$ $\mathbf{a} = \nabla_{\mathbf{u}}\mathbf{u}$

local rotation vector

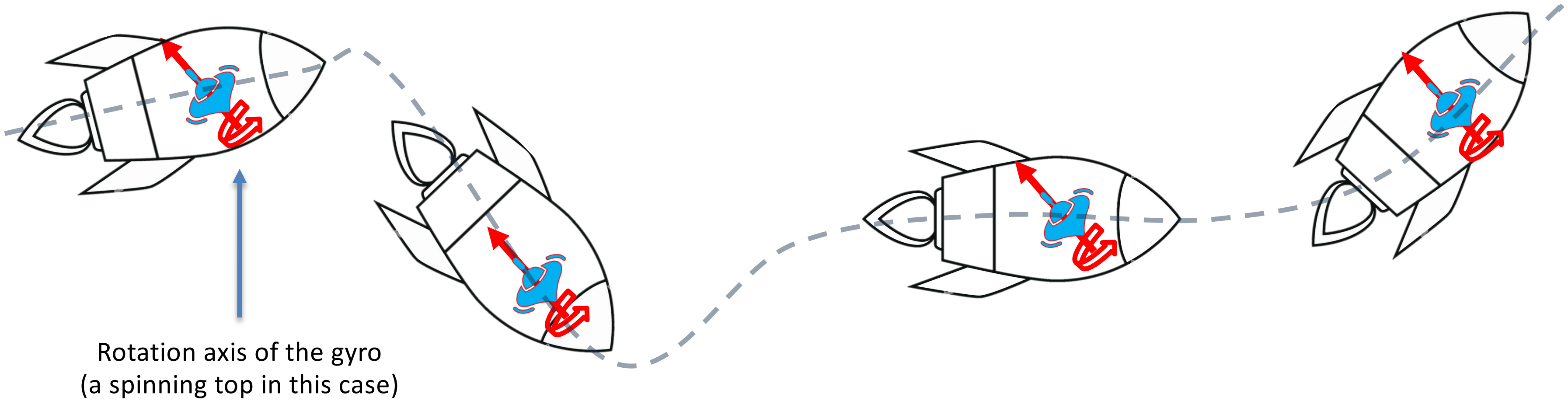
$$ds^2 \approx -(dx^{\hat{0}})^2\left[1+\frac{2}{c^2}\vec{a}\cdot\vec{x}\right] + \delta_{\hat{i}\hat{j}}dx^{\hat{i}}dx^{\hat{j}} + \frac{2}{c}\epsilon_{\hat{i}\hat{j}\hat{k}}\Omega^{\hat{j}}x^{\hat{k}}dx^{\hat{0}}dx^{\hat{i}}$$

$$\Delta\tau_S = -\frac{2}{c}\sqrt{-g_{00}(q)}\int_S \frac{g_{0i}}{g_{00}}dl^i \simeq \frac{4}{c^2}\boxed{\vec{\Omega}}\cdot\vec{\mathcal{A}}$$

How to interpret it?
What kind of information does it provide?

Area vector

Equations of motion of inertial guidance gyros



The simplest model of a gyroscope is a torque-free spinning body.

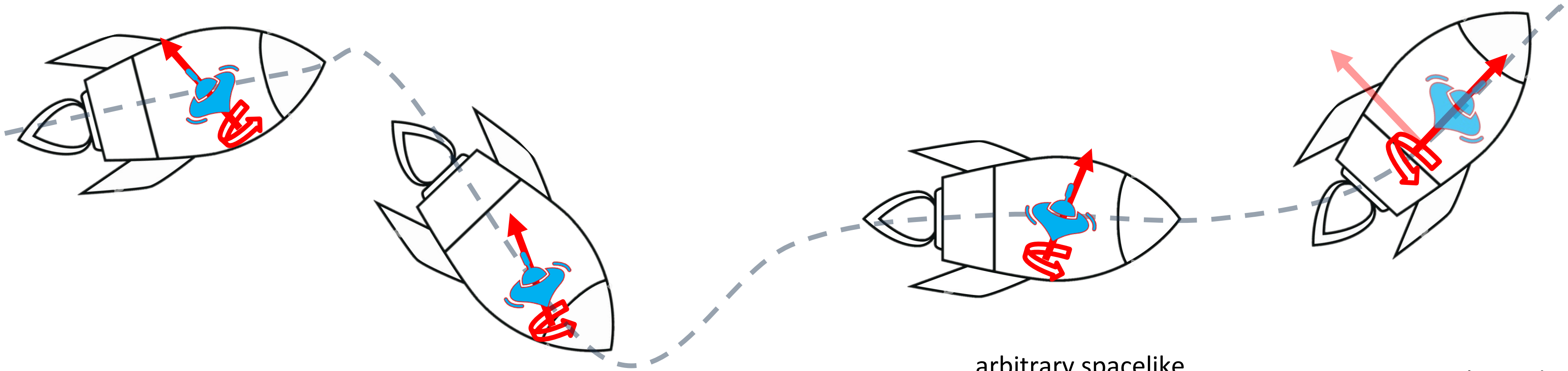
In Newtonian Mechanics, the angular momentum is a three-vector.

Angular momentum conservation implies that the orientation of the rotation axis of a gyro is unaffected by motion, i.e. it does not rotate relative to (the equivalence class of) inertial reference frames.

Angular momentum of the gyro

$$\frac{\mathrm{d}\vec{S}}{\mathrm{d}t} = 0$$

Equations of motion of inertial guidance gyros



In Special Relativity the conserved angular momentum of a body is an antisymmetric rank 2 tensor **J**.

The (intrinsic) angular momentum four-vector **S** (defined in terms of **J**) is Fermi-Walker transported along the worldline of the gyro.

Writing the transport equation is a momentarily comoving Lorentz frame, it is possible to show that gyros precess, in general. The precession rate is, again, relative to inertial frames, identified global Lorentz frames in this context.

arbitrary spacelike hypersurfaces

gyro's 4-velocity

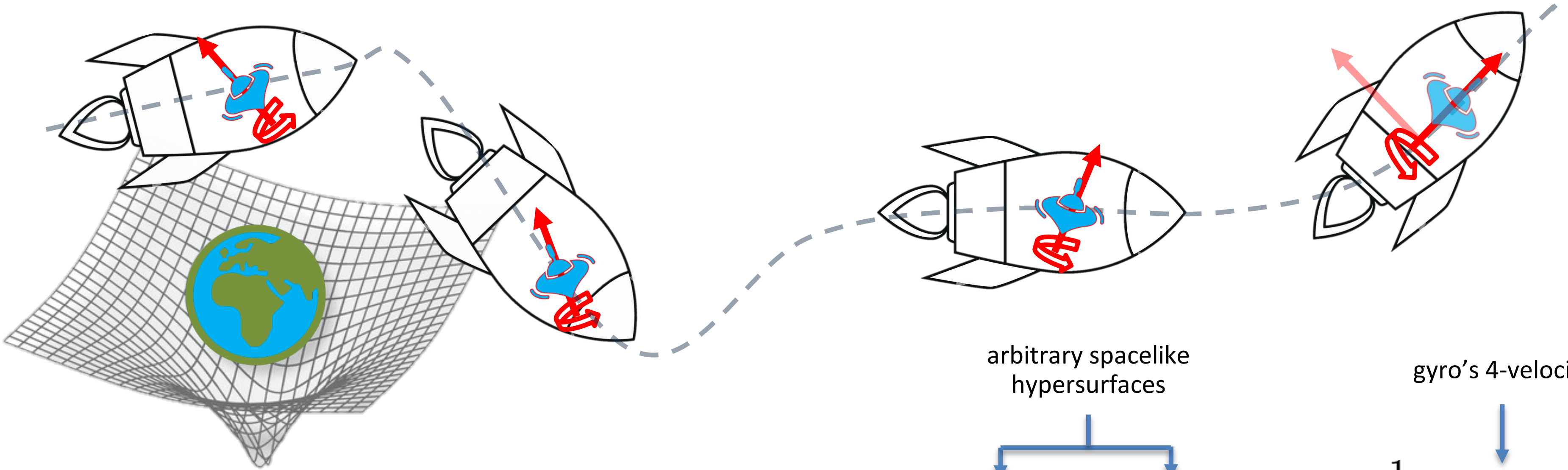
$$J^{\mu\nu}(\Sigma_A) = J^{\mu\nu}(\Sigma_B) \qquad S_\alpha = -\frac{1}{2}\epsilon_{\alpha\beta\mu\nu}u^\beta J^{\mu\nu}$$

$$\frac{\mathrm{d}S_\mu}{\mathrm{d}\tau} - a^\beta S_\beta u_\mu = 0$$

gyro's 4-acceleration

$$\frac{\mathrm{d}\vec{S}}{\mathrm{d}t} = \vec{\Omega}_{prec} \times \vec{S}$$

Equations of motion of inertial guidance gyros



arbitrary spacelike hypersurfaces

gyro's 4-velocity

$$J^{\mu\nu}(\Sigma_A) = J^{\mu\nu}(\Sigma_B) \qquad S_\alpha = -\frac{1}{2}\epsilon_{\alpha\beta\mu\nu}u^\beta J^{\mu\nu}$$

$$u^\alpha \nabla_\alpha S_\mu - a^\beta S_\beta u_\mu = 0$$

covariant derivative

gyro's **proper** 4-acceleration

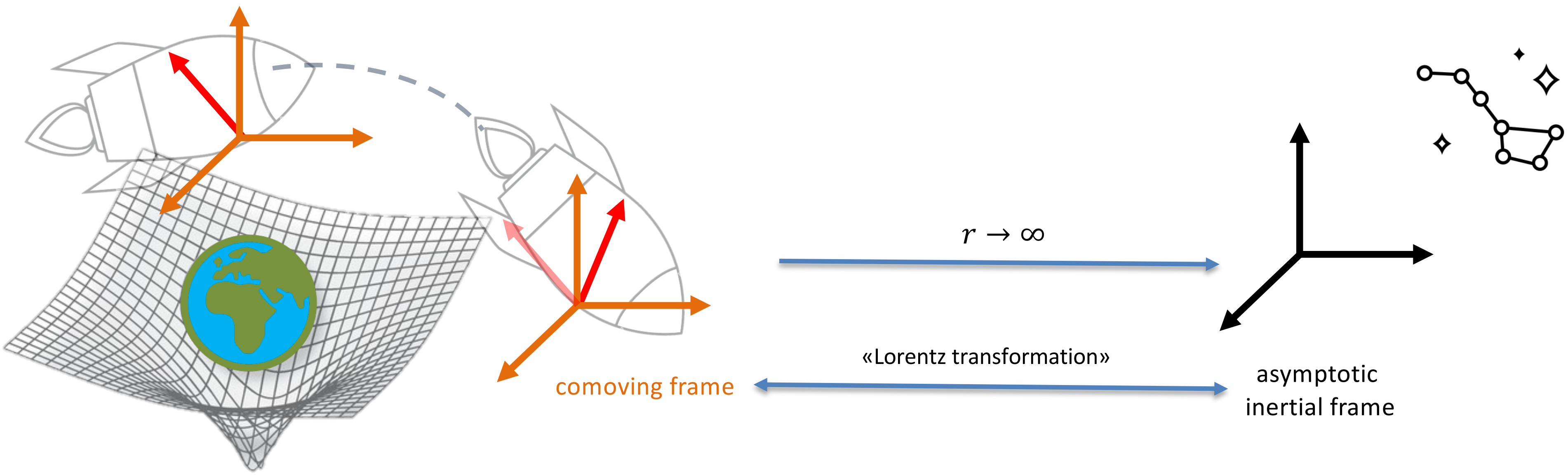
$$\frac{\mathrm{d}\vec{S}}{\mathrm{d}t} = \vec{\Omega}_{prec} \times \vec{S}$$

According to the **Equivalence Principle**, if a gyroscopes moves in curved spacetime, its spin is still **Fermi-Walker transported** along its worldline.

