PhD DAYS
2025

Complexity in the spectra of graphs and strings

Lorenzo Grimaldi
lorenzo.grimaldi@roma2.infn.it

Supervisor: Prof. Massimo Bianchi

Co-supervisor: Prof. Roberto Benzi

TOR VERGATA
UNIVERSITÀ DEGLI STUDI DI ROMA

INFN
Istituto Nazionale di Fisica Nucleare

CENTRO RICERCHE ENRICO FERMI

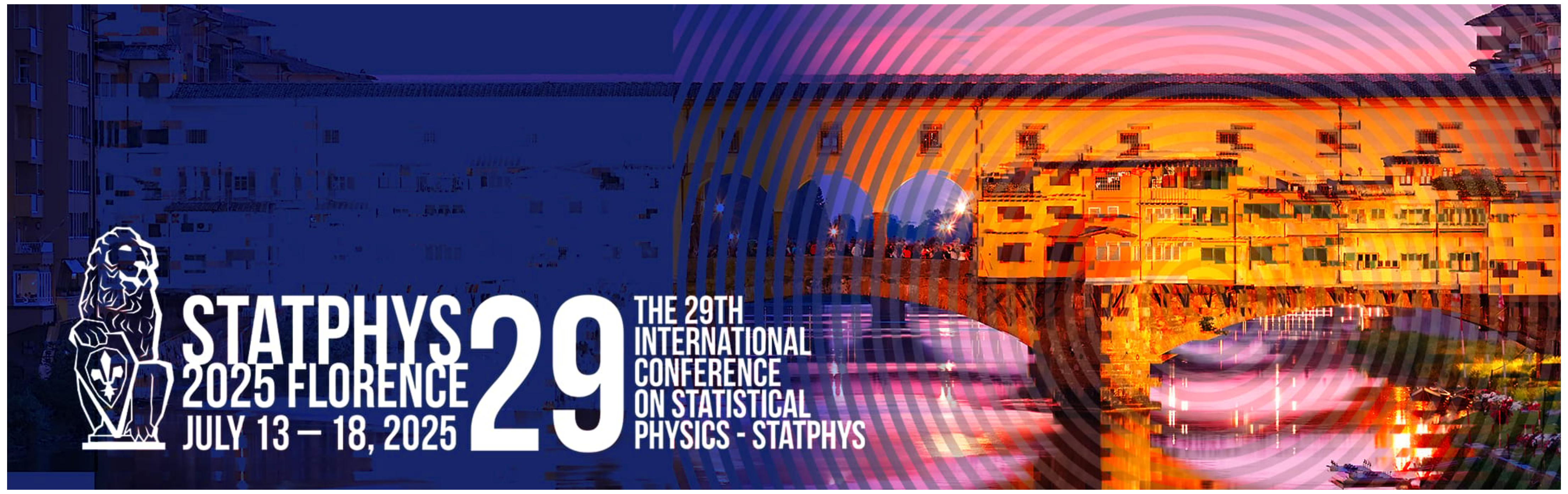
Part I

Unveiling the dimensionality of networks of networks

Based on

LG, P. Villegas, A. Vezzani, R. Burioni, D. Cassi, A. Gabrielli, *in preparation* (2025)

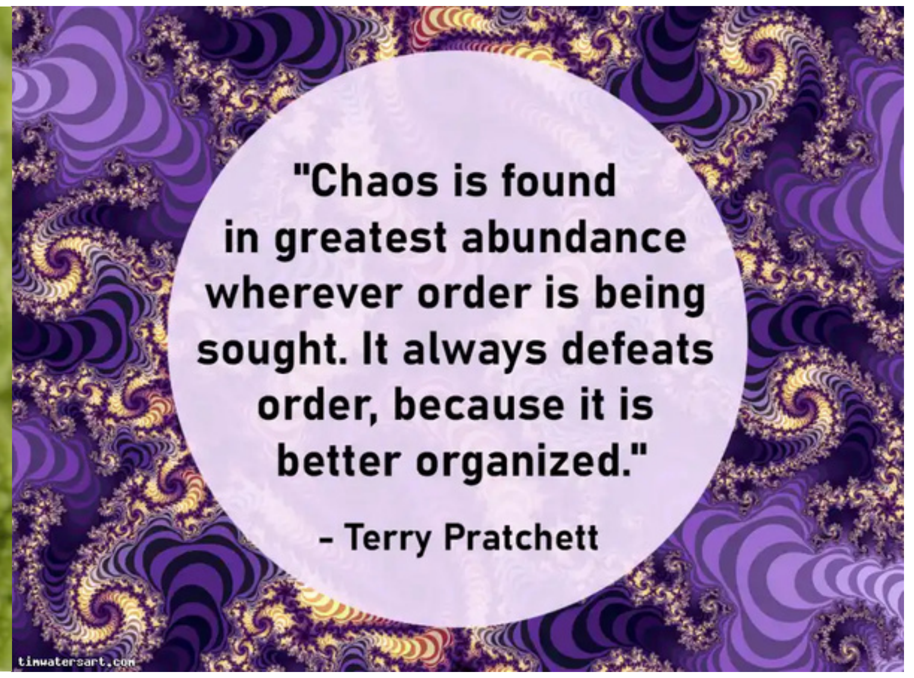
Presented through oral contributions in



International Workshop on Complex Networks, satellite event to StatPhys29, Venice, Italy

Complex Networks: from socio-economic systems to biology and the brain, Lipari, Italy

NATURE IS FRACTAL...

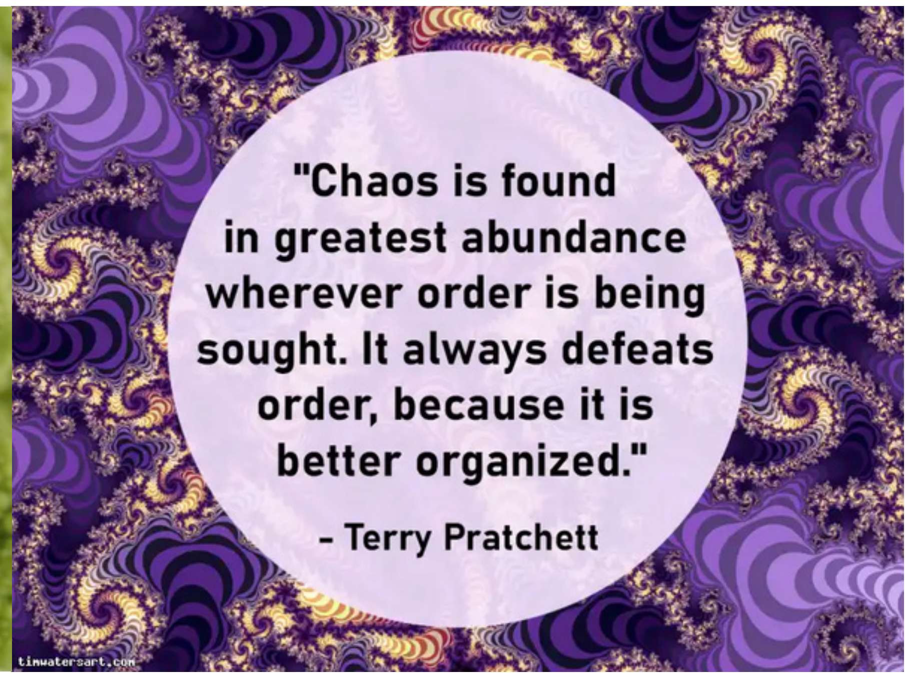


[https://www.treehugger.com/amazing-fractals-found-in-nature-4868776]

[https://timwatersart.com]

MOTIVATIONS

NATURE IS FRACTAL... AND COMPOSITE!



[https://www.treehugger.com/amazing-fractals-found-in-nature-4868776]

[https://timwatersart.com]

Evolutionary tinkering [Jacobs, 1977]

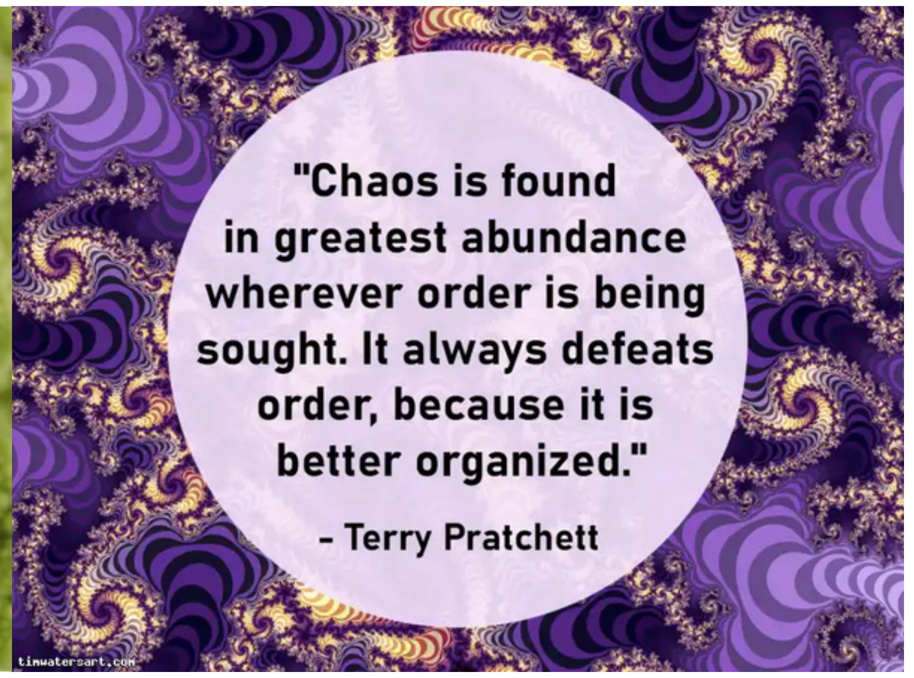
Nature functions by integration. Whatever the level, the objects analyzed by natural sciences are always organizations, or systems. Each system at a given level uses as ingredients some systems of the simpler level, but some only. The hierarchy in the complexity of objects is thus accompanied by a series of restrictions and limitations. At each level, new

environmental challenges. It is natural selection that gives direction to changes, orients chance, and slowly, progressively produces more complex structures, new organs, and new species. Novelties come from previously unseen association of old material. To create is to recombine.

Evolution does not produce novelties from scratch. It works on what already exists, either transforming a system to give it new functions or combining several systems to produce a more elaborate one. This happened, for instance, during one of the main events of cellular evolution: namely, the passage from unicellular to multicellular forms. This was a partic-

MOTIVATIONS

NATURE IS FRACTAL... AND COMPOSITE!



[https://www.treehugger.com/amazing-fractals-found-in-nature-4868776]   [https://timwatersart.com]

Diffusion in bundles

Random Walks on Bundled Structures

Davide Cassi and Sofia Regina

Show more

Phys. Rev. Lett. **76**, 2914 – **Published 15 April, 1996**

DOI: https://doi.org/10.1103/PhysRevLett.76.2914

Anomalous diffusion and response in branched systems: a simple analysis

Giuseppe Forte, Raffaella Burioni, Fabio Cecconi and Angelo Vulpiani

Published 23 October 2013 • © 2013 IOP Publishing Ltd

Journal of Physics: Condensed Matter, Volume 25, Number 46

Citation Giuseppe Forte *et al* 2013 *J. Phys.: Condens. Matter* **25** 465106

DOI 10.1088/0953-8984/25/46/465106

Article   https://doi.org/10.1038/s41567-022-01866-8

Laplacian renormalization group for heterogeneous networks

Received: 20 March 2022

Accepted: 2 November 2022

Pablo Villegas[1], Tommaso Gili[2], Guido Caldarelli[3,4,5,6,7] & Andrea Gabrielli[1,8]

Evolutionary tinkering [Jacobs, 1977]

Nature functions by integration. Whatever the level, the objects analyzed by natural sciences are always organizations, or systems. Each system at a given level uses as ingredients some systems of the simpler level, but some only. The hierarchy in the complexity of objects is thus accompanied by a series of restrictions and limitations. At each level, new

environmental challenges. It is natural selection that gives direction to changes, orients chance, and slowly, progressively produces more complex structures, new organs, and new species. Novelties come from previously unseen association of old material. To create is to recombine.

Evolution does not produce novelties from scratch. It works on what already exists, either transforming a system to give it new functions or combining several systems to produce a more elaborate one. This happened, for instance, during one of the main events of cellular evolution: namely, the passage from unicellular to multicellular forms. This was a partic-

We address typical **anomalous behaviours of bundled structures** through *notions* of **dimensionality** of a graph.

THE MATHEMATICAL TOOLKIT

LAPLACIAN RENORMALISATION GROUP

[Villegas, Gili, Caldarelli, Gabrielli, 2023]

Laplacian $\hat{L} = \hat{D} - \hat{A}$

THE MATHEMATICAL TOOLKIT

LAPLACIAN RENORMALISATION GROUP

[Villegas, Gili, Caldarelli, Gabrielli, 2023]

Laplacian $\hat{L} = \hat{D} - \hat{A}$

Heat kernel $e^{-\tau\hat{L}}$

THE MATHEMATICAL TOOLKIT

LAPLACIAN RENORMALISATION GROUP

[Villegas, Gili, Caldarelli, Gabrielli, 2023]

Laplacian $\hat{L} = \hat{D} - \hat{A}$

Heat kernel $e^{-\tau \hat{L}}$

Canonical ensemble $\hat{\rho}(\tau) = \dfrac{e^{-\tau \hat{L}}}{Z(\tau)}$

THE MATHEMATICAL TOOLKIT

LAPLACIAN RENORMALISATION GROUP

[Villegas, Gili, Caldarelli, Gabrielli, 2023]

Laplacian $\hat{L} = \hat{D} - \hat{A}$

Heat kernel $e^{-\tau \hat{L}}$

Canonical ensemble $\hat{\rho}(\tau) = \dfrac{e^{-\tau \hat{L}}}{Z(\tau)}$

Entropy $S(\tau) = -\,\mathrm{Tr}[\hat{\rho}(\tau) \log \hat{\rho}(\tau)]$

THE MATHEMATICAL TOOLKIT

LAPLACIAN RENORMALISATION GROUP

[Villegas, Gili, Caldarelli, Gabrielli, 2023]

Laplacian $\hat{L} = \hat{D} - \hat{A}$

Heat kernel $e^{-\tau \hat{L}}$

Canonical ensemble $\hat{\rho}(\tau) = \dfrac{e^{-\tau \hat{L}}}{Z(\tau)}$

Entropy $S(\tau) = -\,\mathrm{Tr}[\hat{\rho}(\tau) \log \hat{\rho}(\tau)]$

Heat capacity $C(\tau) \equiv -\dfrac{\mathrm{d}S}{\mathrm{d}\log\tau}$

THE MATHEMATICAL TOOLKIT

LAPLACIAN RENORMALISATION GROUP

[Villegas, Gili, Caldarelli, Gabrielli, 2023]

