Special Challenges in QCD
GGI Florence, 8-9/1/2026

Resummation for Heavy Quark Fragmentation in e+e- Collisions

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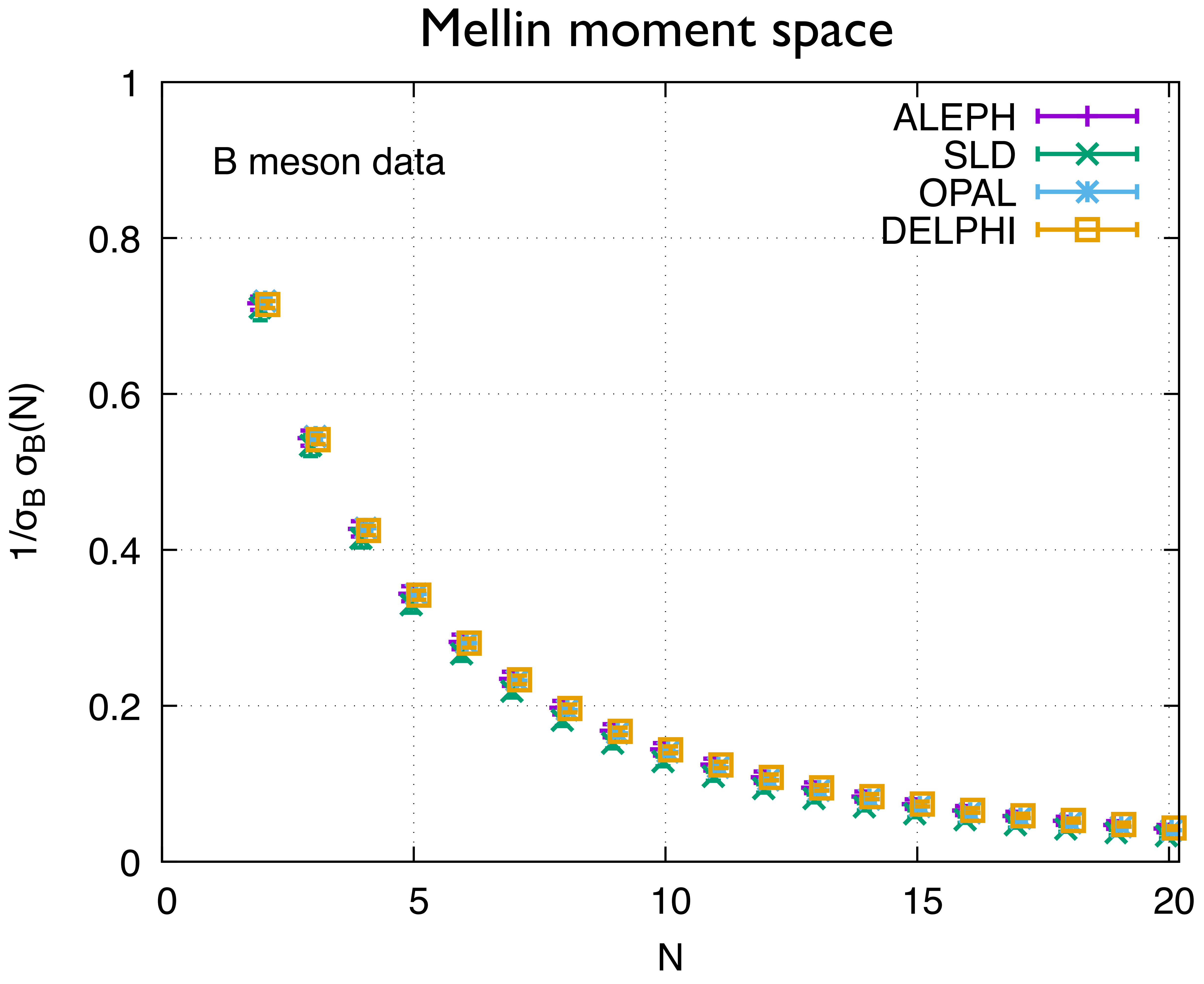
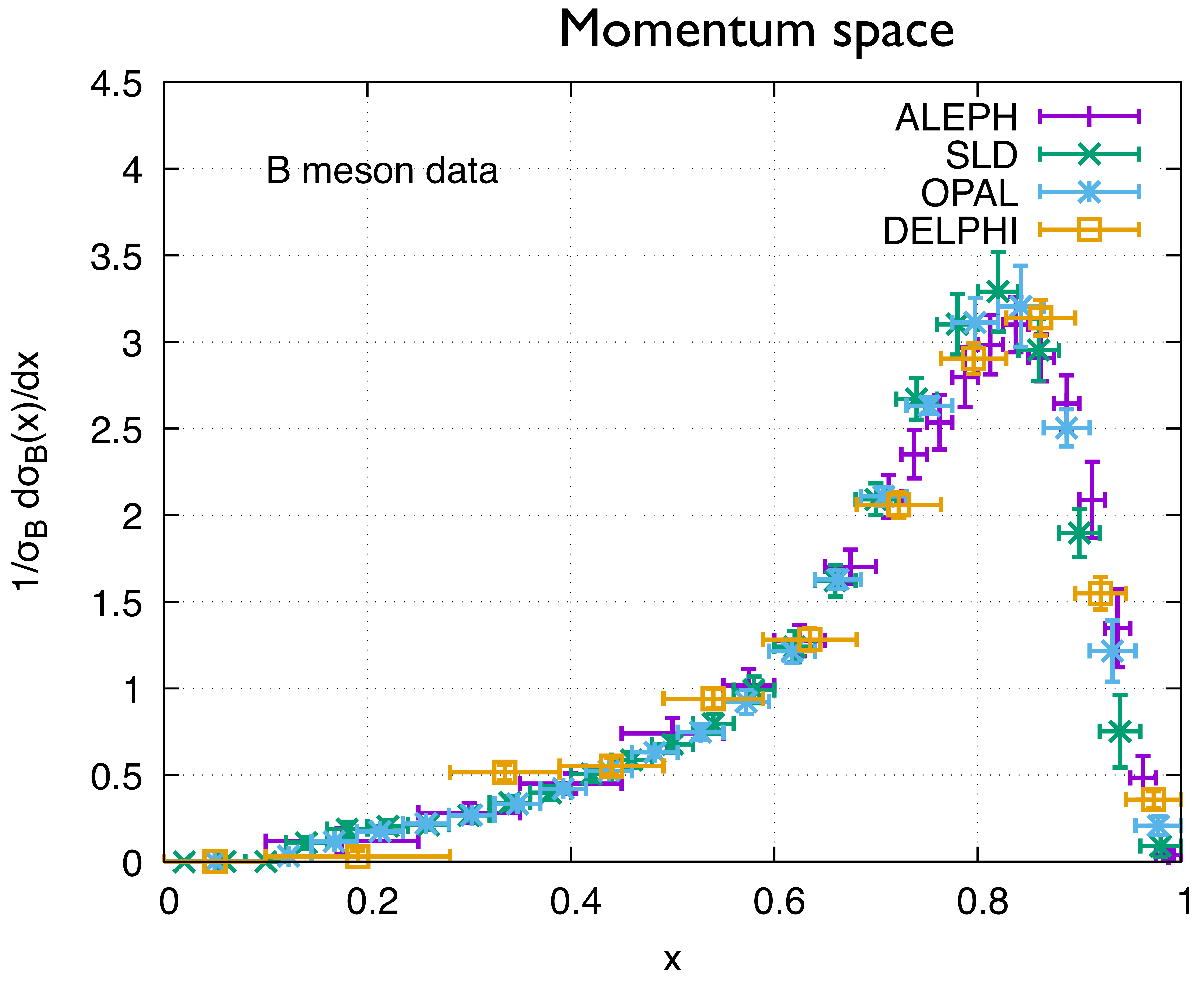
From work in collaboration with Leonardo Bonino, Giovanni Stagnitto, Andrea Ghira, Simone Marzani, Giovanni Ridolfi

LEP and SLC data





Approximate relation : 1-x ~ 1/N

The problem

Hard interaction at some large scale,
observation of a heavy hadron with a given momentum

Multiple scales : at least the **large scale Q**, the **heavy quark mass m**, the momentum of the heavy hadron (which can constrain the **energy of emitted gluons**), the **non-perturbative scale Λ** of the hadronisation of the heavy quark into the heavy hadron

Problem addressed multiple times, using multiple languages:
phenomenological models, pQCD, renormalons, effective coupling, HQET, SCET, bHQET,…

50 years

Heavy quark discovery

Now

A personal (and certainly biased) view of what happened in between

1974

2024

Outline

A very logarithmic view of these 50 years

- An overview of some of the papers that addressed this problem over the years (personal selection)

- The first results of another calculation/implementation of $e^+e^- \to \gamma, Z \to H_Q + X$ to NNLO+NNLL

Bonino, MC, Stagnitto, 2312.12519

- Some phenomenological results obtained with the joint mass/soft resummation

Ghira, Marzani, Ridolfi, 2309.06139
MC, Ghira, Marzani, Ridolfi, 2406.04173
Ghira, Mai, Marzani, 2412.13261

Once upon a time

Bjorken and Suzuki, 1977

involving the same produced partons (with the same momenta), but not involving a cascade decay. (ii) For neutrino production, electroproduction, and e^+e^- annihilation, at energies far above threshold, the inclusive momentum distribution of a stable hadron H containing the Q peaks near the maximum momentum, i.e., at values of the scaling variable $z \sim 1$. (iii) For events containing a nonleptonic decay of Q into ordinary quarks

Bjorken

A model is presented to describe hadron fragmentation off light and heavy partons. Fragmentation functions are parametrized by onc variable. When a heavy parton of a new flavour fragments, a heavy hadron tends to carry away most of the parton momentum, leaving light hadron (π and K) spectra softer than those from light partons.

Suzuki

A heavy quark to heavy hadron
fragmentation function f(z) will be peaked near z=1:

$$\langle z \rangle \simeq 1 - \frac{1\ GeV}{m}$$

Once upon a time

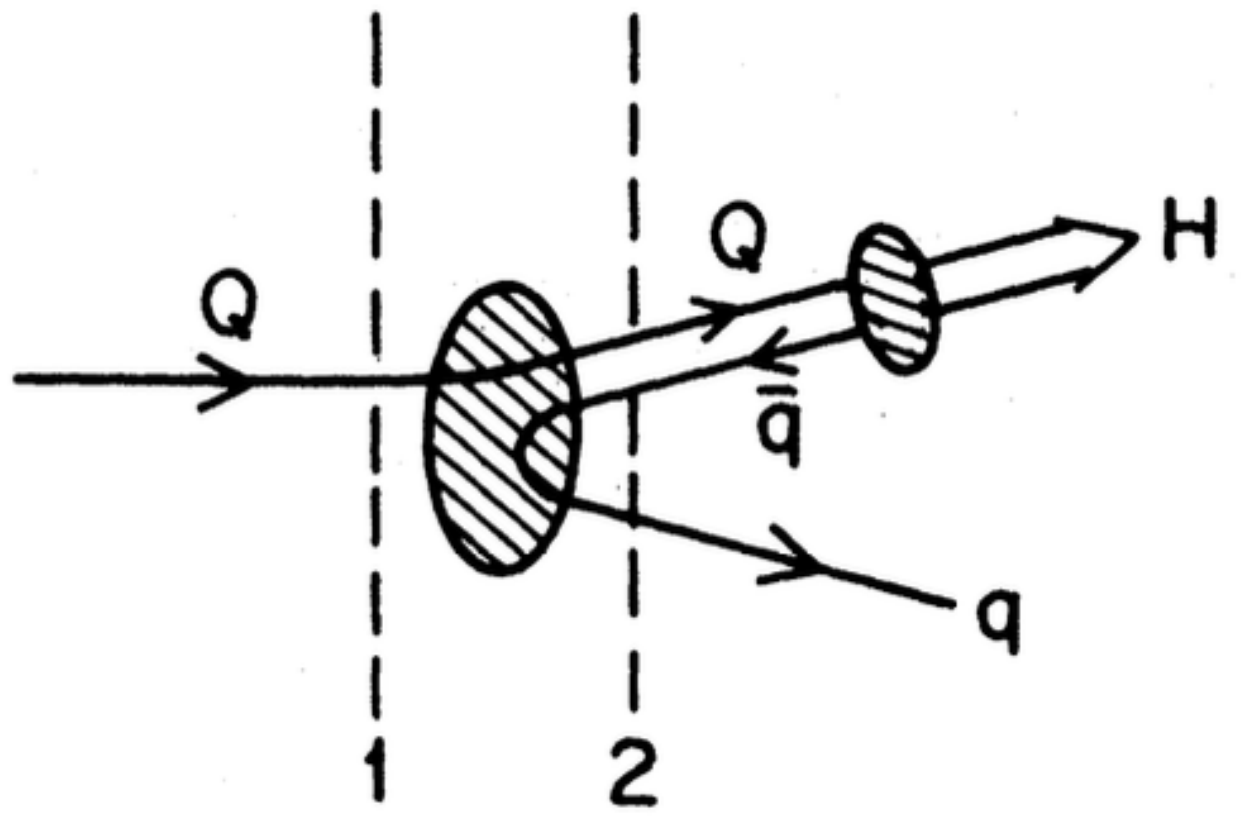
Heavy quark discovery

Peterson, Schlatter, Schmitt and Zerwas

Bjorken
Suzuki

1974 1977 1982 2024

A phenomenological model for heavy quark hadronisation



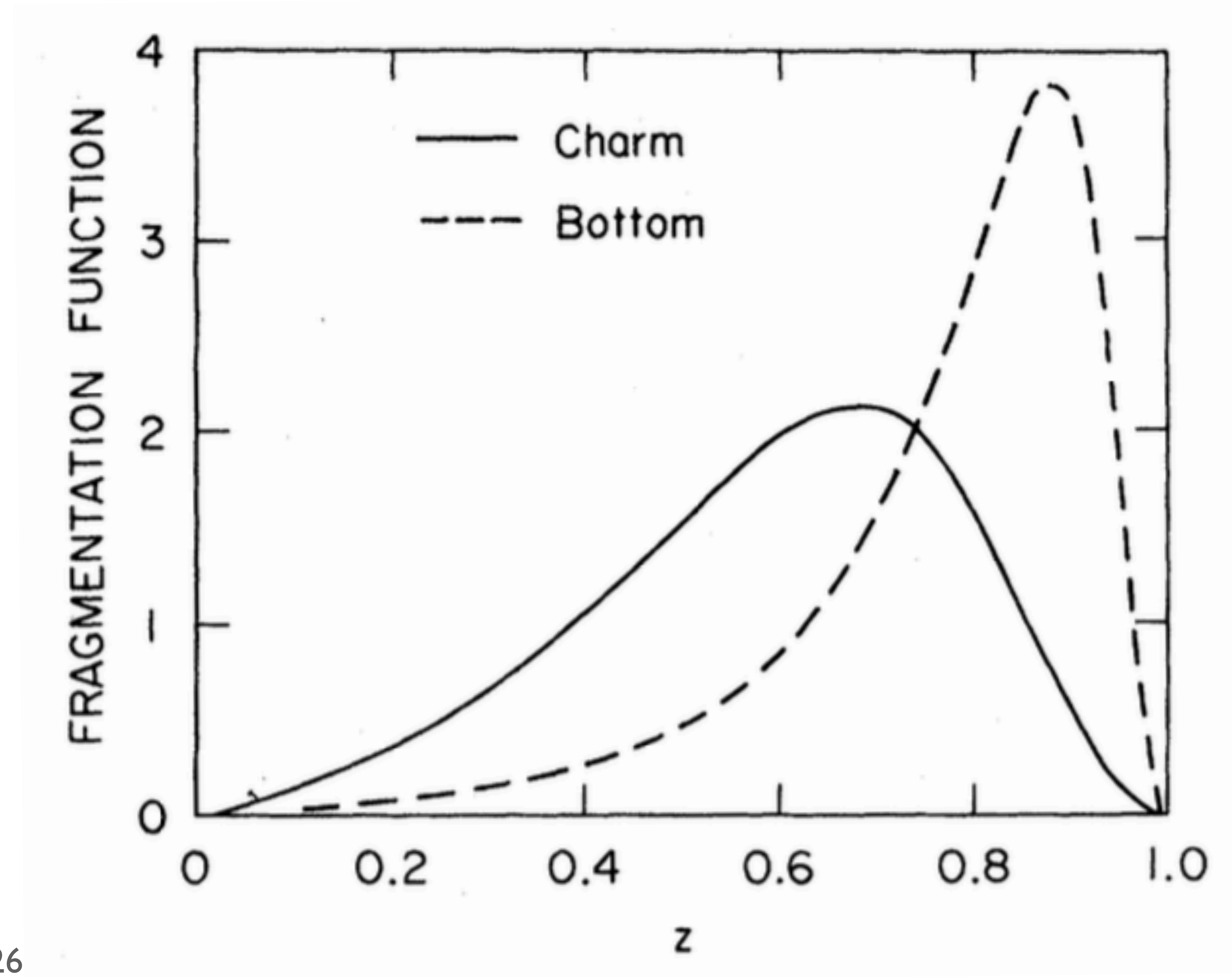
amplitude $(Q \to H + q) \propto \Delta E^{-1}$

$$\Delta E = (m_Q{}^2 + z^2 P^2)^{1/2} + (m_q{}^2 + (1-z)^2 P^2)^{1/2} - (m_Q{}^2 + P^2)^{1/2}$$

$$\propto 1 - (1/z) - (\epsilon_Q/1-z)$$

with $\epsilon_Q \sim m_q^2/m_Q^2$

$$D_Q^H(z) = \frac{N}{z[1-(1/z)-\epsilon_Q/(1-z)]^2}$$



Once upon a time

Heavy quark discovery

Peterson, Schlatter, Schmitt and Zerwas

Bjorken
Suzuki

Mele, Nason

1974 1977 1982 1991 2024

Mele, Nason, 1991

Nuclear Physics B361 (1991) 626–644
North-Holland

THE FRAGMENTATION FUNCTION FOR HEAVY QUARKS IN QCD

B. MELE

CERN, Geneva, Switzerland

P. NASON

INFN, Gruppo Collegato di Parma, Parma, Italy

Received 13 February 1991
(Revised 26 March 1991)

Mele, Nason, 1991

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An erratum in 2017, almost 30 years later

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Nuclear Physics B 921 (2017) 841–842
NUCLEAR PHYSICS B
www.elsevier.com/locate/nuclphysb

Corrigendum

Corrigendum to "The fragmentation function for heavy quarks in QCD" [Nucl. Phys. B 361 (1991) 626–644]

B. Mele [a,*], P. Nason [b]

[a] *INFN, Sezione di Roma, Rome, Italy*
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Received 11 May 2017; accepted 11 May 2017
Available online 24 May 2017

(no worry, just some typos)

Mele, Nason, 1991

PSSZ (and certainly other papers at the time) was already addressing the issue of **perturbative QCD evolution** on top of heavy quark fragmentation.

Mele-Nason takes this to full next-to-leading accuracy, and also resums soft logarithms with leading accuracy

$$\sigma_N(Q) = \hat{\sigma}_N(Q,\mu)\exp\left\{P_N^{(0)}t + \frac{1}{4\pi^2 b_0}(\alpha_S(\mu_0) - \alpha_S(\mu))\left(P_N^{(1)} - \frac{2\pi b_1}{b_0}P_N^{(0)}\right)\right\}\underline{\hat{D}_N^{[1]}(\mu_0,m)}$$

Perturbatively calculable initial condition (or fragmentation function) : pFF

$$\hat{D}^{[1]}(x,\mu,m) = \delta(1-x) + \frac{\alpha_S C_F}{2\pi}\left[\frac{1+x^2}{1-x}\left(\log\frac{\mu^2}{m^2} - 2\log(1-x) - 1\right)\right]_+$$

Overall accuracy: NLO + $\mathrm{NLL_{coll}}$ + $\mathrm{LL_{soft}}$

The factorised picture introduced by Mele and Nason means that the **universal and calculable perturbative initial condition** and its DGLAP evolution can be used to obtain heavy quark production cross sections and resum large collinear log(Q/m) terms in photon and hadron collisions using only massless coefficient functions

$$d\sigma_Q(p_T) \sim \underbrace{d\hat{\sigma}_j(p_T, \mu_F)}_{\text{Coefficient functions}} \otimes \underbrace{E_{ij}(\mu_F, \mu_{0F})}_{\text{DGLAP evolution}} \otimes \underbrace{D_{j\to Q}(m, \mu_{0F})}_{\text{Initial conditions (Decay functions) (Fragmentation functions)}}$$

First used in MC, Greco ’93 to calculate large-pT production of heavy quarks in pp collisions

Once upon a time

Jaffe, Randall

Heavy quark discovery

Peterson, Schlatter, Schmitt and Zerwas

Bjorken
Suzuki

Mele, Nason

1974 1977 1982 1991 1993 2024

Jaffe, Randall, 1993

Analyse heavy quark fragmentation in e+e- collisions using HQET, obtaining a boundary condition containing also a parameterisation of non-perturbative effects

$$D_{Q\to H_Q}(z,\mu_0\sim m) = \frac{1}{\epsilon}\hat{a}\left[\frac{1}{\epsilon}\left(\frac{1}{z}-\frac{m}{M}\right)\right] + \hat{b}\left[\frac{1}{\epsilon}\left(\frac{1}{z}-\frac{m}{M}\right)\right]$$

with $\epsilon = 1 - m/M$

This form means that the fragmentation function at the mass scale shrinks **linearly** in 1/M towards z=1, consistently with Bjorken and Suzuki's argument

Once upon a time

Jaffe, Randall

Heavy quark discovery

Dokshitzer, Khoze, Troyan

Peterson, Schlatter, Schmitt and Zerwas

Bjorken
Suzuki

Mele, Nason

1974 1977 1982 1991 1993 1995 2024

Dokshitzer, Khoze, Troyan, 1995

The Abstract says it all:

Perturbative QCD formula for inclusive energy spectra of heavy quarks from heavy quark initiated jets which takes into account collinear and/or soft logarithms in all orders, the exact first order result and two-loop effects is applied to distributions of heavy flavoured hadrons in the framework of the LPHD concept.