However, we have lost the possibility to define global inertial (Lorentz) frame, in general.

We expect the gyroscopes to precess, but relative to what? How to describe it? Is it useful?

Equations of motion of inertial guidance gyros



At least in asymptotically flat spacetimes, it is still possible to define unambiguously the gyroscopic precession.

From the Fermi-Walker transport equation it is possible to derive a precession equation for the angular momentum of a gyro, where the rotation vector is relative to an ideal inertial frame located at infinity. This frame is usually assumed to be tied, i.e. at rest, relative to far-away fixed stars.

$$u^\alpha \nabla_\alpha S_\mu - a^\beta S_\beta u_\mu = 0$$

$$\frac{\mathrm{d}\vec{S}}{\mathrm{d}t} = \vec{\Omega}_{prec} \times \vec{S}$$

contains (also) info on spacetime geometry

Equations of motion of inertial guidance gyros

The precession angular velocity can be derived in various way, for example adopting the following procedure:

1. Write an asymptotically flat metric (e.g. Lense-Thirring or PPN metrics) adopting rest frame coordinates of the source:

$$\mathrm{d}s^2 = -\left(1 - \frac{2GM}{c^2 r}\right)c^2\mathrm{d}t^2 + \frac{4G}{c^2}\frac{\vec{x}\times\vec{J}}{r^3}\cdot\mathrm{d}\vec{x}\mathrm{d}t + \left(1 + \frac{2GM}{c^2 r}\right)\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j + O(\frac{1}{r^3})$$

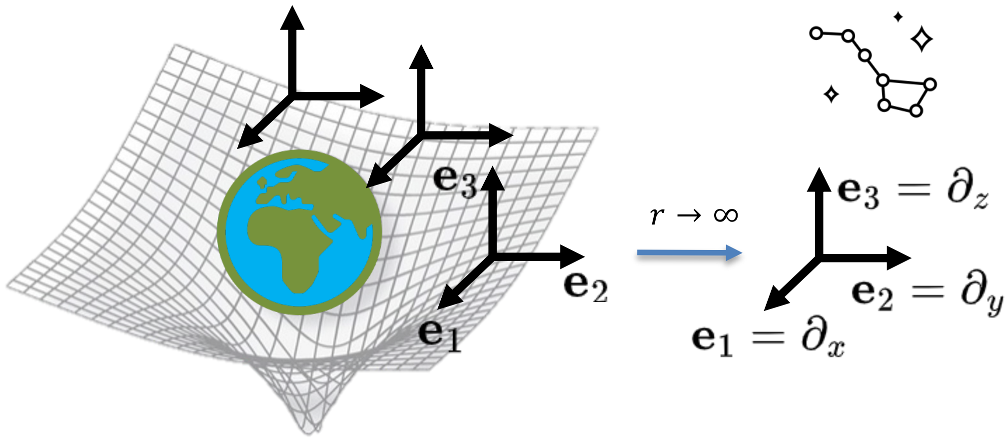
2. Introduce an orthonormal frame field associated to the metric. It is possible to show that the spatial vectors of the frames are just rescaled by this transformation (up to some order in the multipolar or PN expansion), and they are not rotated relative to the coordinate frame spatial vectors.

$$\mathrm{d}s^2 = \eta_{ab}\mathbf{e}^a\mathbf{e}^b$$

“tied” to asymptotic inertial frame, hence non-rotating relative to distant stars

$$\mathbf{e}^a = e^a{}_\mu \mathrm{d}x^\mu$$

they are functions of spacetime coordinates and depends potentials



3. A gyro is Fermi-Walker transported along its worldline. Call $\vec{v}$ its velocity relative to the coordinate grid.

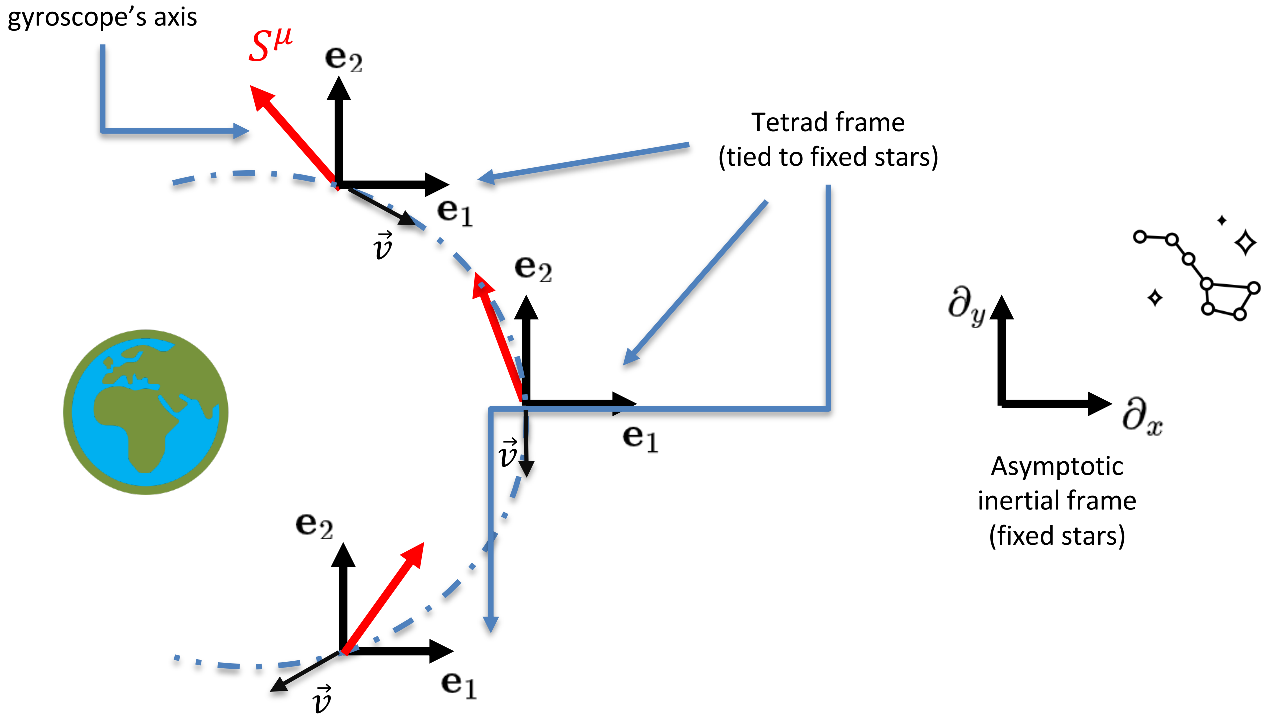
gyroscope's angular momentum

gyroscope's four-velocity

$$\mathbf{S} \cdot \mathbf{u} = 0$$

Fermi-Walker transport equation of a gyro

$$\boldsymbol{\nabla}_{\mathbf{u}} \mathbf{S} = \mathbf{u}\,(\mathbf{a} \cdot \mathbf{S})$$



3. A gyro is Fermi-Walker transported along its worldline. Call $\vec{v}$ its velocity relative to the coordinate grid.

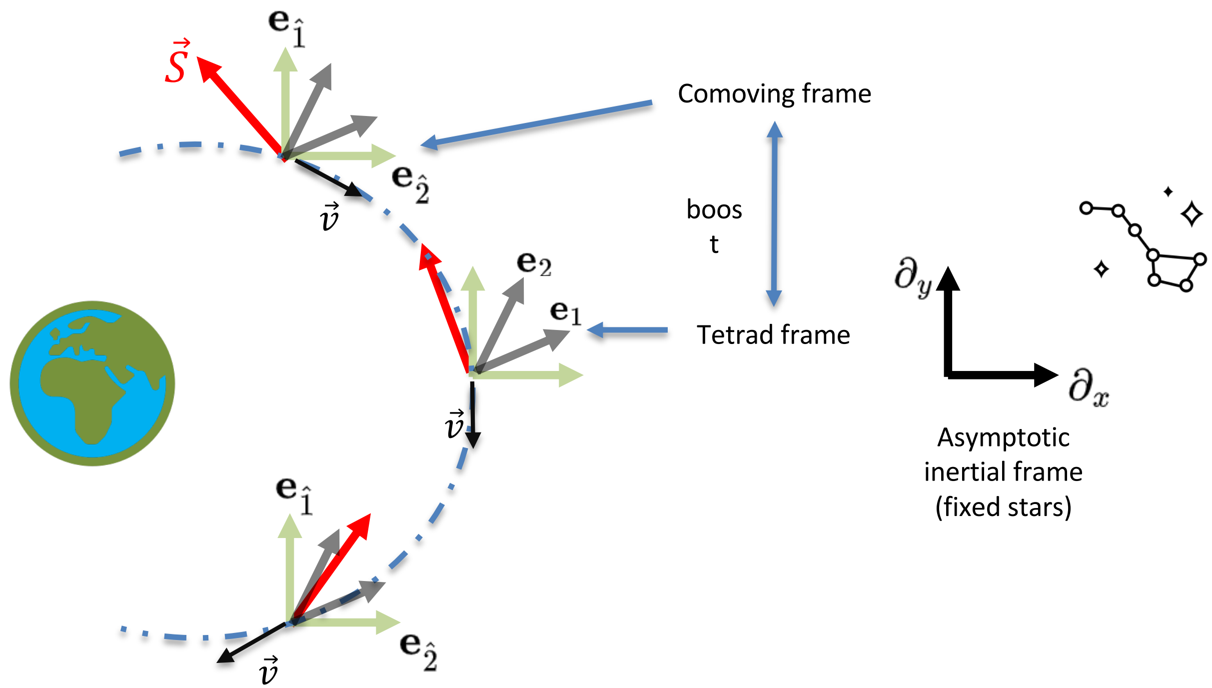
4. In order to compare gyro's orientation with the asymptotic frame, we define a comoving frame: point by point, this frame is related to the tetrad frame introduced before by a pure boost of velocity $\vec{v}$. The comoving frame is non-rotating relative to both the not-comoving tetrad frame and the coordinate frame.

$$u^{\hat{\mu}} = (1, \vec{0}) \;\Rightarrow\; S^{\hat{\mu}} = (0, \vec{S})$$

comoving
frame index

Fermi-Walker transport
equation of a gyro

$$\boldsymbol{\nabla}_{\mathbf{u}}\mathbf{S} = \mathbf{u}\,(\mathbf{a}\cdot\mathbf{S})$$