Laplacian $\hat{L} = \hat{D} - \hat{A}$

Heat kernel $e^{-\tau \hat{L}}$

Canonical ensemble $\hat{\rho}(\tau) = \dfrac{e^{-\tau \hat{L}}}{Z(\tau)}$

Entropy $S(\tau) = -\,\mathrm{Tr}[\hat{\rho}(\tau) \log \hat{\rho}(\tau)]$

Heat capacity $C(\tau) \equiv -\dfrac{\mathrm{d}S}{\mathrm{d}\log\tau}$

Peak in $C(\tau)$ $\Rightarrow$ structural transition

Ability to reveal multi-scale organisation

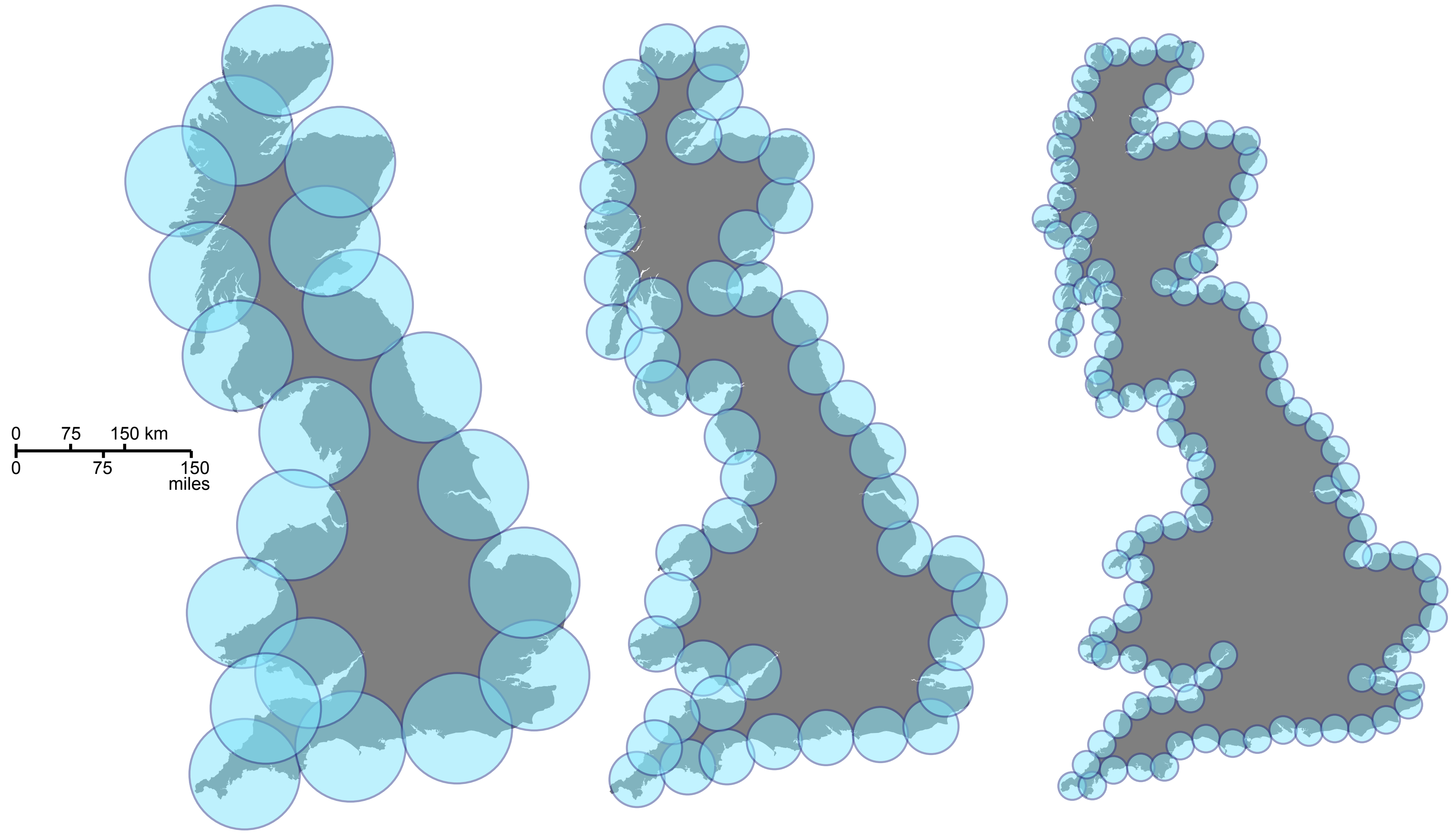
Extension of RG to complex networks

TWO DIFFERENT DIMENSIONS

Fractal dimension d_f

- Related to the **growth rate of neighbouring nodes**.
- The mean mass N of a cluster with radius r satisfies

$$N(r) \sim r^{d_f}$$



[https://commons.wikimedia.org/w/index.php?curid=12042048]

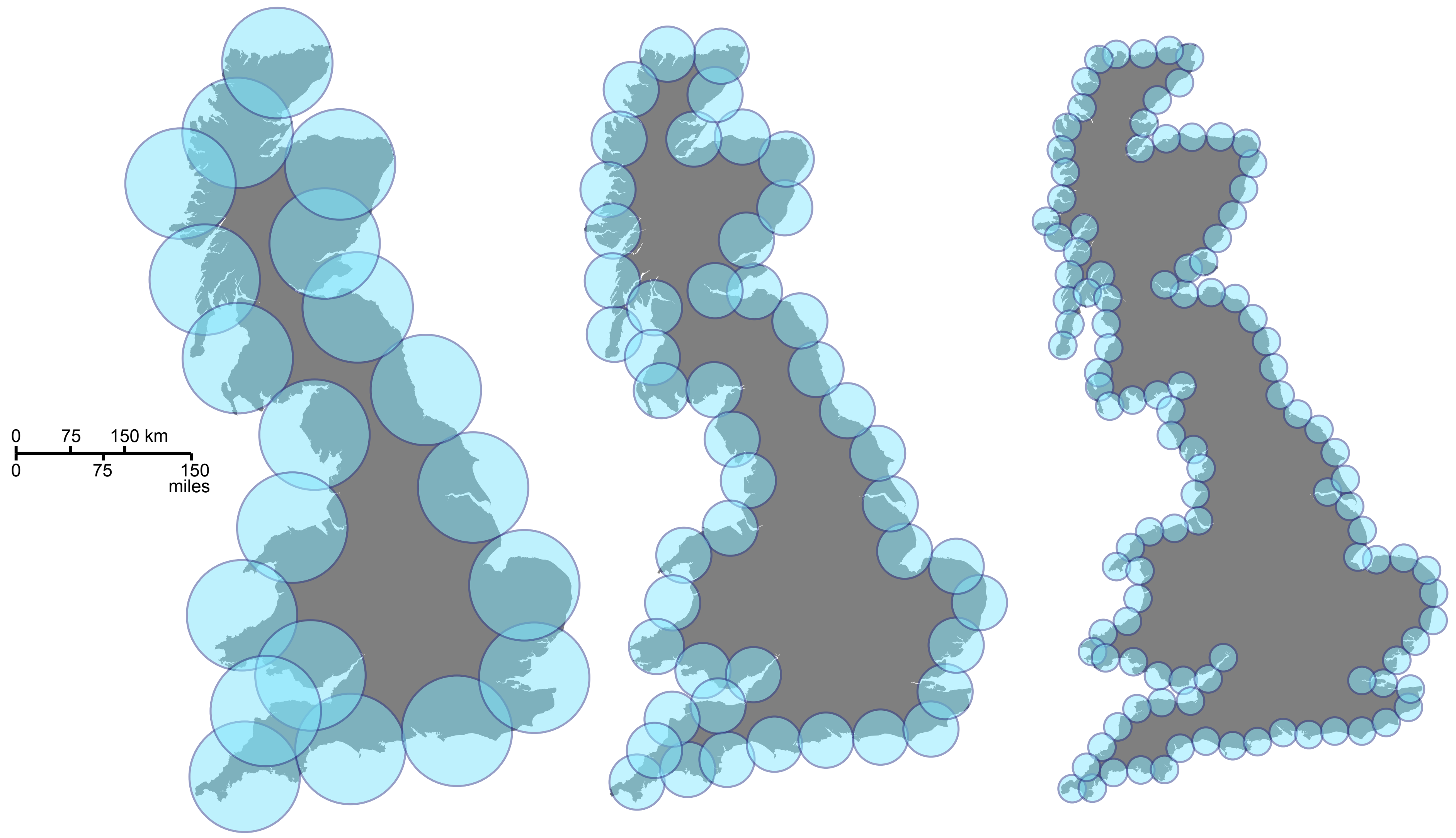
THE MATHEMATICAL TOOLKIT

TWO DIFFERENT DIMENSIONS

Fractal dimension d_f

- Related to the **growth rate of neighbouring nodes**.
- The mean mass N of a cluster with radius r satisfies

$$N(r) \sim r^{d_f}$$



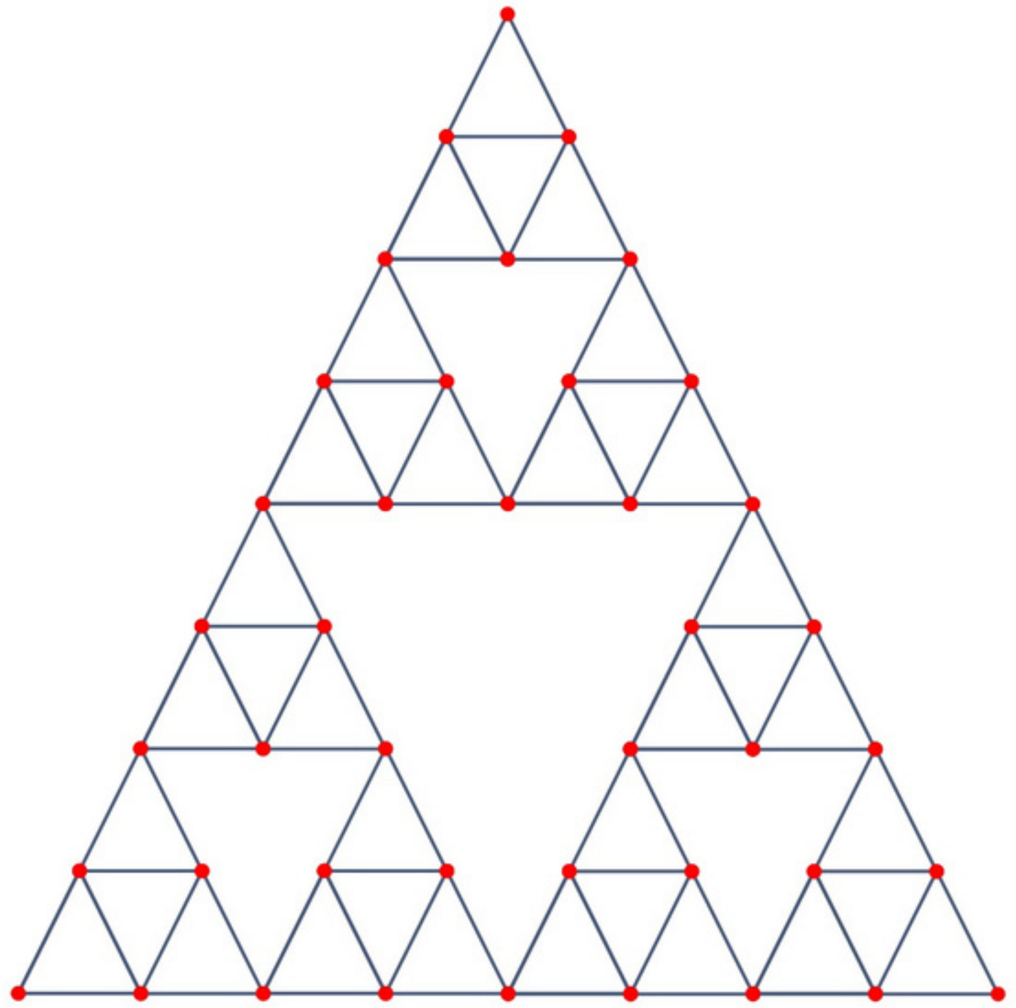
[https://commons.wikimedia.org/w/index.php?curid=12042048]

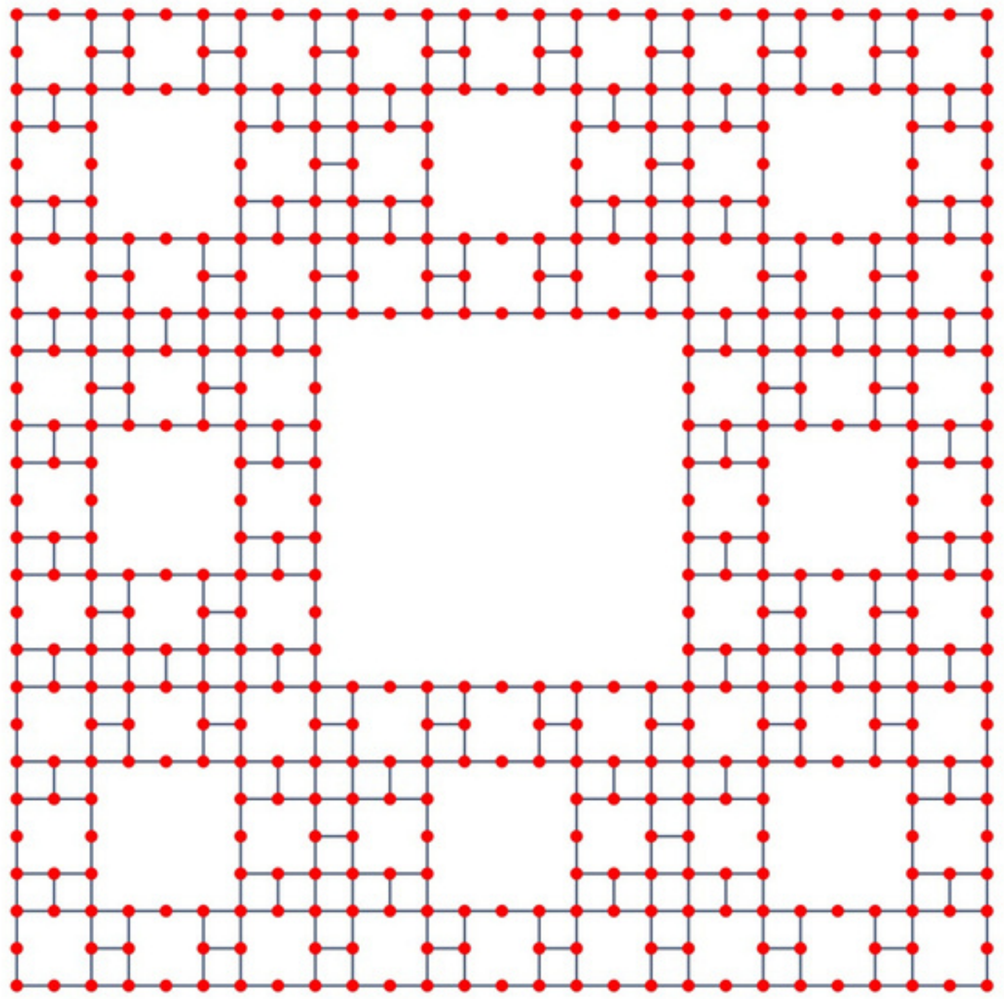
Spectral dimension d_s

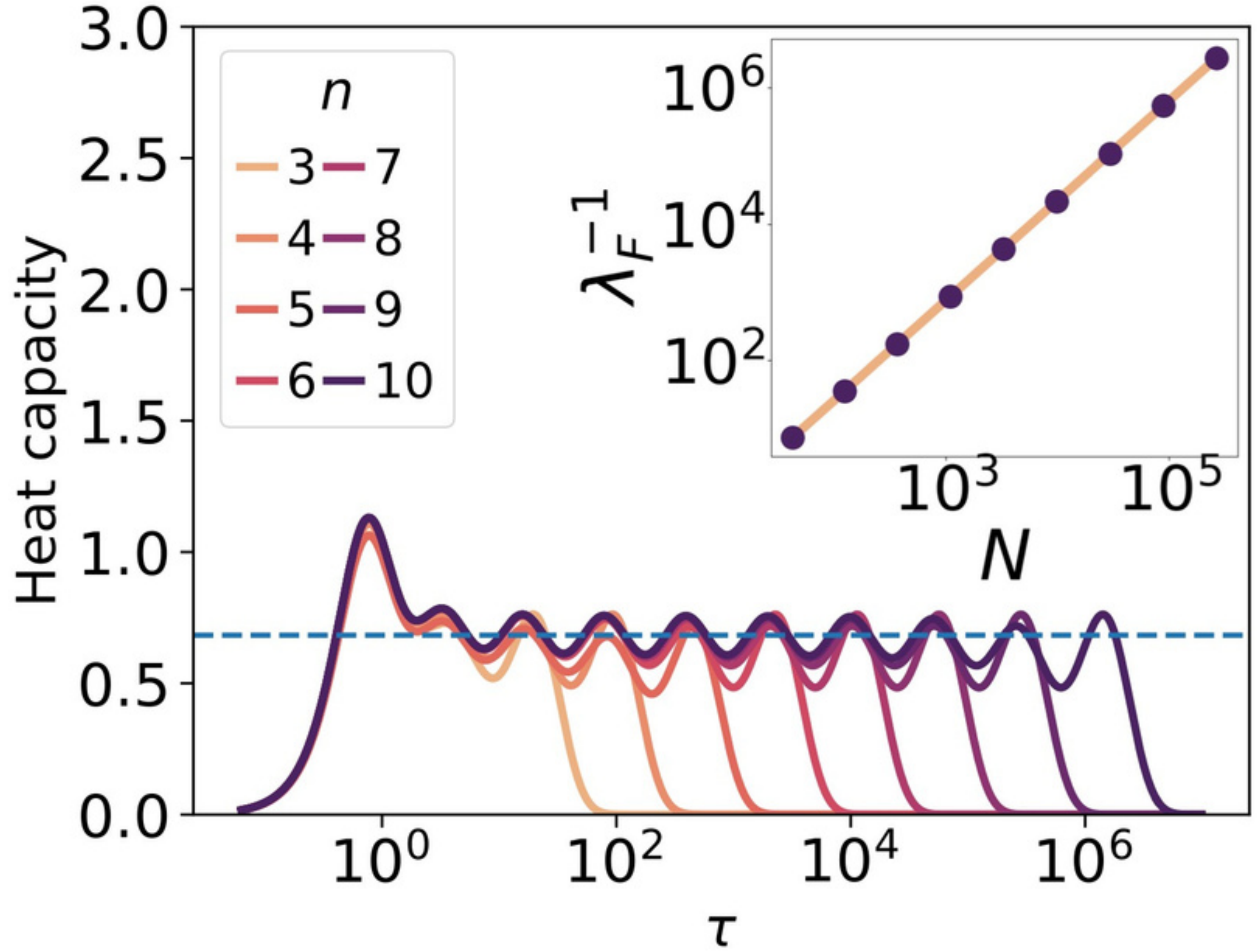
- Related to how fast or slow **diffusion** happens over the manifold.
- The spectral density $P(\lambda)$ of the Laplacian for small eigenvalues satisfies

