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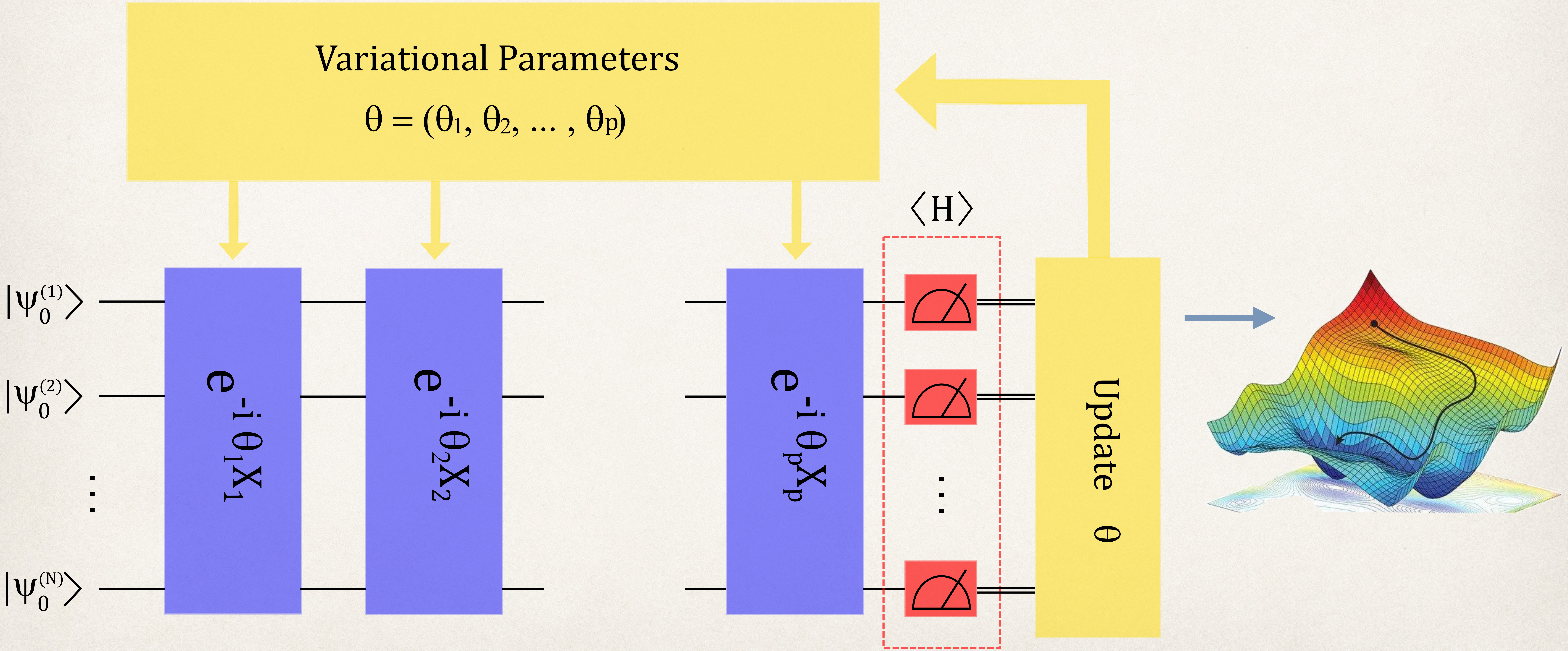
Leonardo Banchi

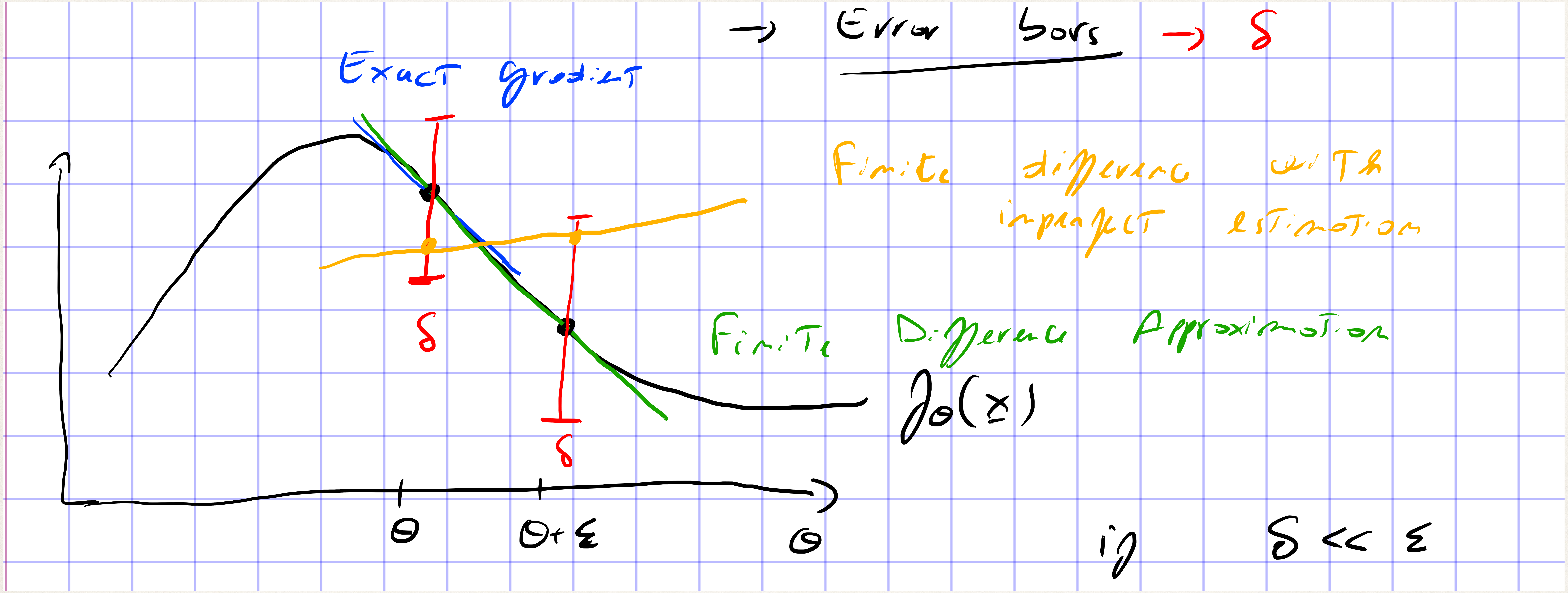
Optimising Complex Quantum Systems with Few Measurements