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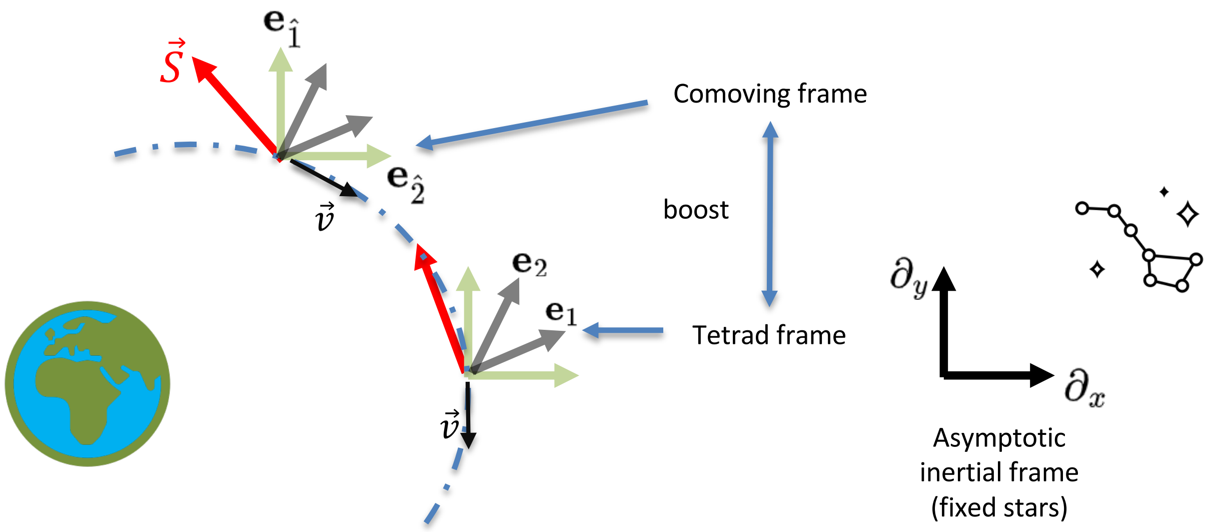
$$u^{\hat{\mu}} = (1, \vec{0}) \Rightarrow S^{\hat{\mu}} = (0, \vec{S})$$

$$\frac{\mathrm{d}S_{\hat{i}}}{\mathrm{d}\tau} \equiv \boldsymbol{\nabla}_{\mathbf{u}} \left(\mathbf{S} \cdot \mathbf{e}_{\hat{i}} \right) = \mathbf{S} \cdot \boldsymbol{\nabla}_{\mathbf{u}} \mathbf{e}_{\hat{i}}$$

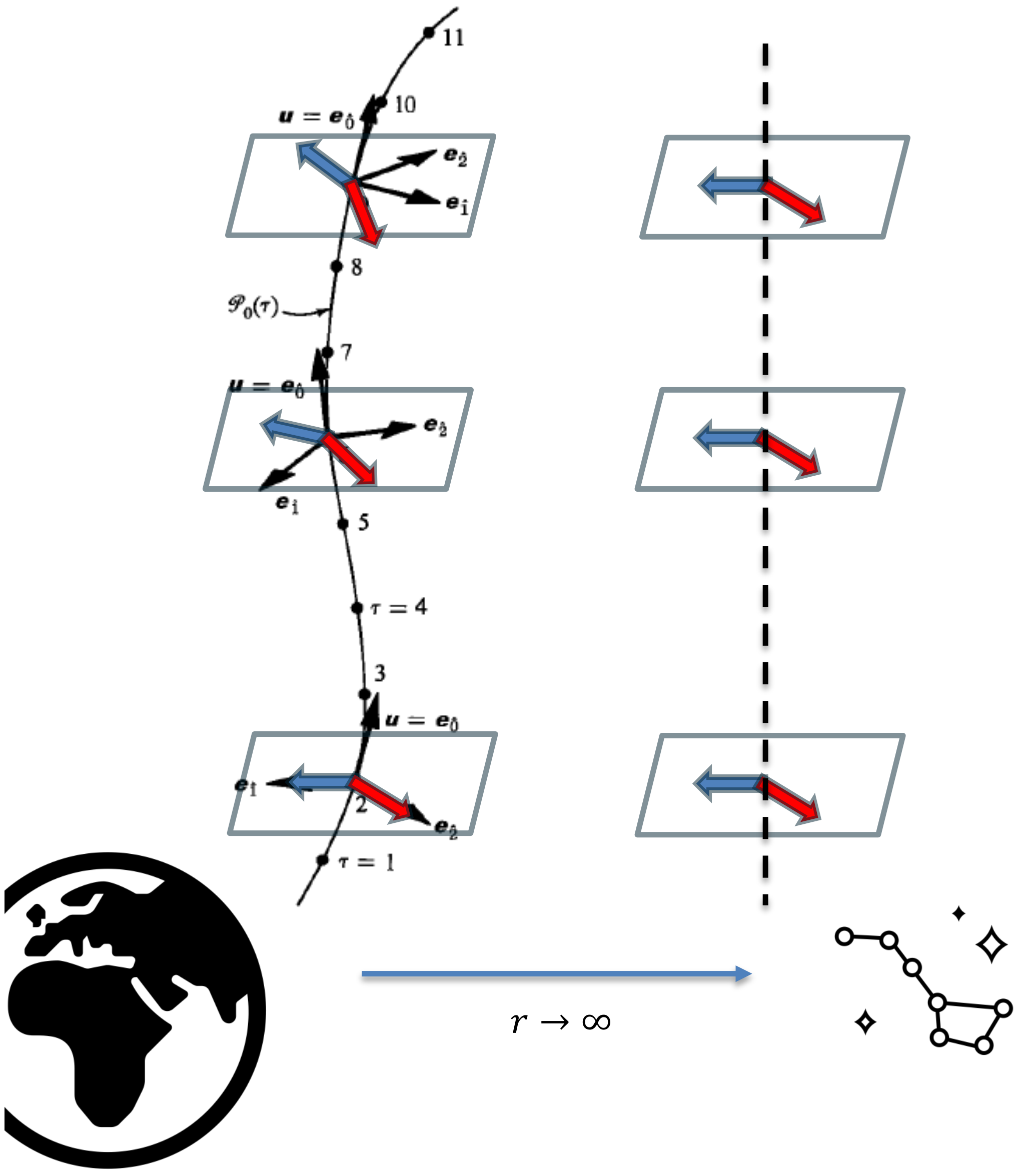
up to v^3 or r^{-2} order $\Downarrow$



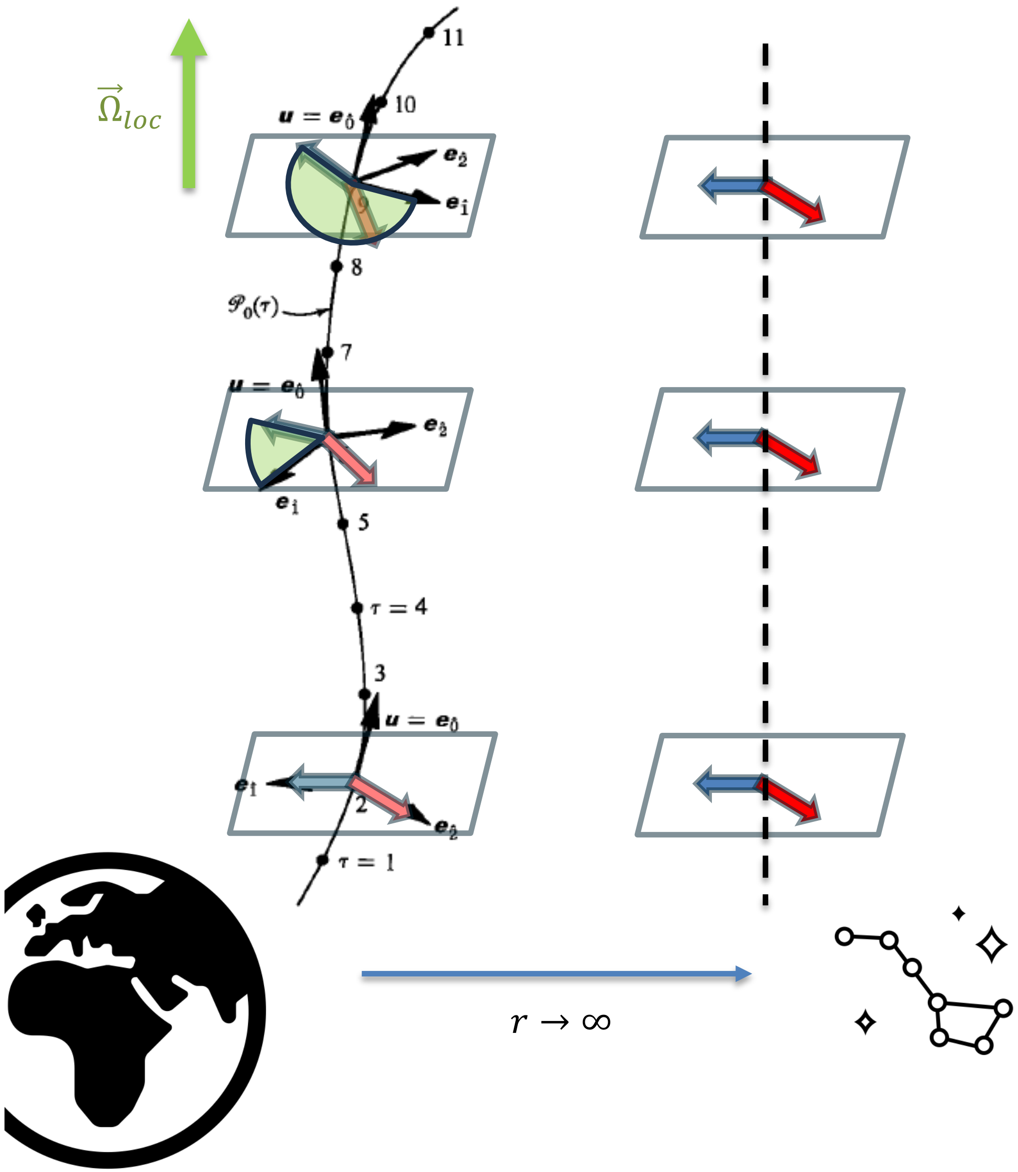
$$\frac{\mathrm{d}(\vec{S} \cdot \vec{S})}{\mathrm{d}\tau} = 0 \qquad \frac{\mathrm{d}\vec{S}}{\mathrm{d}t} = \vec{\Omega}_{prec} \times \vec{S}$$