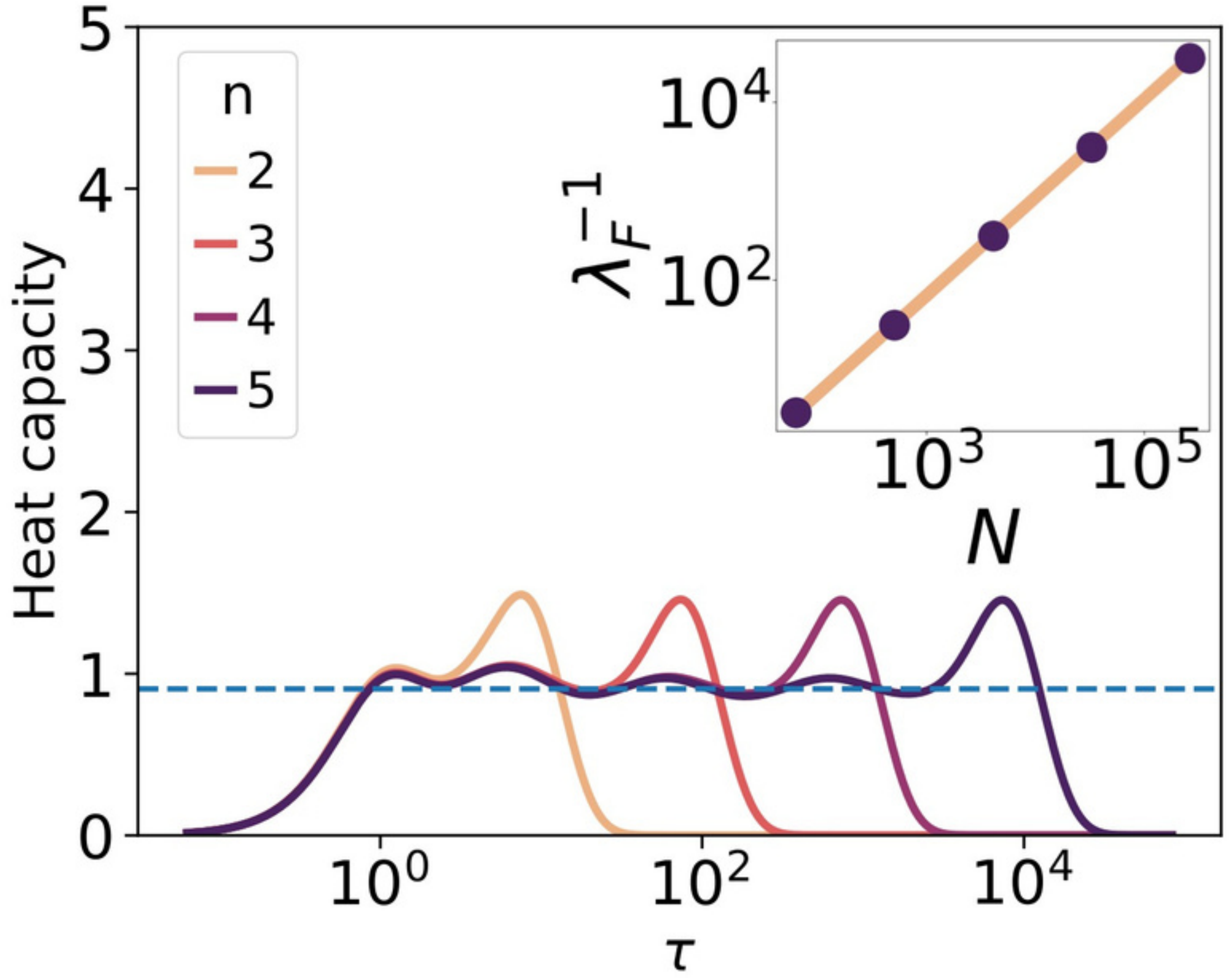
$$P(\lambda) \sim \lambda^{\frac{d_s}{2}-1}$$

THE MATHEMATICAL TOOLKIT
SCALE INVARIANCE









Constant heat capacity $\longrightarrow$ **scale-invariant sector**

Value of the plateau $\longrightarrow$ $\dfrac{d_s}{2}$

Scaling of the **Fiedler eigenvalue**:

$$\lambda_F \sim N^{-\frac{2}{d_s}} \quad \text{as} \quad N \to +\infty$$

[Burioni, Cassi, Fontana, Vulpiani, 2004]

Sierpinski gasket

$$N(n) = \frac{3}{2}(3^n - 1)$$

$d_s = 2\dfrac{\log 3}{\log 5} \simeq 1.37$ $\qquad d_{s,\text{Fiedler}} = 1.37(1)$

[Hilfer, Blumen, 1984]

Sierpinski carpet

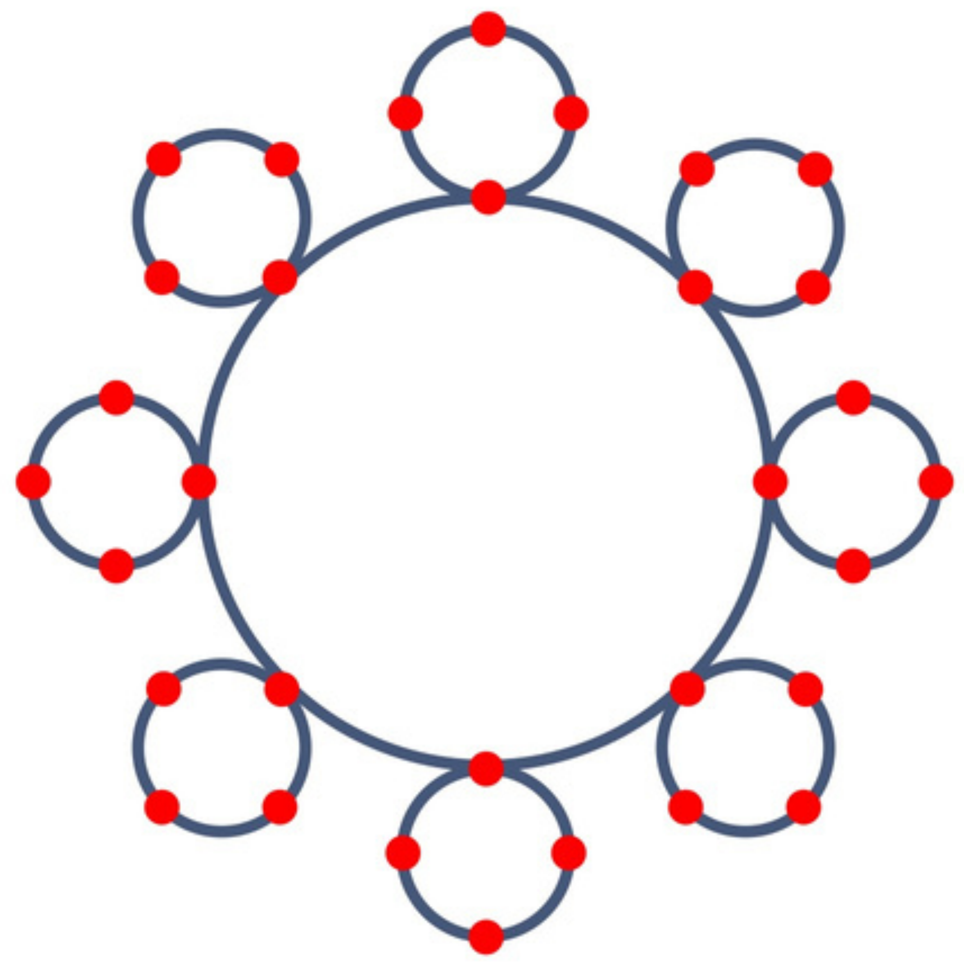
$$N(n) = 2^{n+3}$$

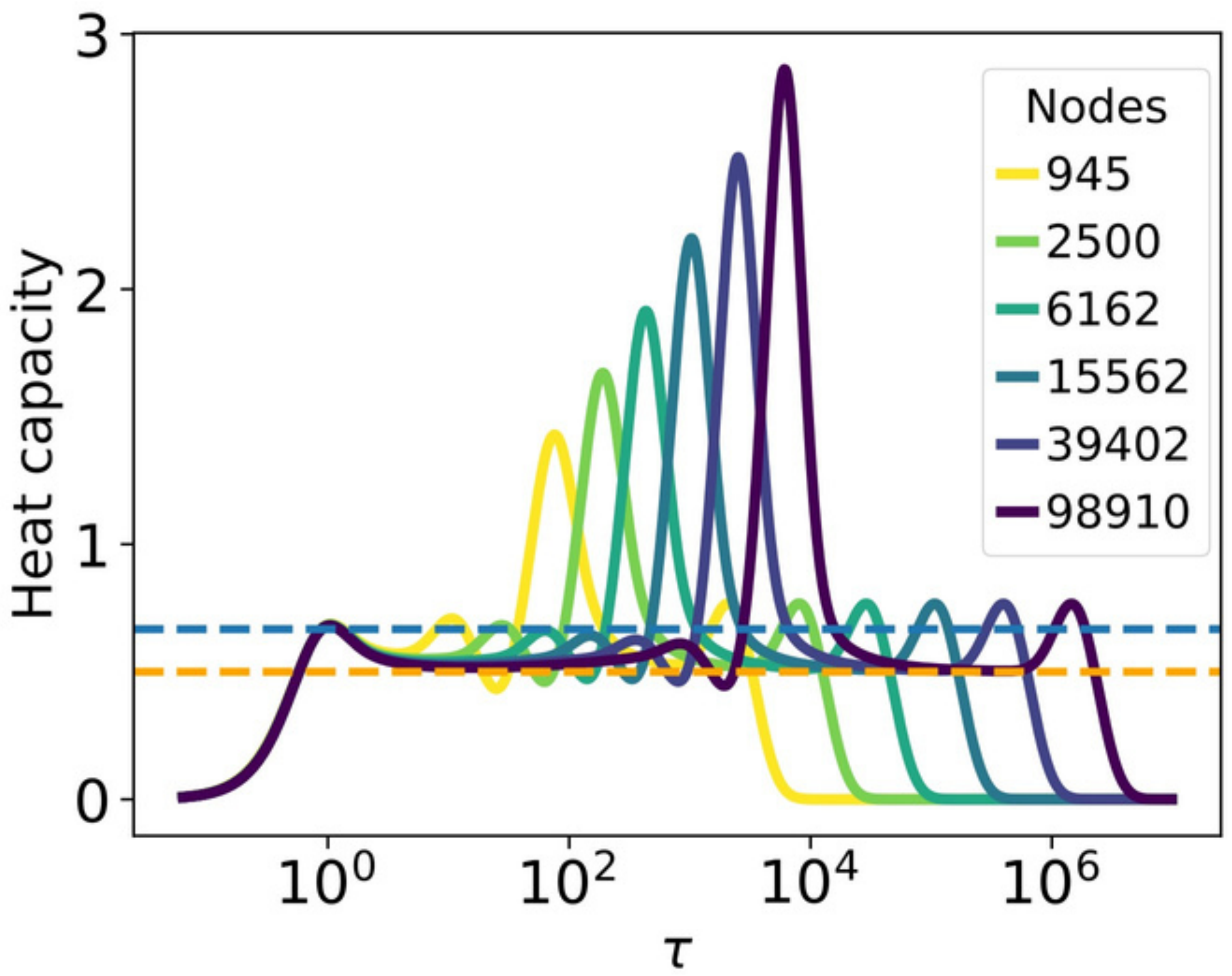
$d_s \approx 1.81$ $\qquad d_{s,\text{Fiedler}} = 1.81(1)$

[Barlow, Bass, Sherwood, 1990]