Variational Quantum Algorithms

$$C(\boldsymbol{\theta}) := \langle \psi(\boldsymbol{\theta}) | \hat{H} | \psi(\boldsymbol{\theta}) \rangle$$



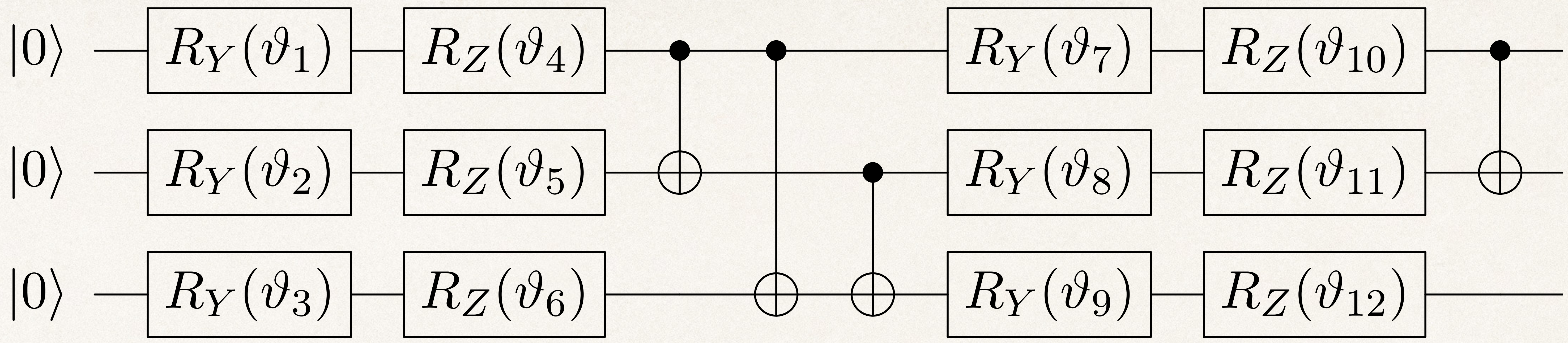
Gradients in Variational Quantum Algorithms



Finite energy approximation is good only if

$$\delta \ll \epsilon \quad \Longrightarrow \quad \text{shots} \gg \epsilon^{-2}$$