gyros spin at a constant rate

this precession angular velocity is relative to distant stars

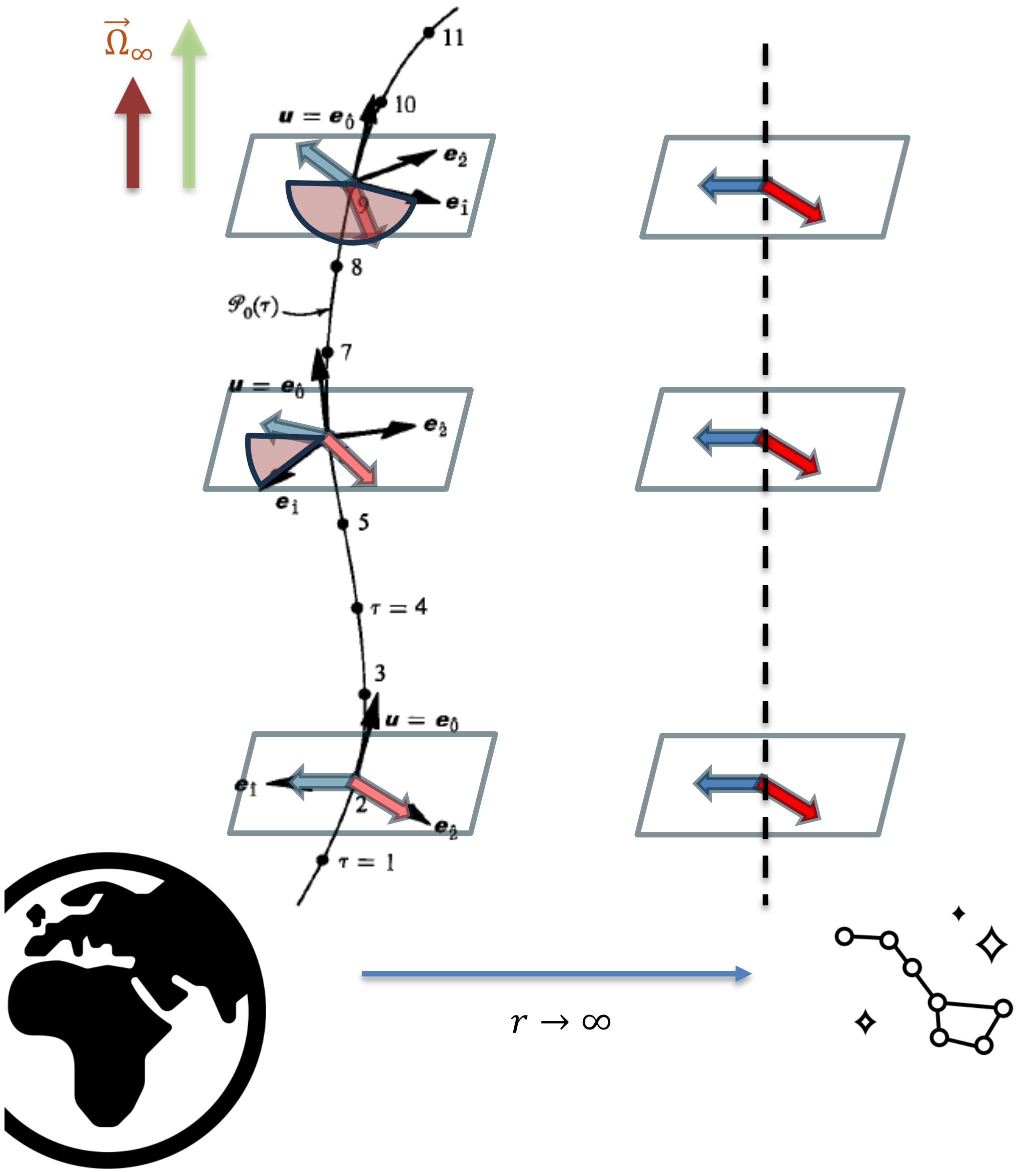


Consider an arbitrary observer carrying a set of inertial guidance gyros with mutually orthogonal axis.



Consider an arbitrary observer carrying a set of inertial guidance gyros with mutually orthogonal axis.

The local rotation vector of the observer's proper frame, point by point along the worldline, is defined as the rotation vector of its spatial axis relative to the inertial guidance gyros, i.e. rotation of an arbitrary observer is unambiguously locally defined as the failure of the proper frame basis vectors to be Fermi-Walker transported along its worldline.



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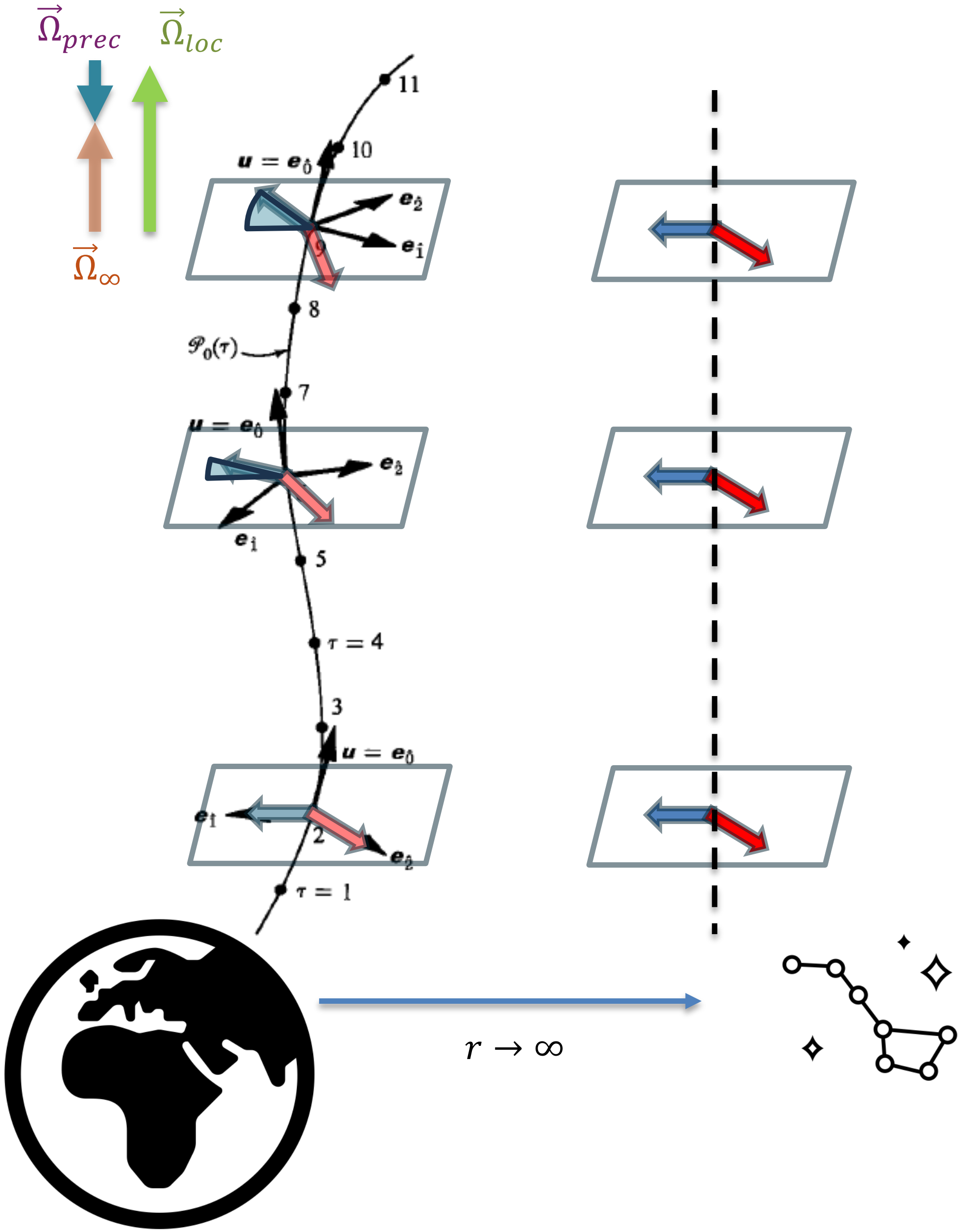
If the spacetime is asymptotically flat, we have an alternative «non-local» definition of rotation relative to the class of ideal asymptotic inertial frames.



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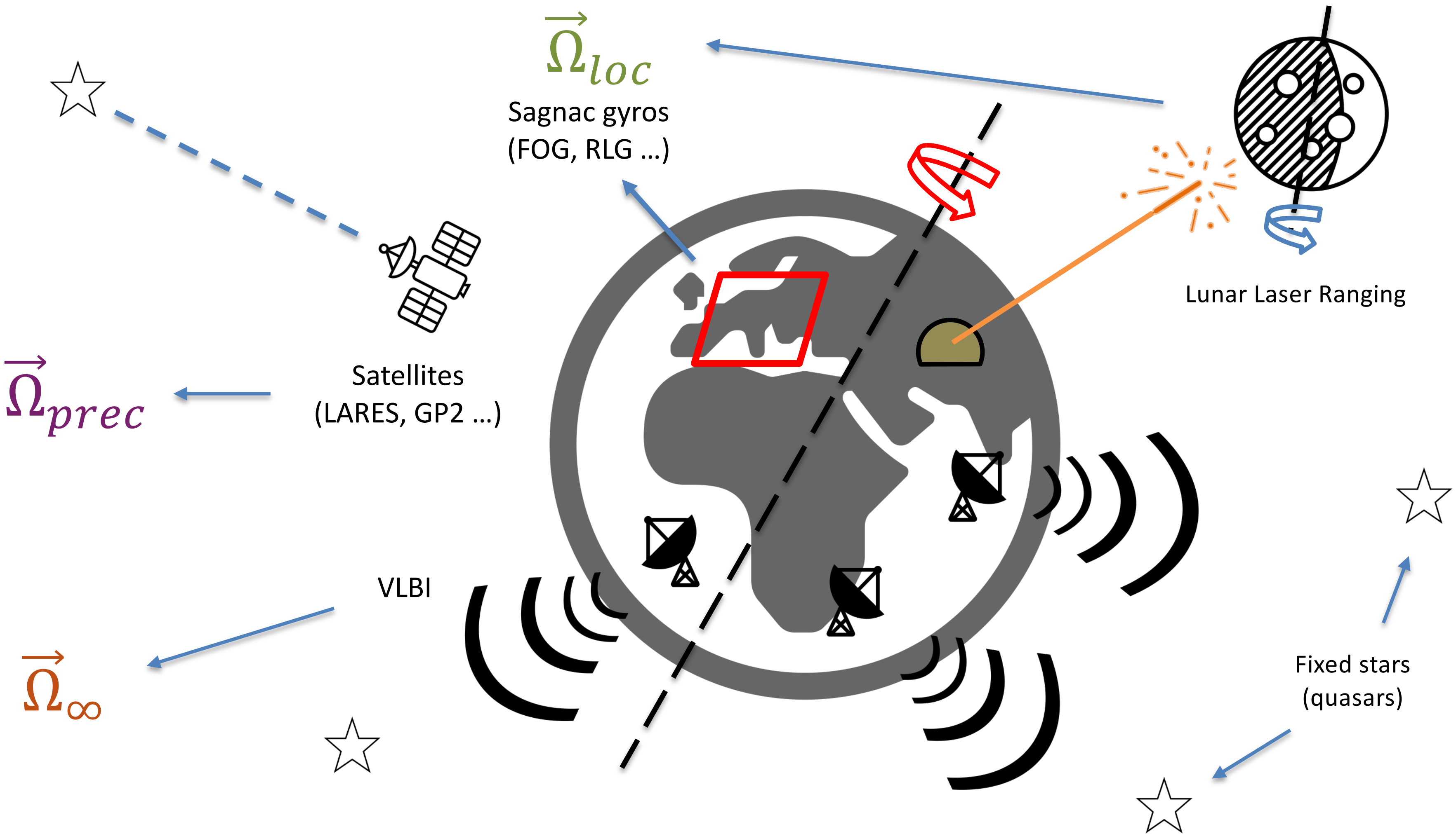
The two definitions are inequivalent:

$$\vec{\Omega}_{\infty} - \vec{\Omega}_{loc} = \vec{\Omega}_{prec}$$



$\overrightarrow{\Omega}_{loc}$
Sagnac gyros
(FOG, RLG …)
Lunar Laser Ranging
$\overrightarrow{\Omega}_{prec}$
Satellites
(LARES, GP2 …)
VLBI
$\overrightarrow{\Omega}_{\infty}$
Fixed stars
(quasars)

Gyroscope precession

$$\vec{\Omega}_{prec} = \vec{\Omega}_{Th} + \vec{\Omega}_{dS} + \vec{\Omega}_{LT}$$

Thomas precession

de Sitter precession

Lense-Thirring precession

This effect is present also in flat spacetime. It is a manifestation of **Wigner rotation** due the composition of two infinitesimally separated Lorentz boosts in different spatial directions.

$$\vec{\Omega}_{Th} = -\frac{1}{2c^2}\vec{v} \times \vec{a}$$

velocity relative to coordinate grid

proper acceleration

It is also called **geodesic effect**. It manifests itself when the gyroscope **moves** in the **gravitational field** of a massive body. It only depends on the mass multipoles, so it is present also for **static** sources.

It is a manifestation of **frame-dragging** , i.e. a distortion of the spacetime that modifies the inertia of test-bodies nearby **rotating sources**. It depends on current multipoles and it affects also gyros placed at a fixed location.