More in detail:

Our approximation includes the two-loop anomalous dimension, keeps track of the *collinear* logarithms $a \ln W$ and $a \ln M$, *soft* double-logarithmic $a \ln^2(1-x_Q)$ and essential single-logarithmic $a \ln(1-x_Q)$ contributions in all orders. At the same time, it embodies the exact first order result $\mathcal{O}(a)$ for the inclusive energy distribution

In modern language, this is (or is very close to) NLO + $\mathrm{NLL_{coll}}$ + $\mathrm{NLL_{soft}}$ (+ study of an effective coupling)

Once upon a time

Rijken, van Neerven, 1996

NNLO massless coefficient functions for fragmentation in e+e- collisions

$$\frac{d\sigma_k^H}{dx} = \int_x^1 \frac{dz}{z} \left[\sigma_{\rm tot}^{(0)}(Q^2) \left\{ D_{\rm S}^H\left(\frac{x}{z}, M^2\right) \mathbb{C}_{k,q}^{\rm S}(z, Q^2/M^2) + D_g^H\left(\frac{x}{z}, M^2\right) \right. \right.$$

$$\left. \left. \cdot\, \mathbb{C}_{k,g}^{\rm S}(z, Q^2/M^2) \right\} + \sum_{p=1}^{n_f} \sigma_p^{(0)}(Q^2)\, D_{{\rm NS},p}^H\left(\frac{x}{z}, M^2\right) \mathbb{C}_{k,q}^{\rm NS}(z, Q^2/M^2) \right]$$

Once upon a time

Rijken, van Neerven

Jaffe, Randall

Heavy quark discovery

Dokshitzer, Khoze, Troyan

Peterson, Schlatter, Schmitt and Zerwas

MC, Catani

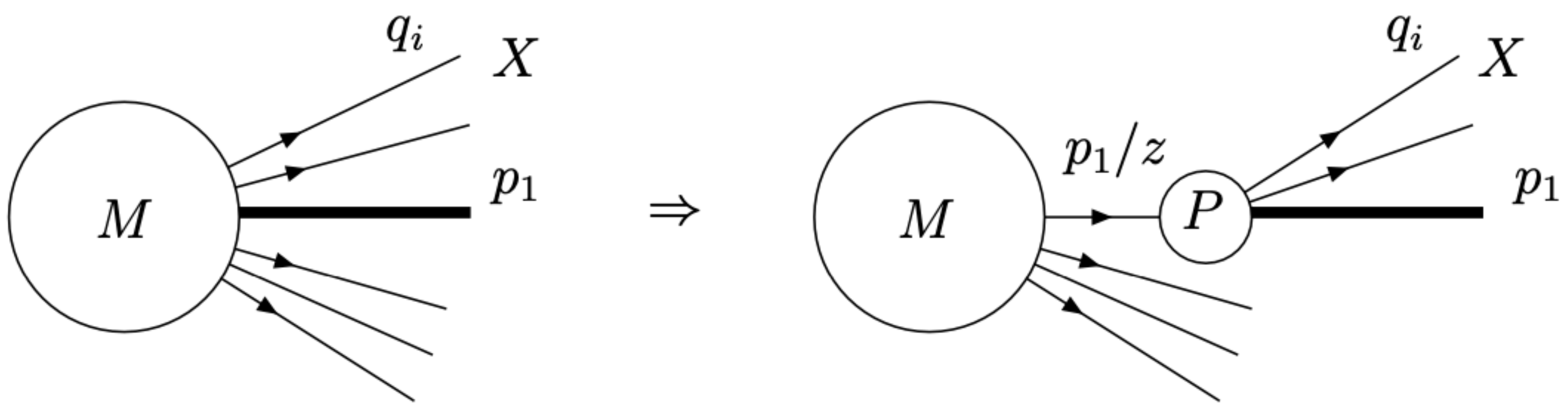
Bjorken Suzuki

Mele, Nason

1974 1977 1982 1991 1993 1995 1996 2001 2024

MC, Catani, 2001

Direct calculation of universal perturbative initial condition in a process-independent way, resummation of soft logarithms to next-to-leading order



$$|M(p_1,q;\ldots)|^2 \simeq |M(p_1/z;\ldots)|^2 \, \frac{4\pi\alpha_S}{p_1\cdot q}\,\hat{P} = |M(p_1/z;\ldots)|^2 \, 8\pi\alpha_S \, \frac{z(1-z)}{\mathbf{q}_\perp^2 + (1-z)^2 m^2}\,\hat{P}$$

$$\hat{P}_{Qg}(z;m^2/\mathbf{q}_\perp^2) = C_F\left[\frac{1+z^2}{1-z} - \frac{m^2}{p_1\cdot q}\right] = C_F\left[\frac{1+z^2}{1-z} - \frac{2z(1-z)m^2}{\mathbf{q}_\perp^2+(1-z)^2m^2}\right]$$

Massive AP splitting function

Combined with the soft resummation of the coefficient function, this allows for e+e- heavy quark fragmentation at accuracy

NLO + NLL$_{\text{coll}}$ + NLL$_{\text{soft}}$

Once upon a time

Rijken,
van Neerven

Jaffe, Randall

Heavy quark
discovery

Dokshitzer, Khoze, Troyan

MC, Gardi

Peterson, Schlatter,
Schmitt and Zerwas

MC, Catani

Bjorken
Suzuki

Mele, Nason

Gardi

1974 1977 1982 1991 1993 1995 1996 2001 2003 2005 2024

MC, Gardi 2003, Gardi 2005

Resummation of soft logarithms and of running coupling effects
in the large-β_0 limit, in the Dressed Gluon Exponentiation approach

$$\tilde{\sigma}(N, q^2, M^2) \simeq \tilde{\sigma}^{\rm PT}(N, q^2, m^2)\, \underline{\tilde{D}^{\rm NP}_{\{\epsilon_n\}}((N-1)\Lambda/m)}$$

Non-perturbative shape function
predicted by the renormalon ambiguity

$$\tilde{D}^{\rm NP}_{\{\epsilon_n\}}((N-1)\Lambda/m) = \exp\left\{-\sum_{n=1}^{\infty} \epsilon_n \left(\frac{(N-1)\Lambda}{m}\right)^n\right\}$$

Gardi 2005 extends this to NNLL accuracy for the soft-gluon resummation

Once upon a time

Rijken, van Neerven

Jaffe, Randall

Melnikov, Mitov

Heavy quark discovery

Dokshitzer, Khoze, Troyan

MC, Gardi

Peterson, Schlatter, Schmitt and Zerwas

MC, Catani

Bjorken Suzuki

Mele, Nason

Gardi

1974 1977 1982 1991 1993 1995 1996 2001 2003 2004 2005 2024

Melnikov, Mitov, 2004

Calculation of NNLO contribution to perturbative initial condition

$$D_a^{\rm ini}\left(z,\frac{\mu_0}{m}\right) = \sum_{n=0}\left(\frac{\alpha_s(\mu_0)}{2\pi}\right)^n d_a^{(n)}\left(z,\frac{\mu_0}{m}\right)$$

$$d_a^{(0)}(z) = \delta_{a\mathcal{Q}}\delta(1-z),$$

$$d_{a=\mathcal{Q}}^{(1)}\left(z,\frac{\mu_0}{m}\right) = C_F\left[\frac{1+z^2}{1-z}\left(\ln\left(\frac{\mu_0^2}{m^2(1-z)^2}\right)-1\right)\right]_+$$

$$d_{a=g}^{(1)}\left(z,\frac{\mu_0}{m}\right) = T_R\left(z^2+(1-z)^2\right)\ln\left(\frac{\mu_0^2}{m^2}\right),$$

$$d_{a\neq\mathcal{Q},g}^{(1)}\left(z,\frac{\mu_0}{m}\right) = 0,$$

NLO contributions from Mele-Nason

$$d_a^{(2)}\left(z,\frac{\mu_0}{m}\right) = \left[\frac{P_{ba}^{(0)}\otimes P_{\mathcal{Q}b}^{(0)}(z)}{2} + \frac{\beta_0}{2}P_{\mathcal{Q}a}^{(0)}(z)\right]\ln^2\left(\frac{\mu_0^2}{m^2}\right)$$
$$+\left[P_{\mathcal{Q}a}^{(1)}(z) + P_{ba}^{(0)}\otimes d_b^{(1)}(z,1) + \beta_0 d_a^{(1)}(z,1)\right]\ln\left(\frac{\mu_0^2}{m^2}\right) + d_a^{(2)}(z,1)$$

$d_a^{(2)}(z,1)$ calculated and given in paper, marvellous result which this margin is too narrow to contain

Once upon a time

Rijken, van Neerven

MC, Nason, Oleari

Jaffe, Randall

Melnikov, Mitov

Heavy quark discovery

Dokshitzer, Khoze, Troyan

Peterson, Schlatter, Schmitt and Zerwas

MC, Gardi

MC, Catani

Bjorken

Suzuki

Mele, Nason

Gardi

1974 1977 1982 1991 1993 1995 1996 2001 2003 2004 2005 2024

MC, Nason, Oleari, 2005

A significant milestone (at the NLO+NLL level) of the approach bootstrapped by the 1991 Mele-Nason paper, allowing for a maximally perturbative description of heavy quark production in e+e- collisions (i.e. added non-perturbative corrections are of order Λ/m)

Now including:

- Full mixings with gluons and light flavours in NLL DGLAP evolution
- NLL soft resummation, regularised in a sensible way to avoid the Landau pole
- Deconvolution of electromagnetic initial state radiation
- Analytical modelling of electroweak decays of D mesons
- Non-perturbative fragmentation functions fitted to data

→ good description of CLEO/BELLE and LEP data, **up to one puzzling issue**

Once upon a time

Rijken,
van Neerven

Mitov, Moch
Blümlein, Ravindran

MC, Nason, Oleari

Jaffe, Randall

Melnikov, Mitov

Heavy quark
discovery

Dokshitzer, Khoze, Troyan

MC, Gardi

Peterson, Schlatter,
Schmitt and Zerwas

MC, Catani

Bjorken
Suzuki

Mele, Nason

Gardi

1974 1977 1982 1991 1993 1995 1996 2001 2003 2004 2005 2006 2024

Mitov, Moch, 2006; Blümlein, Ravindran, 2006

NNLO massless coefficient functions for fragmentation in e+e- collisions

- Recalculation in Mellin space (MM)
- Calculation of Mellin moments from RvN results (BR)

Since 2006, all ingredients available for NNLO+NNLL analysis of e+e- fragmentation in Mellin moments space

Once upon a time

Rijken, van Neerven

Mitov, Moch

Blümlein, Ravindran

MC, Nason, Oleari

Jaffe, Randall

Melnikov, Mitov

Heavy quark discovery

Dokshitzer, Khoze, Troyan

Peterson, Schlatter, Schmitt and Zerwas

MC, Gardi

MC, Catani

Bjorken

Suzuki

Mele, Nason

Gardi

Aglietti, Corcella, Ferrera

Corcella, Ferrera

1974 1977 1982 1991 1993 1995 1996 2001 2003 2004 2005 2006 2007 2024

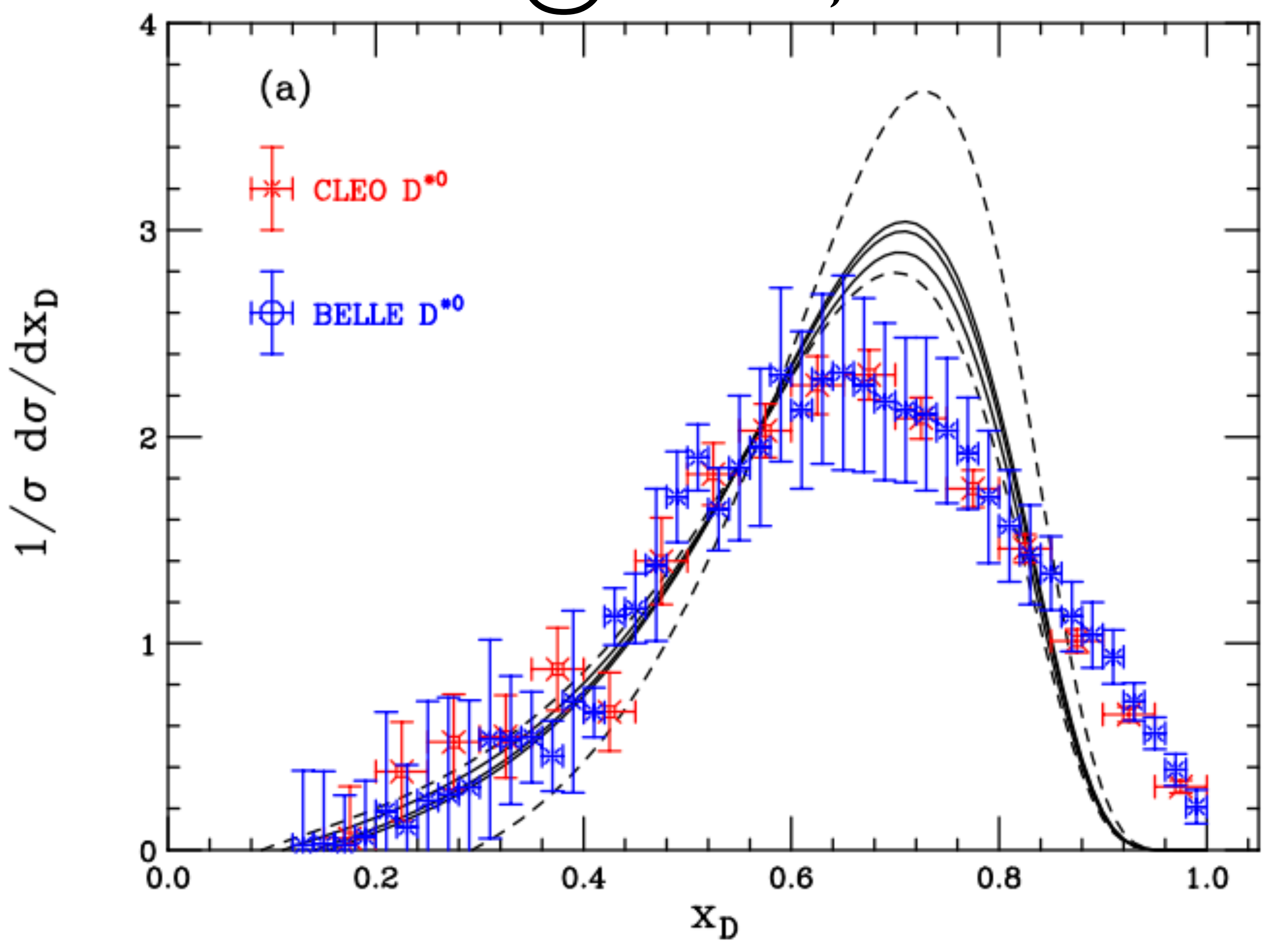
Aglietti, Corcella, Ferrera, 2006+2007

- NNLL soft resummation of initial condition

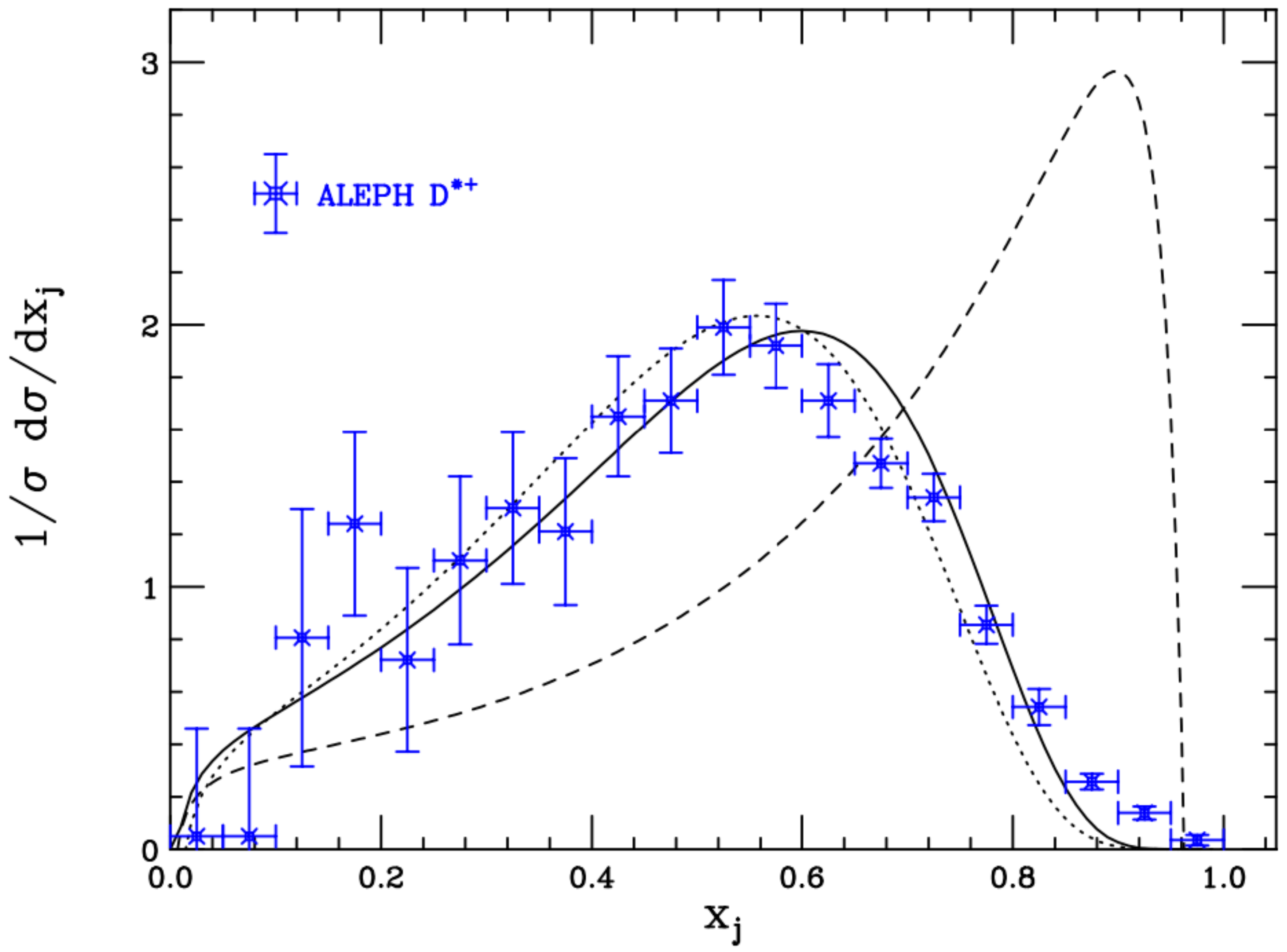
- Inclusion of non-perturbative power corrections via an effecting strong coupling

$$\tilde{\alpha}_S(k^2) = \frac{i}{2\pi} \int_0^{k^2} ds \, \mathrm{Disc}_s \, \frac{\bar{\alpha}_S(-s)}{s}$$

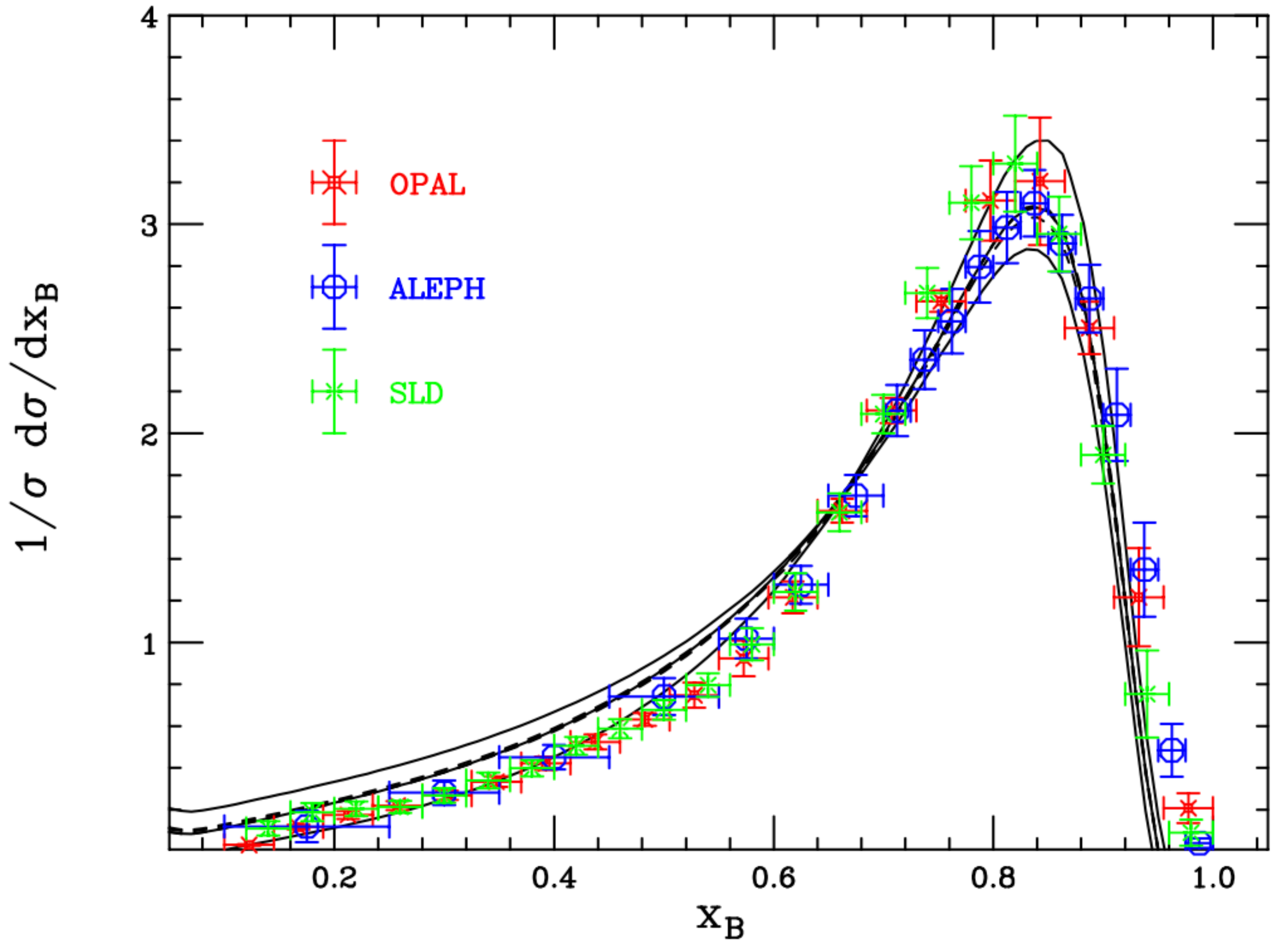
D* @ CLEO, Belle



D* @ LEP



B @ LEP



Model works well for D* and B at LEP. Not so much for D and D* at CLEO, Belle

Once upon a time

Rijken,
van Neerven

Mitov, Moch
Blümlein, Ravindran

MC, Nason, Oleari

Jaffe, Randall

Melnikov, Mitov

Heavy quark
discovery

Dokshitzer, Khoze, Troyan

Peterson, Schlatter,
Schmitt and Zerwas

MC, Gardi

Neubert

MC, Catani

Bjorken
Suzuki

Mele, Nason

Gardi

Aglietti, Corcella, Ferrera
Corcella, Ferrera

1974 1977 1982 1991 1993 1995 1996 2001 2003 2004 2005 2006 2007 2024

Define a new "partially quenched" scheme:

Only the heavy quark (no pairs) allowed here.
All NP information in this function

$$\frac{d\sigma_H}{dx} = \sum_a \frac{d\hat{\sigma}_a^{(n_l+1)}}{dx}(x, \sqrt{s}, \mu) \otimes C_{a/Q}(x, m_Q, \mu) \otimes D_{Q/H}^{(n_l)}(x, m_Q, \mu)$$

with

$$\frac{dC_{a/Q}}{d\ln\mu^2} = \sum_b P_{a\to b}^{(n_l+1)} \otimes C_{b/Q} - C_{a/Q} \otimes P_{Q\to Q}^{(n_l)}, \qquad \frac{dD_{Q/H}^{(n_l)}}{d\ln\mu^2} = P_{Q\to Q}^{(n_l)} \otimes D_{Q/H}^{(n_l)}$$