Parameter Shift Rule



- Parameter shift rules allow the evaluation of gradients directly in hardware

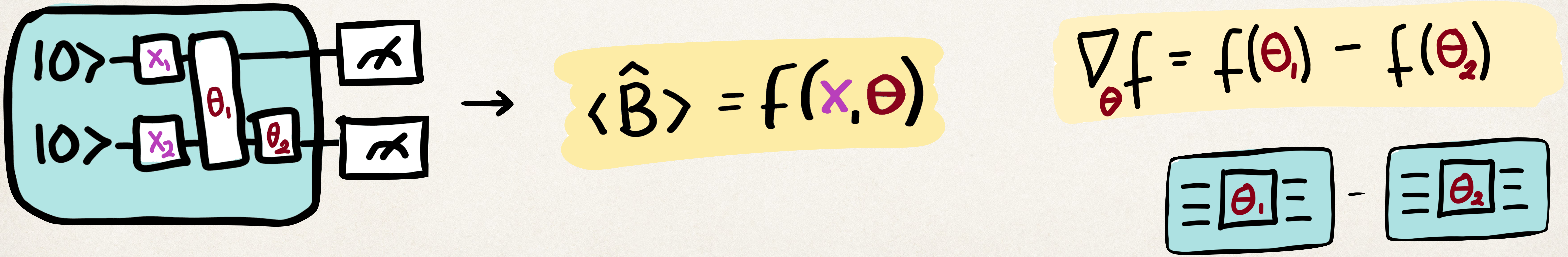
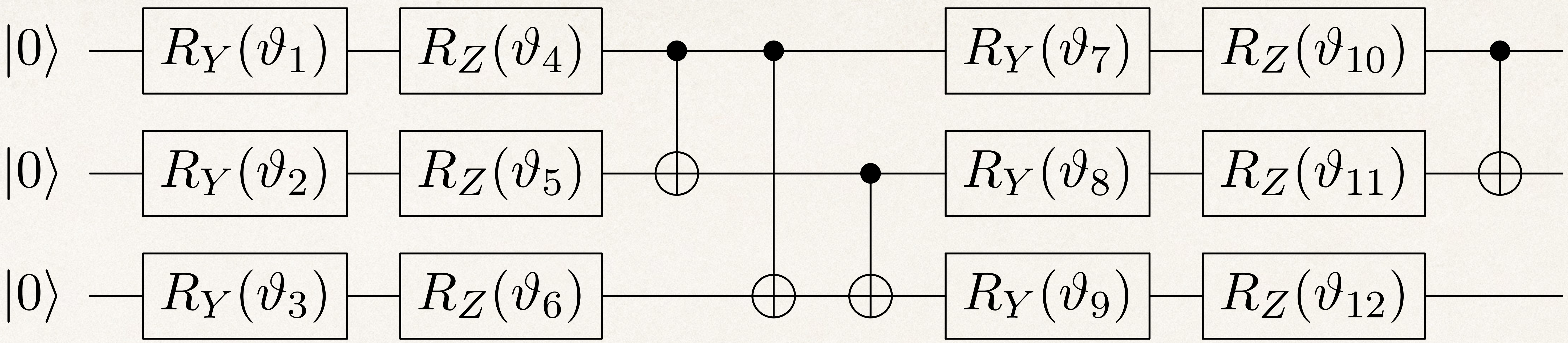