Gravitomagnetic effects

Gravitoelectromagnetism (GEM)

Linearized General Relativity
in Lorenz gauge

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G_N}{c^4} T_{\mu\nu}$$

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

$$\partial_\nu \left(h^{\mu\nu} - \frac{1}{2} \eta^{\mu\nu} h \right) = 0$$

$$\Rightarrow$$

$$\vec{\nabla} \cdot \vec{\mathrm{E}} = -4\pi G_N \rho_m$$

$$\vec{\nabla} \times \vec{\mathrm{E}} + \frac{1}{c} \frac{\partial}{\partial t} \left(\frac{1}{2} \vec{\mathrm{B}} \right) = 0$$

$$\vec{\nabla} \cdot \left(\frac{1}{2} \vec{\mathrm{B}} \right) = 0$$

$$\vec{\nabla} \times \left(\frac{1}{2} \vec{\mathrm{B}} \right) - \frac{1}{c} \frac{\partial \vec{\mathrm{E}}}{\partial t} = -\frac{4\pi G_N}{c} \vec{j}_m$$

$$\vec{\mathrm{E}} = -\vec{\nabla}\Phi - \frac{1}{c} \frac{\partial}{\partial t} \left(\frac{1}{2} \vec{\mathrm{A}} \right) \qquad \vec{\mathrm{B}} = \vec{\nabla} \times \vec{\mathrm{A}}$$

$$\mathrm{d}s^2 = -\left(1 + \frac{2\Phi}{c^2} \right) c^2 \mathrm{d}t^2 + \frac{4}{c} \vec{\mathrm{A}} \cdot \mathrm{d}\vec{x}\mathrm{d}t + \left(1 - \frac{2\Phi}{c^2} \right) \delta_{ij} \mathrm{d}x^i \mathrm{d}x^j$$

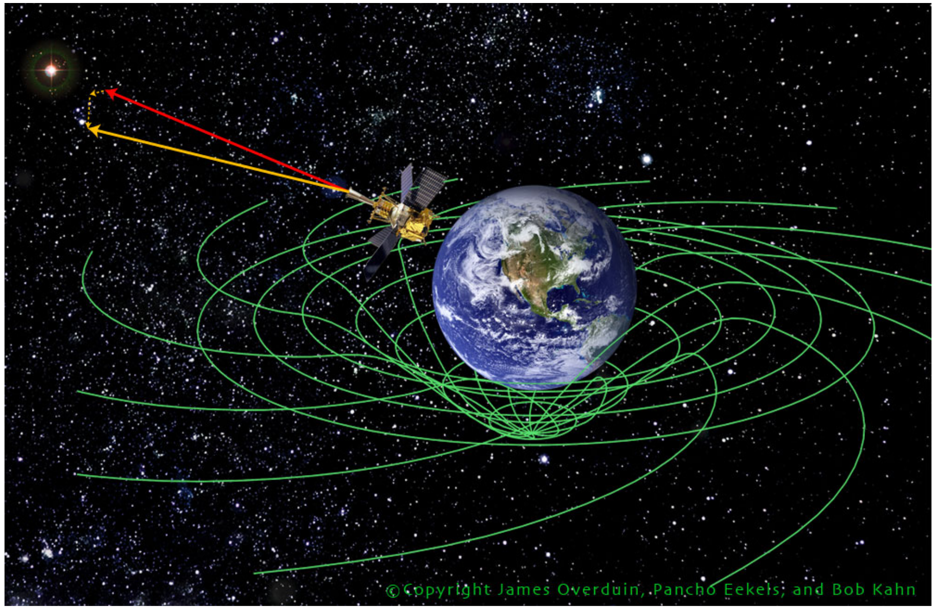
Gravitoelectromagnetism

GEM precession terms are a gravitational analogue of the Larmor precession:

Gravitomagnetic field for an observer at rest

Induced gravitomagnetic field in the comoving frame + spatial curvature correction

$$\vec{\Omega}_{grav} = \underbrace{-\frac{1}{c}\vec{\nabla}\times\vec{A}}_{\vec{\Omega}_{LT}} - \underbrace{\frac{3}{2}\frac{\vec{v}\times\vec{\nabla}\Phi}{c^2}}_{\vec{\Omega}_{dS}}$$



Gravitoelectromagnetism

Discarding all except the lowest non-trivial orders in the multipolar expansion of the source, **General Relativity** the precession vectors assume the following form:

$$\mathbf{\Omega}_{DS}^{(GR)} = -\left(1+\frac{1}{2}\right)\frac{G_N M}{c^2}\frac{\mathbf{v}\times\mathbf{r}}{r^3}$$

→ Vanishing for a gyroscope at rest in the coordinate grid

$$\mathbf{\Omega}_{LT}^{(GR)} = \frac{G_N}{c^2 r^3}\left[-\mathbf{J}+3\frac{(\mathbf{J}\cdot\mathbf{r})\mathbf{r}}{r^2}\right]$$

→ Precession in the same direction of rotation of the source on the axis of rotation of the source, in the opposite direction in the equatorial plane

For a gyroscope at rest **on Earth's surface** both the vectors are contained in the **meridian plane**

For a gyro in free-fall the Thomas precession is vanishing. For a gyroscope at rest on Earth's surface:

$$\mathbf{\Omega}_{Th}^{(GR)} = -\frac{1}{2}\frac{\mathbf{v}}{c^2}\times\nabla\Phi = -\frac{1}{2}\frac{G_N M}{c^2}\frac{\mathbf{v}\times\mathbf{r}}{r^3} \quad \propto \quad \mathbf{\Omega}_{DS}^{(GR)}$$

M	Mass of the source
$\mathbf{r}$	Position relative to the center of mass
$\mathbf{J}$	Angular momentum

Gravitoelectromagnetism

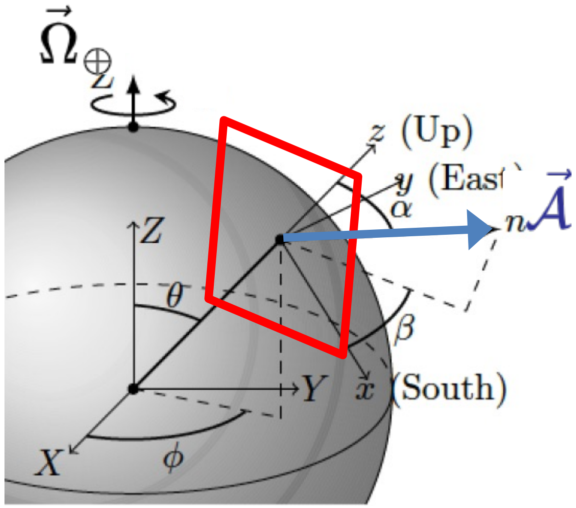
The **Sagnac time delay** predicted by **General Relativity** for an interferometer at rest on Earth's surface is given by:

moment of inertia

$$\Delta\tau_{(GR)} = \frac{4\Sigma}{c^2}\Omega_{\oplus}\left[\cos(\theta+\alpha) + 2\frac{G_N M}{c^2 R_{\oplus}}\sin\theta\sin\alpha - \frac{G_N I_{\oplus}}{c^2 R_{\oplus}^3}(2\cos\theta\cos\alpha + \sin\theta\sin\alpha)\right]$$

Earth's rotation rate as measured at spatial infinity

Earth's radius



In ideal conditions, i.e. neglecting all sources of error (e.g. seismic noise, instrumental instabilities, environmental conditions ecc.) and if Earth's rotation vector is known by other means (e.g. VLBI) **a system comprising two independent Sagnac interferometers is enough to determine and decouple the de Sitter and Lense-Thirring effects** (since their precession vectors are contained in a single plane).

Gravitoelectromagnetism

For a gyroscope at rest on Earth's surface, adopting the **PPN formalism** we have:

$$\mathbf{\Omega}_{LT}^{(PPN)} = -\frac{\mathrm{C}}{4}\mathbf{\Omega}_{LT}^{(GR)} = -\frac{\mathrm{C}}{4}\frac{G_N}{c^2 r^3}\left[-\mathbf{J} + 3\frac{(\mathbf{J}\cdot\mathbf{r})\mathbf{r}}{r^2}\right]$$

$$\mathbf{\Omega}_{DS}^{(PPN)} = -\left(\gamma + \frac{1}{2}\right)\frac{G_N M}{c^2}\frac{\mathbf{v}\times\mathbf{r}}{r^3}$$

$$\mathbf{\Omega}_{Th}^{(PPN)} = \mathbf{\Omega}_{Th}^{(GR)} = -\frac{1}{2}\frac{\mathbf{v}}{c^2}\times\nabla\Phi = -\frac{1}{2}\frac{G_N M}{c^2}\frac{\mathbf{v}\times\mathbf{r}}{r^3}$$

M	Mass of the source
$\mathbf{r}$	Position relative to the center of mass
$\mathbf{J}$	Angular momentum

$$C = 1 + \gamma_{PPN}$$

$$\Delta\tau_{(PPN)} = \frac{4\Sigma}{c^2}\Omega_{\oplus}\left[\cos(\theta+\alpha) + (1+\gamma)\frac{G_N M}{c^2 R_{\oplus}}\sin\theta\sin\alpha + \frac{\mathrm{C}}{4}\frac{G_N I_{\oplus}}{c^2 R_{\oplus}^3}(2\cos\theta\cos\alpha + \sin\theta\sin\alpha)\right]$$

Gravitoelectromagnetism

In order to measure corrections to General Relativity, we need a very high-sensitivity.

Note that the total contribution of de-Sitter and Lense-Thirring effects, as predicted by General Relativity, to the Sagnac time delay in a closed interferometer is nearly 9 orders of magnitude lower than the kinematic effect due to Earth's rotation:

Schwarzschild radius

$$\left|\frac{\Delta\tau^{(RG)}_{(DS)}}{\Delta\tau^{(RG)}_{\oplus}}\right| \approx \frac{1}{2}\frac{r_s}{R_\oplus} \approx 6.96\times10^{-10}$$

$$\left|\frac{\Delta\tau^{(RG)}_{(LT)}}{\Delta\tau^{(RG)}_{\oplus}}\right| \approx \frac{0.3307}{4}\frac{r_s}{R_\oplus} \approx 1.15\times10^{-10}$$

(calculated assuming $\theta = -\alpha\ 45°$)

Case Studies

Horava-Lifshits Gravity

$$S=\int d^3x\,dt\,\sqrt{-g}\left\{\frac{2}{\kappa^2}\left(K_{ij}K^{ij}-\lambda K^2\right)\right.$$
$$\left.-\frac{\kappa^2}{2w^4}\left(\nabla_i R_{jk}\nabla^i R^{jk}-\nabla_i R_{jk}\nabla^j R^{ik}-\frac{1}{8}\nabla_i R\nabla^i R\right)\right\}$$

$$ds^2=N^2dt^2-g_{ij}\left(dx^i+N^idt\right)\left(dx^j+N^jdt\right)$$

$$K_{ij}=\frac{1}{2N}\left(\dot{g}_{ij}-\nabla_i N_j-\nabla_j N_i\right)\qquad K^2=g_{ij}K^{ij}$$

General Scalar-Tensor Gravity

$$S=\int\sqrt{-g}\left[f\left(R,R_{\alpha\beta}R^{\alpha\beta},\phi\right)\right.$$
$$\left.+\,\omega(\phi)\nabla_\alpha\phi\nabla^\alpha\phi\right]d^4x,$$

$$Y\equiv R^{\mu\nu}R_{\mu\nu}$$

$\omega(\phi) \longrightarrow$ Kinetic term: general function of ϕ

Both provide Schwarzschild solution as a particular case

$$ds^2=\left(1-\frac{2GM}{c^2r}\right)c^2dt^2-\left(1-\frac{2GM}{c^2r}\right)^{-1}dr^2-r^2d\theta^2-r^2\mathrm{sen}^2\theta d\phi^2$$