HETEROGENEOUS STRUCTURES

BUNDLED LATTICES



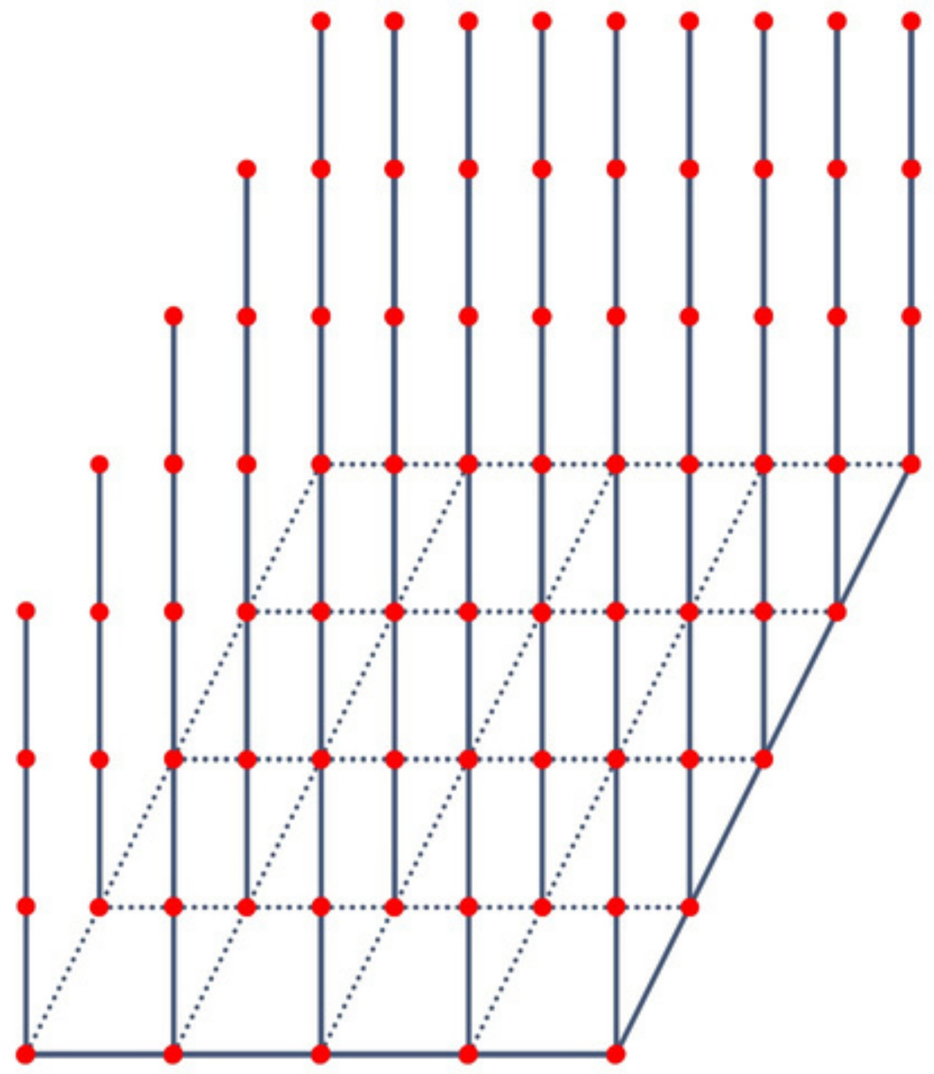


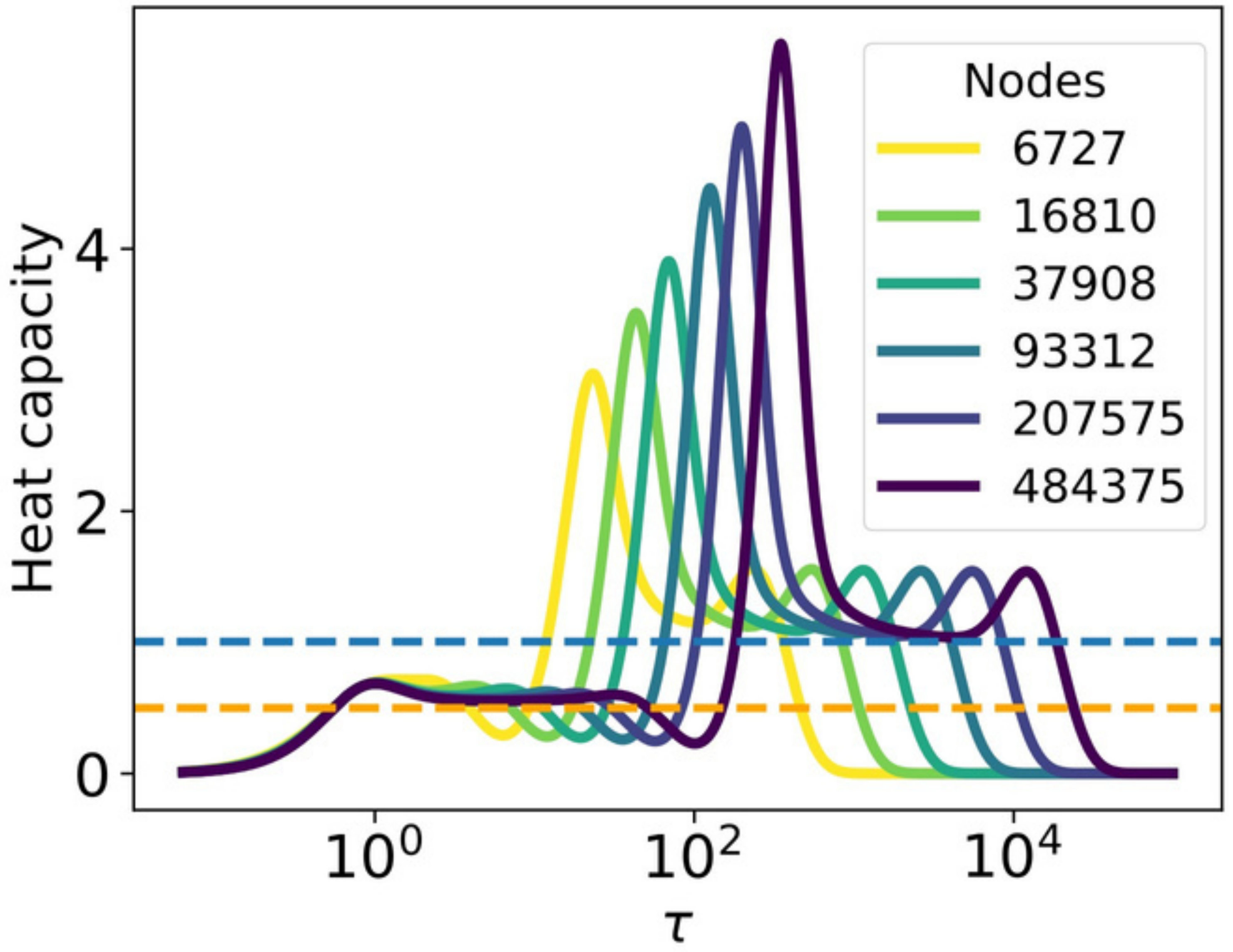
Dirac comb

Two 1-dimensional sectors ⟶ **two distinct plateaux**

Global Fiedler dimension

$$d_g = 1.334(1)$$





Dirac brush

2-dimensional bulk ⟶ **2-dimensional plateau**

1-dimensional fiber ⟶ **1-dimensional plateau**

Global Fiedler dimension

$$d_g = 2.01(1)$$

HETEROGENEOUS STRUCTURES

BUNDLED NETWORKS

Write the eigenvalue equation in a basis where the Laplacian of the base is diagonal.

$$\sum_{j_2} \hat{L}^f_{i_2,\,j_2}\psi^k_{n_1,\,j_2} + l^b_{n_1}\delta_{0,\,i_2}\psi^k_{n_1,\,i_2} = \lambda_k\psi^k_{n_1,\,i_2} \tag{1}$$

HETEROGENEOUS STRUCTURES

BUNDLED NETWORKS

Write the eigenvalue equation in a basis where the Laplacian of the base is diagonal.

$$\sum_{j_2} \hat{L}^f_{i_2, j_2} \psi^k_{n_1, j_2} + l^b_{n_1} \delta_{0, i_2} \psi^k_{n_1, i_2} = \lambda_k \psi^k_{n_1, i_2} \tag{1}$$

Implement a **perturbative approach** to extract the Fiedler eigenvalue of the graph.

$$\psi^k_{n_1, i_2} = C + l^b_{n_1} \psi^{k\,(1)}_{n_1, i_2}, \qquad \lambda_{n_1} = A l^b_{n_1} \tag{2}$$

$$\Rightarrow \qquad \lambda_F = \frac{l^b_F}{N_f} \tag{3}$$

HETEROGENEOUS STRUCTURES

BUNDLED NETWORKS

Write the eigenvalue equation in a basis where the Laplacian of the base is diagonal.

$$\sum_{j_2} \hat{L}^f_{i_2,j_2}\psi^k_{n_1,j_2} + l^b_{n_1}\delta_{0,i_2}\psi^k_{n_1,i_2} = \lambda_k \psi^k_{n_1,i_2} \tag{1}$$

Implement a **perturbative approach** to extract the Fiedler eigenvalue of the graph.

$$\psi^k_{n_1,i_2} = C + l^b_{n_1}\psi^{k\,(1)}_{n_1,i_2}, \qquad \lambda_{n_1} = A l^b_{n_1} \tag{2}$$

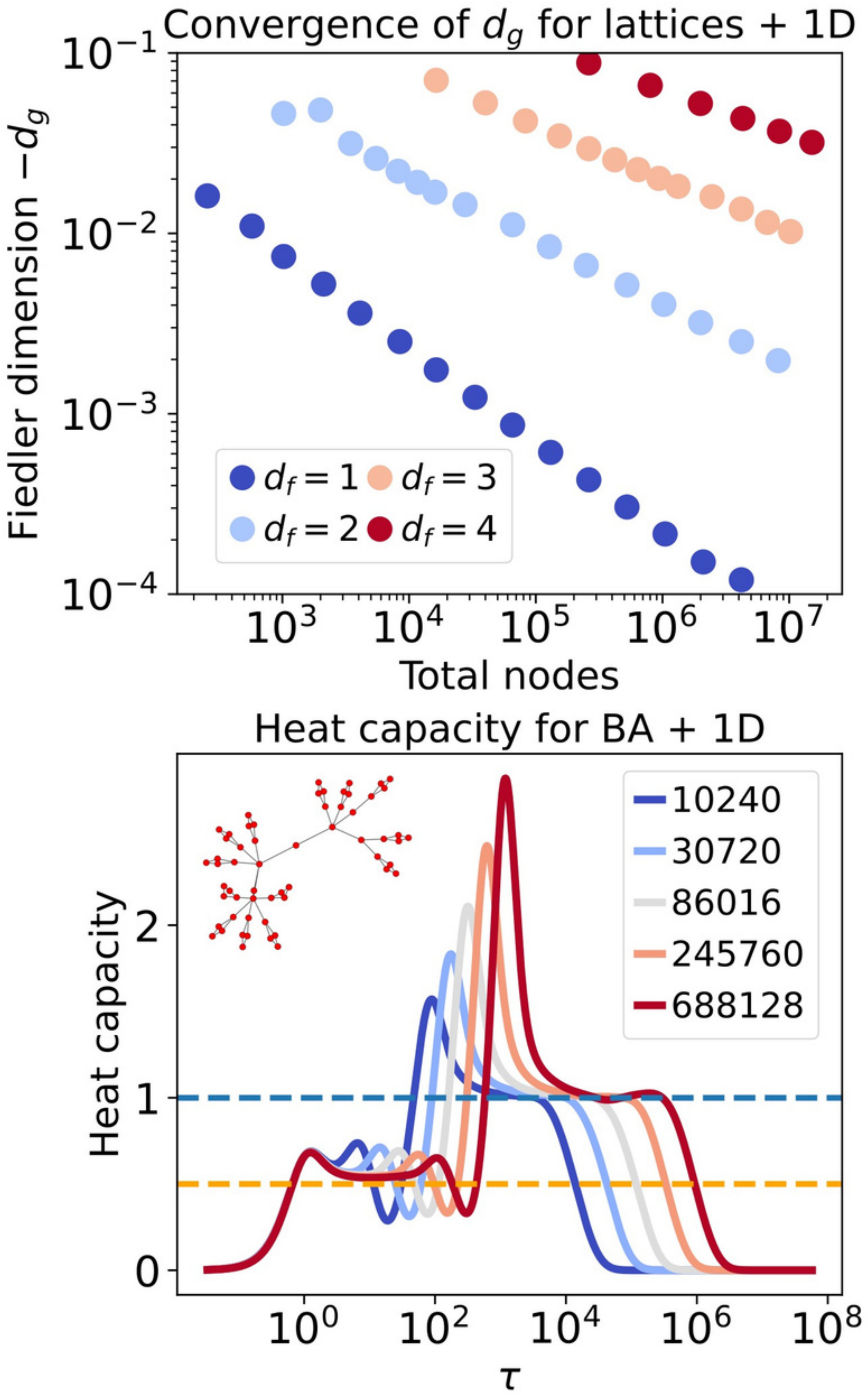
$$\Rightarrow \qquad \lambda_F = \frac{l^b_F}{N_f} \tag{3}$$

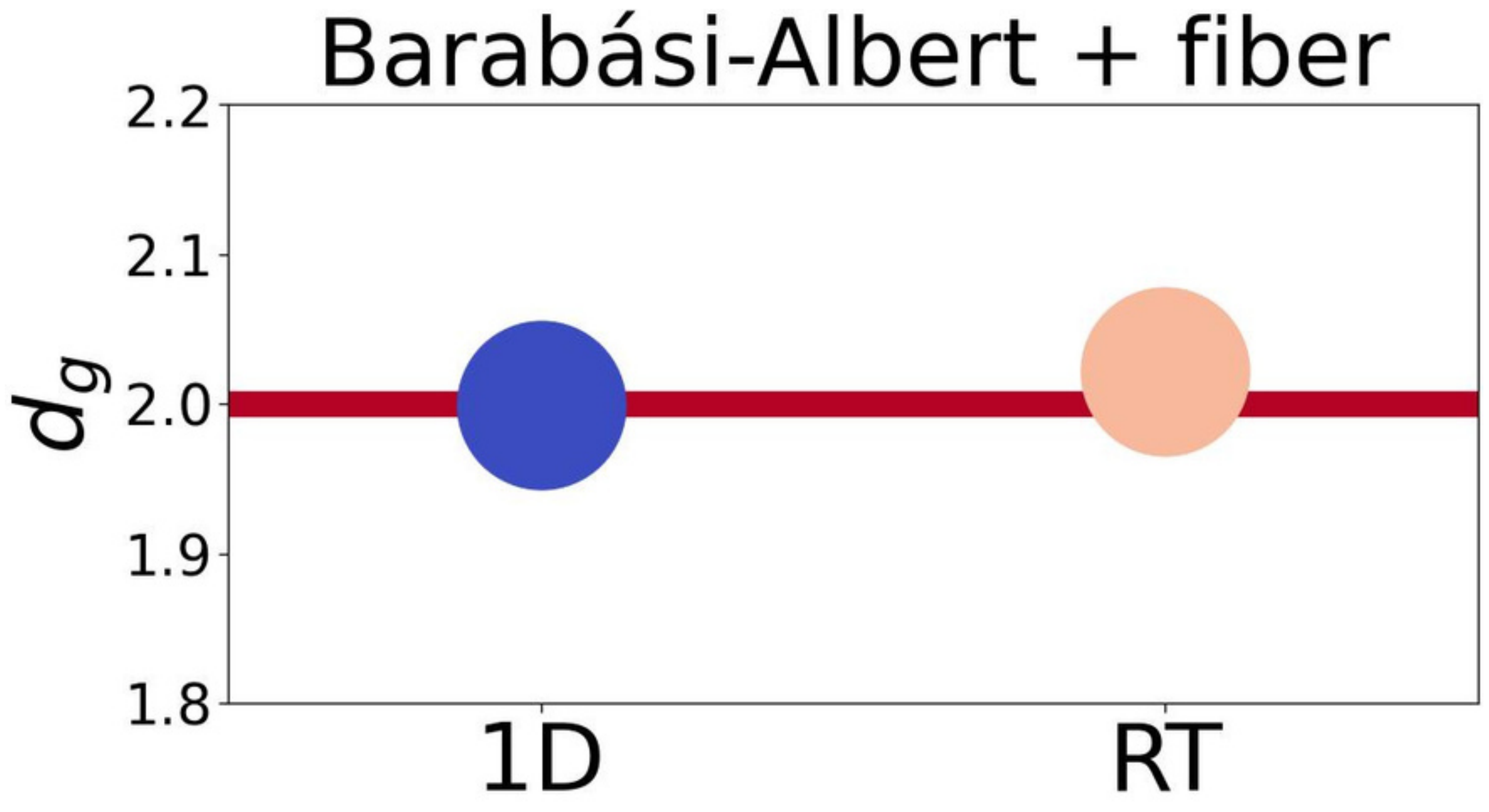
Exploit the scaling of the Fiedler eigenvalue and of the size of the graph.

$$\begin{cases} l^b_F \sim L^{-2\frac{d_{f,b}}{d_{g,b}}} \\ N_f \sim L^{d_{f,f}} \\ N \sim L^{d_{f,f}+d_{f,b}} \end{cases} \Rightarrow \quad \lambda_F \sim N^{-\frac{d_{f,f}+2\frac{d_{f,b}}{d_{g,b}}}{d_{f,f}+d_{f,b}}} \quad \Rightarrow \quad d_g = 2\frac{d_{f,f}+d_{f,b}}{d_{f,f}+2\frac{d_{f,b}}{d_{g,b}}} \tag{4}$$

HETEROGENEOUS STRUCTURES

BUNDLED NETWORKS





SUMMARY

What we have done so far

- Study of **anomalous diffusive behaviours**.
- Introduction of a **non-trivial notion of dimension**.
- **Analytical characterisation** of the Fiedler dimension for a wide class of graphs.

SUMMARY

What we have done so far

- ⁕ Study of **anomalous diffusive behaviours**.
- ⁕ Introduction of a **non-trivial notion of dimension**.
- ⁕ **Analytical characterisation** of the Fiedler dimension for a wide class of graphs.

Possible developments

- ⁕ What happens when the fractal dimension of a sector of the graph is **infinite**?

Recursive use of our result allows for the understanding of even more complex structures, provided that we know the fundamental constituents.

- ⁕ **Inverse problem**: can we use the knowledge of the whole graph to identify individual sectors?