Image from: https://pennylane.ai/qml/glossary/variational_circuit

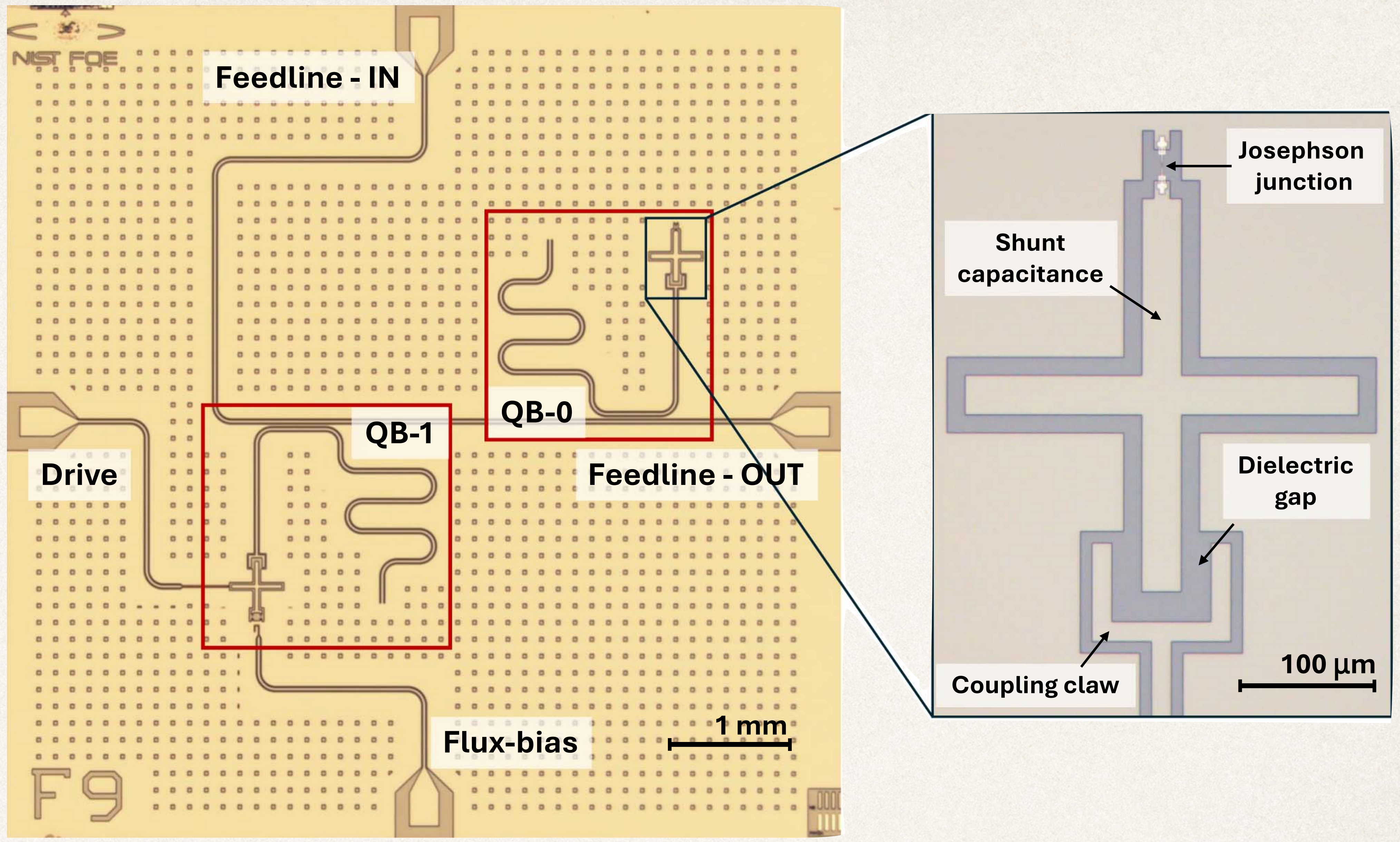
Variational Quantum Algorithms



- Why do we fix a particular structure? (Most of the times) because we can!
- Analogy with neural networks: “hardware friendly-ness” is more important than clever designs
- But then why do we have to use qubits? Perfect qubits do not exist yet!

Transmon Architecture

- Nonlinear resonator (≈ quDit) + cavity
- Unbounded spectrum, but low energy approximations are good



Quantum Sensing and Metrology

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Transmon Qubit Modeling and Characterization for Dark Matter Search

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Gradients of Parametric Quantum Circuits

$$|\psi(\boldsymbol{\theta})\rangle = \hat{W}_L e^{i\theta_L \hat{H}_L} \dots e^{i\theta_2 \hat{H}_2} \hat{W}_1 e^{i\theta_1 \hat{H}_1} |\psi_0\rangle,$$

$$f(\theta) \equiv f(\boldsymbol{\theta}) \Big|_{\substack{\theta_k = \theta, \\ \theta_{j \neq k} = \text{const}}} = \langle \psi | e^{-i\hat{H}_k \theta} \hat{M} e^{i\hat{H}_k \theta} | \psi \rangle = \sum_{\omega \in \Omega} M_\omega e^{i\omega\theta}$$

$$\Omega = \{E_j^{(k)} - E_i^{(k)} \text{ for } i, j = 1, \dots, N_k\},$$

$$f'(\theta) \equiv \frac{\partial f(\boldsymbol{\theta})}{\partial \theta_k} = \sum_{\omega \in \Omega} M_\omega e^{i\omega\theta} i\omega \,.$$

Parameter Shift Rules for Arbitrary Spectrum

We look for expressions of the derivative as a linear combination of function evaluations

$$\frac{df(\theta)}{d\theta} = \sum_{p=1}^{P} c_p f(\theta + \vartheta_p),$$

This is true provided that the (unknown) coefficients satisfy

$$\sum_{p=1}^{P} c_p e^{i\omega\vartheta_p} = i\omega, \qquad \forall \omega \in \Omega\,.$$

For $P > |\Omega|$ the linear system might have infinitely many solutions arXiv > quant-ph > arXiv:2510.05289

Quantum Physics

[Submitted on 6 Oct 2025]

Overshifted Parameter-Shift Rules: Optimizing Complex Quantum Systems with Few Measurements

Leonardo Banchi, Dominic Branford, Chetan Waghela

$$\frac{df(\theta)}{d\theta} = \sum_{p=1}^{P} c_p f(\theta + \vartheta_p),$$

When $P > |\Omega|$ the rule is "over shifted" and among the infinitely many solutions we can choose the ones with desired properties

If each term $f(\theta + \vartheta_p)$ is experimentally measured using S_p shots then the optimal shot allocation is