General Scalar-Tensor Gravity

$$S = \int \sqrt{-g}\left[f\left(R, R_{\alpha\beta}R^{\alpha\beta}, \phi\right) + \omega(\phi)\nabla_\alpha\phi\nabla^\alpha\phi\right] d^4x$$

Field equations

$$f_R R_{\mu\nu} - \frac{f + \omega(\phi)\nabla^\alpha\phi\nabla_\alpha\phi}{2} g_{\mu\nu} - \nabla_\mu\nabla_\nu f_R + g_{\mu\nu}\Box f_R + 2 f_Y R_\mu^\alpha R_{\alpha\nu}$$
$$-2 f_Y \left(\nabla_\alpha\nabla_\nu R_\mu^\alpha + \nabla_\alpha\nabla_\mu R_\nu^\alpha\right) + \Box\left(f_Y R_{\mu\nu}\right) + g_{\mu\nu}\nabla_\beta\nabla_\alpha\left(f_Y R^{\alpha\beta}\right) + \omega(\phi)\nabla_\mu\phi\nabla_\nu\phi = 0$$

Klein-Gordon equation

$$2\omega(\phi)\Box\phi + \omega_\phi(\phi)\nabla_\alpha\phi\nabla^\alpha\phi - f_\phi = 0$$

Properties:

- **Explain late and early time evolution without DM and DE**
- **Fit the experimental observations at the astrophysical level**

Horava-Lifshitz Gravity

Properties:

- **Theory of quantum gravity capable of solving some small-scale shortcomings of GR**
- **Lorentz-Invariance emerges at large distances**
- **Saccessfully passes the Solar System Tests**

***A possible* spherically symmetric solution:**

$$g_{00} = (g_{11})^{-1} = 1 + \omega r^2 - \omega r^2 \sqrt{1 + \frac{4M}{\omega r^3}}$$

$$\omega \longrightarrow \text{Constant}$$

Schwarzschild solution: $4M/\omega r^3 \ll 1$

However…..

Exact spherically symmetric solutions in ETGs are very rare

Weak Field Limit

The Weak-Field limit

Motivations:

Often exact solutions in MOGs cannot be found analytically

The typical values of the Newtonian gravitational potential Φ are larger than 10^{-5} in the Solar System (in geometrized units, Φ is dimensionless).

Scheme:

Linearization of the metric tensor

$$g_{\mu\nu} \sim \begin{pmatrix} 1 + g_{00}^{(2)} + g_{00}^{(4)} + \dots & g_{0i}^{(3)} + \dots \\ g_{0i}^{(3)} + \dots & -\delta_{ij} + g_{ij}^{(2)} + \dots \end{pmatrix}$$

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu}$$

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \qquad |h_{\mu\nu}| \ll 1.$$

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{1}{2}(\partial_\sigma \partial_\mu h^\sigma_\nu + \partial_\sigma \partial_\nu h^\sigma_\mu - \partial_\mu \partial_\nu h - \Box h_{\mu\nu} - \eta_{\mu\nu} \partial_\rho \partial_\lambda h^{\rho\lambda} + \eta_{\mu\nu} \Box h)$$

Some restults provided by PN limit in ETGs

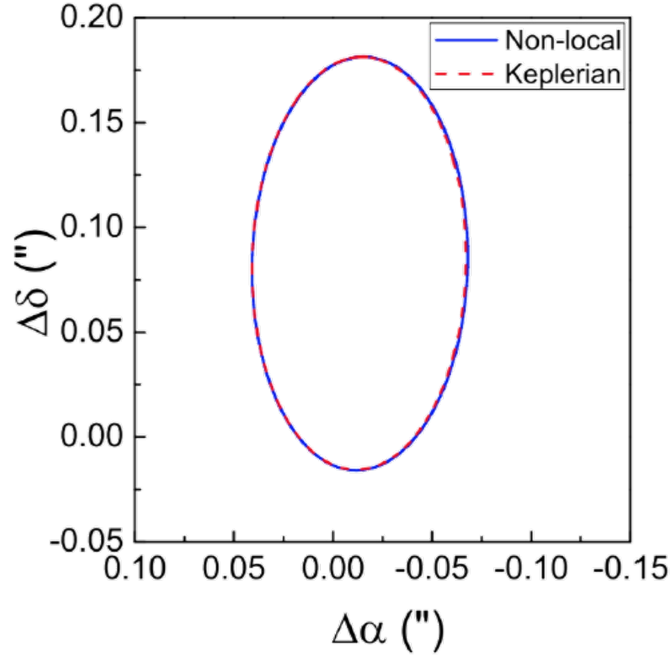
$$g_{00} \sim \mathcal{O}(6), g_{0i} \sim \mathcal{O}(5) \text{ and } g_{ij} \sim \mathcal{O}(4)$$

$$S = \frac{1}{2\kappa^2}\int \sqrt{-g}\left\{R[1+f(\phi)] + \varepsilon(r,t)(\Box\phi - R)\right\}d^4x$$

$$\phi \equiv \Box^{-1}R$$

Non-Local Gravity: after constraining the free parameters, it fits the Keplerian orbit better than GR

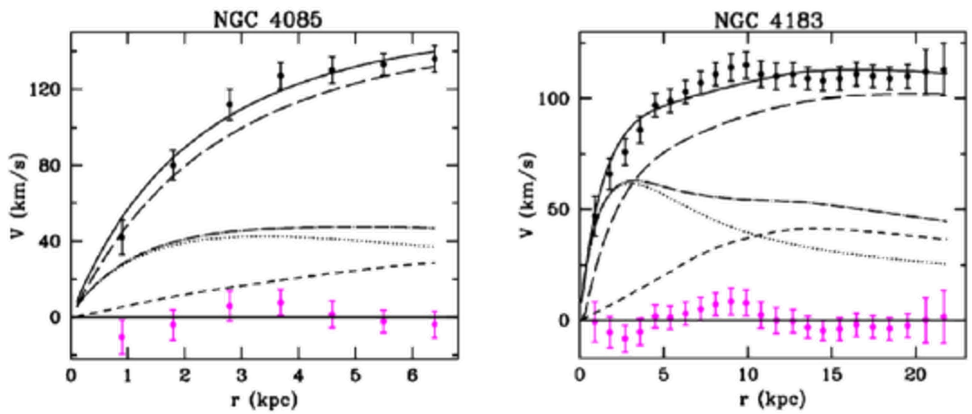
*K. F. Dialektopoulos, D. Borka, S. Capozziello, V. Borka Jovanovic and P. Jovanovic Phys. Rev. D **99** (2019) no.4, 044053 doi:10.1103/PhysRevD.99.044053*



$$g_{00} \sim \mathcal{O}(6), g_{0i} \sim \mathcal{O}(5) \text{ and } g_{ij} \sim \mathcal{O}(4)$$

$$S = \frac{c^4}{16\pi G}\int \sqrt{-g}R^k\, d^4x$$

Galaxy rotation curve for specific values of free parameters. Solid line is the best fit line of the total circular velocity, the dotted and the dashed lines refer to Newtonian contributions of star and gaseous components respectively respectively. The non-Newtonian contribution is labeled by the long-dashed line.



*C. F. Martins and P. Salucci. Mon. Not. Roy. Astron. Soc **381**(2007), 1103-1108 doi:10.1111/j.1365-2966.2007.12273.x*

First case: $f(R, R^{\mu\nu}R_{\mu\nu}, \varphi)$ gravity

Weak field limit for $f(R, R^{\mu\nu}R_{\mu\nu}, \varphi)$ *gravity*

Linearization of the metric tensor

$$g_{\mu\nu} \sim \begin{pmatrix} 1 + g^{(2)}_{00} + g^{(4)}_{00} + \ldots & g^{(3)}_{0i} + \ldots \\ g^{(3)}_{0i} + \ldots & -\delta_{ij} + g^{(2)}_{ij} + \ldots \end{pmatrix} = \begin{pmatrix} 1 + 2\phi + 2\Xi & 2A_i \\ 2A_i & -\delta_{ij} + 2\Psi\delta_{ij} \end{pmatrix}$$

- *Three potentials arise: two scalar potentials and one vector potential*
- *Φ, Ψ are proportional to the power* c^{-2} *(Newtonian limit) while* A_i *is proportional to* c^{-3} *and Ξ to* c^{-4} *(post-Newtonian limit)*

$$\boxed{ds^2 = \mathcal{A}(t,r,\theta)dt^2 + \mathcal{B}(t,r,\theta)dr^2 + \mathcal{C}(t,r,\theta)d\theta^2 + \mathcal{D}(t,r,\theta)\sin^2\theta d\phi^2 + \mathcal{E}(t,r,\theta)dt\,d\phi}$$

$$g_{00} \equiv \mathcal{A}(t,r,\theta)$$
$$g_{0i} = \mathcal{E}(t,r,\theta)$$
$$g_{ij}\delta^{ij} = \mathcal{B}(t,r,\theta) + \mathcal{C}(t,r,\theta) + \mathcal{D}(t,r,\theta)$$

Kerr spacetime

Application of the PN limit to $f(R, R^{\mu\nu}R_{\mu\nu}, \varphi)$ gravity

By means of the decomposition of the metric

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \qquad |h_{\mu\nu}| \ll 1.$$

$$\begin{array}{ccc} h_{00} & \sim & \mathcal{O}(2) \\ h_{0i} & \sim & \mathcal{O}(3) \\ h_{ij} & \sim & \mathcal{O}(2), \end{array}$$

The function f, up to the c^{-4} order, can be developed as:

$$f(R, R_{\alpha\beta}R^{\alpha\beta}, \phi) = f_R(0,0,\phi^{(0)})R + \frac{f_{RR}(0,0,\phi^{(0)})}{2}R^2 + \frac{f_{\phi\phi}(0,0,\phi^{(0)})}{2}(\phi - \phi^{(0)})^2 + f_{R\phi}(0,0,\phi^{(0)})R\phi + f_Y(0,0,\phi^{(0)})R_{\alpha\beta}R^{\alpha\beta},$$

Weak field limit for $f(R, R^{\mu\nu}R_{\mu\nu}, \varphi)$ gravity

***Result**:*

- *Form of the vector potential* →

$$\mathbf{A}(\mathbf{x}) = \frac{G}{|\mathbf{x}|^2}\left[1-(1+m_Y|\mathbf{x}|)\,e^{-m_Y|\mathbf{x}|}\right]\hat{\mathbf{x}}\times\mathbf{J}$$

- *Form of the scalar potential* →

$$\phi(r) = -\frac{GM}{r}\Big[1+g(\xi,\eta)e^{-m_R\tilde{k}_R r} + [1/3-g(\xi,\eta)]e^{-m_R\tilde{k}_\phi r} - \frac{4}{3}e^{-m_Y r}\Big]$$

with the definitions:

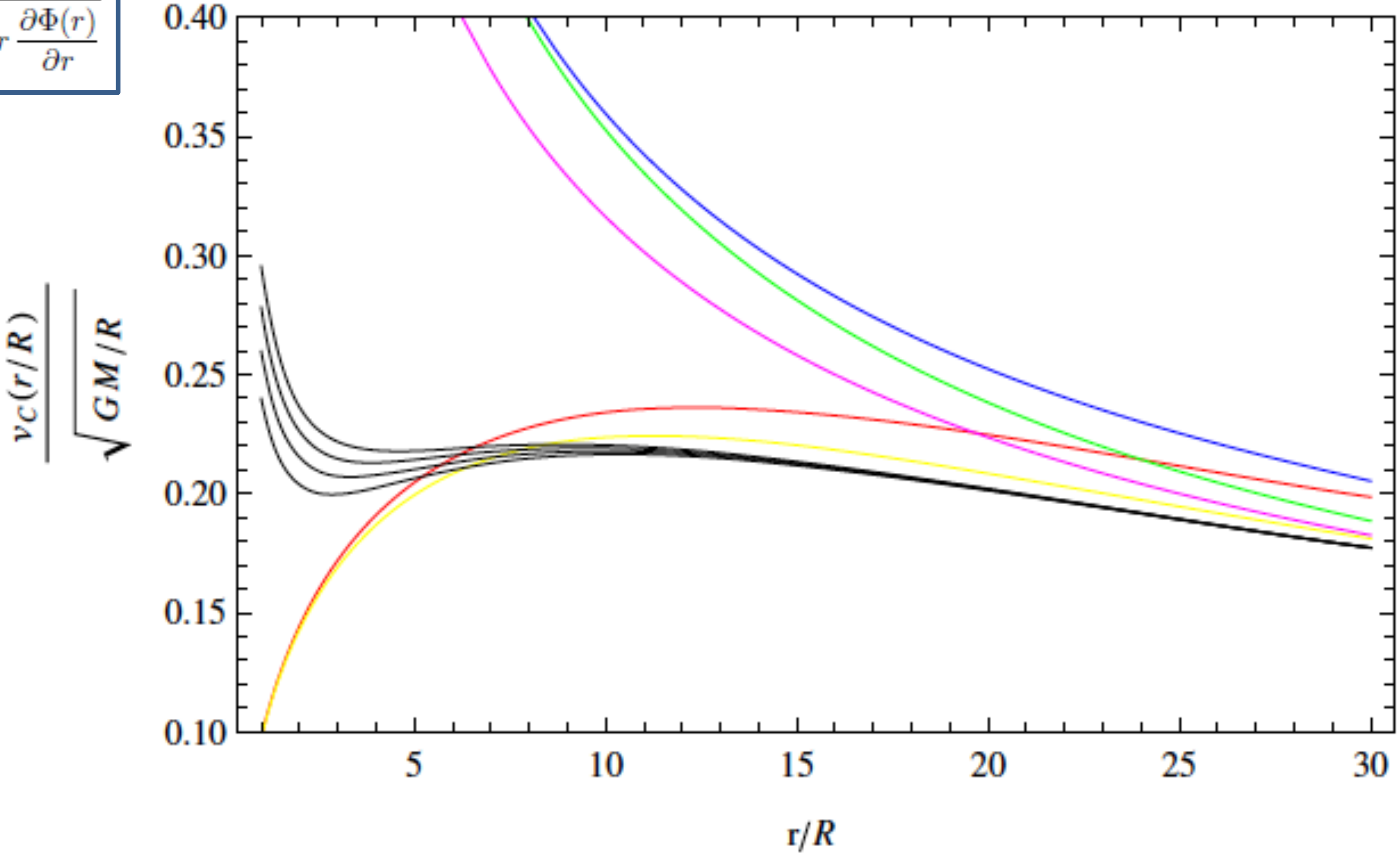
$$m_R^2 = -\frac{1}{3f_{RR}\left(0,0,\phi^{(0)}\right)+2f_Y\left(0,0,\phi^{(0)}\right)}$$

$$m_Y^2 = \frac{1}{f_Y\left(0,0,\phi^{(0)}\right)} \qquad \eta = \frac{m_\phi}{m_R}$$

$$m_\phi^2 = -\frac{f_{\phi\phi}\left(0,0,\phi^{(0)}\right)}{2\omega\left(\phi^{(0)}\right)} \qquad g(\xi,\eta) = \frac{1-\eta^2+\xi+\sqrt{\eta^4+(\xi-1)^2-2\eta^2(\xi+1)}}{6\sqrt{\eta^4+(\xi-1)^2-2\eta^2(\xi+1)}}$$

$$\xi = \frac{3f_{R\phi}\left(0,0,\phi^{(0)}\right)^2}{2\omega\left(\phi^{(0)}\right)} \qquad \tilde{k}_{R,\phi}^2 = \frac{1-\xi+\eta^2\pm\sqrt{\left(1-\xi+\eta^2\right)^2-4\eta^2}}{2}$$

$$v_c(r) = \sqrt{r \frac{\partial \Phi(r)}{\partial r}}$$



The circular velocity of a ball source of mass M and radius R, with the potentials of Table I. We indicate case A by a green line, case B by a yellow line, case D by a red line, case C by a blue line, and the GR case by a magenta line. The black lines correspond to the Sanders model for $-0.95<a<-0.92$. The values of free parameters are $\omega(\phi^{(0)})$.. . $-1/2$,
$\Xi = -5$, $\eta = .3$, $m_Y = 1.5 * m_R$, $m_S = 1.5 * m_R$, $m_R = .1 * R^{-1}$.

Lense-Thirring precession in $f(R, R^{\mu\nu}R_{\mu\nu}, \varphi)$ gravity

$$\boxed{\mathbf{\Omega}_{\mathrm{LT}}^{(\mathrm{EG})} = \frac{1}{2}(\epsilon^{ijk}\partial_i A_k)(\epsilon_{\ell nk}\partial^\ell A^k) = \frac{G}{r^3}\sqrt{(\epsilon_{\ell km}\partial^m \epsilon^{ijk} J_i x_j)^2} = -e^{-m_Y r}\left(1 + m_Y r + m_Y^2 r^2\right)\mathbf{\Omega}_{\mathrm{LT}}^{(\mathrm{GR})}}$$

$$\mathbf{A}(\mathbf{x}) = \frac{G}{|\mathbf{x}|^2}\left[1 - (1 + m_Y|\mathbf{x}|)\, e^{-m_Y|\mathbf{x}|}\right]\hat{\mathbf{x}} \times \mathbf{J} \qquad Y \equiv R^{\mu\nu}R_{\mu\nu}$$

$$\Omega_{LT}^{(GR)} = \frac{r_S}{4Mr^3} J; \qquad m_Y^2 = \frac{1}{f_Y\left(0, 0, \phi^{(0)}\right)}$$

For $f_Y \rightarrow 0$ i.e. $m_Y \rightarrow \infty$, we obtain the same outcome for the gravitational potential of $f(R, \phi)$-theory

Gravitoelectromagnetism in $f(R, R^{\mu\nu}R_{\mu\nu}, \varphi)$ *gravity*

For a gyroscope at rest on Earth's surface in a **higher-order scalar tensor theory** we have:

$$
\begin{aligned}
&\Omega_{DS}^{(ST)} - \Omega_{DS}^{(GR)} = \\
&-\Big[g(\xi,\eta)\left(m_R\tilde{k}_R r + 1\right)F\left(m_R\tilde{k}_R R_\oplus\right)e^{-m_R\tilde{k}_R r} + \frac{8}{3}(m_Y r + 1)\,F\,(m_Y R_\oplus)\,e^{-m_Y r} \\
&+\left(\frac{1}{3} - g(\xi,\eta)\right)\left(m_R\tilde{k}_\phi r + 1\right)F\left(m_R\tilde{k}_\phi R_\oplus\right)e^{-m_R\tilde{k}_\phi r}\Big]\frac{\Omega_{DS}^{(GR)}}{3},
\end{aligned}
$$

$$
\Omega_{LT}^{(ST)} - \Omega_{LT}^{(GR)} = -G_N\frac{e^{-m_Y r}}{r^3}\left[-\mathbf{J}\left(1 + m_Y r + m_Y^2 r^2\right) + 3\frac{(\mathbf{J}\cdot\mathbf{r})\mathbf{r}}{r^2}\left(1 + m_Y r + \frac{1}{3}m_Y^2 r^2\right)\right]
$$

$$
\begin{aligned}
&\Omega_{Th}^{(ST)} - \Omega_{Th}^{(GR)} = \\
&\Big[g(\xi,\eta)\left(m_R\tilde{k}_R r + 1\right)F\left(m_R\tilde{k}_R R_\oplus\right)e^{-m_R\tilde{k}_R r} - \frac{4}{3}(m_Y r + 1)\,F\,(m_Y R_\oplus)\,e^{-m_Y r} \\
&+\left(\frac{1}{3} - g(\xi,\eta)\right)\left(m_R\tilde{k}_\phi r + 1\right)F\left(m_R\tilde{k}_\phi R_\oplus\right)e^{-m_R\tilde{k}_\phi r}\Big]\,\Omega_{Th}^{(GR)}
\end{aligned}
$$

Capozziello et al. PRD, **91** (2015) 044012

Weak-field limit in Horava-Lifshitz Gravity

$$S=\int d^3x\,dt\,\sqrt{-g}\left\{\frac{2}{\kappa^2}\left(K_{ij}K^{ij}-\lambda K^2\right)-\frac{\kappa^2}{2w^4}\left(\nabla_i R_{jk}\nabla^i R^{jk}-\nabla_i R_{jk}\nabla^j R^{ik}-\frac{1}{8}\nabla_i R\nabla^i R\right)\right\}$$

$$K_{ij}=\frac{1}{2N}\left(\dot{g}_{ij}-\nabla_i N_j-\nabla_j N_i\right)\qquad K^2=g_{ij}K^{ij}$$

Linearization of the metric tensor

$$g_{\mu\nu}\sim\begin{pmatrix}1+g_{00}^{(2)}+g_{00}^{(4)}+\dots & g_{0i}^{(3)}+\dots\\ g_{0i}^{(3)}+\dots & -\delta_{ij}+g_{ij}^{(2)}+\dots\end{pmatrix}$$

With similar computations as the previous case, the ratio between the Horava-Lifshitz and General Relativity Gyroscopic precession is

$$\frac{\Omega^G_{HL}}{\Omega^G_{GR}}=\frac{1}{3}\left(1+2\frac{G}{G_N}a_1-2\frac{a_2}{a_1}\right)$$

with $\rightarrow$ a_1, a_2 constants *to be constrained*

$\Omega^G_{HL} \rightarrow$ *Gyroscopic precession*

$G \rightarrow$ effective gravitational constant

Constraining a_1, a_2

$$\frac{\Omega^G_{HL}}{\Omega^G_{GR}} = \frac{1}{3}\left(1 + 2\frac{G}{G_N}a_1 - 2\frac{a_2}{a_1}\right)$$

It has been shown that, in order for the matter coupling to be consistent with solar system tests, the gauge field and the Newtonian potential must be coupled to matter in a specific way, but there are no indication on how to obtain the precise prescription from the action principle. Recently such a prescription has been generalised and a scalar-tensor extension of the theory has been developed to allow the needed coupling to emerge in the IR without spoiling the power-counting renormalizability of the theory.