Multifractals are known to display a range of fractal dimensions, changing with the considered scale.

- ⁕ Is there a notion of **multifractal spectral dimension**?

Part II

One-loop mass corrections of interacting string states

Based on

M. Bianchi, M. Firrotta, LG, *in preparation* (2025)

Presented through an oral contribution in

INTERNATIONAL PHYSICISTS NETWORK

YRM YOUNG RESEARCHER MEETING

XIV Young Researcher Meeting, L'Aquila, Italy

and soon in

XXI Avogadro Meeting on Strings, Supergravity and Gauge Theories, Catania, Italy

MOTIVATIONS

FERTILE GROUND FOR COMPLEXITY

String Theory was born as a theory of nuclear interactions, which display **level repulsion**. Resonance energies are described by **random matrix statistics**.

Similar behaviour for strings $\longrightarrow$ new *holographic descriptions* of hadrons?

Highly excited string states are suitable candidates for **microstates of black holes**. The great degeneracy of such states might play a role in the study of the *complexity of black holes*.

[https://www.pinterest.com/pin/276338127110271146/]

MOTIVATIONS

FERTILE GROUND FOR COMPLEXITY

String Theory was born as a theory of nuclear interactions, which display **level repulsion**. Resonance energies are described by **random matrix statistics**.

Similar behaviour for strings $\longrightarrow$ new *holographic descriptions* of hadrons?

Highly excited string states are suitable candidates for **microstates of black holes**. The great degeneracy of such states might play a role in the study of the *complexity of black holes*.

[https://www.pinterest.com/pin/276338127110271146/]

Measure for Chaotic Scattering Amplitudes

Massimo Bianchi[*] and Maurizio Firrotta[†]
Jacob Sonnenschein[‡]
Dorin Weissman[§]

Show more

Phys. Rev. Lett. **129**, 261601 – **Published 22 December, 2022**
DOI: https://doi.org/10.1103/PhysRevLett.129.261601

Measuring chaos in string scattering processes

Massimo Bianchi[1,2,*], Maurizio Firrotta[1,2,†], Jacob Sonnenschein[3,‡], and Dorin Weissman[4,‖]

Show more

Phys. Rev. D **108**, 066006 – **Published 7 September, 2023**
DOI: https://doi.org/10.1103/PhysRevD.108.066006

JHEP
Published for SISSA by Springer
Received: April 23, 2024
Accepted: June 11, 2024
Published: June 27, 2024

From spectral to scattering form factor

Massimo Bianchi[a,b] Maurizio Firrotta,[c] Jacob Sonnenschein[d] and Dorin Weissman[e]

SELECTION OF STATES

NS-NS STRING SPECTRUM

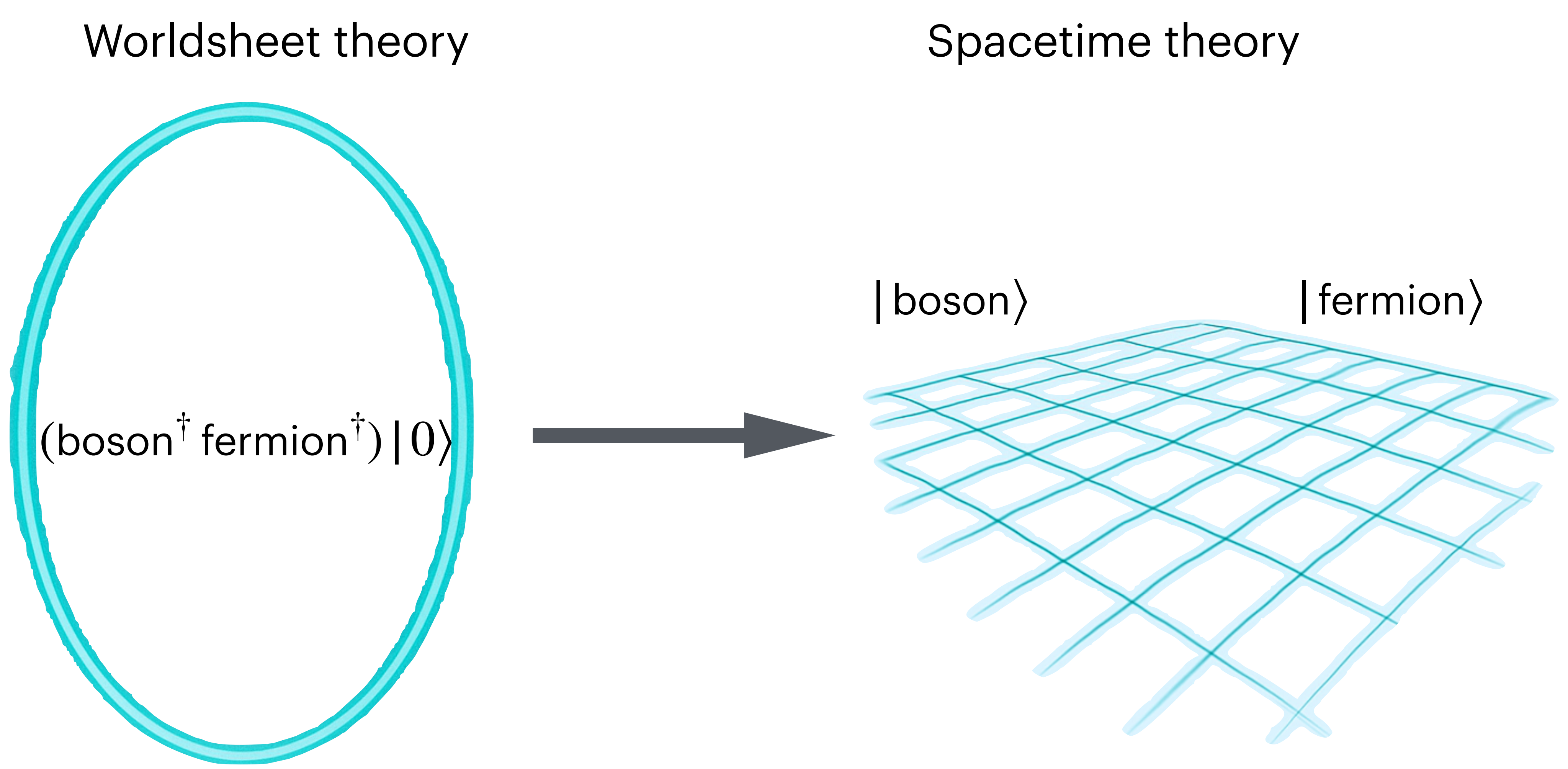
Worldsheet scalar $X^M \longleftrightarrow$ creation and annihilation operators, $n \in \mathbb{N}$

$$\alpha^M_{-n}, \tilde{\alpha}^M_{-n} \qquad \alpha^M_n, \tilde{\alpha}^M_n$$

Worldsheet spinor $\psi^M_i \longleftrightarrow$ creation and annihilation operators, $r \in \mathbb{N} + \frac{1}{2}$

$$b^M_{-r}, \tilde{b}^M_{-r} \qquad b^M_r, \tilde{b}^M_r$$

Neveau-Schwarz (NS) sector $\longleftrightarrow$ antiperiodic boundary conditions for ψ



SELECTION OF STATES

NS-NS STRING SPECTRUM

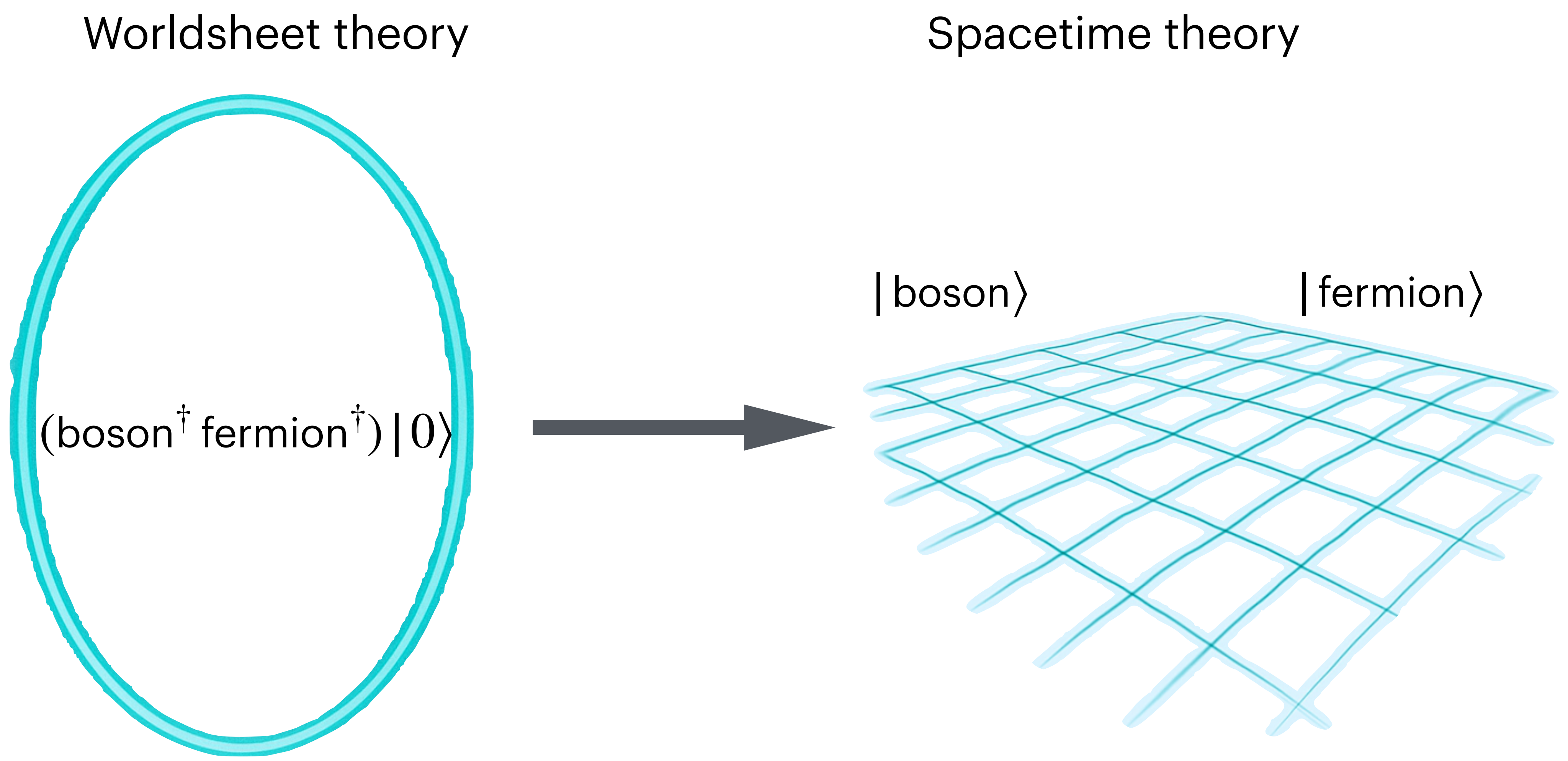
Worldsheet scalar $X^M \longleftrightarrow$ creation and annihilation operators, $n \in \mathbb{N}$

$$\alpha^M_{-n}, \tilde{\alpha}^M_{-n} \qquad \alpha^M_n, \tilde{\alpha}^M_n$$

Worldsheet spinor $\psi^M_i \longleftrightarrow$ creation and annihilation operators, $r \in \mathbb{N} + \frac{1}{2}$

$$b^M_{-r}, \tilde{b}^M_{-r} \qquad b^M_r, \tilde{b}^M_r$$

Neveau-Schwarz (NS) sector $\longleftrightarrow$ antiperiodic boundary conditions for ψ



Bosonic occupation number: $\quad N^b \equiv \sum_{n>0} \alpha^M_{-n}\alpha^M_n \quad$ (and similarly for N^f)

Level matching condition: $\quad N^b_{\text{left}} + N^f_{\text{left}} = N^b_{\text{right}} + N^f_{\text{right}}$

Mass of a state: $\quad M^2 = \dfrac{4}{\alpha'}\left(N^b_{\text{left}} + N^f_{\text{left}} - 1\right) \equiv \dfrac{4}{\alpha'}(N-1)$

Spectrum

Massless ground state $\equiv$ {*graviton* g_{MN}*, Kalb-Ramond field* B_{MN}*, dilaton* ϕ}

Infinite **tower of massive states**, with increasing degeneracy $\sim e^{\beta_H \sqrt{N}}$

AMPLITUDES IN STRING THEORY

TREE-LEVEL INTERACTIONS

Quantum Field Theory

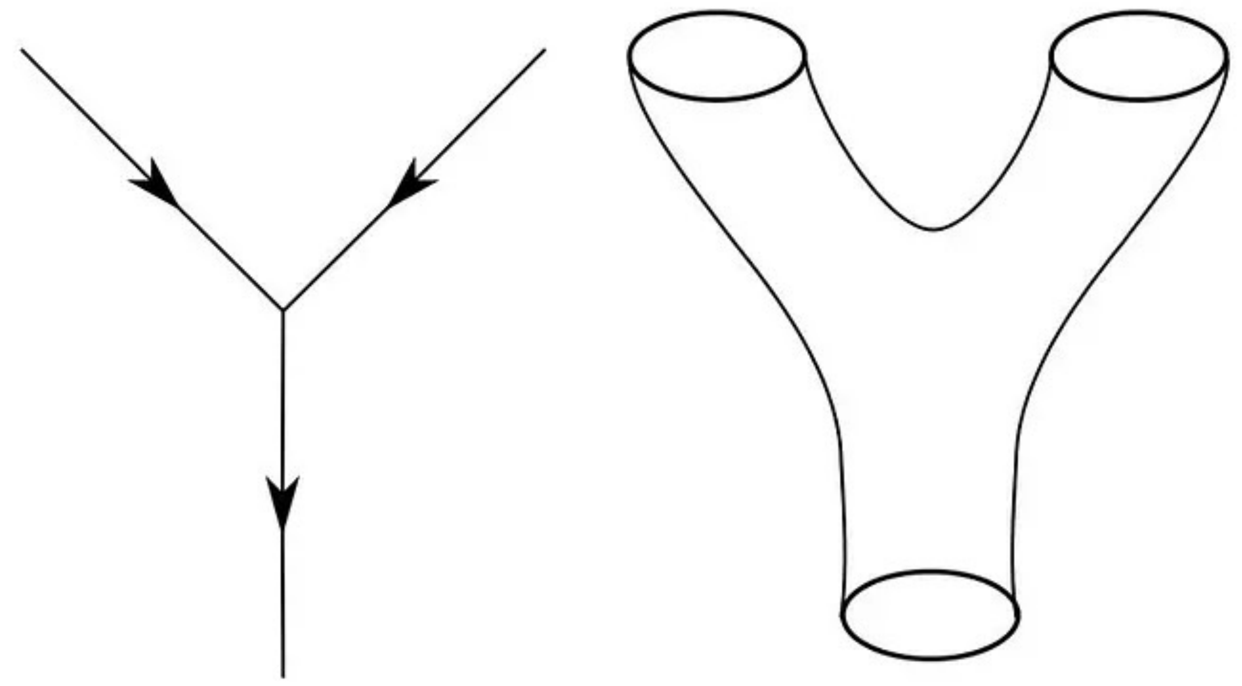
The interactions are *pointlike*. We integrate over the coordinates of the vertex.

The *number of legs* of a vertex depends on the Lagrangian.

String Theory

The interactions are *spread* over the worldsheet, acting as a **natural UV cutoff**. We integrate over the related coordinates.

The vertices have **3 legs**.



[https://bigthink.com/starts-with-a-bang/what-every-layperson-should-know-about-string-theory/]

AMPLITUDES IN STRING THEORY

TREE-LEVEL INTERACTIONS

Quantum Field Theory

The interactions are *pointlike*. We integrate over the coordinates of the vertex.

The *number of legs* of a vertex depends on the Lagrangian.