$D_{Q/H}$ terms of a primordial low-energy shape function $S_{Q/H}$:

$$D_{Q/H}(x, m_Q, \mu) = \frac{M_H\, g(x)}{x} \exp\left[2S(\mu_0, \mu_h) + 2a_{\gamma^S}(\mu_0, \mu_h) + 2a_{\gamma^\phi}(\mu, \mu_h)\right] \left(\frac{\mu_0}{m_Q}\right)^{2a_\Gamma(\mu,\mu_h)}$$

$$\times\, C_D(m_Q, \mu_h)\, \frac{e^{-\gamma_E \eta}}{\Gamma(\eta)} \int_0^{\hat{\omega}} d\hat{\omega}'\, \frac{S_{Q/H}(\hat{\omega}', \mu_0)}{\mu_0^\eta (\hat{\omega} - \hat{\omega}')^{1-\eta}}\,; \qquad \eta = 2a_\Gamma(\mu, \mu_0)$$

Once upon a time

Rijken, van Neerven

Mitov, Moch

Blümlein, Ravindran

MC, Nason, Oleari

Flickinger, Fleming, Kim, Mereghetti

Jaffe, Randall

Melnikov, Mitov

Heavy quark discovery

Dokshitzer, Khoze, Troyan

MC, Gardi

Peterson, Schlatter, Schmitt and Zerwas

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Neubert

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Aglietti, Corcella, Ferrera

Corcella, Ferrera

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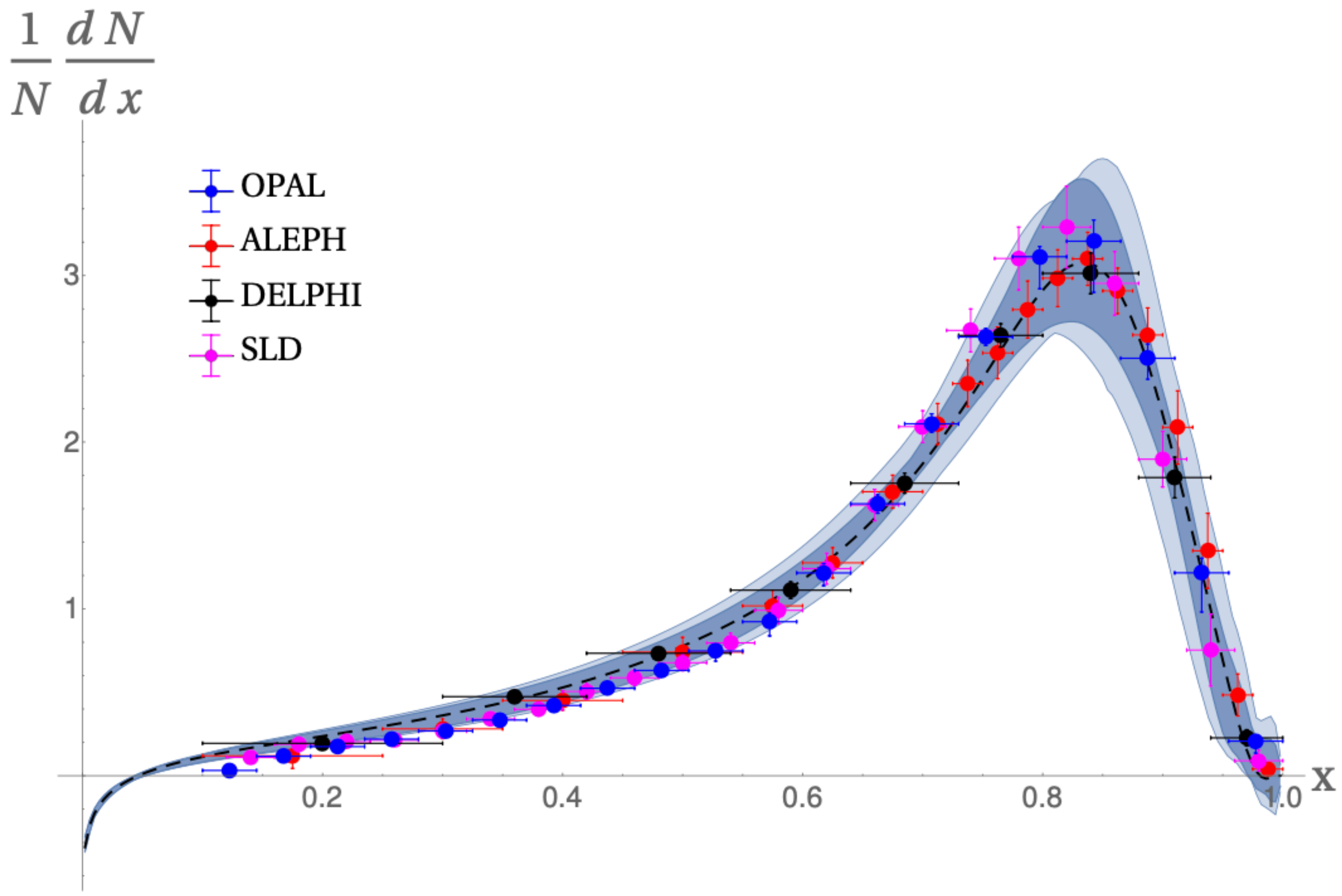
Flickinger, Fleming, Kim, Mereghetti, 2016

Heavy quark fragmentation function in e+e- collisions to NNLO+NNNLL using SCET and bHQET



Good fits to B fragmentation data in e+e- collisions

Once upon a time

Maltoni, Ridolfi, Ubiali, Zaro, 2019+2022

- Calculation and implementation of Mellin transforms of NNLO initial conditions from Melnikov-Mitov 2004

- NNLL collinear evolution

- Calculation of NNLL soft-resummation of initial condition

Once upon a time

Heavy quark discovery

Bjorken
Suzuki

Peterson, Schlatter, Schmitt and Zerwas

Dokshitzer, Khoze, Troyan

Jaffe, Randall

Mele, Nason

Rijken, van Neerven

MC, Nason, Oleari

Melnikov, Mitov

MC, Gardi

MC, Catani

Aglietti, Corcella, Ferrera
Corcella, Ferrera

Mitov, Moch
Blümlein, Ravindran

Neubert

Gardi

Flickinger, Fleming, Kim, Mereghetti

Ridolfi, Ubiali, Zaro

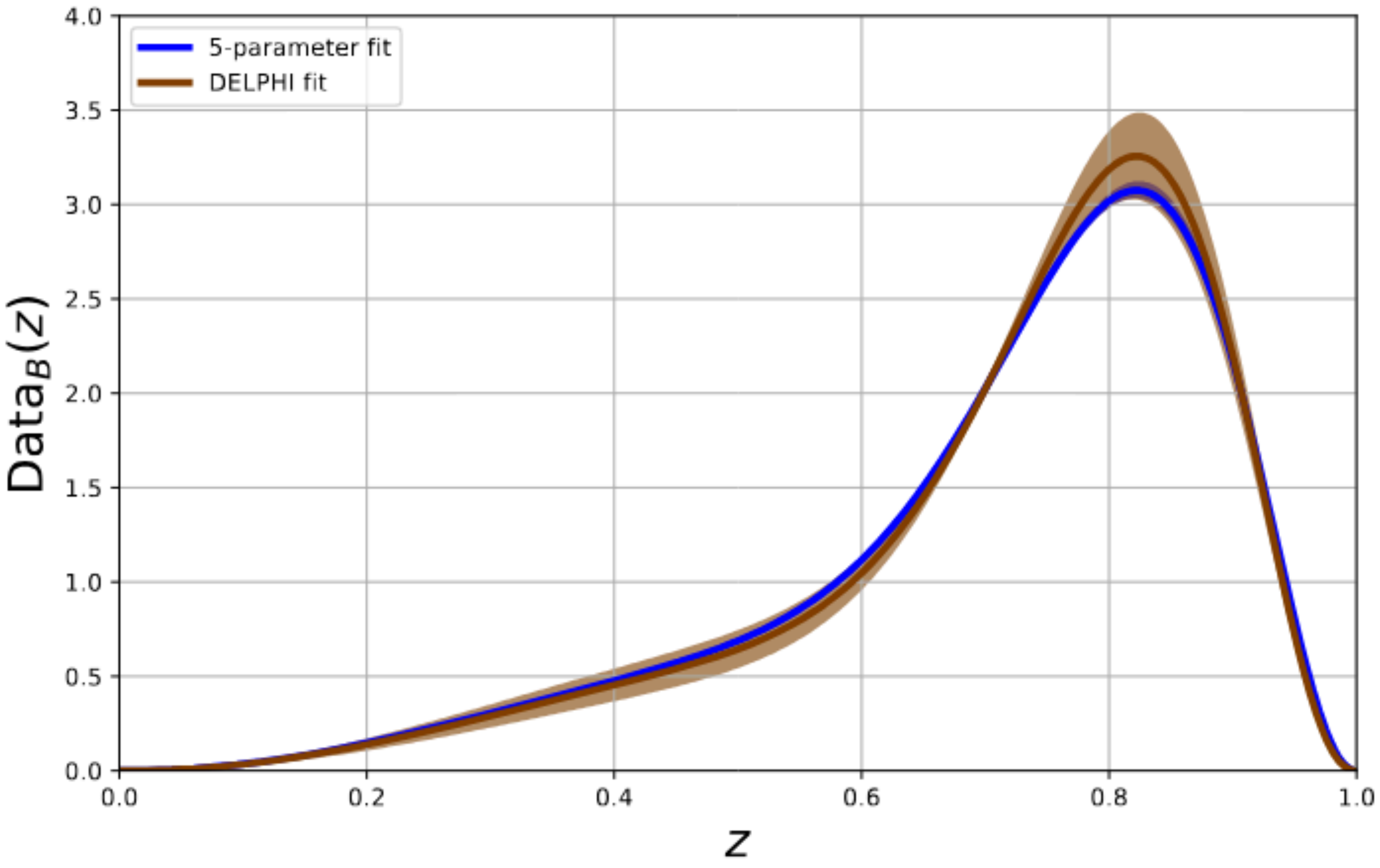
Maltoni, Ridolfi, Ubiali, Zaro

Czakon, Generet, Mitov, Poncelet

1974 1977 1982 1991 1993 1995 1996 2001 2003 2004 2005 2006 2007 2016 2019 2022 2024

Czakon, Generet, Mitov, Poncelet, 2022

B meson fragmentation in e+e- collisions



Implementation of NNLL-resummed fragmentation function, with numerical NNLO coefficient function

Fits to data, and use in hadronic collisions

Once upon a time

Rijken, van Neerven

Mitov, Moch

Blümlein, Ravindran

Czakon, Generet, Mitov, Poncelet

MC, Nason, Oleari

Flickinger, Fleming, Kim, Mereghetti

Jaffe, Randall

Melnikov, Mitov

Heavy quark discovery

Dokshitzer, Khoze, Troyan

Ridolfi, Ubiali, Zaro

MC, Gardi

Peterson, Schlatter, Schmitt and Zerwas

MC, Catani

Neubert

Maltoni, Ridolfi, Ubiali, Zaro

Bjorken

Suzuki

Mele, Nason

Gardi

Bonino, MC, Stagnitto

Aglietti, Corcella, Ferrera

Corcella, Ferrera

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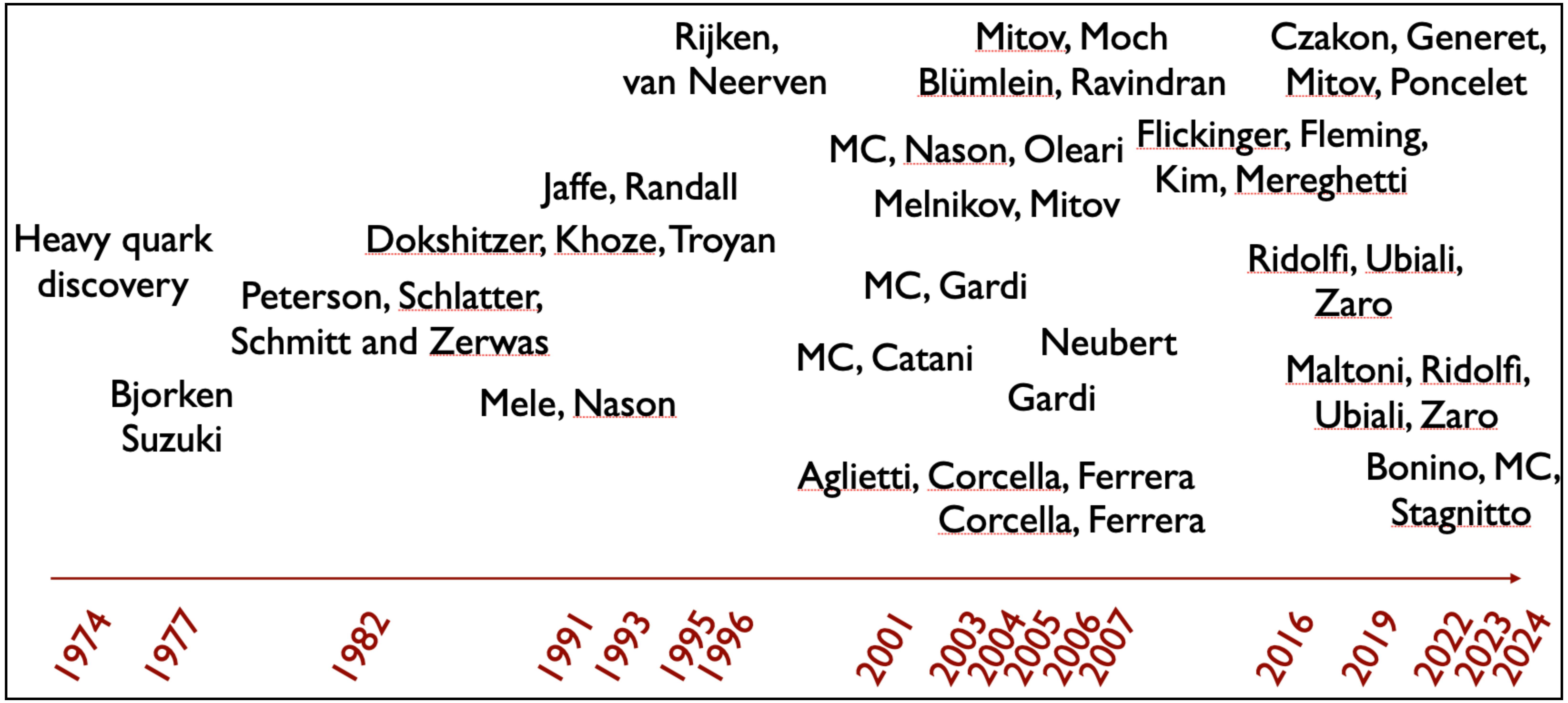
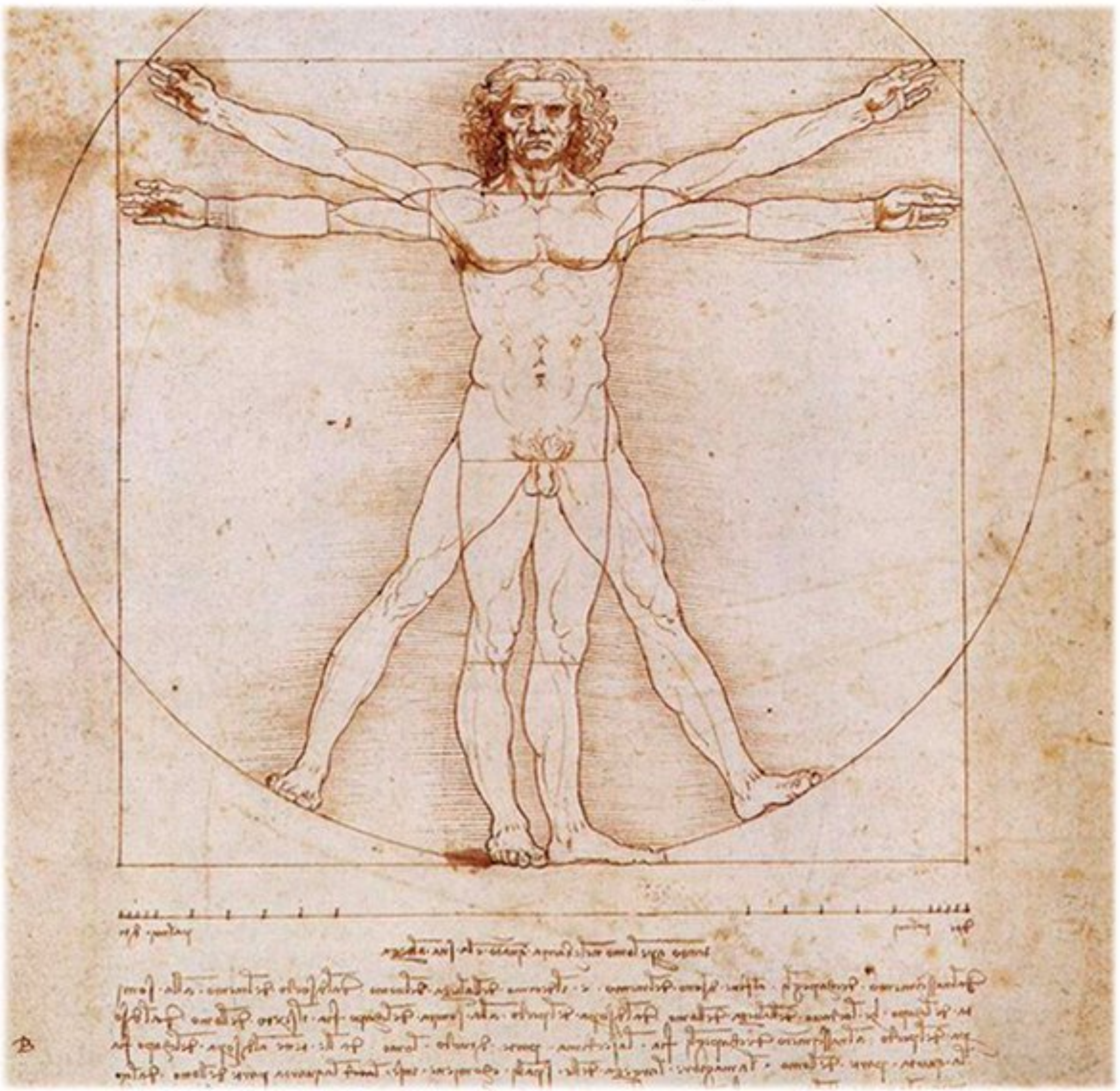
e+e- to heavy quark fragmentation



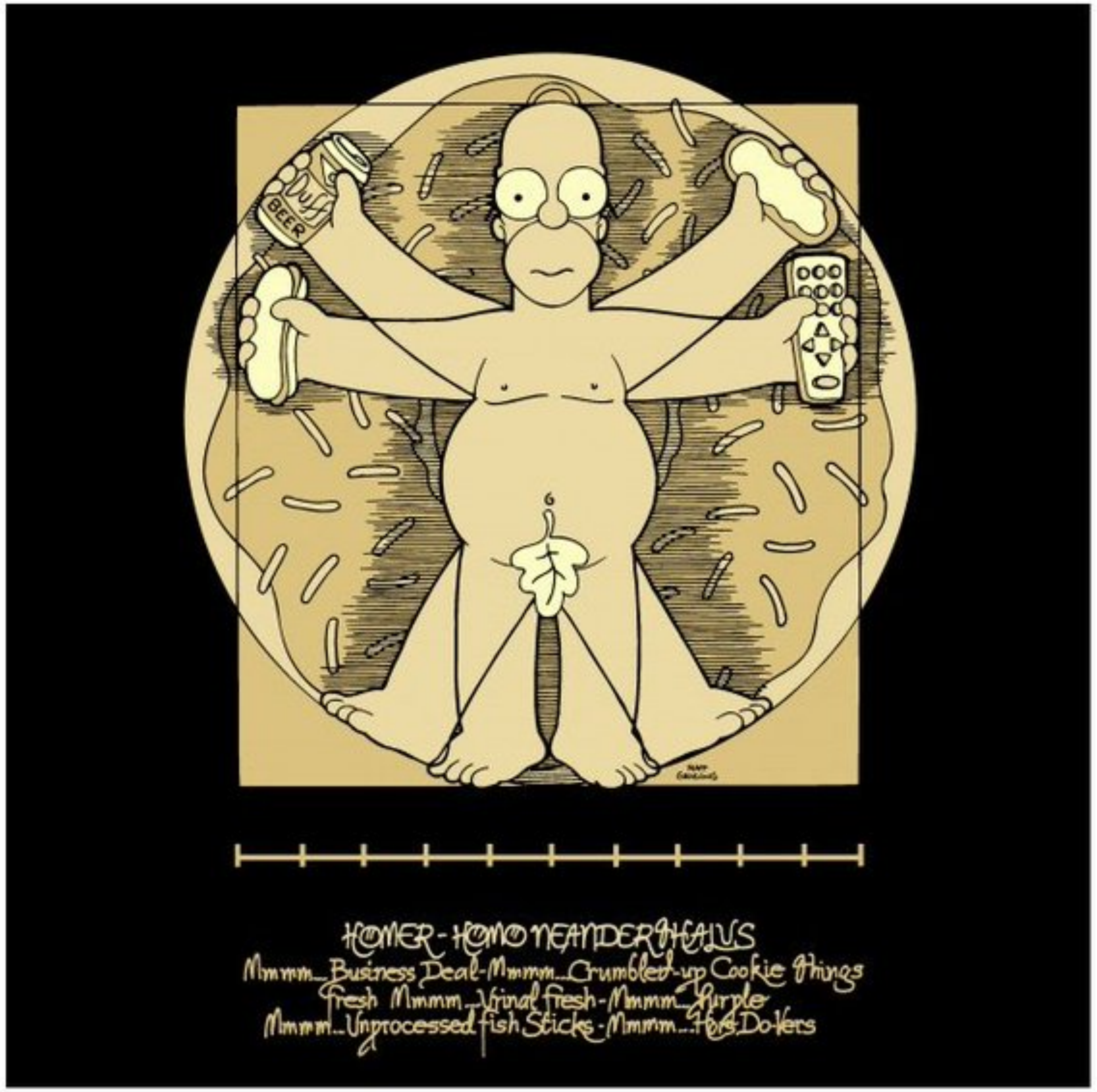
Thanks to these and many other papers, we can now (re)consider heavy quark fragmentation to overall **N**NLO + **N**NLL$_{coll}$ + **N**NLL$_{soft}$ accuracy

What kind of precision do we expect ?