Matter action

$$S_M = \int dt d^3x \tilde{N}\sqrt{\tilde{g}}\ \mathcal{L}_M\ (\tilde{N}, \tilde{N}_i, \tilde{g}_{ij}; \psi_n)$$

Lapse function

Scalar Potential

$$\begin{aligned} \tilde{N} &= (1 - a_1\sigma)N, \\ \tilde{N}^i &= N^i + N g^{ij}\nabla_j\phi, \\ \tilde{g}_{ij} &= (1 - a_2\sigma)^2 g_{ij}, \end{aligned}$$

Vector

$$\sigma = \frac{A - \mathcal{A}}{N}, \quad \text{with} \quad \mathcal{A} = -\dot{\phi} + N^i\nabla_i\phi + \frac{1}{2}N\nabla^i\phi\nabla_i\phi.$$

a_1, a_2 are then related to the potentials and can be constrained by GINGER measurements

Gravitoelectromagnetism in H-L gravity

For a gyroscope at rest on Earth's surface in **Horava-Lifshitz gravity** we have:

$$\mathbf{\Omega}_{LT}^{(HL)} = \frac{G_{HL}}{G_N}\mathbf{\Omega}_{LT}^{(GR)} = \frac{G_{HL}}{c^2 r^3}\left[-\mathbf{J} + 3\frac{(\mathbf{J}\cdot\mathbf{r})\mathbf{r}}{r^2}\right]$$

$$\mathbf{\Omega}_{DS}^{(HL)} = -\left(\frac{1}{2} + \frac{G_{HL}}{G_N}a_1 - \frac{a_2}{a_1}\right)\frac{G_N M}{c^2}\frac{\mathbf{v}\times\mathbf{r}}{r^3}$$

$$\mathbf{\Omega}_{Th}^{(HL)} = \mathbf{\Omega}_{Th}^{(GR)} = -\frac{1}{2}\frac{G_N M}{c^2}\frac{\mathbf{v}\times\mathbf{r}}{r^3}$$

S. Capozziello, et al., *Eur. Phys. J. Plus* 136 (2021), p. 394

Experimental constraints



Experimental constrains: GP-B

$$\Omega_{\rm LT}^{\rm (EG)} = -e^{-m_Y r}(1 + m_Y r + {m_Y}^2 r^2)\Omega_{\rm LT}^{\rm (GR)}$$ ***and*** $$\Omega_{\rm LT}^{\rm (GR)} = \frac{G}{2r^3}\mathbf{J}$$

$$\boxed{\mathbf{\Omega}_{\rm LT} = \mathbf{\Omega}_{\rm LT}^{\rm (GR)} + \mathbf{\Omega}_{\rm LT}^{\rm (EG)}}$$

The Gravity Probe B (GP-B) four gyroscopes aboard an Earth-orbiting satellite allowed to measure the frame-dragging effect with an error of about 19%. Extended Gravity(EG).

Effect	Measured (mas/y)	Predicted (mas/y)
Geodesic precession	6602 ± 18	6606
Lense-Thirring precession	37.2 ± 7.2	39.2

$$\left|\frac{\Omega^{LT}_{obs} - \Omega^{LT}_{GR}}{\Omega^{LT}_{GR}}\right| = 0.05$$

The changes in the direction of spin gyroscopes, contained in the satellite orbiting at h = 650 km of altitude and crossing directly over the poles, have been measured with extreme precision

Experimental constrains: GP-B

Results:

1) $(1+m_Y r^* + m_Y{}^2 r^{*2})\, e^{-m_Y r^*} \lesssim \frac{\delta|\mathbf{\Omega}_{\rm LT}|}{|\mathbf{\Omega}_{\rm LT}^{\rm (GR)}|} \simeq 0.19$

2) $m_Y \geq 7.3 \times 10^{-7} m^{-1}$

$$\boxed{m_Y^2 = \frac{1}{f_Y\left(0,0,\phi^{(0)}\right)}}$$

$$\mathbf{\Omega}_{\rm LT} = \mathbf{\Omega}_{\rm LT}^{\rm (GR)} + \mathbf{\Omega}_{\rm LT}^{\rm (EG)}$$

$$\Omega_{\rm LT}^{\rm (EG)} = -e^{-m_Y r}(1 + m_Y r + m_Y{}^2 r^2)\Omega_{\rm LT}^{\rm (GR)}$$

Capozziello et al. PRD, **91** (2015) 044012

Experimental constrains: LARES

The Laser Relativity Satellite (LARES) mission of the Italian Space Agency is designed to test the frame dragging and the Lense-Thirring effect, to within 1% of the value predicted in the framework of GR

The body of this satellite has a diameter of about 36.4 cm and weights about 400 kg

It was inserted in an orbit with 1450 km of perigee, an inclination of 69.5 ± 1 degrees and eccentricity 9.54×10^{-4}

It allows to obtain a stronger constraint for m_Y:

$$\boxed{(1 + m_Y r^* + {m_Y}^2 r^{*2}) e^{-m_Y r^*} \lesssim \frac{\delta|\Omega_{\rm LT}|}{|\Omega_{\rm LT}^{\rm (GR)}|} \simeq 0.01}$$

From which we obtain $m_Y \geq 1.2 \times 10^{-6} m^{-1}$

LARES vs GP-B

Summing up, using data from the Gravity Probe B and LARES missions, we obtain constraints on m_Y.

$$\boldsymbol{\Omega}_{\rm LT} = \boldsymbol{\Omega}_{\rm LT}^{(\rm GR)} + \boldsymbol{\Omega}_{\rm LT}^{(\rm EG)} \qquad \Omega_{\rm LT}^{(\rm EG)} = -e^{-m_Y r}(1 + m_Y r + {m_Y}^2 r^2)\Omega_{\rm LT}^{(\rm GR)} \qquad m_Y^2 = \frac{1}{f_Y\left(0,0,\phi^{(0)}\right)}$$

GP-B

$$\dot{m}_Y \geq 7.3 \times 10^{-7} m^{-\dot{1}}$$

LARES

$$m_Y > 1.2 \times 10^{-6} \mathrm{m}^{-1}$$

This result shows that space-based experiments can be used to test extensively parameters of fundamental theories

Perspective:

Put a further limit to the mass by GINGER

GINGER: the case of Horava-Lifshitz Gravity

Terrestrial experiment: GINGER

GINGER measures the difference in frequency of light of two beams circulating in a laser cavity in opposite directions. This translates into a time difference between the right-handed beam propagation time and the left-handed one

$$\boxed{\delta\tau = -2\sqrt{g_{00}}\oint \frac{g_{0i}}{g_{00}}\, ds^i}$$

The difference in time can be linked to ***the Sagnac frequency*** $\boldsymbol{\Omega_S}$, *measured by GINGER*

Perimeter

Laser wavelength

$$c\delta\tau = N\underbrace{(\lambda_+ - \lambda_-)}_{\text{Wavelength difference}} = Nc\left(\frac{f_- - f_+}{f^2}\right) = \frac{P\lambda}{c}\,\delta f \equiv \frac{P\lambda}{c}\,\Omega_S$$

Splitting in terms of frequency between the two beams

GINGER in Horava-Lifshitz Gravity

$$\delta\tau = -2\sqrt{g_{00}}\oint \frac{g_{0i}}{g_{00}}\,ds^i \Longleftrightarrow \Omega_S = -\frac{2c^2\sqrt{g_{00}}}{P\lambda}\oint \frac{g_{0i}}{g_{00}}\,ds^i$$

In Horava-Lifshitz gravity, it is

$$\Omega_S = \frac{4A}{P\lambda}\Omega_E\left[\cos(\theta+\alpha) - \left(1+\frac{G}{G_N}a_1 - \frac{a_2}{a_1}\right)\frac{GM}{c^2R}\sin\alpha\sin\theta - \frac{GI_E}{c^2R^3}(2\cos\theta\cos\alpha + \sin\theta\sin\alpha)\right]$$

Sagnac term

Lense-Thirring term

- A → Area encircled by the light beams
- α → Angle between the local radial direction and the normal to the plane of the array-laser ring
- θ → Colatitude of the laboratory
- Ω_E → Rotation rate of the Earth as measured in the local reference frame
- I_E → Momentum of Inertia
- P → Perimeter
- λ → Laser wavelength

Horava-Lifshitz vs General Relativity

$G = G_N$

General Relativity

$$\Omega_S = \frac{4A}{P\lambda}\,\Omega_E \quad \left[\cos(\theta+\alpha) - \frac{G_N M}{c^2 R}\sin\alpha\sin\theta - \frac{G I_E}{c^2 R^3}\left(2\cos\theta\cos\alpha + \sin\theta\sin\alpha\right)\right]$$

$G = G_N$

$G = G_N$

Horava-Lifshitz Gravity

$$\Omega_S = \frac{4A}{P\lambda}\,\Omega_E \quad \left[\cos(\theta+\alpha) - \left(1 + \frac{G}{G_N}\,a_1 - \frac{a_2}{a_1}\right)\frac{GM}{c^2 R}\sin\alpha\sin\theta - \frac{G I_E}{c^2 R^3}\left(2\cos\theta\cos\alpha + \sin\theta\sin\alpha\right)\right]$$

Perspectives:

Free parameters constrained by GINGER

$$\Omega_S = \frac{4A}{P\lambda}\,\Omega_E\left[\cos(\theta+\alpha) - \left(1 + \frac{G}{G_N}\,a_1 - \frac{a_2}{a_1}\right)\frac{GM}{c^2R}\sin\alpha\sin\theta - \frac{GI_E}{c^2R^3}\left(2\cos\theta\cos\alpha + \sin\theta\sin\alpha\right)\right]$$

GINGER advantages

- The actual precision of GINGERINO is 1/1000 in the geodesic term, 1/100 in the LT term
- GINGER experiment should overcome such uncertainty providing a precision of 1/1000 in the LT term
- The presence of two rings yields a dynamic measure of the angle **α**

$$\Omega_S = \frac{4A}{P\lambda}\Omega_E\left[\cos(\theta+\alpha) - \underbrace{\left(1+\frac{G}{G_N}a_1 - \frac{a_2}{a_1}\right)\frac{GM}{c^2R}\sin\alpha\sin\theta}_{\text{Geodesic Term}} - \underbrace{\frac{GI_E}{c^2R^3}(2\cos\theta\cos\alpha + \sin\theta\sin\alpha)}_{\text{LT Term}}\right]$$

Notice that:

- *While the measure of the LT term can constrain the value of G, from the measure of the geodesic term we can get the value of a_1 and a_2*
- *The precision of GINGERINO is 10^{-15} rad/s, which corresponds to a precision of 1.4 $\cdot\, 10^{-9}$ with respect to the dominant term.*

Conclusions

- *In the context of metric theories of gravity, linearized field equations can be studied in the weak field limit and small velocities generated by rotating gravitational sources, aimed at constraining the free parameters, which can be modeled as effective masses (or lengths).*

- *The precession of spin of a gyroscope orbiting around a rotating gravitational source can be studied.*

- *Gravitational field gives rise, according to GR predictions, to geodesic and Lense-Thirring processions, the latter being strictly related to the off-diagonal terms of the metric tensor generated by the rotation of the source (Kerr metric)*

- *The gravitational field generated by the Earth can be tested by Gravity Probe B and LARES satellites. These experiments tested the geodesic and Lense-Thirring spin precessions with high precision.*

- *The corrections on the precession induced by scalar, tensor and curvature corrections can be measured and confronted with data.*

- *Earth-based experiments can improve satellite constraints in view to probe alternative theories of gravity (Extended Gravity and Modified Gravity).*

Conclusions

In $f(R, R^{\mu\nu}R_{\mu\nu}, \varphi)$ gravity, GP-B and LARES satellites provide

$$m_Y \geq 7.3 \times 10^{-7} m^{-1} \qquad m_Y > 1.2 \times 10^{-6} \mathrm{m}^{-1}$$

Perspective: constraint on m_y by GINGER

Perspective: constraints on a_1, a_2 by GINGER

In Horava-Lifshitz gravity, the weak-field limit provide

$$c\,\delta\tau = \frac{4A\Omega_E}{c}\left[\cos(\theta+\alpha) - \left(1 + \frac{G}{G_N}a_1 - \frac{a_2}{a_1}\right)\frac{GM}{c^2R}\sin\theta\sin\alpha - \frac{GI_E}{c^2R^3}\left(2\cos\theta\cos\alpha + \sin\theta\sin\alpha\right)\right]$$