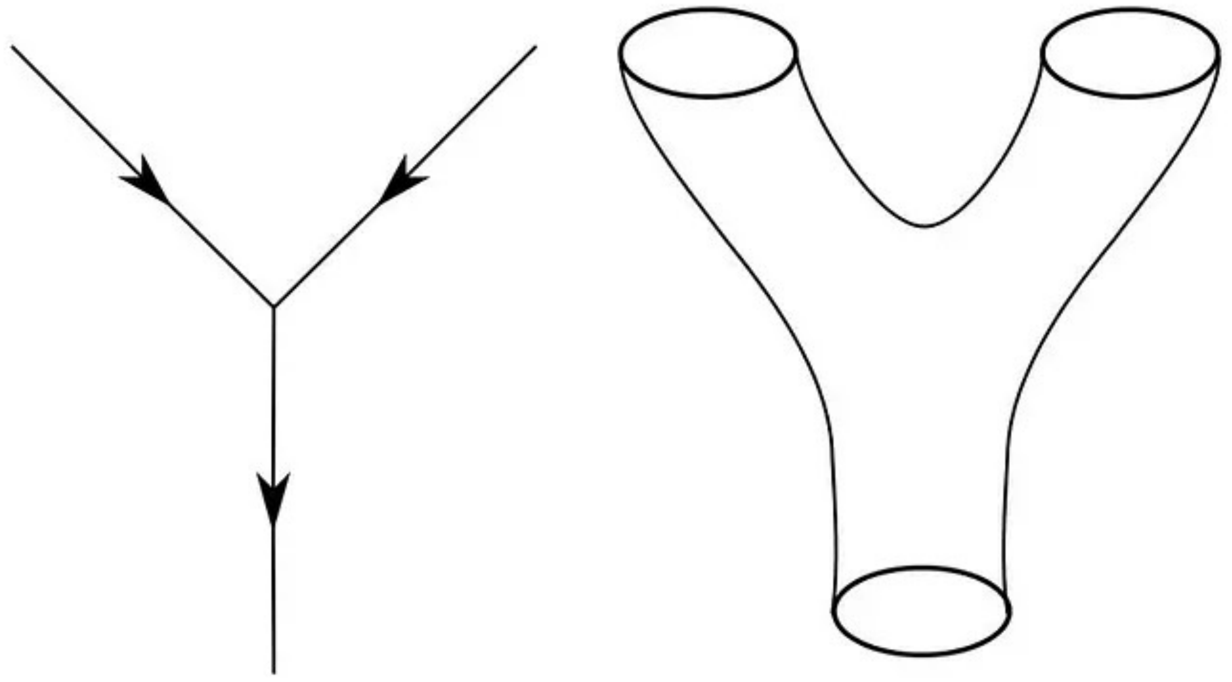
Different *channels* are allowed.

String Theory

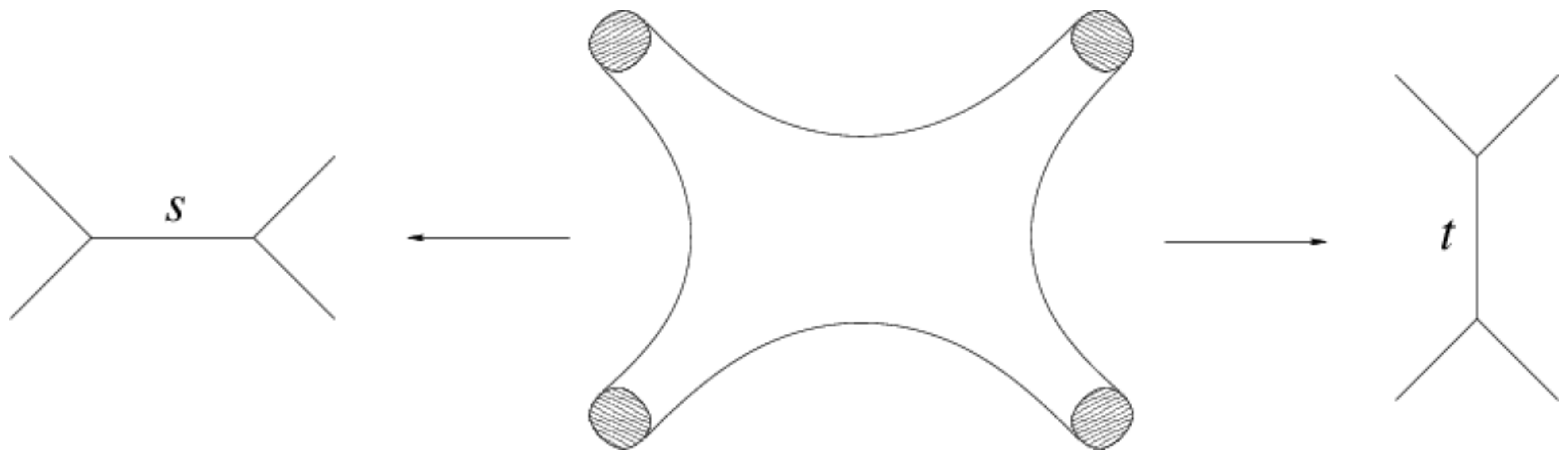
The interactions are *spread* over the worldsheet, acting as a **natural UV cutoff**. We integrate over the related coordinates.

The vertices have **3 legs**.

All the channels are encoded by a **single diagram**.



[https://bigthink.com/starts-with-a-bang/what-every-layperson-should-know-about-string-theory/]



[https://www.researchgate.net/figure/Scattering-string-amplitude-can-be-seen-in-two-ways_fig2_242357918]

AMPLITUDES IN STRING THEORY

TREE-LEVEL INTERACTIONS

Quantum Field Theory

The interactions are *pointlike*. We integrate over the coordinates of the vertex.

The *number of legs* of a vertex depends on the Lagrangian.

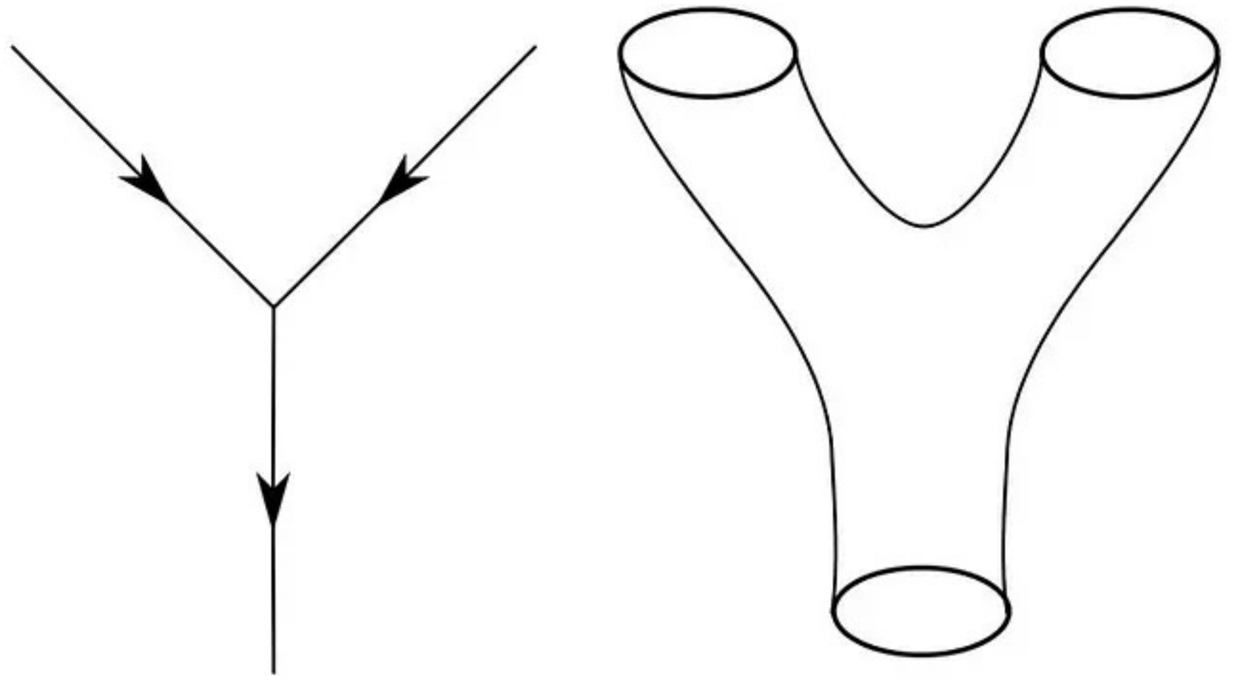
Different *channels* are allowed.

String Theory

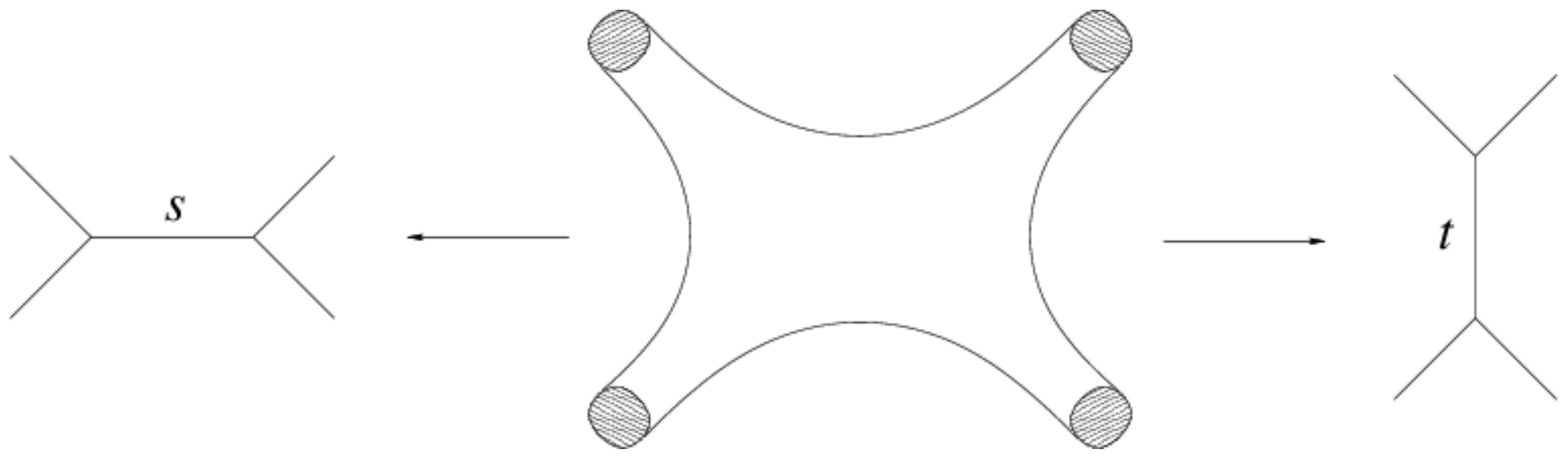
The interactions are *spread* over the worldsheet, acting as a **natural UV cutoff**. We integrate over the related coordinates.

The vertices have **3 legs**.

All the channels are encoded by a **single diagram**.

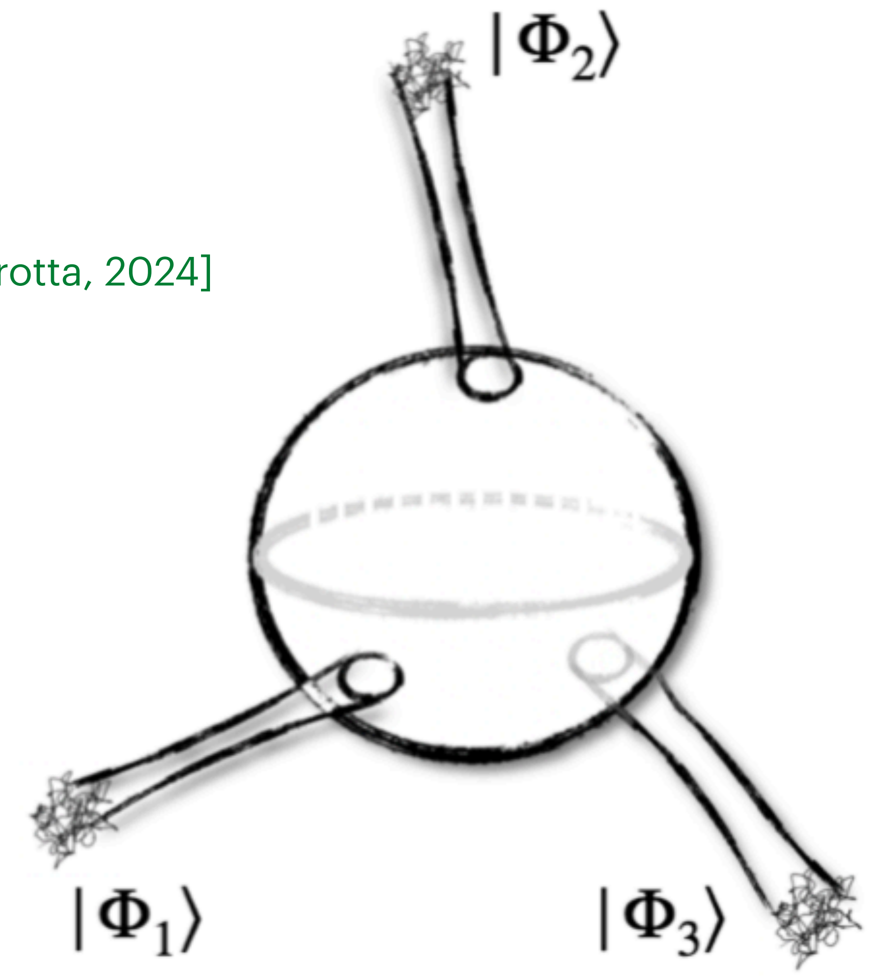


[https://bigthink.com/starts-with-a-bang/what-every-layperson-should-know-about-string-theory/]



[https://www.researchgate.net/figure/Scattering-string-amplitude-can-be-seen-in-two-ways_fig2_242357918]

[Firrotta, 2024]



At the tree-level, **closed strings** can be thought to **hit a D-sphere**. The information on the states is carried in the **vertex operators** $V_i(z, \bar{z})$ entering with each string, where $z, \bar{z} \in \mathbb{C}$ parametrise the worldsheet.

AMPLITUDES IN STRING THEORY

ONE-LOOP AMPLITUDES

Quantum Field Theory

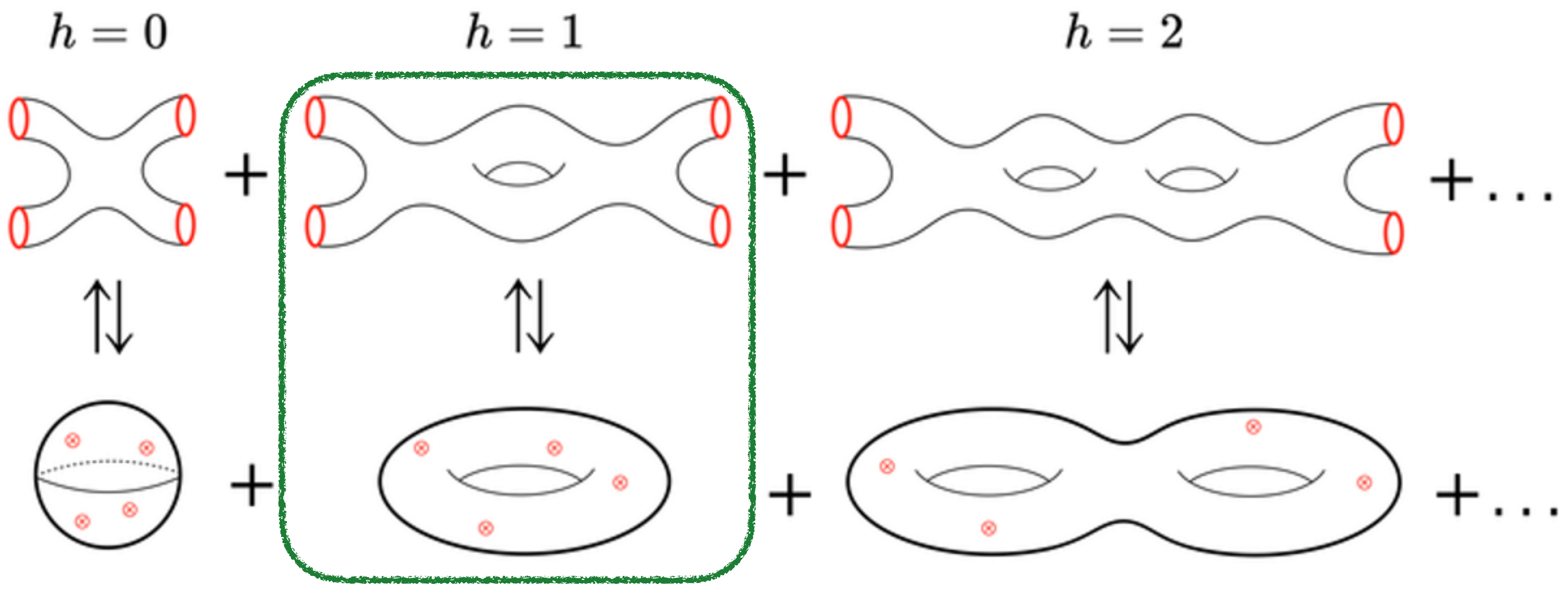
Particles carry momenta along *loops*.