$$S_p = \frac{|c_p|}{\|\boldsymbol{c}\|_1} S, \qquad S = \sum_p S_p$$

arXiv > quant-ph > arXiv:2510.05289

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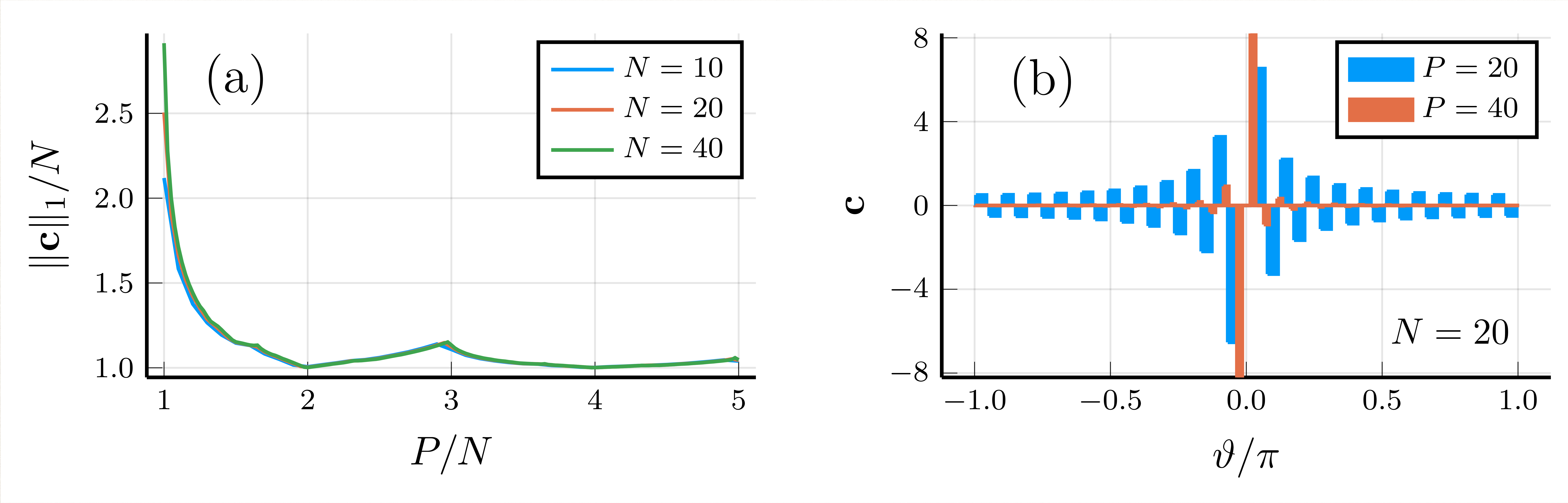
$$\frac{df(\theta)}{d\theta} = \sum_{p=1}^{P} c_p f(\theta + \vartheta_p), \qquad \mathrm{Var}\left[\frac{df(\theta)}{d\theta}\right] \leq \frac{\|\boldsymbol{c}\|_1^2 \sigma^2}{S},$$

We can minimise the total number of measurements by solving the **convex** problem

$$\min_{\boldsymbol{c}} \|\boldsymbol{c}\|_1 \qquad \text{such that} \qquad \sum_p c_p e^{i\omega\vartheta_p} = i\omega, \qquad \forall \omega \in \Omega,$$

Example:



$$N = |\Omega|$$

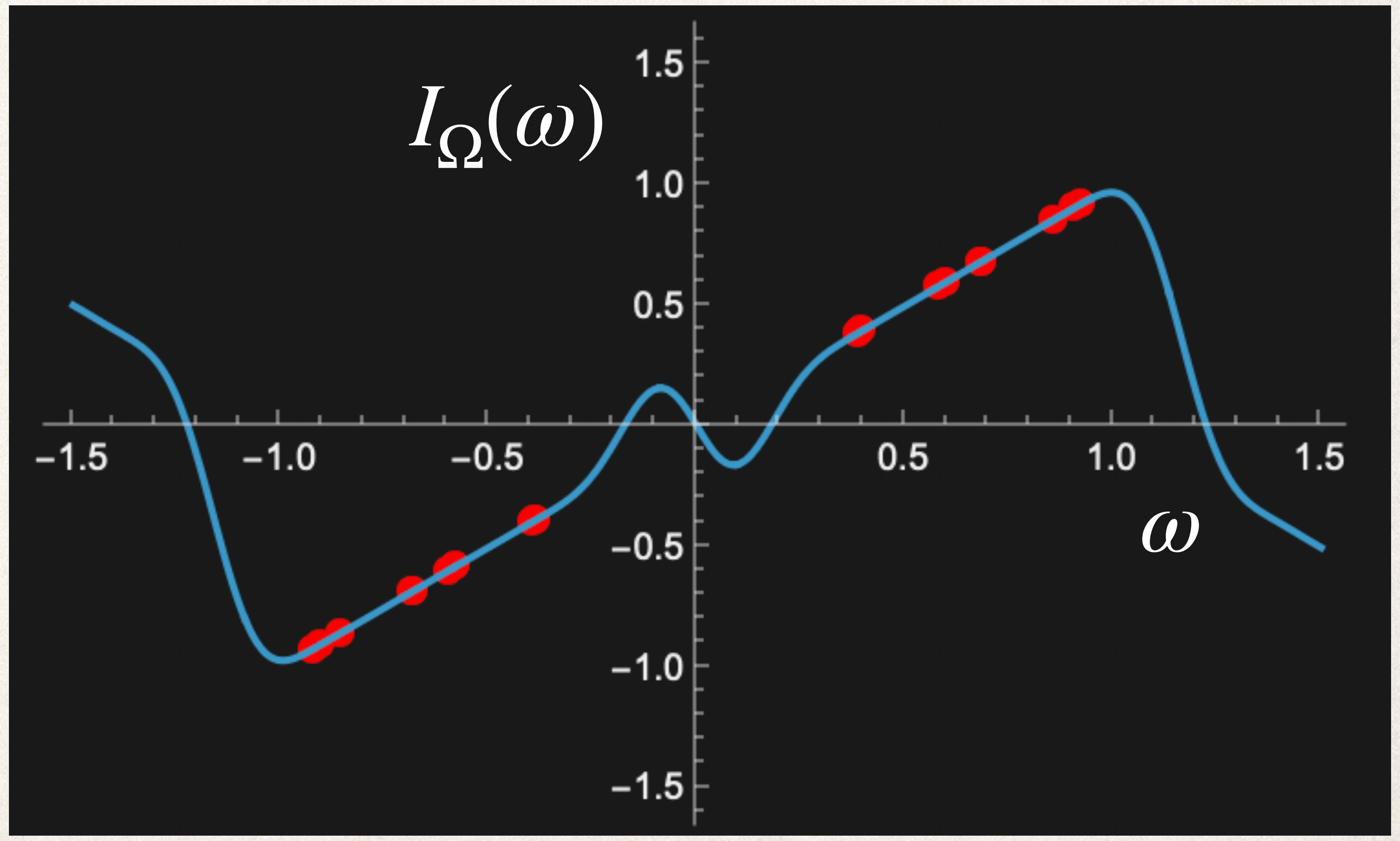
Infinite overshifting, continuous limit

In the limit of infinite overshifting

$$\frac{df(\theta)}{d\theta} = \int d\vartheta\, c(\vartheta) f(\theta + \vartheta), \qquad \int d\vartheta\, c(\vartheta) e^{i\omega\vartheta} = i\omega, \qquad \forall \omega \in \Omega,$$

the coefficients become the Fourier Transform of **any** interpolating function

$$c(\vartheta) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega\vartheta} i I_\Omega(\omega),$$



Infinite overshifting, continuous limit

Infinite overshifting can be expressed as

$$\frac{df(\theta)}{d\theta} = \frac{\|\boldsymbol{c}\|_1}{2}\left(\mathbb{E}_{\vartheta_+\sim p_+,\vartheta_-\sim p_-}\left[f(\theta+\vartheta_+) - f(\theta+\vartheta_-)\right]\right)$$

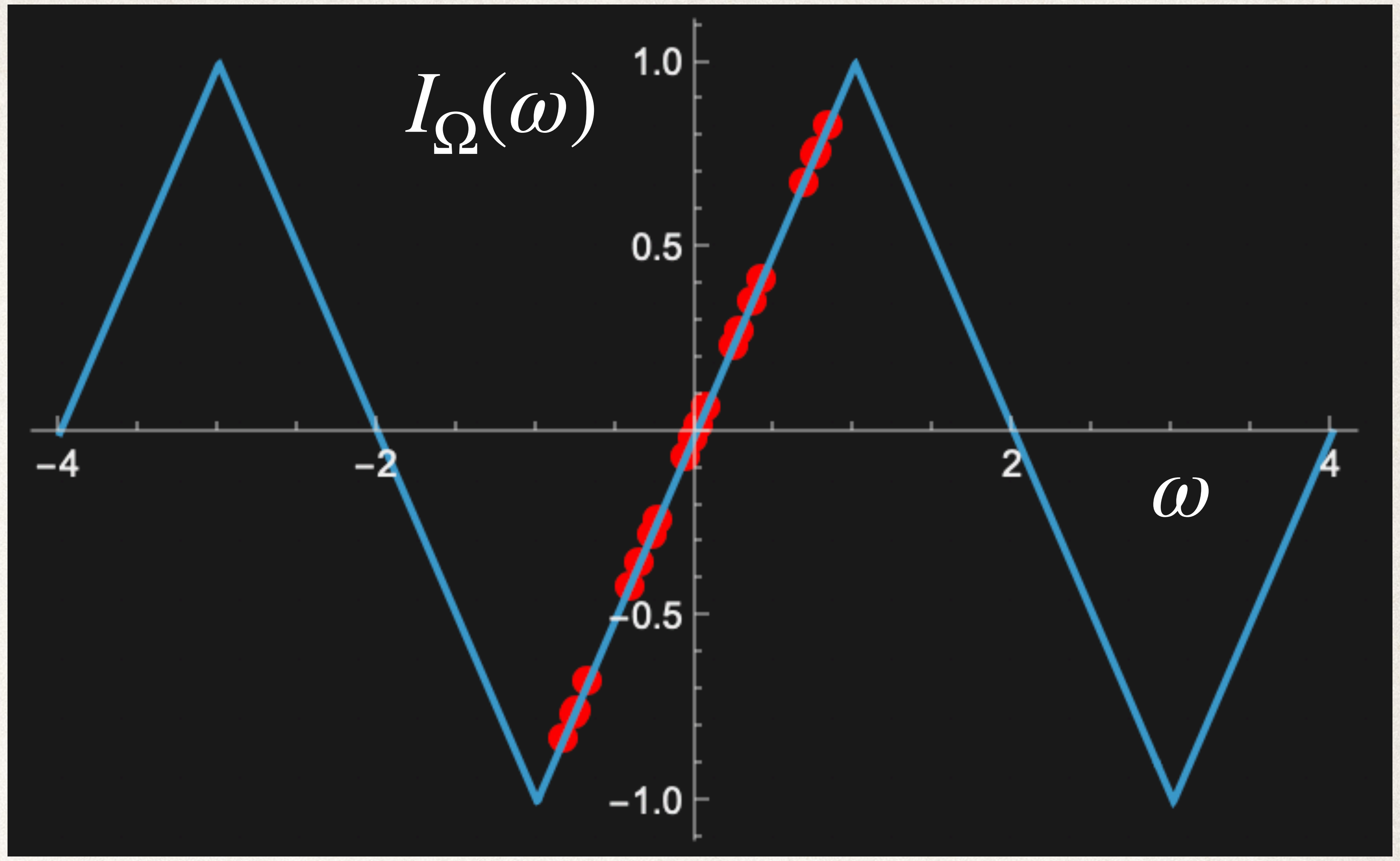
With probability distributions $p_\pm(\vartheta) = 2c_\pm(\vartheta)/\|\boldsymbol{c}\|_1$