Expectation



Reality



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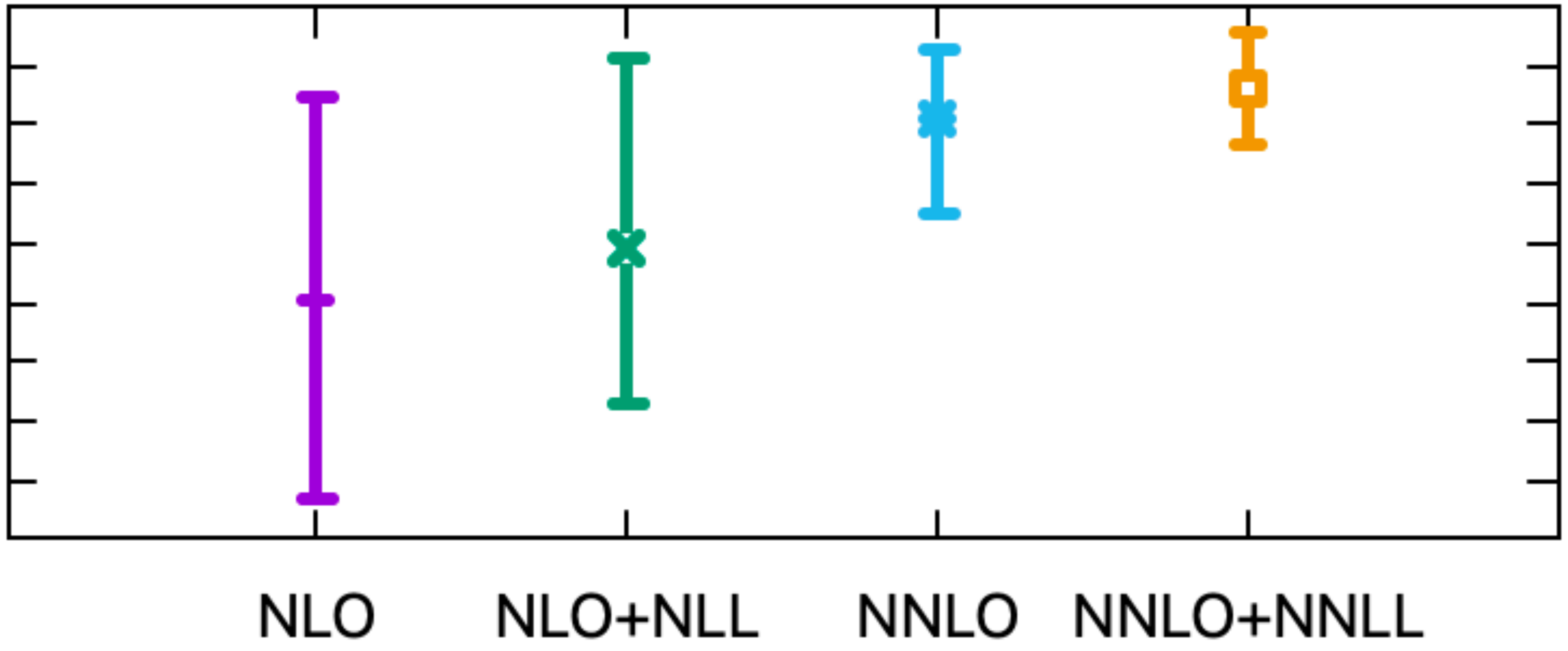
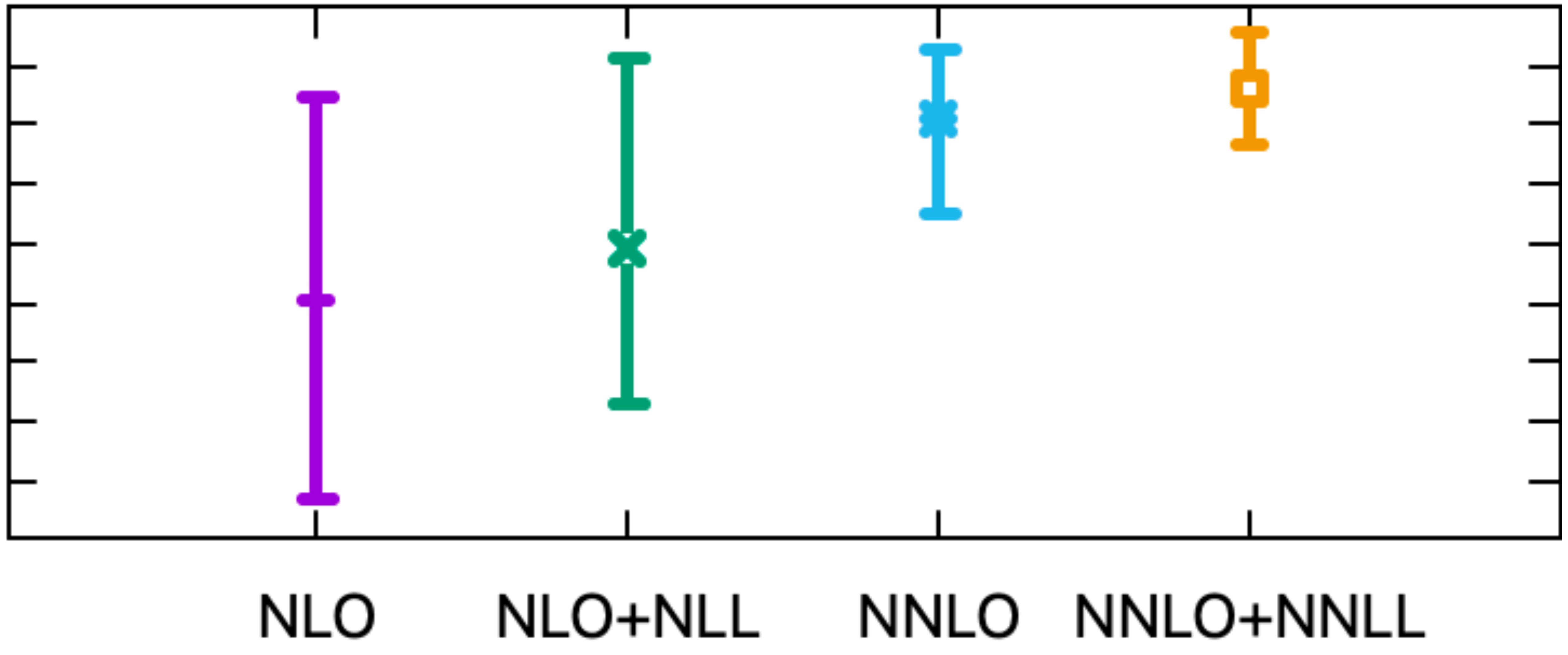
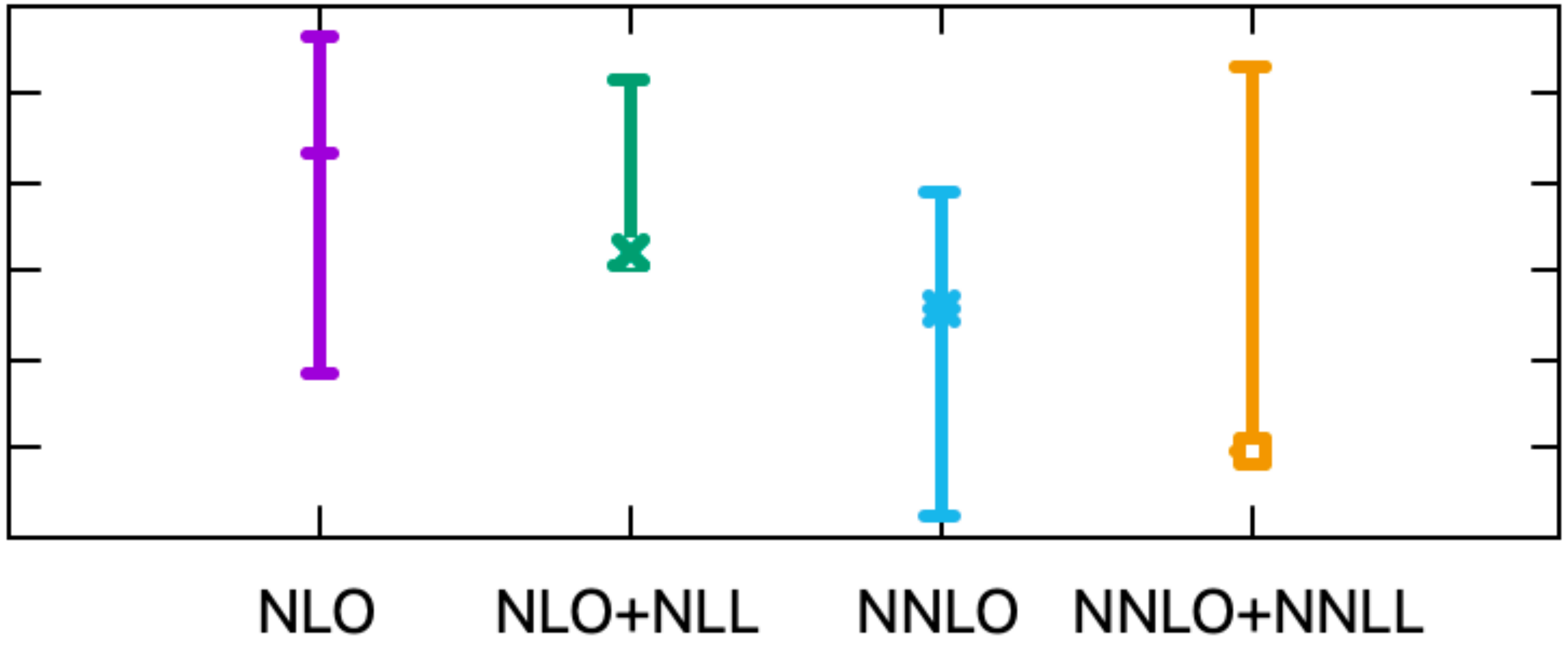
Expectation



Expectation



Reality



Putting it all together

Bonino, MC, Stagnitto, 2312.12519

$$e^+e^- \to \gamma, Z \to H_Q + X$$

- Up to NNLO coefficient function
- Up to NNLO initial condition (= decay function = pFF = FF = …)
- Up to NNLL collinear resummation
- Up to NNLL soft resummation → matching to fixed order
  - Landau pole regularisations
- Phenomenological non-perturbative fragmentation functions
- Modular and (eventually) public C++ library
  - Mellin moments and x-space results
  - Fits to experimental data

The cross section

$$\frac{d\sigma_H}{dx}(x,\sqrt{s}) \simeq \frac{d\sigma_Q}{dx}(x,\sqrt{s},m) \otimes D^{np}_{Q\to H}(x,\{\text{params}\})$$

Perturbative
(with resummations)

Non-perturbative

Moments or x-space distribution

$$\sigma_Q(\sqrt{s},m) = \hat{\sigma}_i(\sqrt{s},\mu_F,\mu_R) \otimes E_{ij}(\mu_F,\mu_{F0}) \otimes D_{j\to Q}(m,\mu_{F0},\mu_{R0})$$

Coefficient functions

DGLAP evolution (MELA)

Initial conditions (Decay functions) (Fragmentation functions)

Thanks to Moch, De Florian, Maltoni, Ridolfi, Ubiali, Zaro, for providing Fortran implementations of the NNLO coefficient functions and initial conditions

The components of σ_Q, the **coefficient functions** $\hat{\sigma}_i$ and the **initial conditions** $D_{j\to Q}$, are calculated to a given **perturbative order**, with or without **soft resummation matched** to the fixed order, with or without **Landau pole regularisation**

$$\sigma_Q^{fo+res,match,reg}(\cdot, \sqrt{s}, \mu_R, \mu_F, \mu_{0R}, \mu_{0F}, m)$$

The final result also has residual factorisation and renormalisation scale dependence

Additive

$$D_{i\to Q}^{fo+res\,\boxed{\mathrm{add}}\,reg} = D_{i\to Q}^{fo} + D_{i\to Q}^{res,reg} - [D_{i\to Q}^{res(,reg)}]_{\alpha_s^p}$$

log-R

$$\log D_{i\to Q}^{fo+res\,\boxed{\mathrm{logR}}\,reg} = \log D_{i\to Q}^{fo} + \log D_{i\to Q}^{res,reg} - [\log D_{i\to Q}^{res(,reg)}]_{\alpha_s^p}$$

Landau pole

Moments of soft-resummed coefficient functions and initial conditions have poles respectively at

$$N^L = \exp\left(\frac{1}{b_0 \alpha_s(\mu^2)}\right) \qquad \text{and} \qquad N_0^L = \exp\left(\frac{1}{2b_0 \alpha_s(\mu_0^2)}\right)$$

~ 7 for charm
~ 30 for bottom

- Signal of onset of non-perturbative physics
- Perturbative moments unphysical beyond the Landau poles
- x-space distributions (with Minimal Prescription) highly irregular near x=1

Landau pole regularisation

An ad hoc regularisation allows one to make the resummed moments better behaved, i.e. more **physical-looking** (though not necessarily more physical or "accurate")

CNO

MC, Oleari, Nason 05

$$N \to N \frac{1 + f/N_0^L}{1 + fN/N_0^L}$$

Rescale N so as to shift the pole to higher moments

$$D_{i\to Q}^{fo+res,match,\mathrm{CNO(f)}}$$

CGMP

Czakon, Generet, Mitov, Poncelet 22

$$\begin{aligned}
&\exp\left[\ln N g_0^{(1)}(\lambda_0) + g_0^{(2)}(\lambda_0) + \alpha_s g_0^{(3)}(\lambda_0)\right] \\
&\simeq g_2^{(1)}\alpha_s \ln^2(N) + g_3^{(1)}\alpha_s^2 \ln^3(N) + g_4^{(1)}\alpha_s^3 \ln^4(N) + g_5^{(1)}\alpha_s^4 \ln^5(N) + g_6^{(1)}\alpha_s^5 \ln^6(N) \\
&+ g_1^{(2)}\alpha_s \ln(N) + g_2^{(2)}\alpha_s^2 \ln^2(N) + g_3^{(2)}\alpha_s^3 \ln^3(N) + g_4^{(2)}\alpha_s^4 \ln^4(N) \\
&+ g_1^{(3)}\alpha_s^2 \ln(N) + g_2^{(3)}\alpha_s^3 \ln^2(N) .
\end{aligned}$$

Expand and truncate the Sudakov exponential

$$D_{i\to Q}^{fo+res,match,\mathrm{CGMP}}$$

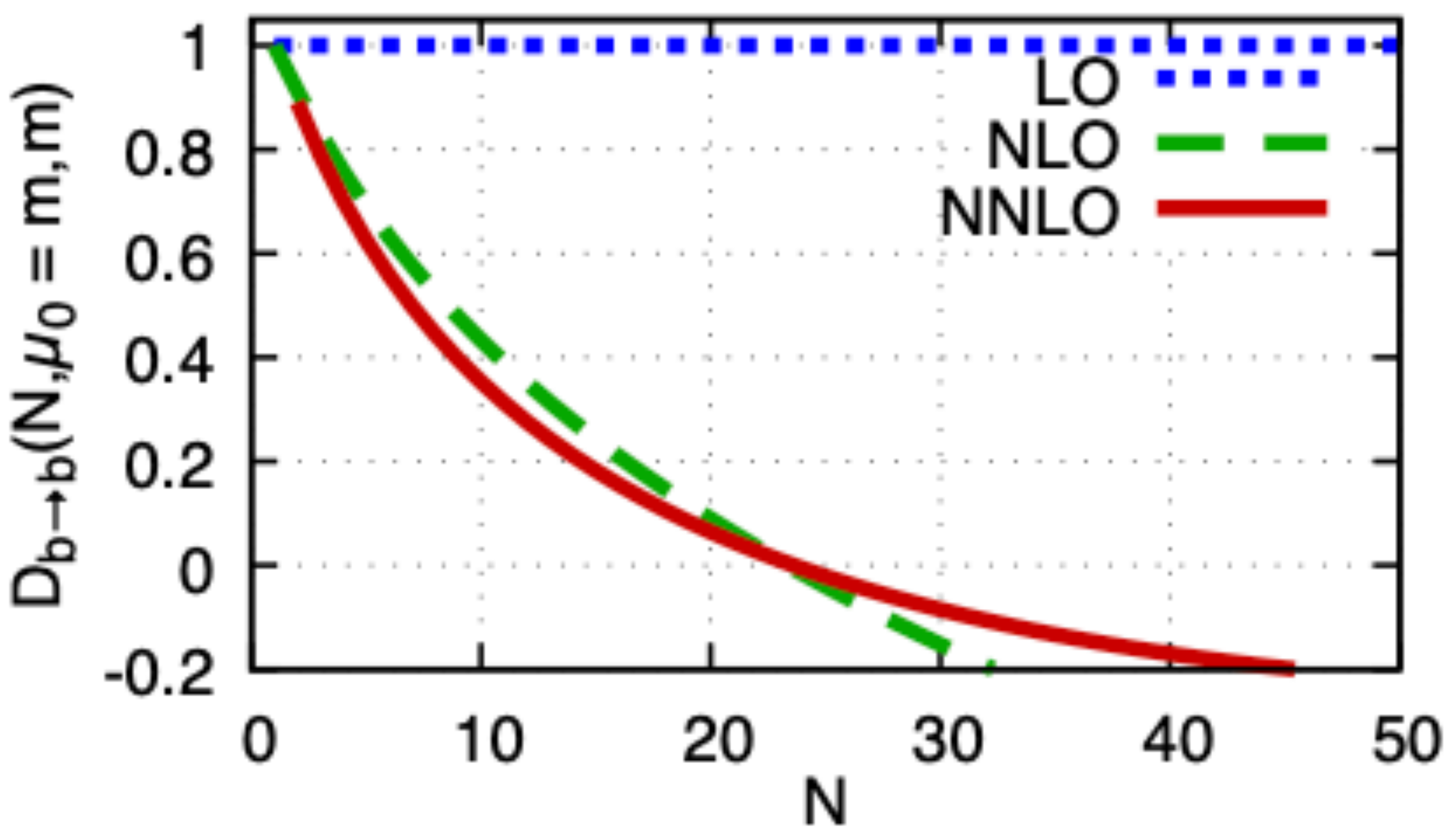
Bottom initial condition

bottom initial condition, m=4.75 GeV

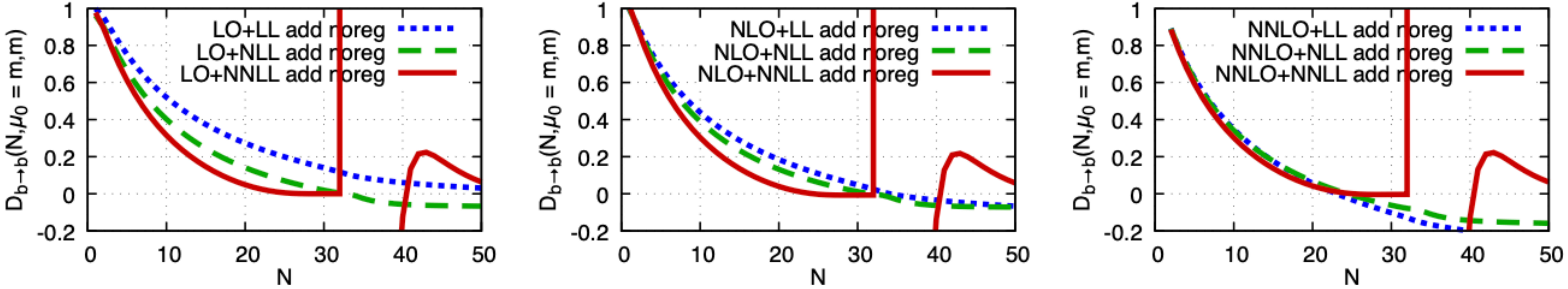
$\mu_0 = \mu_{0R} = \mu_{0F} = m$

$\alpha_S(Q=91.2\text{ GeV}) = 0.118$

$\alpha_S(m) = 0.21593775$



Fixed order



Additive matching, no Landau pole regularisation

Bottom initial condition

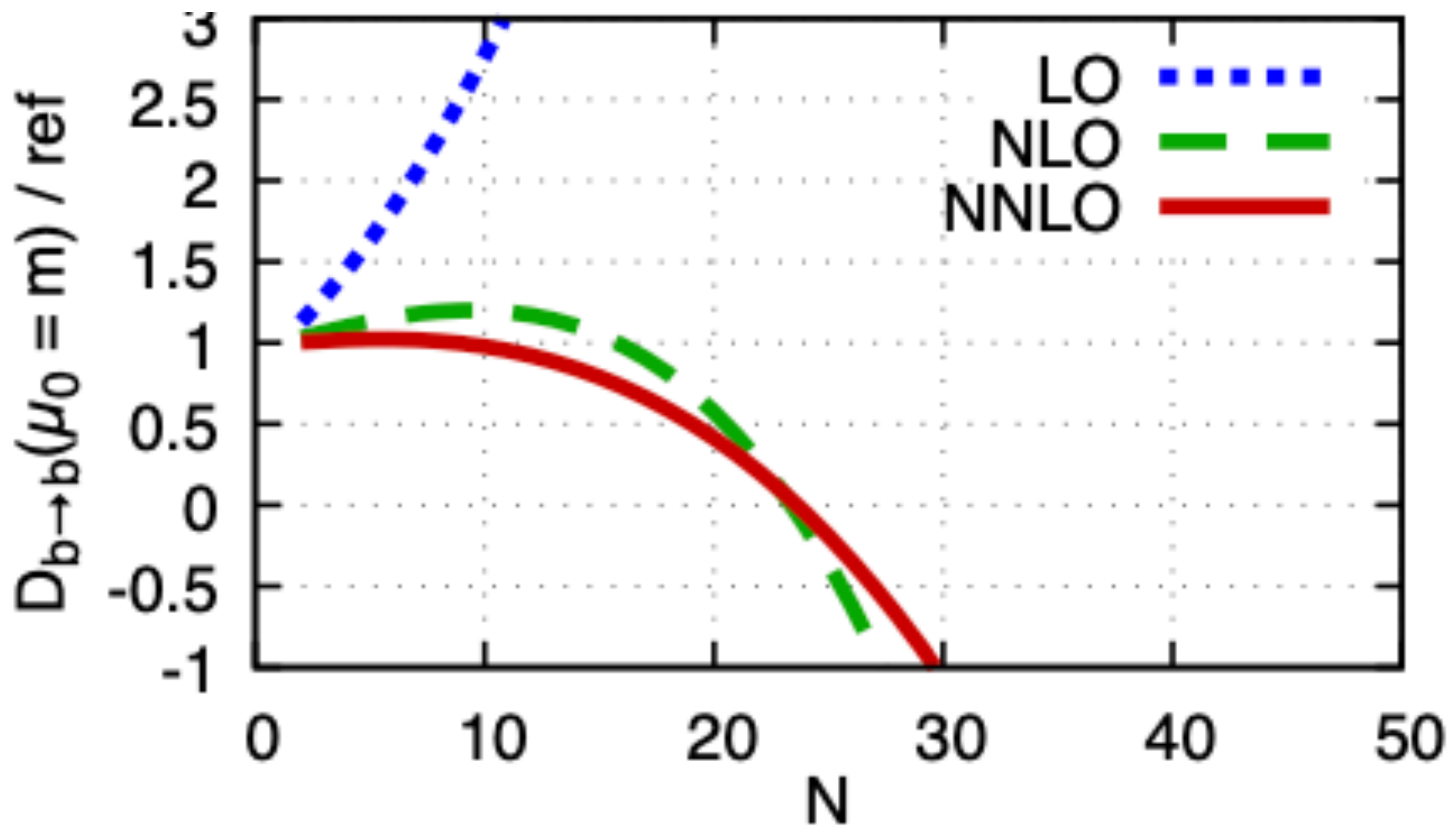
Additive matching,
without Landau pole regularisation

bottom initial condition, m=4.75 GeV

$\mu_0 = \mu_{0R} = \mu_{0F} = m$

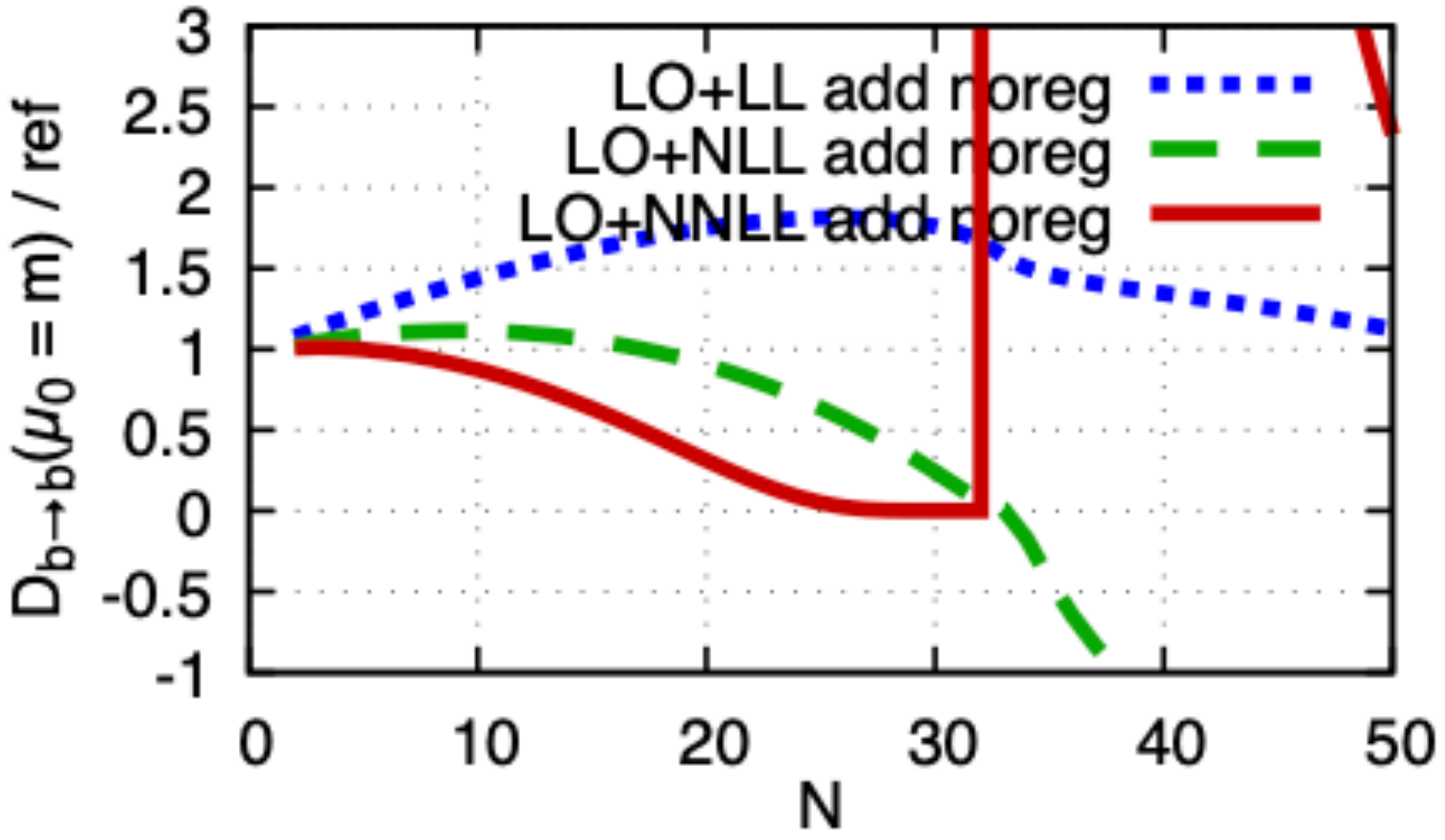
$\alpha_S(Q=91.2\ \mathrm{GeV}) = 0.118$