Many diagrams may arise, depending on the Lagrangian.

String Theory

We consider interaction manifolds with an ever-increasing number of *handles* - that is, an increasing **genus**.

We only have to consider the **inequivalent manifolds** with the same genus, at each order.



[https://www.quantumuniverse.nl/string-gravity]

Closed strings $+$ genus-1 $\longrightarrow$ **torus**

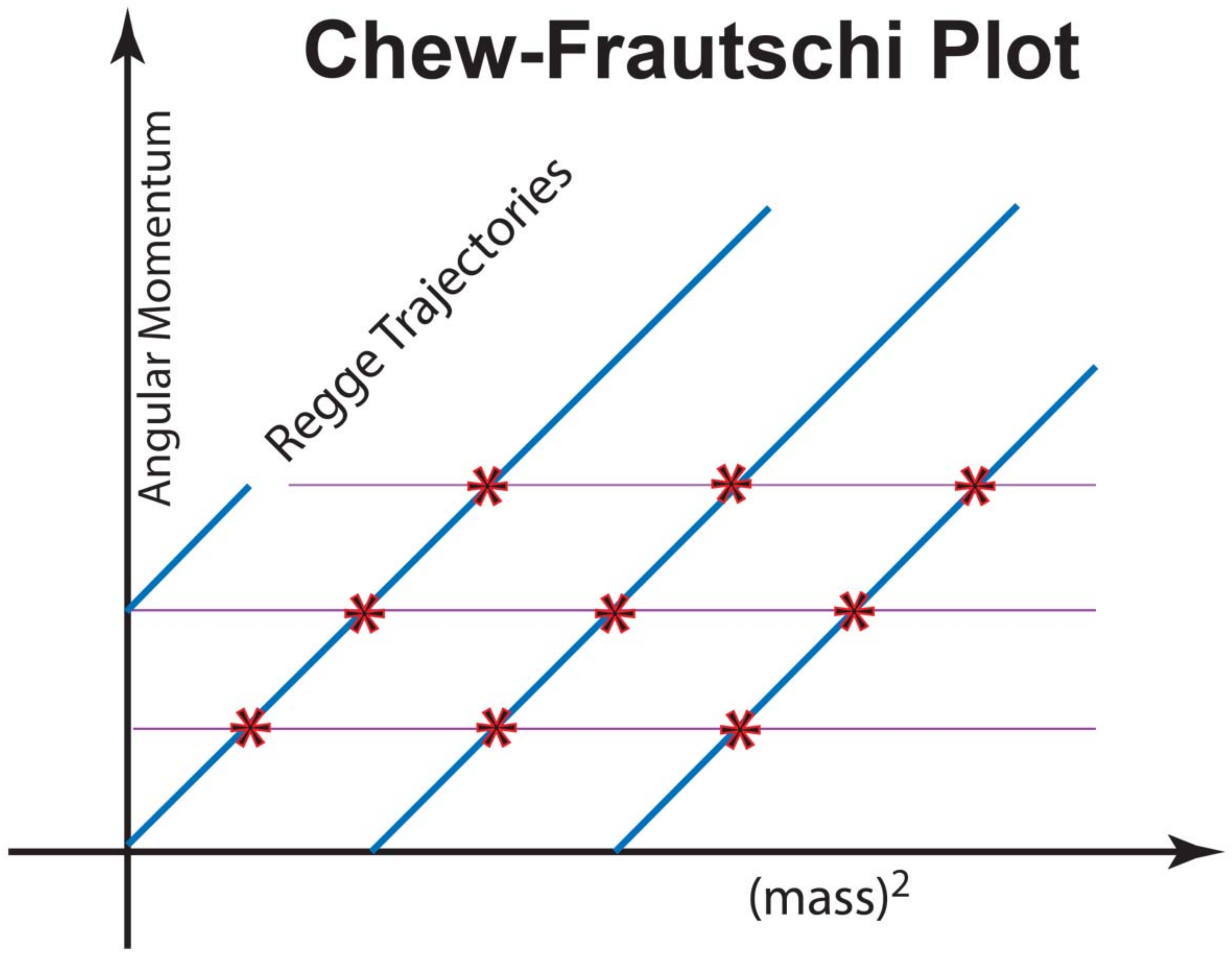
In addition, we have to integrate over the **modular parameter** $\tau \in \mathbb{C}$, characterising the different tori.

ONE-LOOP MASS CORRECTIONS

AN INTEGRAL FOR EVERY MASS LEVEL

We focus on NS-NS states in the **leading Regge trajectory** - that is, with maximal allowed spin.

The aim is to compute **one-loop corrections to the mass** of such states, at a generic mass level N.



[https://ysfine.com/feynman/chewfrau.html]

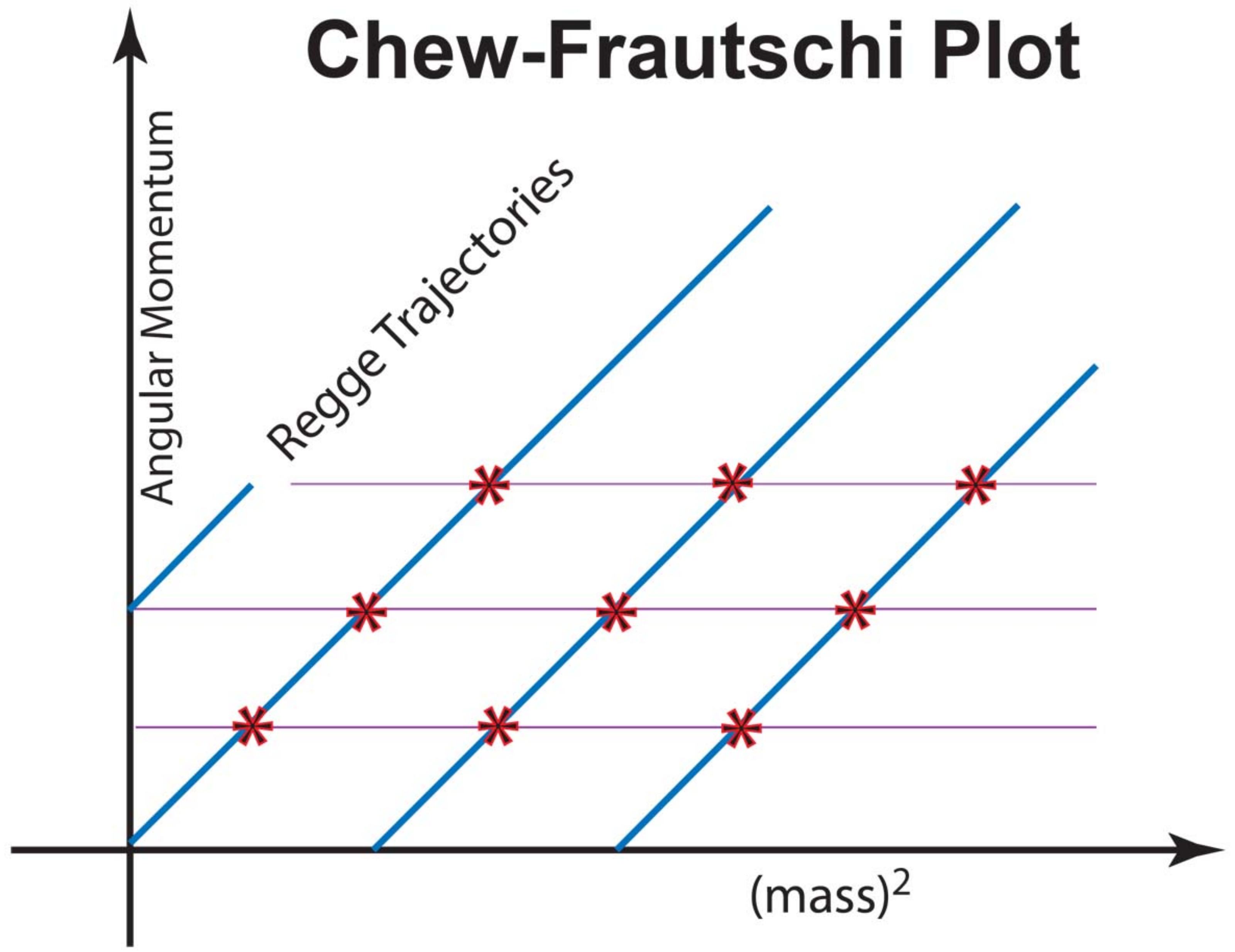
ONE-LOOP MASS CORRECTIONS

AN INTEGRAL FOR EVERY MASS LEVEL

We focus on NS-NS states in the **leading Regge trajectory** - that is, with maximal allowed spin.

The aim is to compute **one-loop corrections to the mass** of such states, at a generic mass level N.

The computation involves one integral over the torus $\mathcal{T}_2$ (after a trivial one) and one integral over the *fundamental domain* $\mathcal{F}$ of modular parameters:



[https://ysfine.com/feynman/chewfrau.html]

$$\mathcal{M}_{\text{1-loop}} \propto \int_{\mathcal{F}} \frac{\mathrm{d}^2\tau}{\tau_2^5} \frac{1}{\eta^{6(N-1)}\overline{\eta}^{6(N-1)}} \int_{\mathcal{T}_2} \mathrm{d}^2 z\, e^{-4\pi\tau_2 y^2(N-1)} \left(\vartheta_1(z)\overline{\vartheta_1(z)}\right)^{2(N-1)} \left(\partial_z^2 G(z)\right)^{N-2} \left(\overline{\partial_z^2 G(z)}\right)^{N-2},$$

where $G(z,\bar{z}) = -\log\left|\frac{\theta_1(z\,|\,\tau)}{\theta_1'(0\,|\,\tau)}\right|^2 + 2\pi\frac{z_2^2}{\tau_2}$ and we have set $z = x + \tau y$, with $x, y \in [0,1]$.

ONE-LOOP MASS CORRECTIONS

CLASS OF INTEGRALS

The worldsheet integral is always in the form

$$I_{N_1,N_2,\overline{N}_1,\overline{N}_2} = \int_{\mathcal{T}_2} \mathrm{d}^2 z\, e^{-4\pi\tau_2(N-1)y^2} \vartheta_1(z)^{2N_1} \vartheta_2(z)^{2N_2} \overline{\vartheta_1(z)}^{2\overline{N}_1} \overline{\vartheta_2(z)}^{2\overline{N}_2}, \text{ with } N_1 + N_2 = \overline{N}_1 + \overline{N}_2 = N - 1.$$

ONE-LOOP MASS CORRECTIONS

CLASS OF INTEGRALS

The worldsheet integral is always in the form

$$I_{N_1,N_2,\overline{N}_1,\overline{N}_2} = \int_{\mathcal{T}_2} \mathrm{d}^2 z\, e^{-4\pi\tau_2(N-1)y^2} \vartheta_1(z)^{2N_1} \vartheta_2(z)^{2N_2} \overline{\vartheta_1(z)}^{2\overline{N}_1} \overline{\vartheta_2(z)}^{2\overline{N}_2}, \text{ with } N_1 + N_2 = \overline{N}_1 + \overline{N}_2 = N - 1.$$

Relations between the ϑ_is and the *lattice sums* $\Lambda_r^{SU(2n)}$ allow for a recasting of $I_{N_1,N_2,\overline{N}_1,\overline{N}_2}$ into a **Gaussian integral**, yielding the result

$$I_{N_1,N_2,\overline{N}_1,\overline{N}_2} = \frac{\sqrt{\tau_2(N-1)}}{2} \sum_{r_1,r_2,\bar{r}_1,\bar{r}_2}^{0,\,2N_i-1} (-)^{r_1+\bar{r}_1} \Lambda_{r_1}^{SU(2N_1)} \Lambda_{r_2}^{SU(2N_2)} \bar{\Lambda}_{\bar{r}_1}^{SU(2\overline{N}_1)} \bar{\Lambda}_{\bar{r}_2}^{SU(2\overline{N}_2)} \frac{1+(-)^{\bar{r}_1+\bar{r}_2-r_1-r_2}}{2} \cdot$$

$$\cdot \sum_{m,\bar{m}\in\mathbb{Z}} q^{\left(\frac{1}{N_1}+\frac{1}{N_2}\right)\Delta^2(m,r_1,r_2)} \bar{q}^{\left(\frac{1}{\overline{N}_1}+\frac{1}{\overline{N}_2}\right)\bar{\Delta}^2(\overline{m},\bar{r}_1,\bar{r}_2)} \sum_{k=1}^{N-1} e^{-\frac{2\pi i}{N-1}k\left(N_1 m + \frac{\bar{r}_1+\bar{r}_2-r_1-r_2}{2} - \overline{N}_1\overline{m}\right)},$$

where $q = e^{2\pi i\tau}$.

ONE-LOOP MASS CORRECTIONS

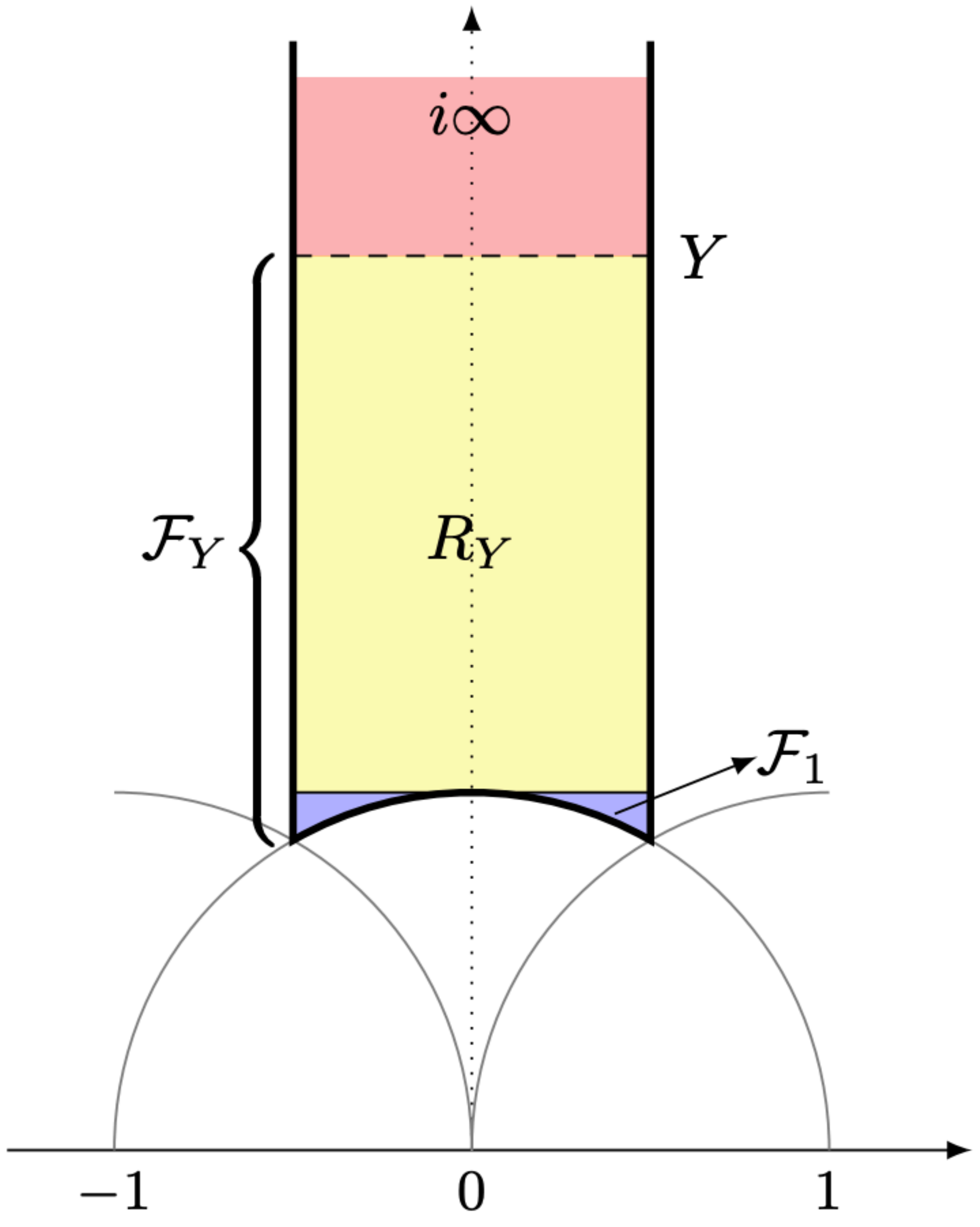
REGULARISATION AND RENORMALISATION

The integral over the fundamental domain is always in the form

$$\mathscr{M}_{\text{1-loop}} \propto \int_{I_2} \frac{\mathrm{d}^2\tau}{\tau_2^5} \frac{F(\tau,\overline{\tau})}{\eta^{6(N-1)}\overline{\eta}^{6(N-1)}} I_{N_1,N_2,\overline{N}_1,\overline{N}_2}(\tau,\overline{\tau}).$$

The real part of the integral diverges. This is cured by an **extension of the $i\varepsilon$-prescription** to String Theory [Manschot, Wang, 2024]. In a nutshell:

- Expand the integrand in powers of q and $\bar{q}$;
- Identify the dangerous region of the domain (the red one, for negative powers);
- Remove the divergent contribution.