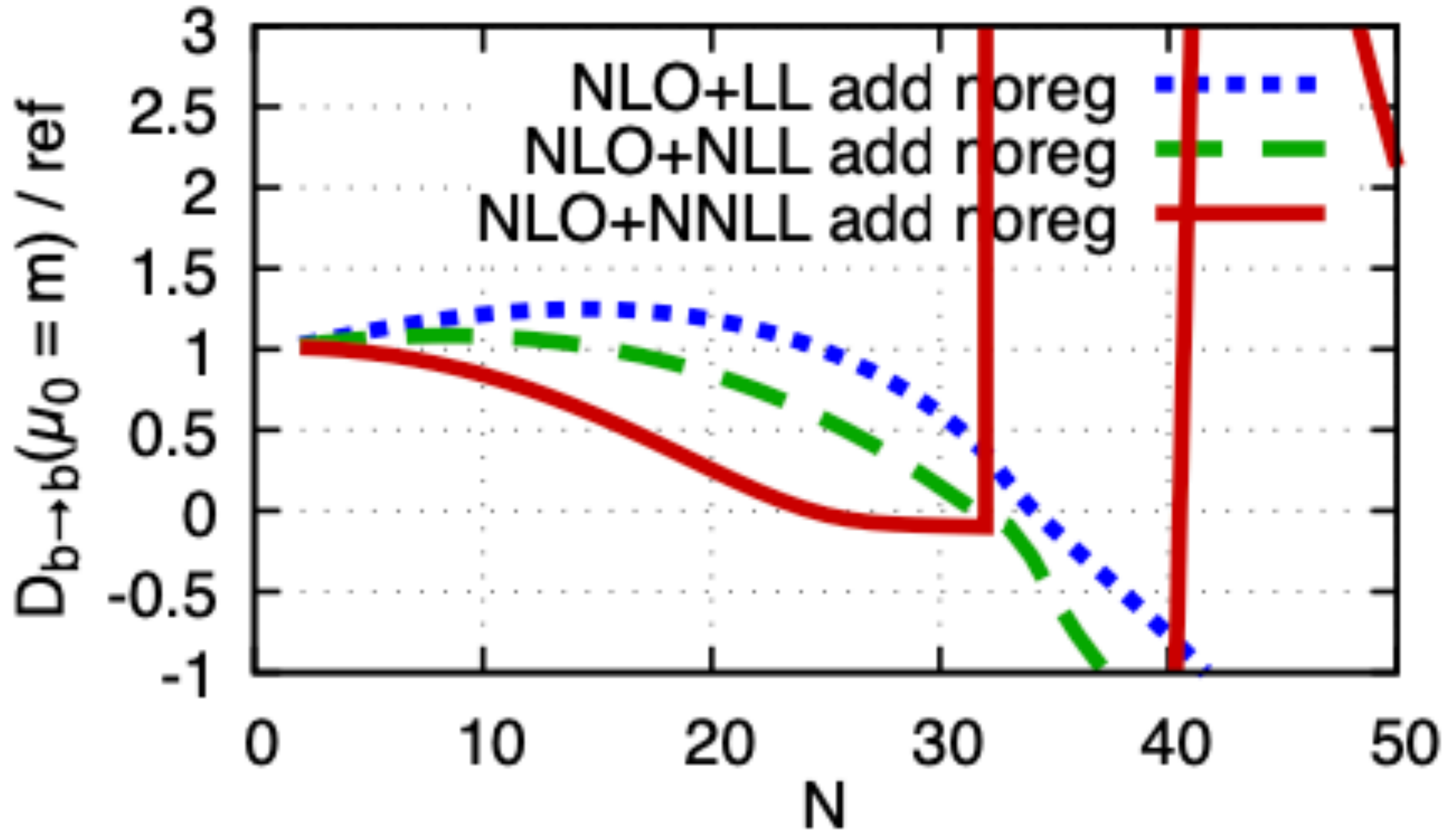
$\alpha_S(m) = 0.21593775$

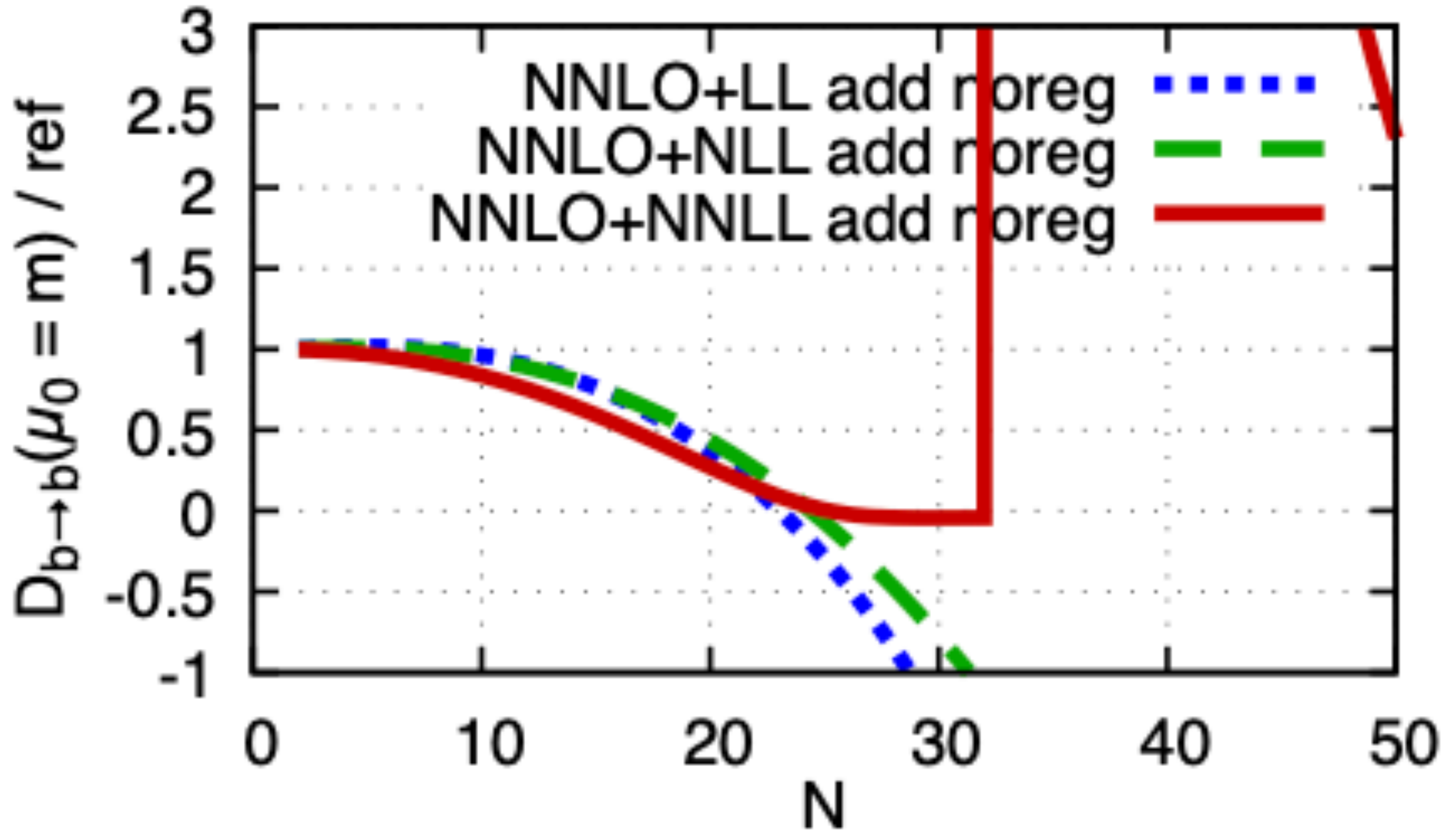


Ratio to a single curve

ref = 'NNLO+NNLL logR CNO(1.25)'



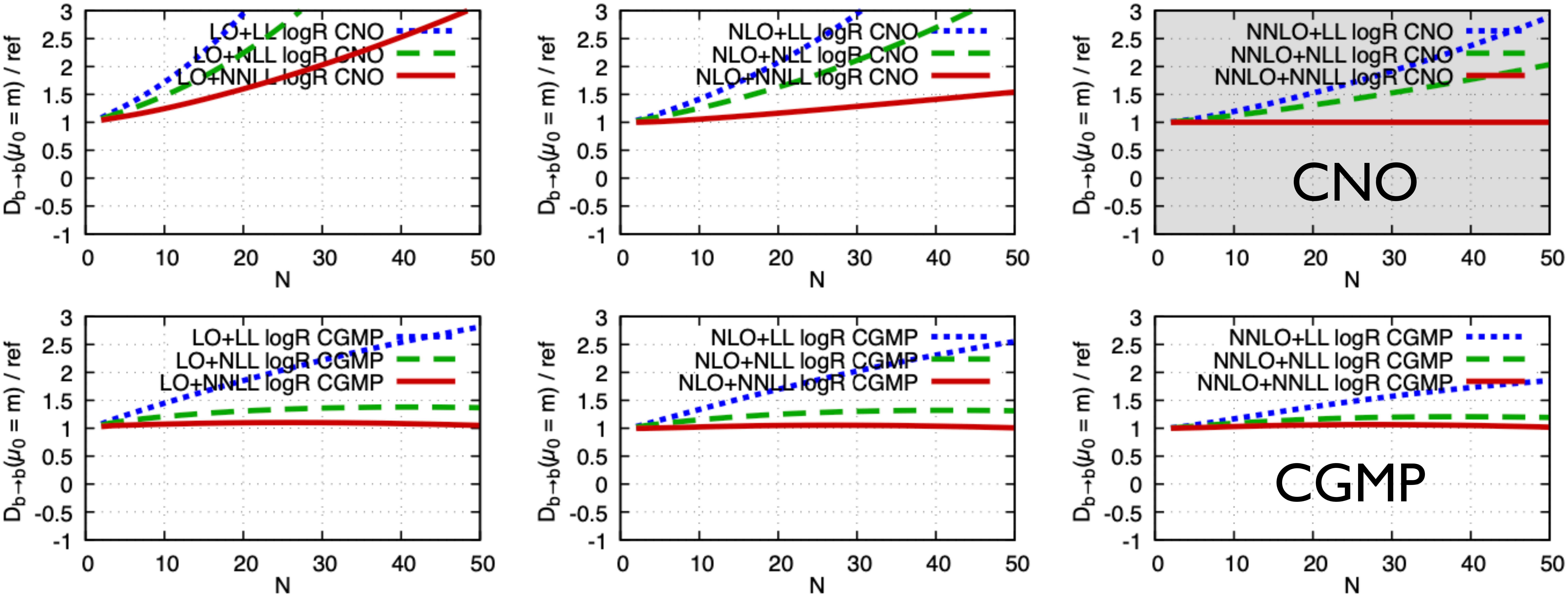




No obvious perturbative hierarchy NNLL < NLL < LL

Log-R matching,
with Landau pole regularisation

Bottom initial condition



log-R matching with CGMP Landau pole regularisation displays the expected hierarchy, NNLL < NLL < LL, but also leads to other problems

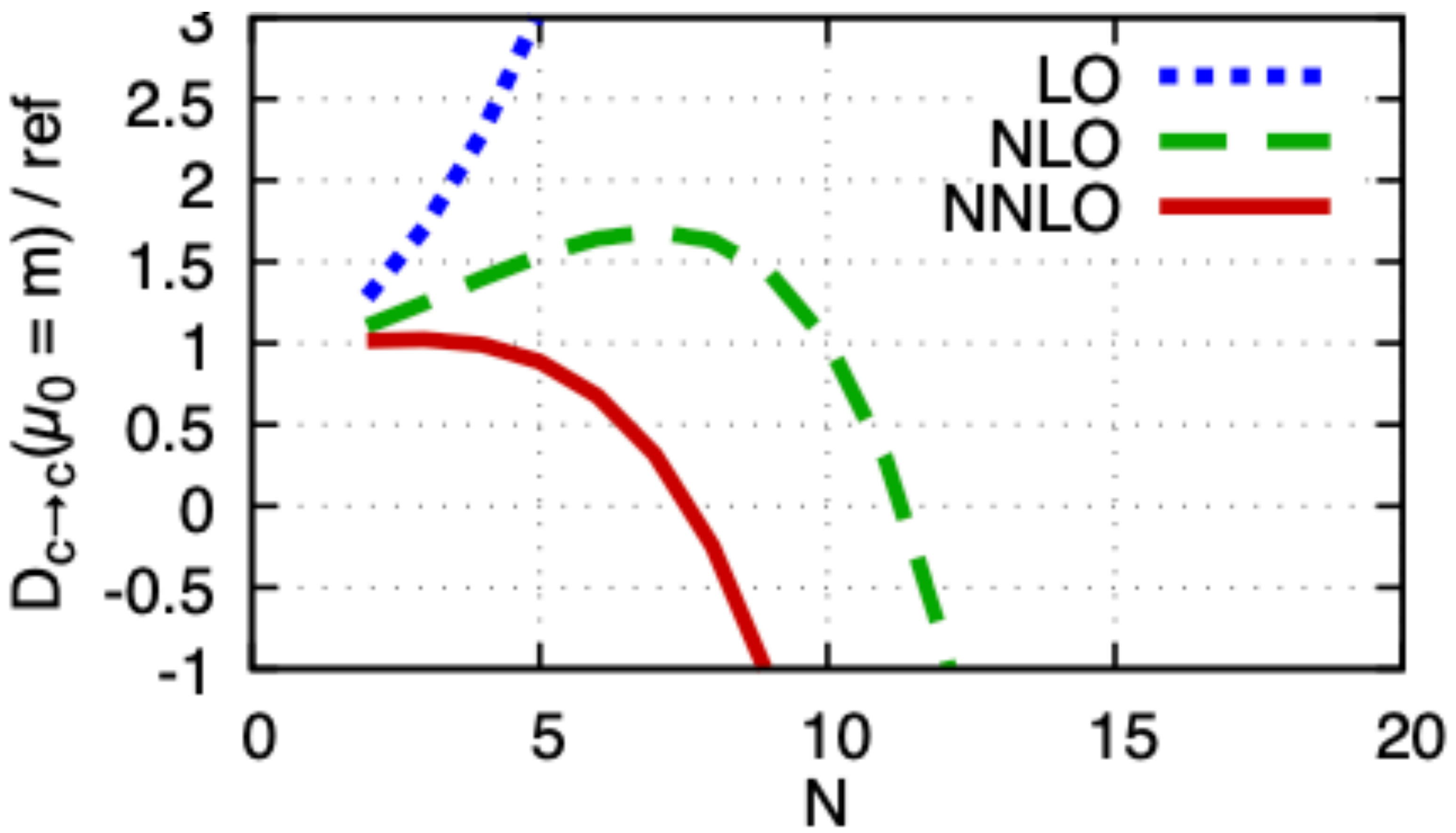
Charm initial condition

charm initial condition, m=1.5 GeV

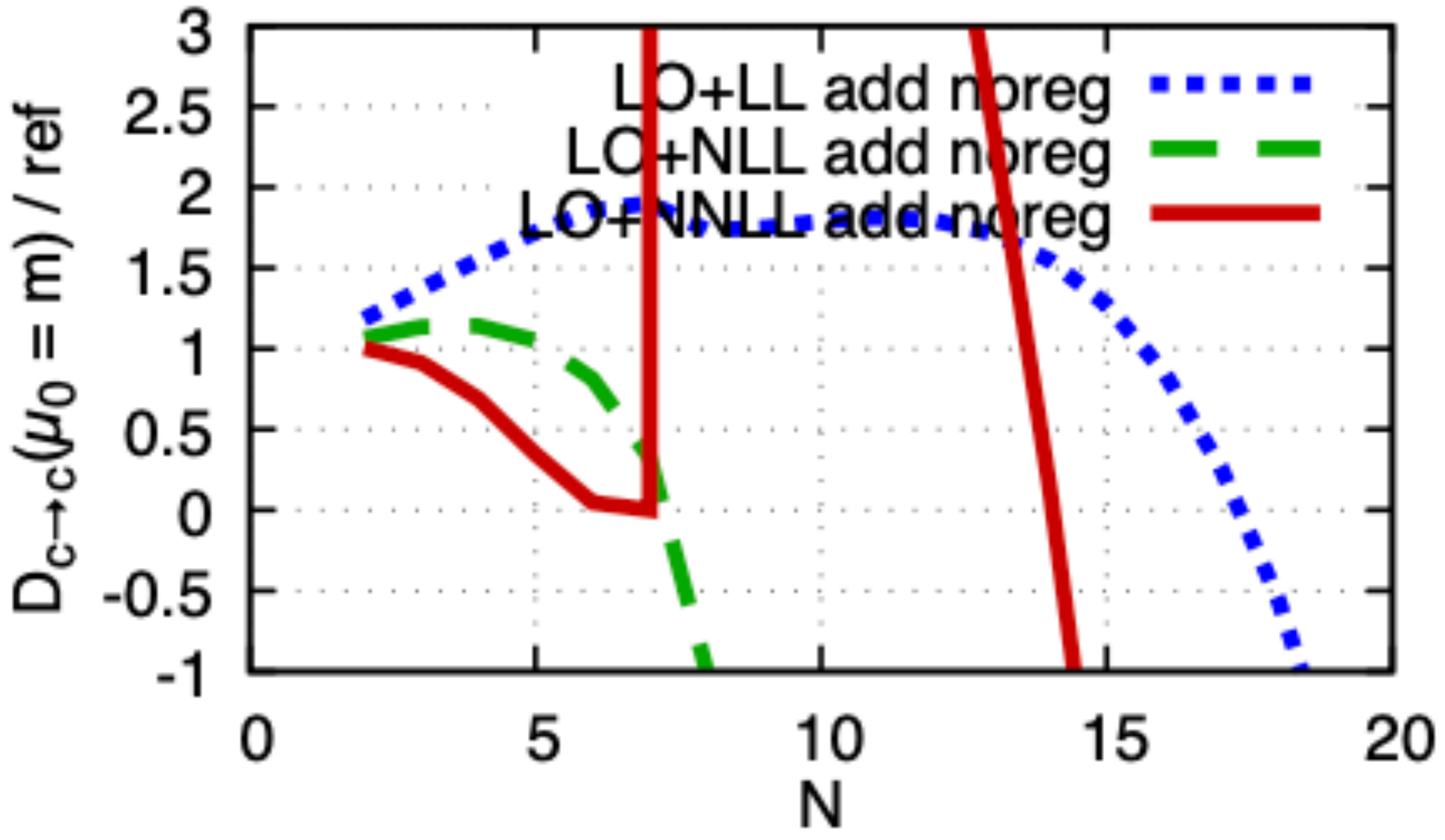
$\mu_0 = \mu_{0R} = \mu_{0F} = m$

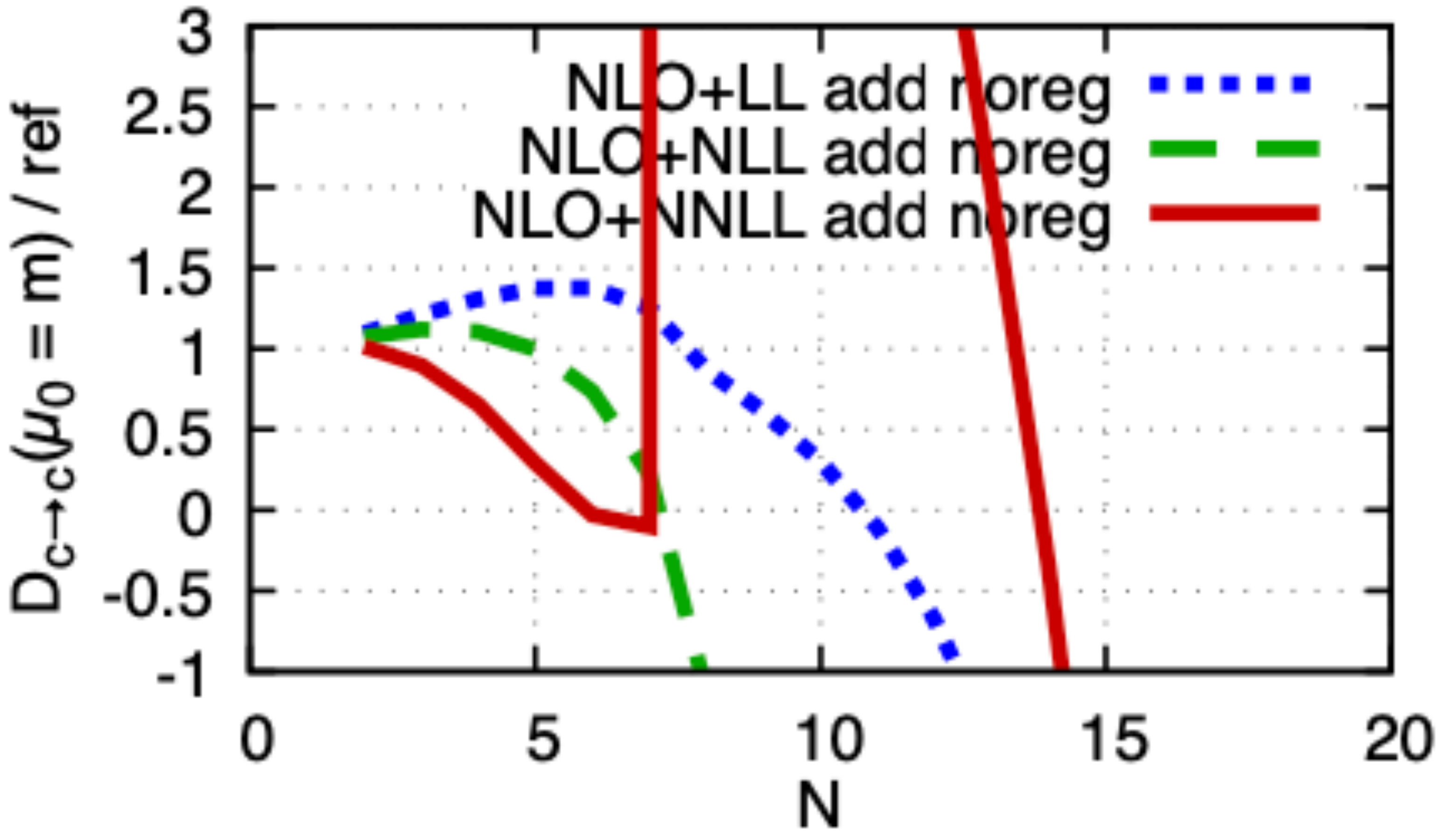
α_S(Q=91.2 GeV) = 0.118

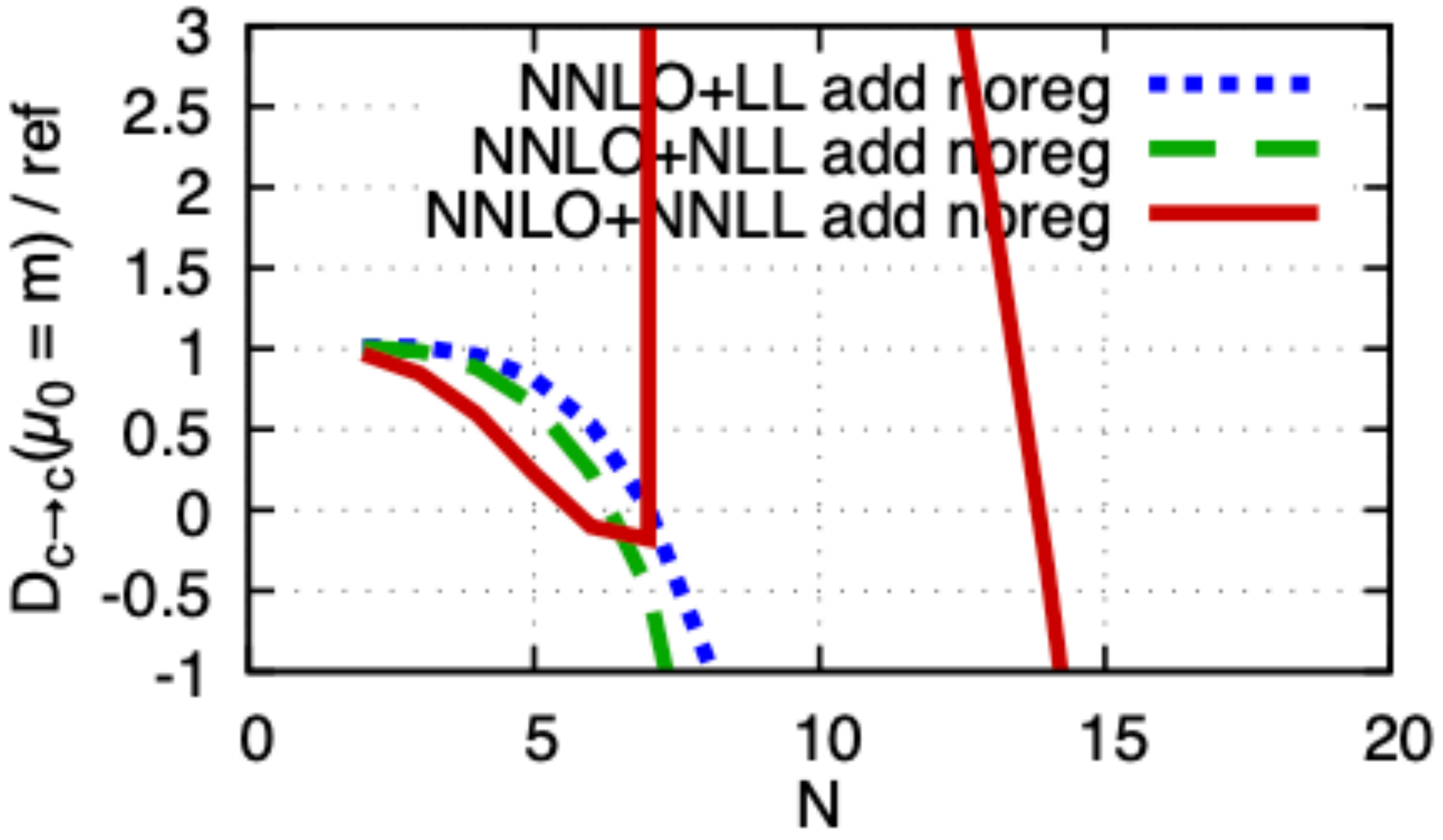
α_S(m) = 0.34731228



ref = 'NNLO+NNLL logR CNO(1.25)'

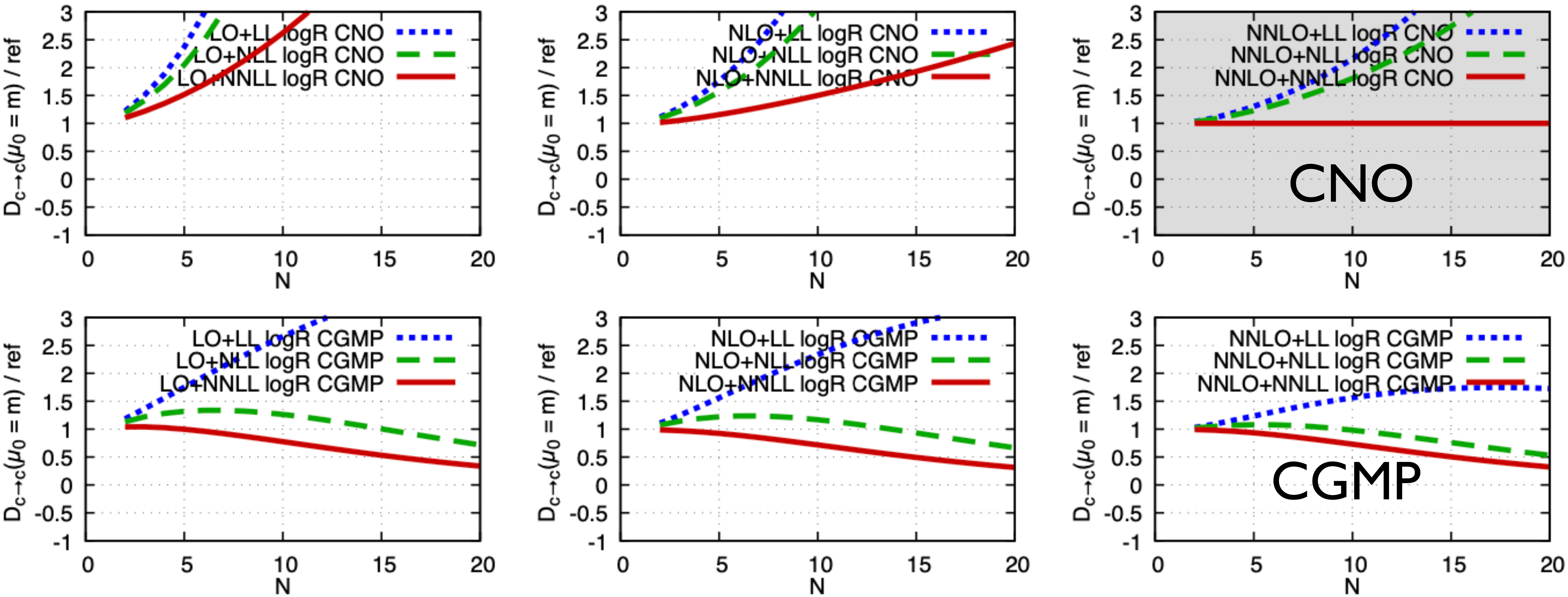




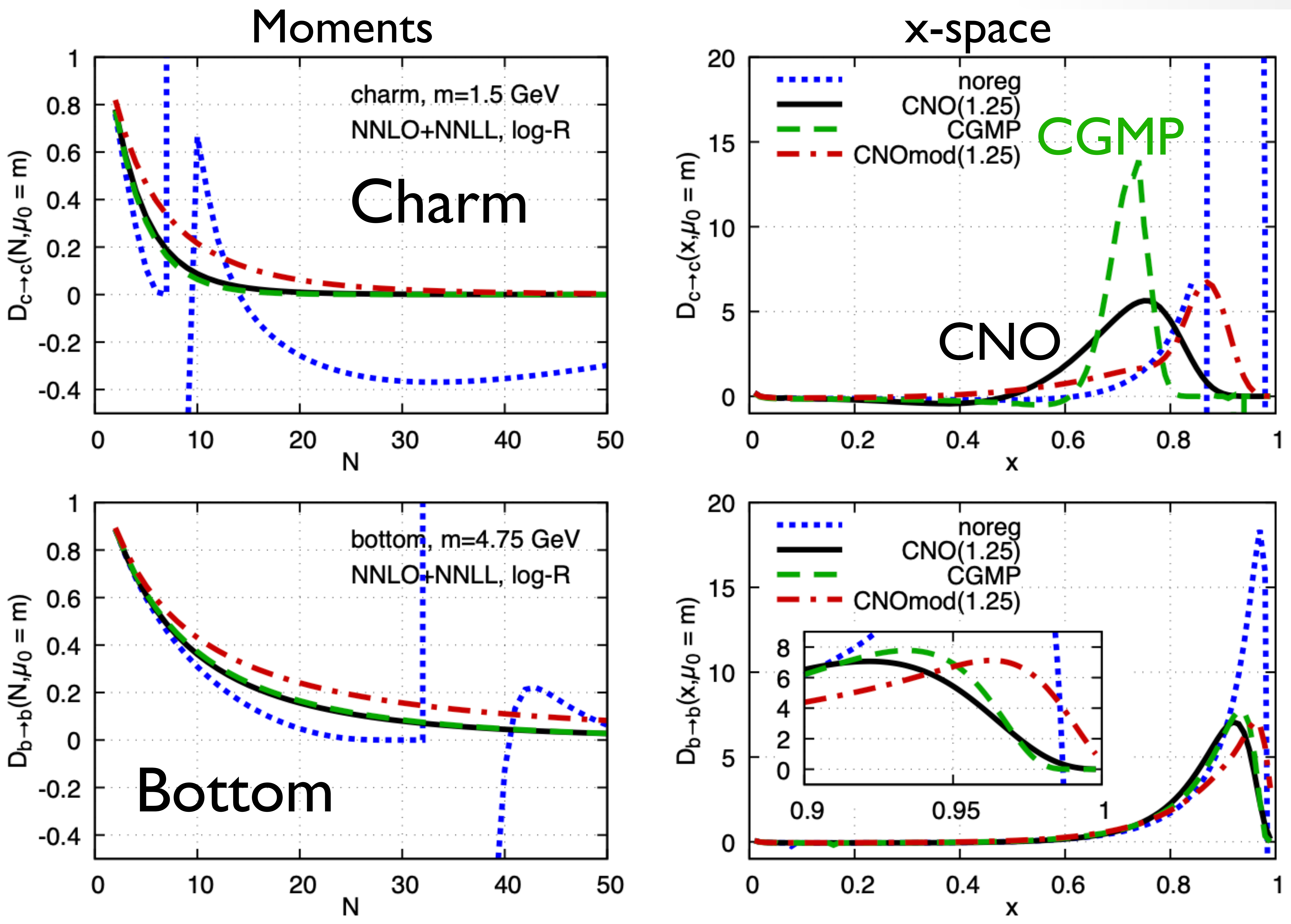


Charm initial condition

Log-R matching,
with Landau pole regularisation



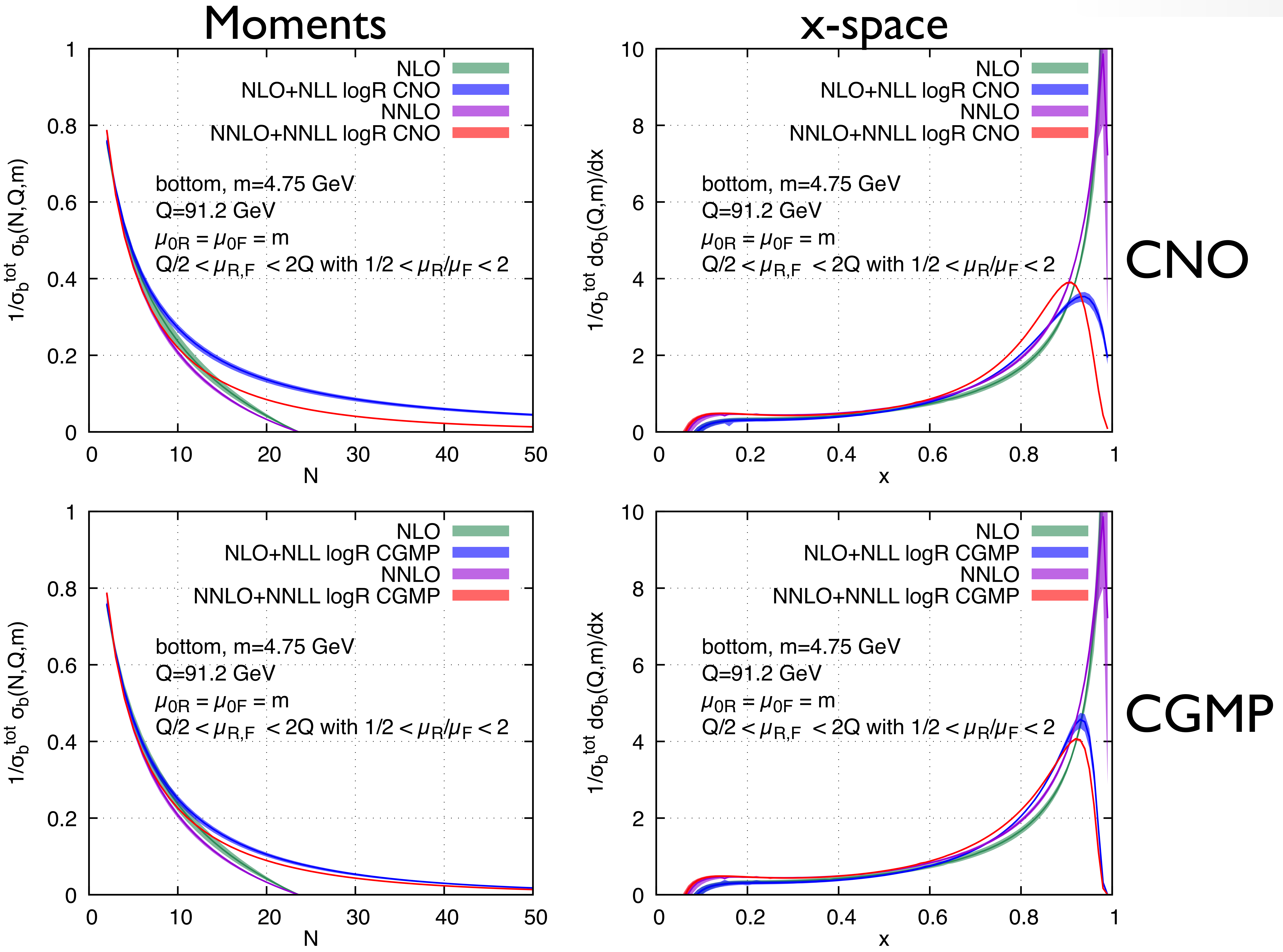
Charm v. Bottom initial conditions



CNO and, to a larger extent, CGMP problematic for charm (too rapid fall-off)

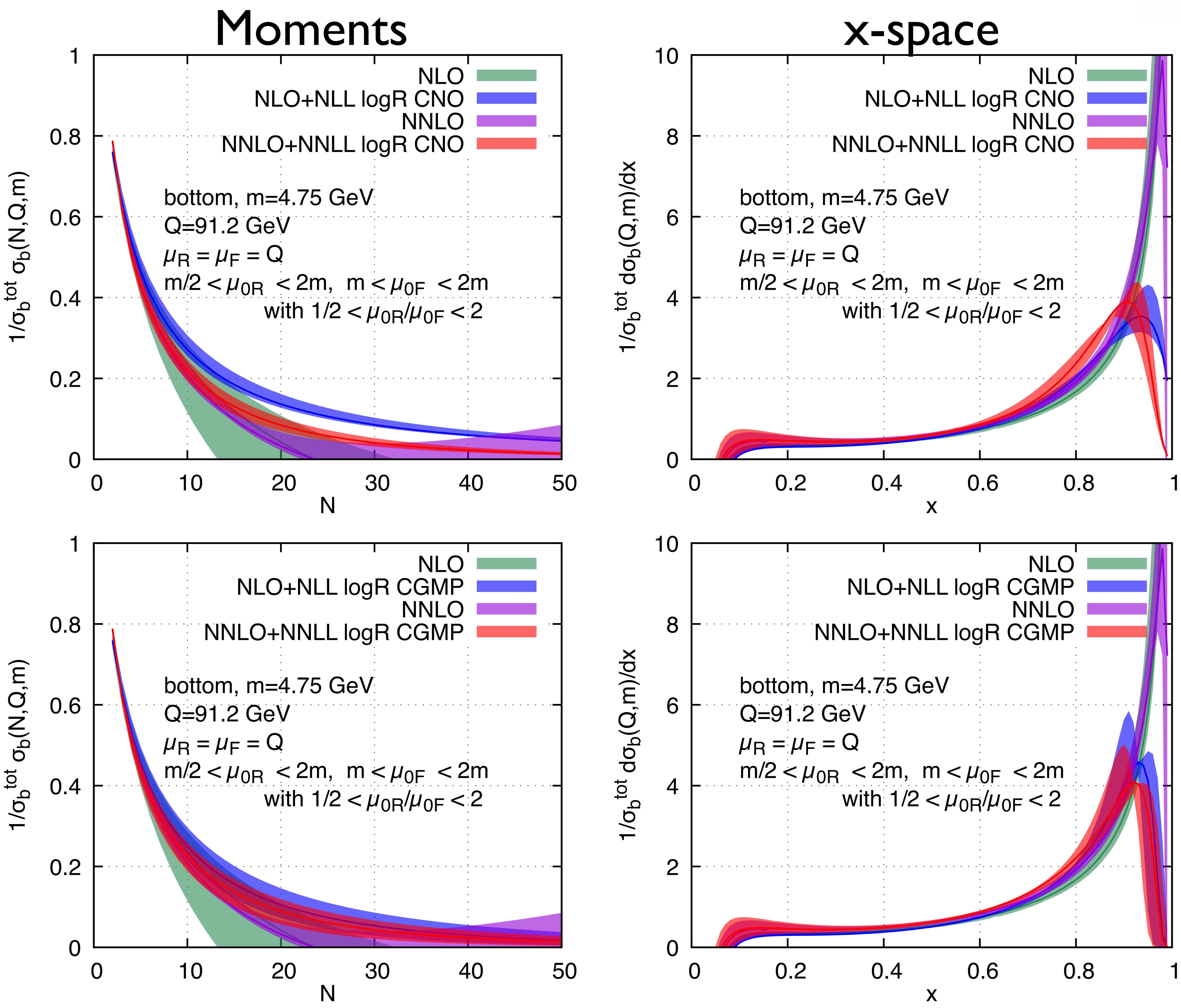
Both CGMP and CNO well behaved for bottom