[Manschot, Wang, 2024]

ONE-LOOP MASS CORRECTIONS

RESULTS AT LOW MASS LEVELS

$\mathfrak{R}(\mathscr{M}_{\text{1-loop}}) \longrightarrow$ mass correction

$\mathfrak{I}(\mathscr{M}_{\text{1-loop}}) \longrightarrow$ decay width

These correction vanish for massless states ($N = 1$), but not for massive states!

Manschot and Wang relied on previous results [Stieberger, 2023] to check their procedure at $N = 2$. Our result agrees with theirs:

$$\mathscr{M}_{\text{1-loop}, N=2} \propto -(27.85 + 59.37i) \qquad ✅$$

ONE-LOOP MASS CORRECTIONS

RESULTS AT LOW MASS LEVELS

$\Re(\mathscr{M}_{\text{1-loop}}) \longrightarrow$ mass correction

$\Im(\mathscr{M}_{\text{1-loop}}) \longrightarrow$ decay width

These correction vanish for massless states ($N = 1$), but not for massive states!

Manschot and Wang relied on previous results [Stieberger, 2023] to check their procedure at $N = 2$. Our result agrees with theirs:

$$\mathscr{M}_{\text{1-loop},\,N=2} \propto -(27.85 + 59.37i)$$ ✅

$\boxed{\text{We carried out the full computation at } N = 3\text{, finding } \mathscr{M}_{\text{1-loop},\,N=3} \propto -(1306 + 3363i).}$

SUMMARY

What we have done so far

- Analysis of **one-loop mass corrections** for a set of NS-NS states.
- Derivation of the mass corrections at **any mass level** N, with the possibility to extract numerical results.

SUMMARY

What we have done so far

- Analysis of **one-loop mass corrections** for a set of NS-NS states.
- Derivation of the mass corrections at **any mass level** N, with the possibility to extract numerical results.

Possible developments

Away from the leading Regge trajectory, *polynomials* in $\partial_z^{2l}G$ appear. The presence of the integral $I_{N_1, N_2, \overline{N}_1, \overline{N}_2}$ is not manifest anymore.

Moreover, **mixing** might be involved! At $N = 4$ we have both $(\partial_z^2 G)^2$ and $\partial_z^4 G$ from two distinct states...

- Possible way out: use the **Weierstrass $\wp(z\,|\,\tau)$ function**! *Work in progress...*

- We expect complexity to emerge at large N. Still, level repulsion at $N = 4$ would be a major hint!

THANK YOU!

Backup slides

HETEROGENEOUS STRUCTURES

BUNDLED LATTICES

Write the eigenvalue equation in a basis where the Laplacian of the base is diagonal.

$$\sum_{j_2} \hat{L}^f_{i_2,j_2}\psi^k_{n_1,j_2} + l^b_{n_1}\delta_{0,i_2}\psi^k_{n_1,i_2} = \lambda_k\psi^k_{n_1,i_2} \quad (1)$$

$$2\psi^k_{n_1,i_2} - \psi^k_{n_1,i_2+1} - \psi^k_{n_1,i_2-1} + \delta_{0,i_2}l^b_{n_1}\psi^k_{n_1,i_2} = \lambda_k\psi^k_{n_1,i_2} \quad (2)$$

The eigenvalues λ_k are those of a ring with an impurity. The spatial frequency ω_k is fixed by a boundary condition.

$$\lambda_k = 2[1-\cos(\omega_k)], \qquad \omega_k \simeq 2\sqrt{\frac{l^b_k}{L}} \quad (3)$$

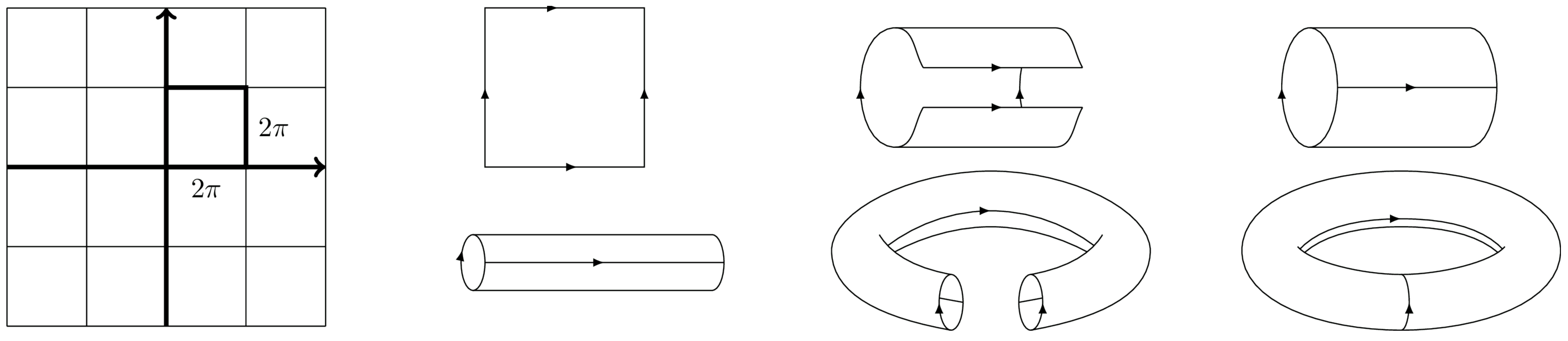
Exploit the scaling of the Fiedler eigenvalue of the size of the graph.

$$l^b_F \sim L^{-2\frac{d_{f,b}}{d_{g,b}}} \quad \Rightarrow \quad \lambda_F \sim L^{-1-2\frac{d_{f,b}}{d_{g,b}}} \sim N^{-\frac{1+2\frac{d_{f,b}}{d_{g,b}}}{1+d_{f,b}}} \equiv N^{-\frac{2}{d_g}} \quad (4)$$

$$d_g = 2\frac{1+d_{f,b}}{1+2\frac{d_{f,b}}{d_{g,b}}} \quad (5)$$

AMPLITUDES IN STRING THEORY

MODULAR INVARIANCE

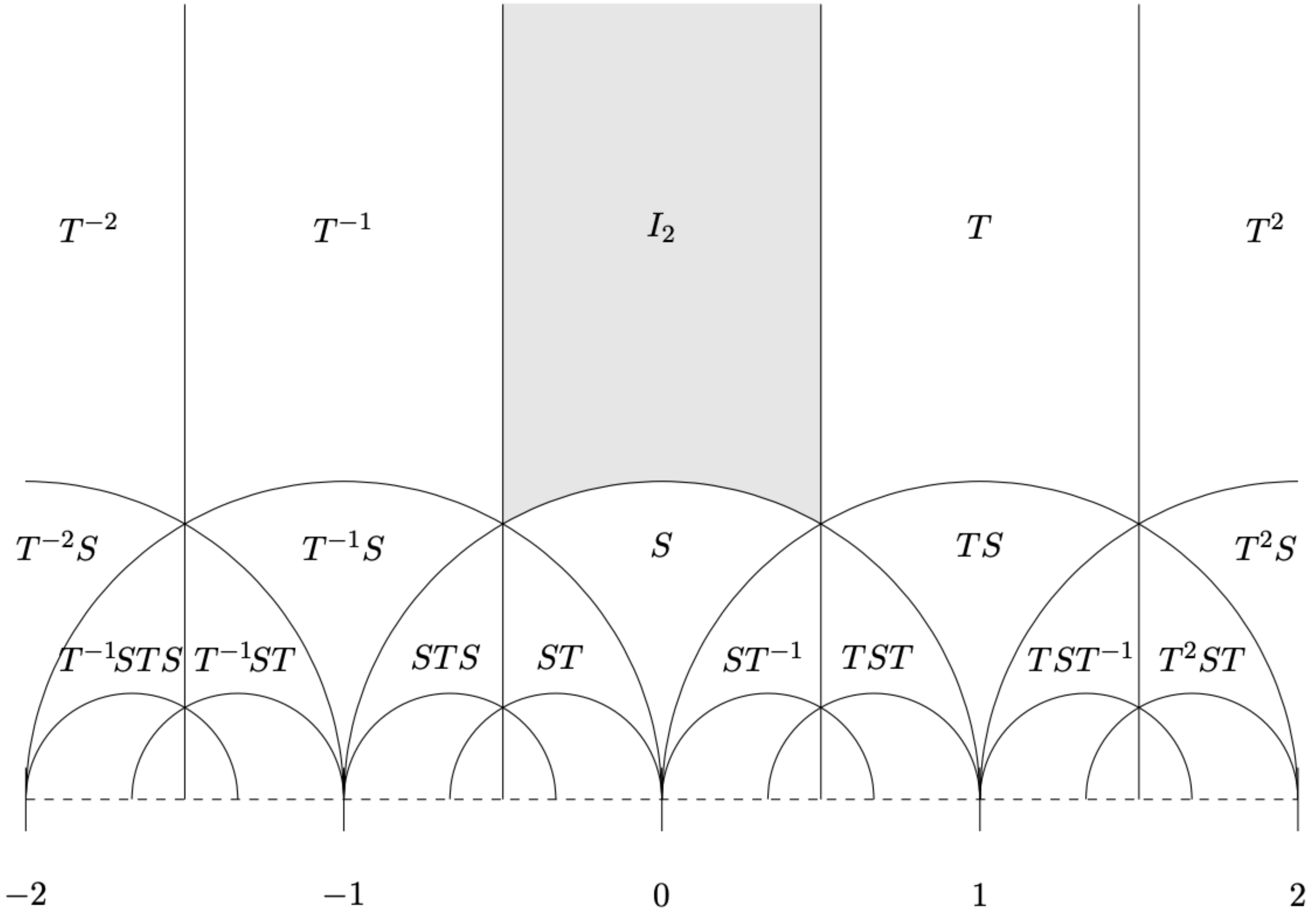


[https://tikz.net/plane-to-torus/]

⟵ Mapping the torus to the fundamental cell in the complex plane

The torus is invariant under the transformations $T: \ \tau \to \tau + 1$ and $S: \ \tau \to -\dfrac{1}{\tau}$, which generate the **modular group** $SL(2,\mathbb{Z})$. Hence, we only have to integrate over the *inequivalent tori* - that is, the **fundamental domain** I_2.

The resulting amplitude must be a **modular invariant**.



[https://reu.dimacs.rutgers.edu/~cjf151/dimacs/]

ONE-LOOP AMPLITUDES
RECURRING FUNCTIONS

Jacobi ϑ functions

$$\vartheta_1(z|\tau) = i\sum_{n\in\mathbb{Z}} (-)^n q^{\frac{1}{2}\left(n+\frac{1}{2}\right)^2} \zeta^{n+\frac{1}{2}}, \qquad \vartheta_3(z|\tau) = \sum_{n=-\infty}^{\infty} q^{\frac{n^2}{2}} \zeta^n,$$

$$\vartheta_2(z|\tau) = \sum_{n\in\mathbb{Z}} q^{\frac{1}{2}\left(n+\frac{1}{2}\right)^2} \zeta^{n+\frac{1}{2}}, \qquad \vartheta_4(z|\tau) = \sum_{n=-\infty}^{\infty} (-)^n q^{\frac{n^2}{2}} \zeta^n,$$

with $q = e^{2i\pi\tau}$ and $\zeta = e^{2i\pi z}$

Dedekind function $\quad \eta(\tau) = q^{\frac{1}{24}} \prod_{n\in\mathbb{N}} (1 - q^n)$

Lattice sums $\quad \Lambda_r^{SU(2n)} = \sum_{m\in\mathbb{Z}^{2n}} q^{\frac{1}{2}\sum_i \left(m_i - \frac{r}{2n}\right)^2} \delta\left(\sum_i m_i = r\right)$

Weierstrass elliptic function $\quad \wp(z|\tau) = \dfrac{1}{z^2} + \sum_{(n,m)\neq 0} \left[\dfrac{1}{(z+n+m\tau)^2} - \dfrac{1}{(n+m\tau)^2}\right]$

ONE-LOOP AMPLITUDES

RECURRING RELATIONS

Differential equation for $\wp(z\,|\,\tau)$

$$(\partial_z\wp)^2 = 4\wp^3 - g_2\wp - g_3, \qquad \text{with } g_2 = \frac{4}{3}\pi^4 E_4(\tau) \text{ and } g_3 = \frac{8}{27}\pi^6 E_6(\tau)$$

Bargman kernel in terms of $\wp(z\,|\,\tau)$ and $\eta(\tau)$

$$\partial_z^2 G = \wp(z) - 4i\pi\partial_\tau \log\left(\eta(\tau)\sqrt{\tau_2}\right)$$

Jacobi $\vartheta_{1,2}$ in terms of lattice sums

$$\vartheta_a^{2n} = \begin{cases} \displaystyle\sum_r (-)^r \Lambda_r^{SU(2n)} \sum_{l\in\mathbb{Z}} q^{\frac{1}{2}2nQ^2} e^{2\pi i z 2nQ} & \text{if } a = 1 \\ \displaystyle\sum_r \Lambda_r^{SU(2n)} \sum_{l\in\mathbb{Z}} q^{\frac{1}{2}2nQ^2} e^{2\pi i z 2nQ} & \text{if } a = 2 \end{cases}, \qquad \text{with } Q = l + \frac{1}{2} - \frac{r}{2n}$$

Derivatives of $\wp(z\,|\,\tau)$

$$\wp^{(2n)}(z\,|\,\tau) = \sum_{k=0}^{n+1} c_k^{(2n)} \wp^k, \qquad \text{with}$$

$$c_k^{(2n+2)} = -g_3(k+1)(k+2)c_{k+2}^{(2n)}\Big|_{k\in[0,n-1]} - \frac{g_2(k+1)(2k+1)}{2}c_{k+1}^{(2n)}\Big|_{k\in[0,n]} + 2(k-1)(2k-1)c_{k-1}^{(2n)}\Big|_{k\in[2,n+2]}$$