Bottom e+e- FF: **final** scales variations



Bands shrink as expected at higher orders, but do not always overlap

Bottom e+e- FF: **initial** scales variations

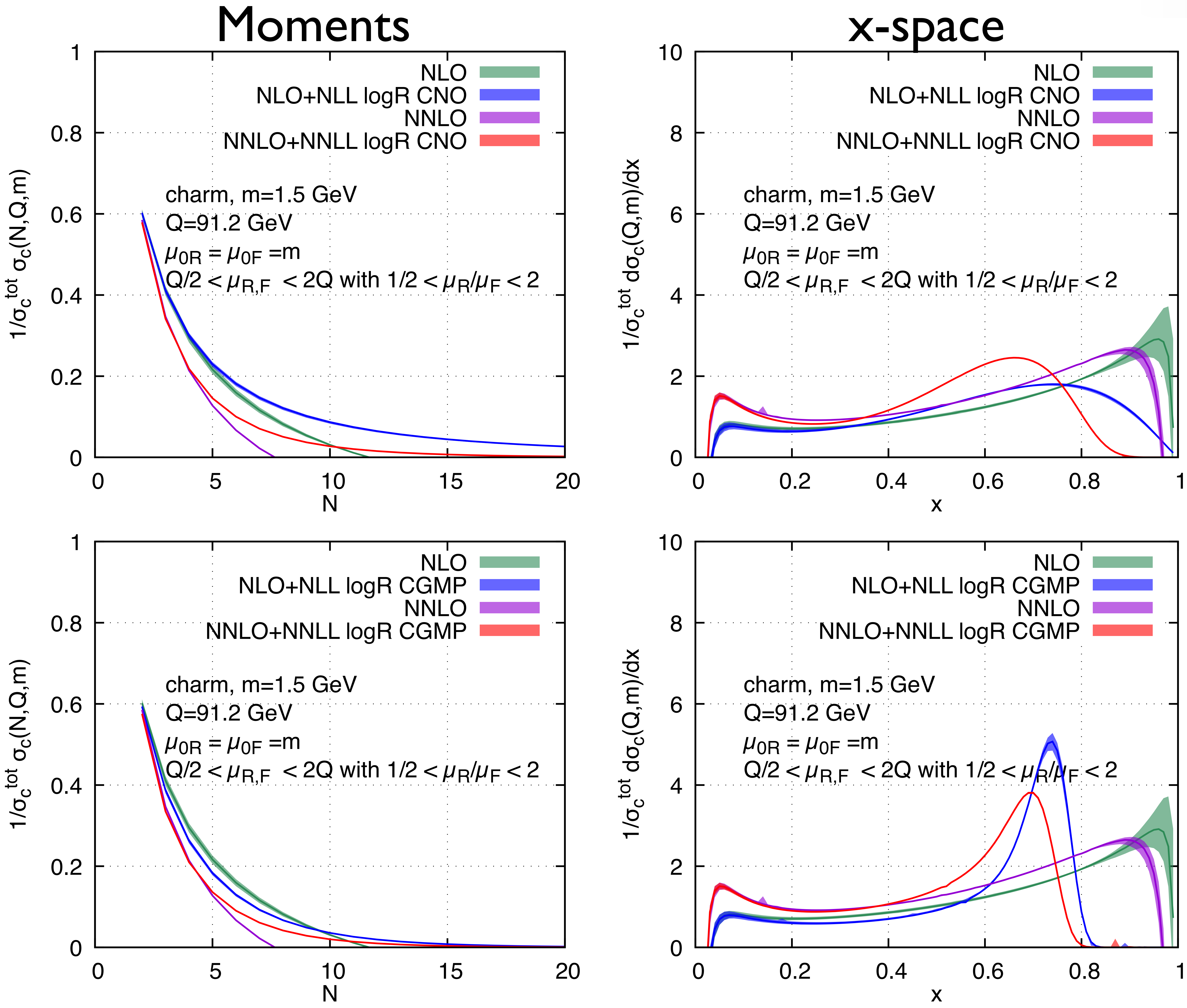


CNO

CGMP

Bands shrink as expected at higher orders, but do not always overlap

Charm e+e- FF: **final** scales variations



CNO

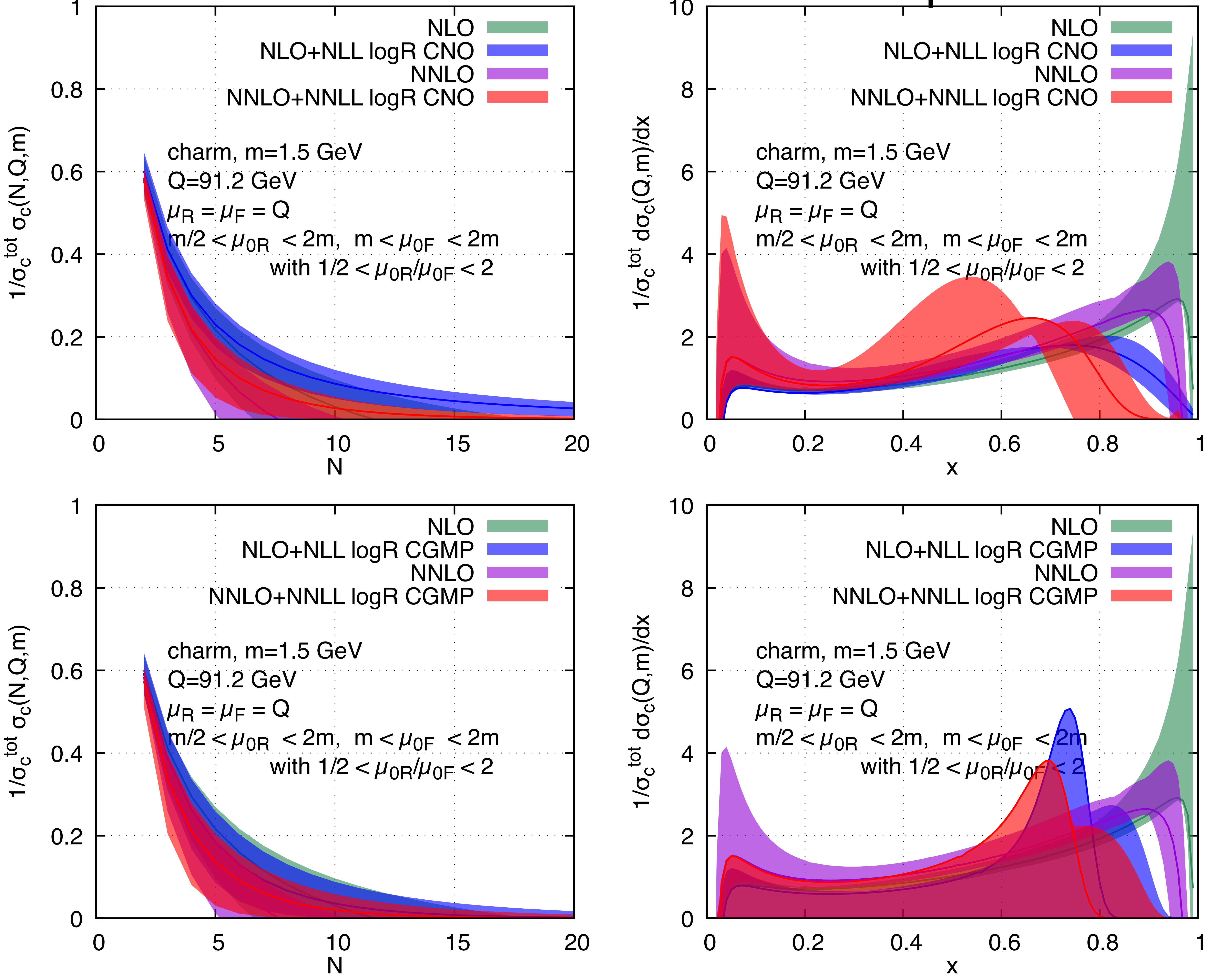
CGMP

Bands a bit all over the place

Charm e+e- FF: **initial** scales variations

Moments

x-space



CNO

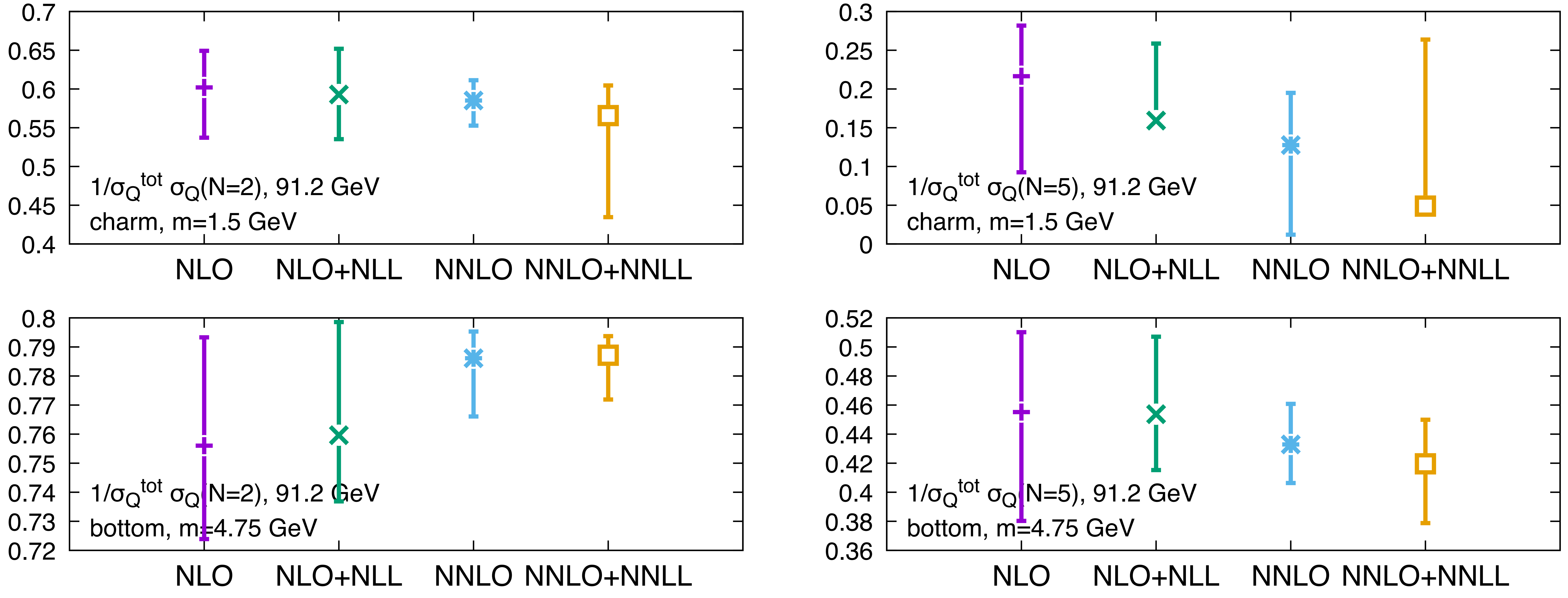
CGMP

Bands a bit all over the place

Convergence (or lack thereof)

N=2

N=5

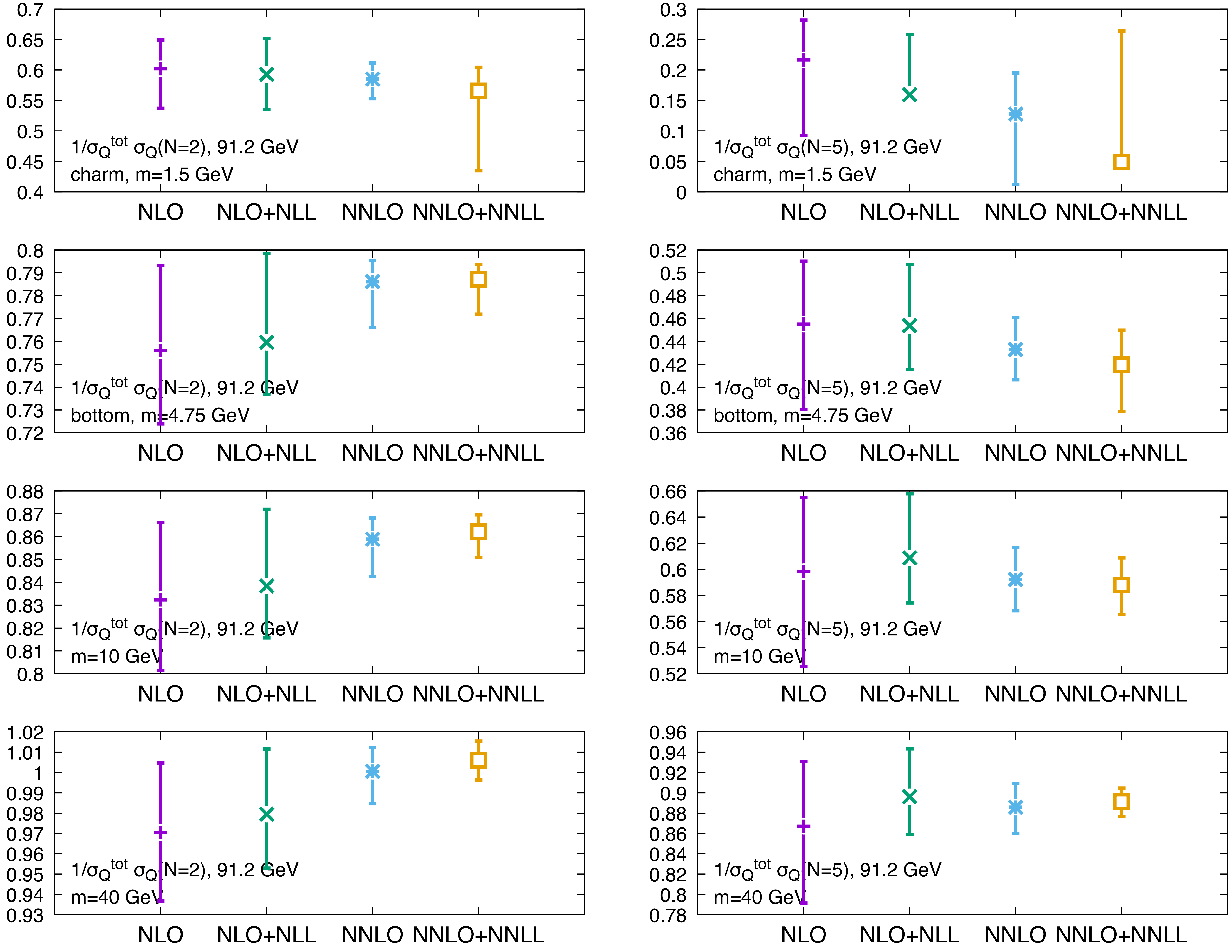


Charm

Bottom

Convergence (or lack thereof)

N=2 N=5



Charm

Bottom

m=10 GeV

m=40 GeV

Nothing wrong with NNLO or NNLL.

It's just that bottom to a certain extent, and certainly charm, are quite light.

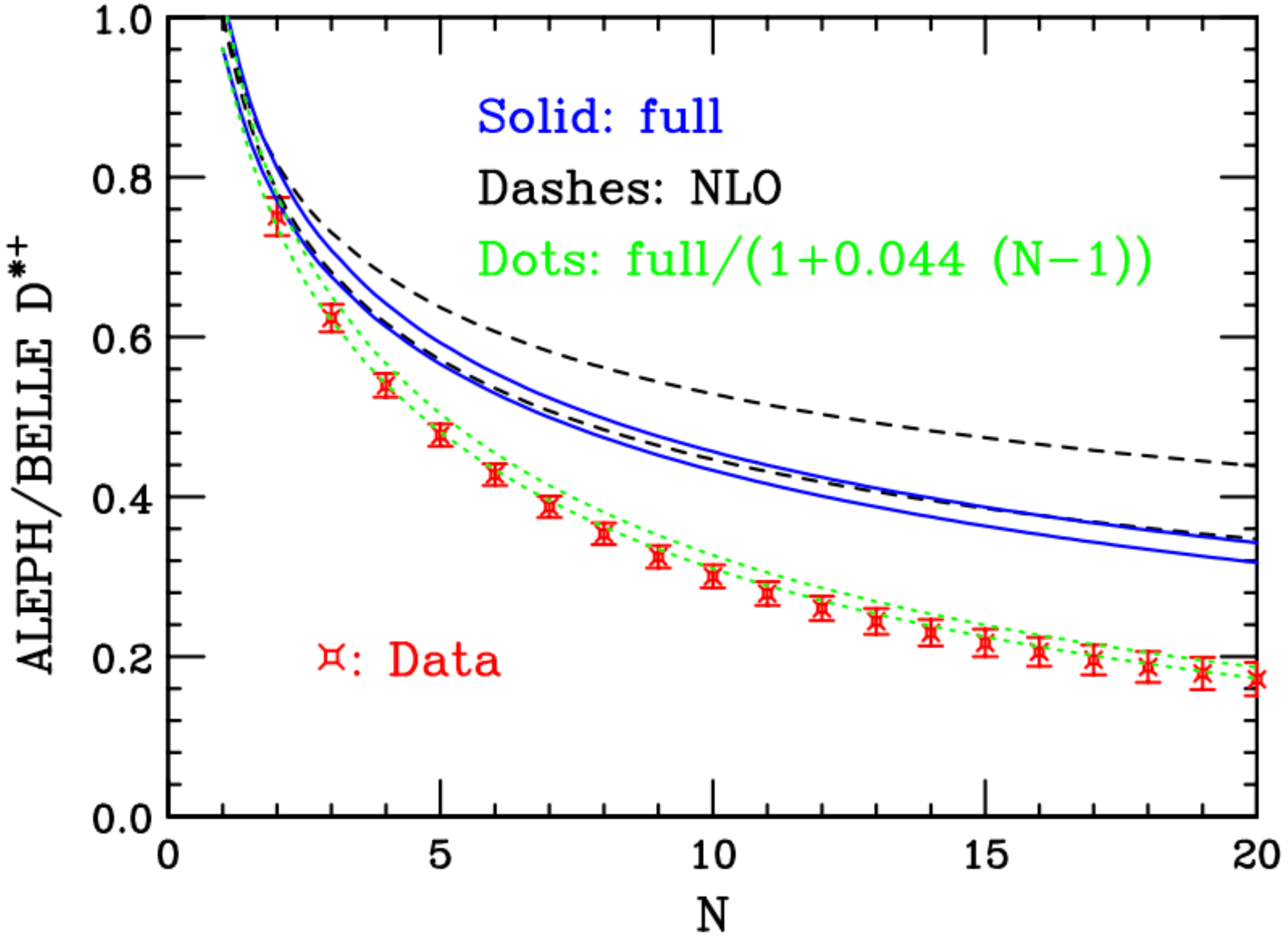
A first observable: charm ratio

$$R_D(N, Q_i, Q_f) \equiv \frac{\sigma_D(N, Q_f = 91.2, m = 1.5, \text{np pars})}{\sigma_D(N, Q_i = 10.6, m = 1.5, \text{np pars})}$$

Ratio of moments of D meson data at two different energies

Essentially independent of non-perturbative and low scales physics.
It tests factorisation and DGLAP evolution from 10.6 GeV to 91.2 GeV

MC, Oleari, Nason 05



Previously calculated at NLO+NLL and compared to data

Sizeable discrepancy observed, likely beyond perturbative uncertainties.

A sign of power corrections at 10.6 GeV ?

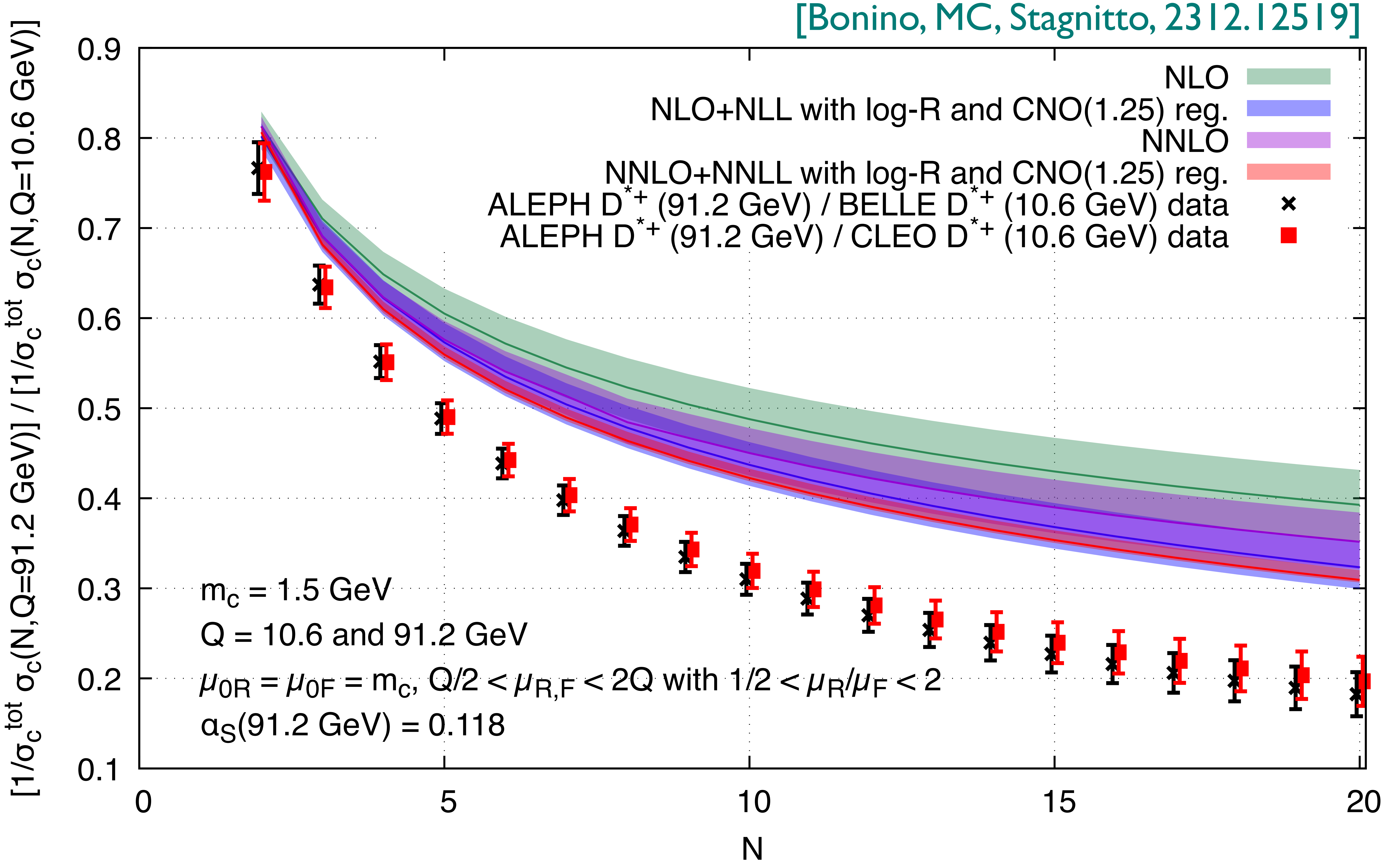
A very big coefficient to a $1/Q^2$ correction, or a reasonably-sized coefficient to an (unexpected) $1/Q$ correction would fit the data

A first observable: charm ratio

$$R_D(N, Q_i, Q_f) \equiv \frac{\sigma_D(N, Q_f = 91.2, m = 1.5, \text{np pars})}{\sigma_D(N, Q_i = 10.6, m = 1.5, \text{np pars})}$$

Ratio of moments of D meson data at two different energies

[Bonino, MC, Stagnitto, 2312.12519]



New evaluation at NNLO+NNLL

As expected, perturbatively compatible with NLO+NLL

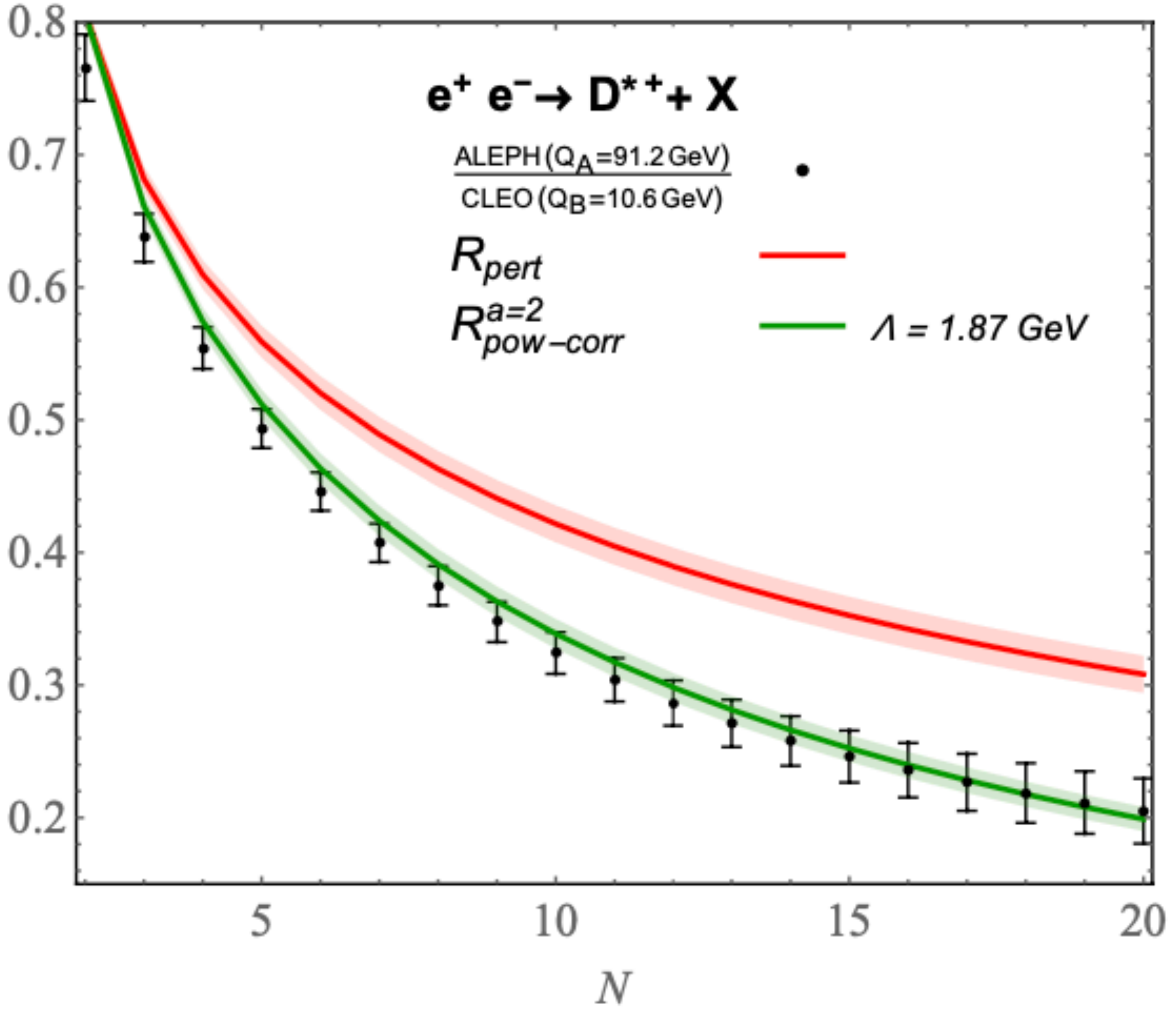
Discrepancy with data unchanged

Things improve a bit when considering mass corrections in resummation

A first observable: charm ratio

$$R_D(N, Q_i, Q_f) \equiv \frac{\sigma_D(N, Q_f = 91.2, m = 1.5, \text{np pars})}{\sigma_D(N, Q_i = 10.6, m = 1.5, \text{np pars})}$$

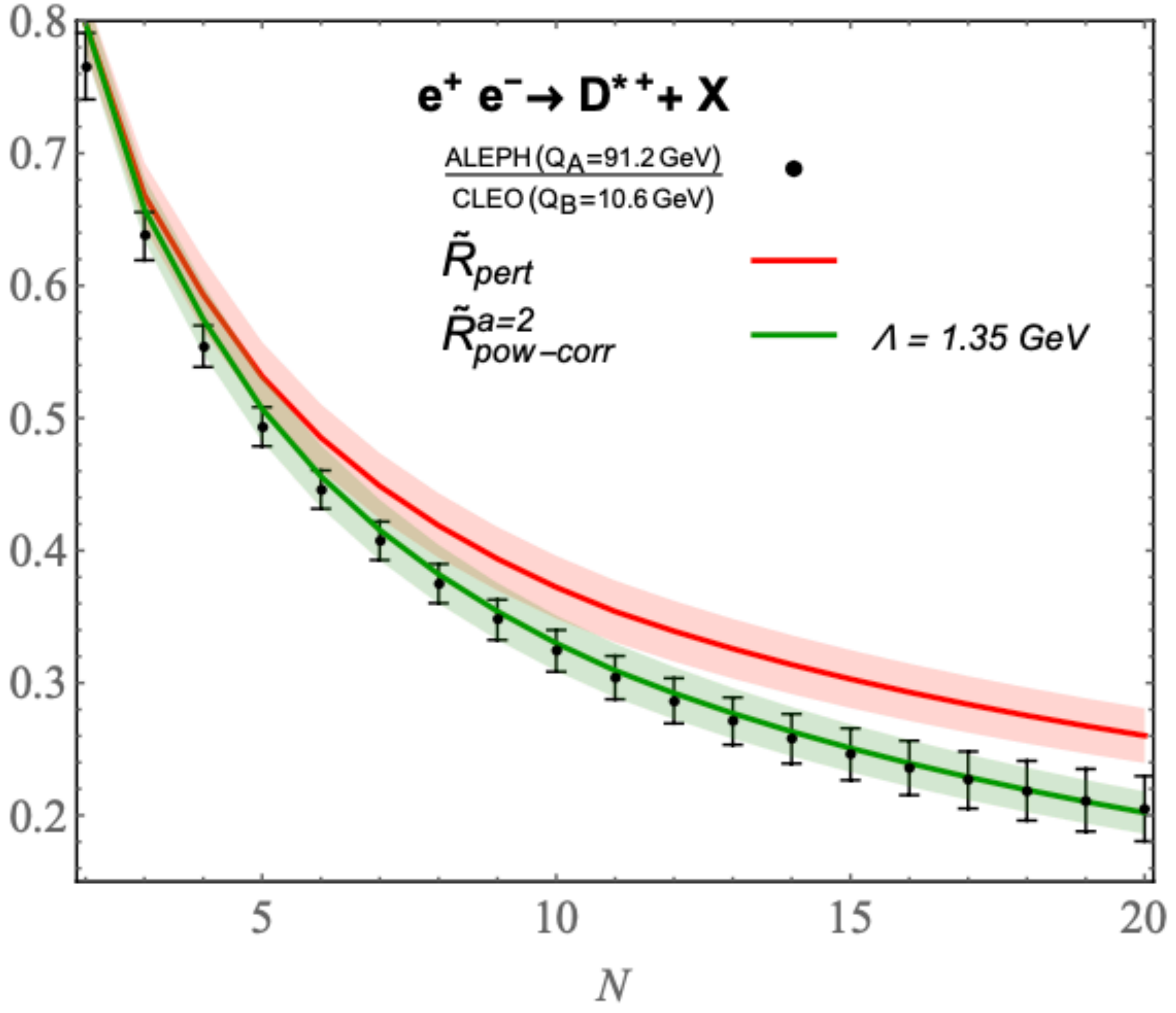
Ratio of moments of D meson data at two different energies



Consistent NLL joint resummation of mass and soft logarithms

[Ghira, Marzani, Ridolfi, 2309.06139]

[MC, Ghira, Marzani, Ridolfi, 2406.04173]

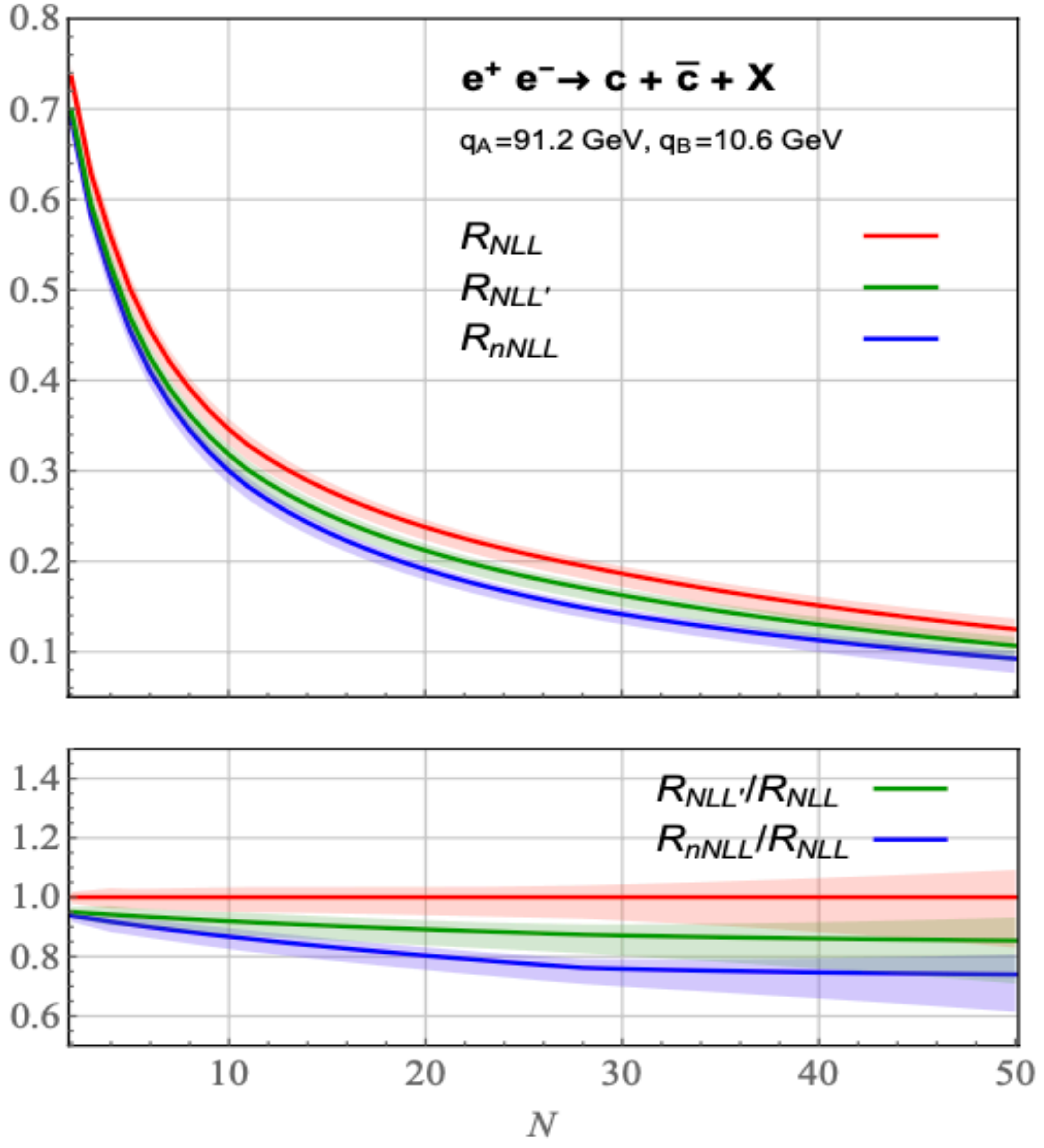


A first observable: charm ratio

$$R_D(N, Q_i, Q_f) \equiv \frac{\sigma_D(N, Q_f = 91.2, m = 1.5, \text{np pars})}{\sigma_D(N, Q_i = 10.6, m = 1.5, \text{np pars})}$$

Ratio of moments of D meson data at two different energies

[Ghira, Mai, Marzani, 2412.13261]

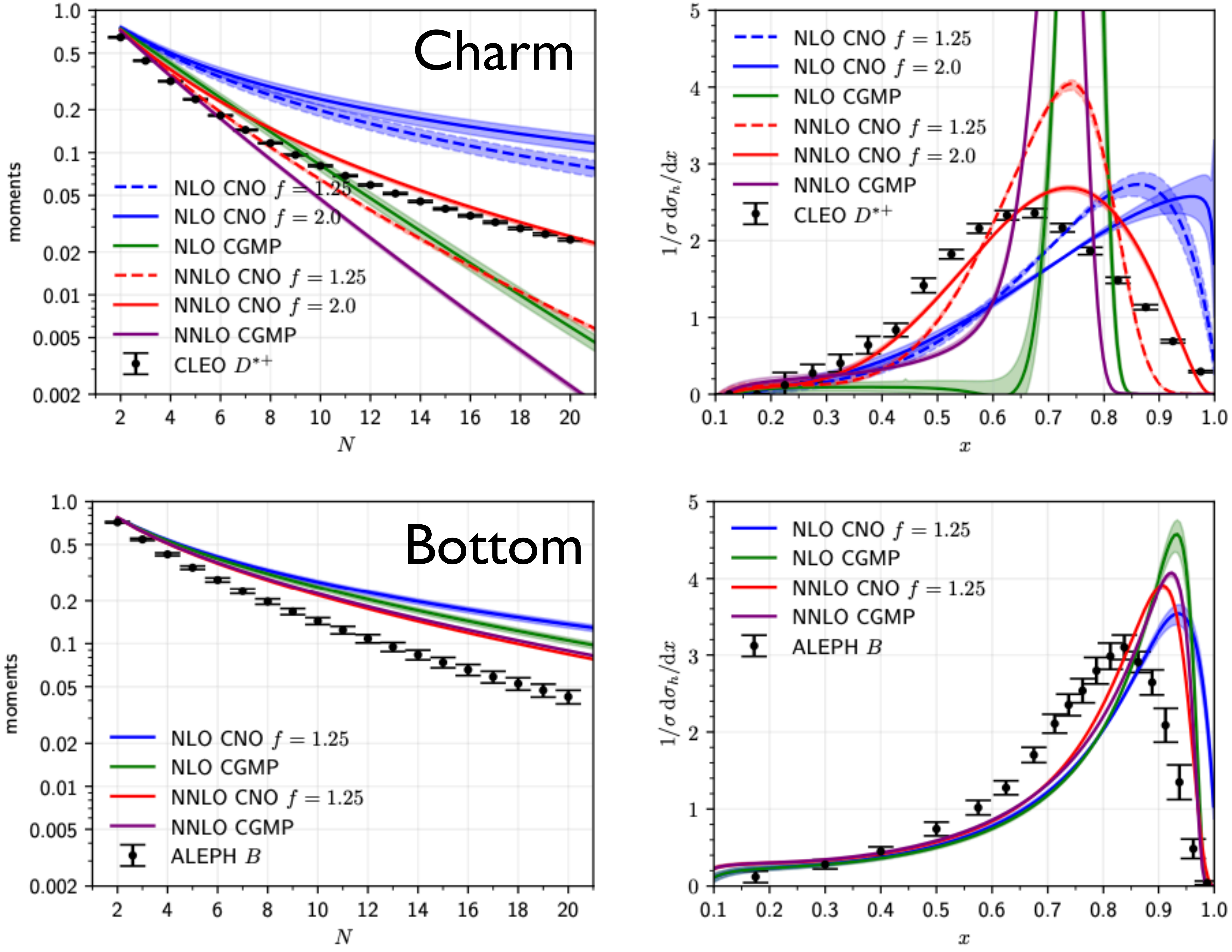


An **approximate nNLL** joint resummation suggests a potential even better agreement with the data

⇩

interest in pushing the joint resummation formalism to full NNLL

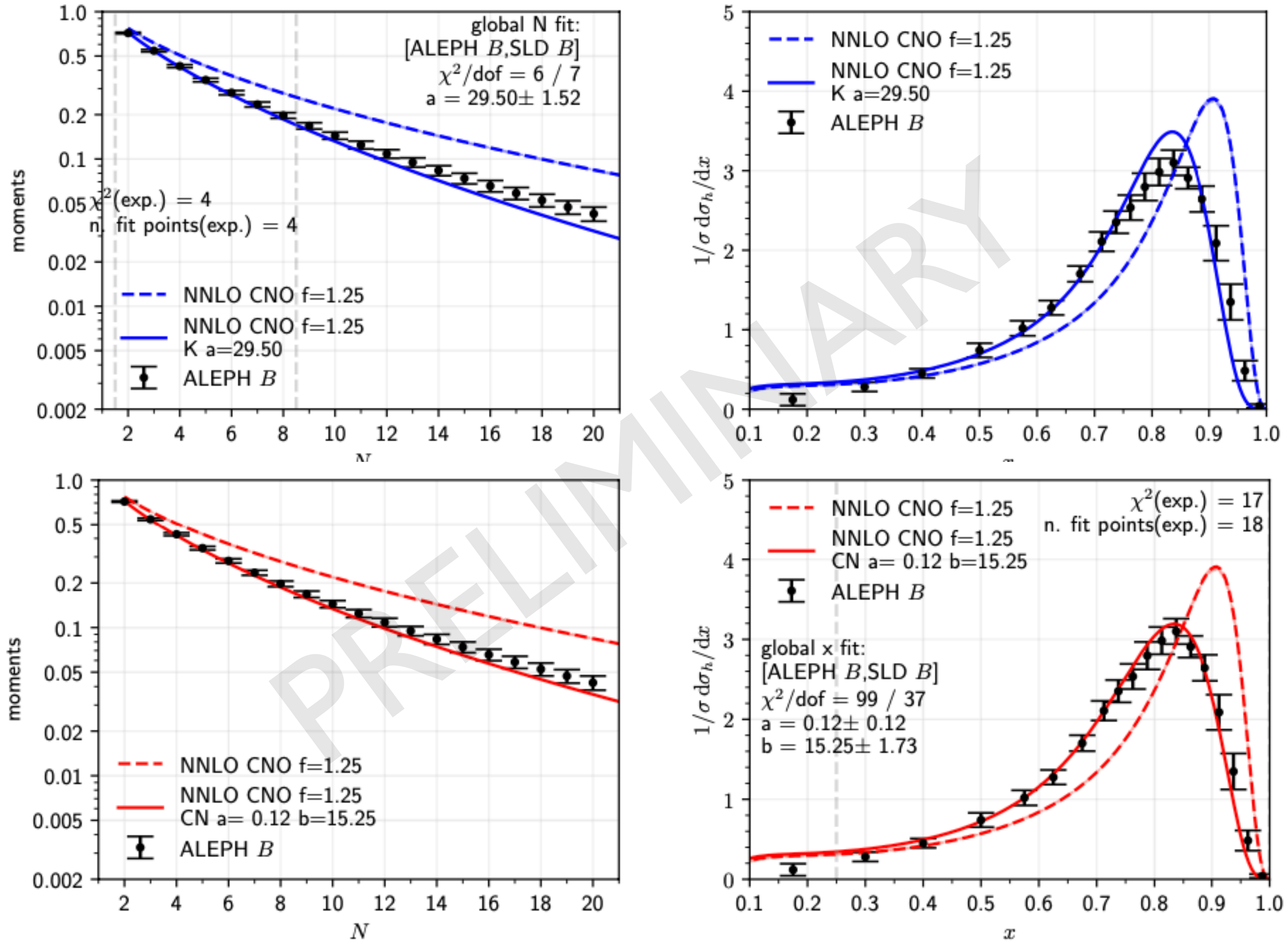
Comparisons to data



pQCD + Landau regularisation only

Challenge of describing data from this baseline clearly larger for charm. The shape of the pQCD+Landau regularisation curve can make fits impossible with a simple non-perturbative parameterisation
=> CNO f=2 needed for charm

Global fits to ALEPH+SLD bottom data



Fit in N-space with

$$D_{\rm NP}^{K}(x) = (a+1)(a+2)x^a(1-x)$$

(Kartvelishvili et al.)

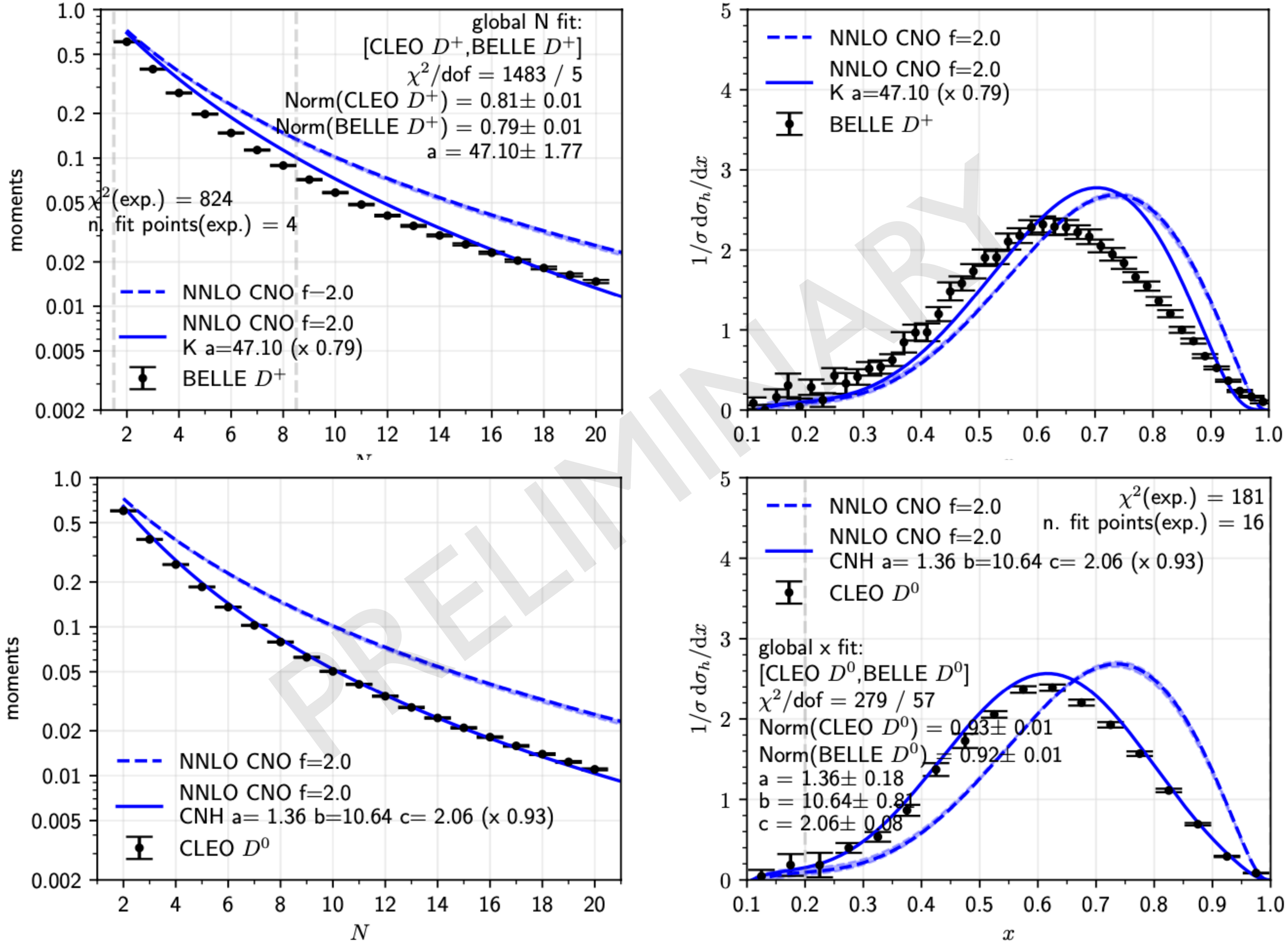
Fit in x-space with

$$D_{\rm NP}^{CN}(x) = \frac{\Gamma(a+b+2)}{\Gamma(a+1)\Gamma(b+1)}(1-x)^a x^b$$

(Colangelo-Nason)

Reasonably good fits can be obtained with few parameters and ‘legacy’ Landau pole regularisation

Global fits to CLEO+Belle charm data



Fit in N-space with

$$D^K_{\rm NP}(x) = (a+1)(a+2)x^a(1-x)$$

(Kartvelishvili et al.)

Fit in x-space with

$$D^{CNH}_{\rm NP}(x) = \frac{1}{1+c}\left[\delta(1-x) + cN^{-1}_{a,b}(1-x)^a x^b\right]$$

(Colangelo-Nason + hard component)

Good fits harder to come by, more parameters and tweaks to Landau pole regularisation (f=2) needed

Conclusions

- Long history of heavy quark fragmentation
  - Renewed recent interest
  - Multiple implementations available, at least at NNLO+NNLL accuracy

- Our own work does not seem to show a systematic improvement going from NL to NNL accuracy. Sometimes, especially for charm, it's quite the contrary.

- A strong dependence on the choice of the regularisation procedure of the Landau pole is observed. This also affects the perturbative convergence of the resummed predictions

- Perspectives:
  - Best case (but unlikely….) scenario: petition for the existence of a new heavy quark with 30 GeV mass
  - Realistic scenario: the interface between the perturbative and non-perturbative regions, and how it affects phenomenology, likely deserves further study