"Doubly-stochastic" gradient descent, with variance overhead $\|\boldsymbol{c}\|_1$

$\Longrightarrow$ we want small $\|\boldsymbol{c}\|_1$ (again)

L_1 Uncertainty principle: sparse coefficients $c(\vartheta) \Longrightarrow$ dense $I_\Omega(\omega)$

Triangle Wave Interpolation

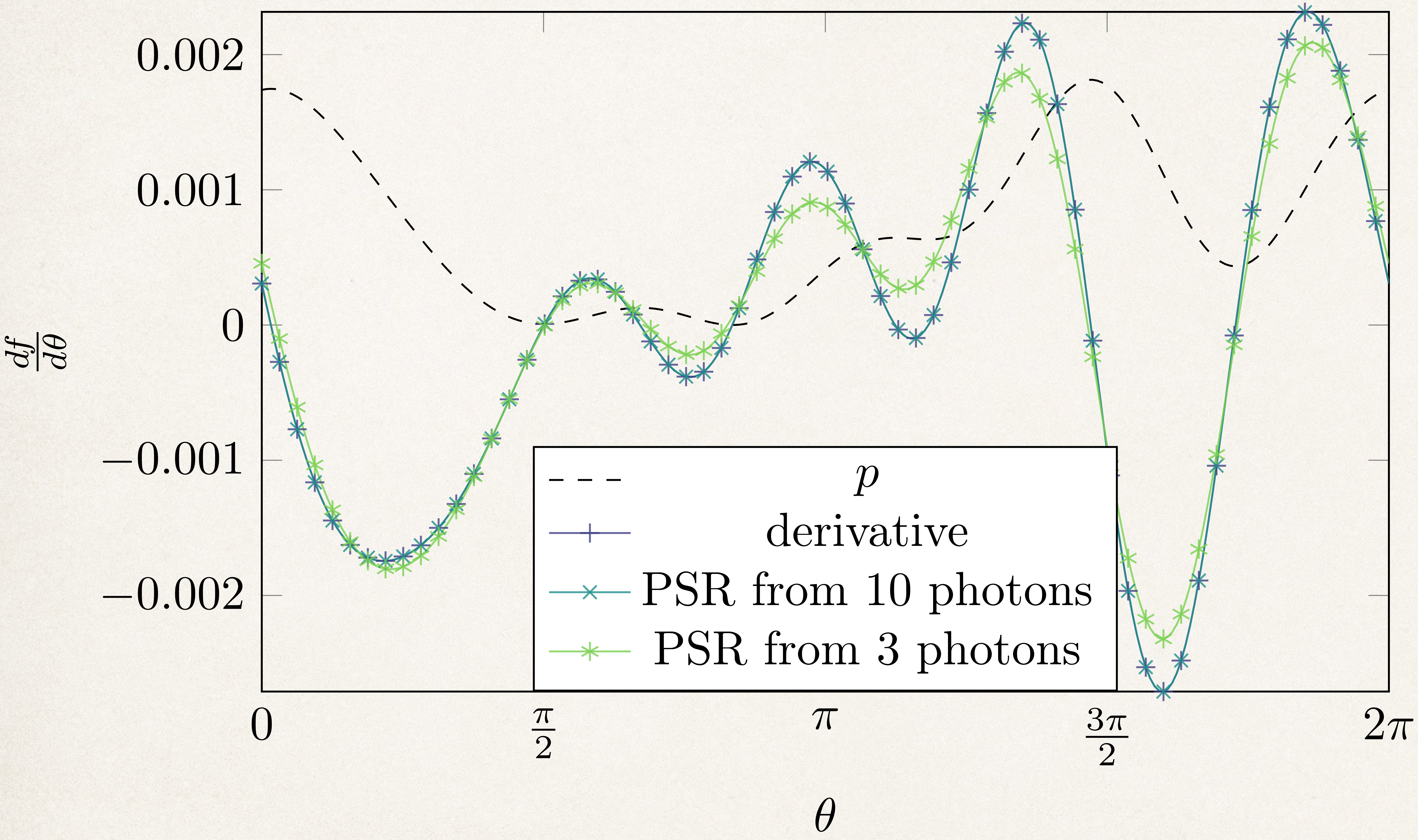


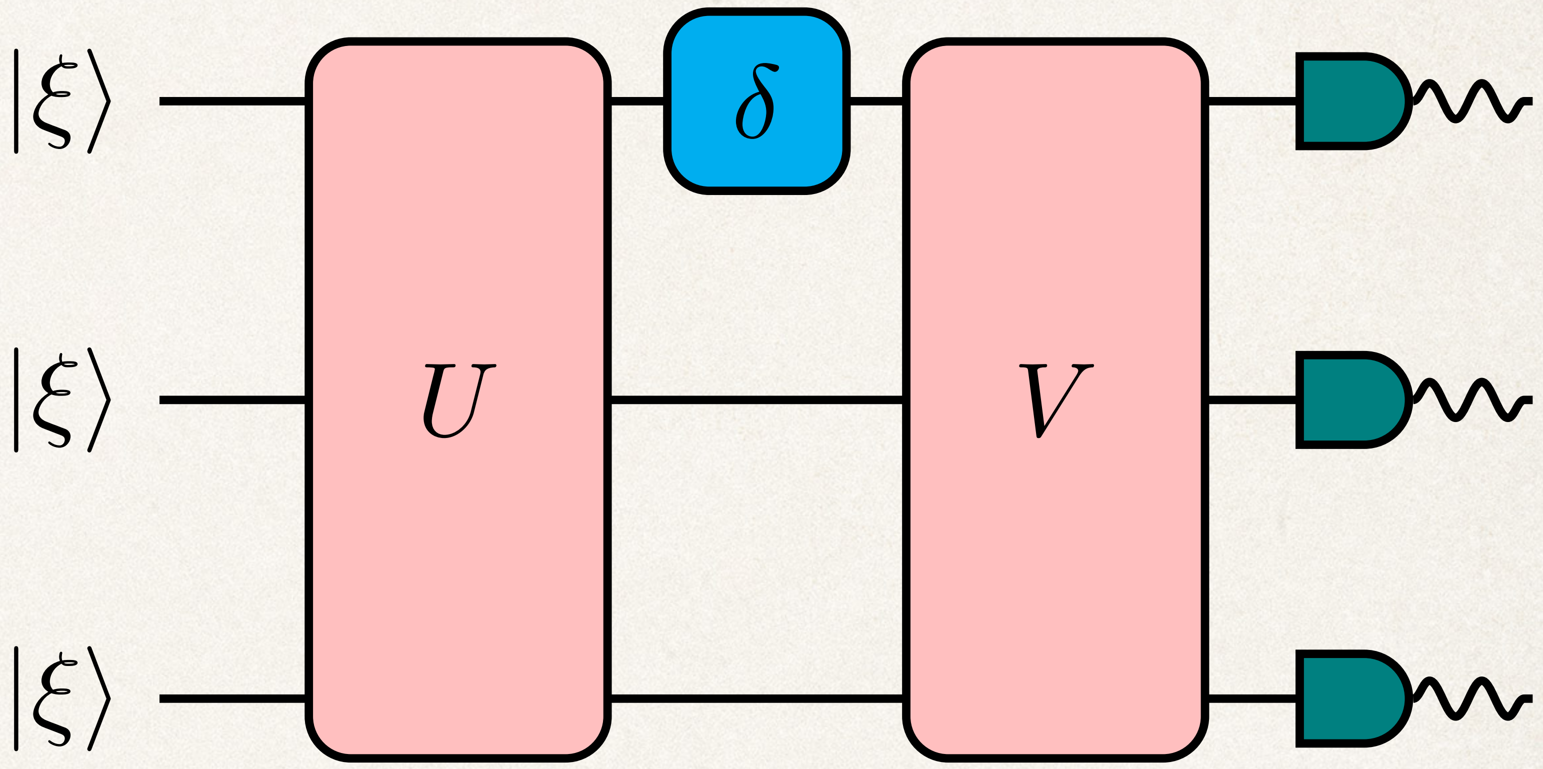
Triangle wave interpolation only requires the bandwidth $\max_{\omega\in\Omega}|\omega|$, applicable also if $|\Omega|$ is exponentially large

Algorithm 3 Triangle Shift Rule: single-shot unbiased estimator of $f'(\theta)$.

1: Fix $\Lambda \geq \max_{\omega\in\Omega}|\omega|$ (see Table I).
2: Sample u uniformly from $[0,1] \subset \mathbb{R}$.
3: Repeat the iteration $q_i = q_{i-1} + 8/\pi^2(2i+1)^{-2}$ with $q_{-1} = 0$ while $q_i \leq u$. Let t be the index such that $q_{t-1} \leq u < q_t$.
4: Sample a fair coin $p \in \{0,1\}$ and set $\vartheta = (-1)^p\pi(2t+1)/(2\Lambda)$.
5: Estimate $f(\theta+\vartheta)$ in a quantum device and call the outcome g.
6: Return $(-1)^{t+p}\Lambda g$.

Application: Gaussian Boson Sampling





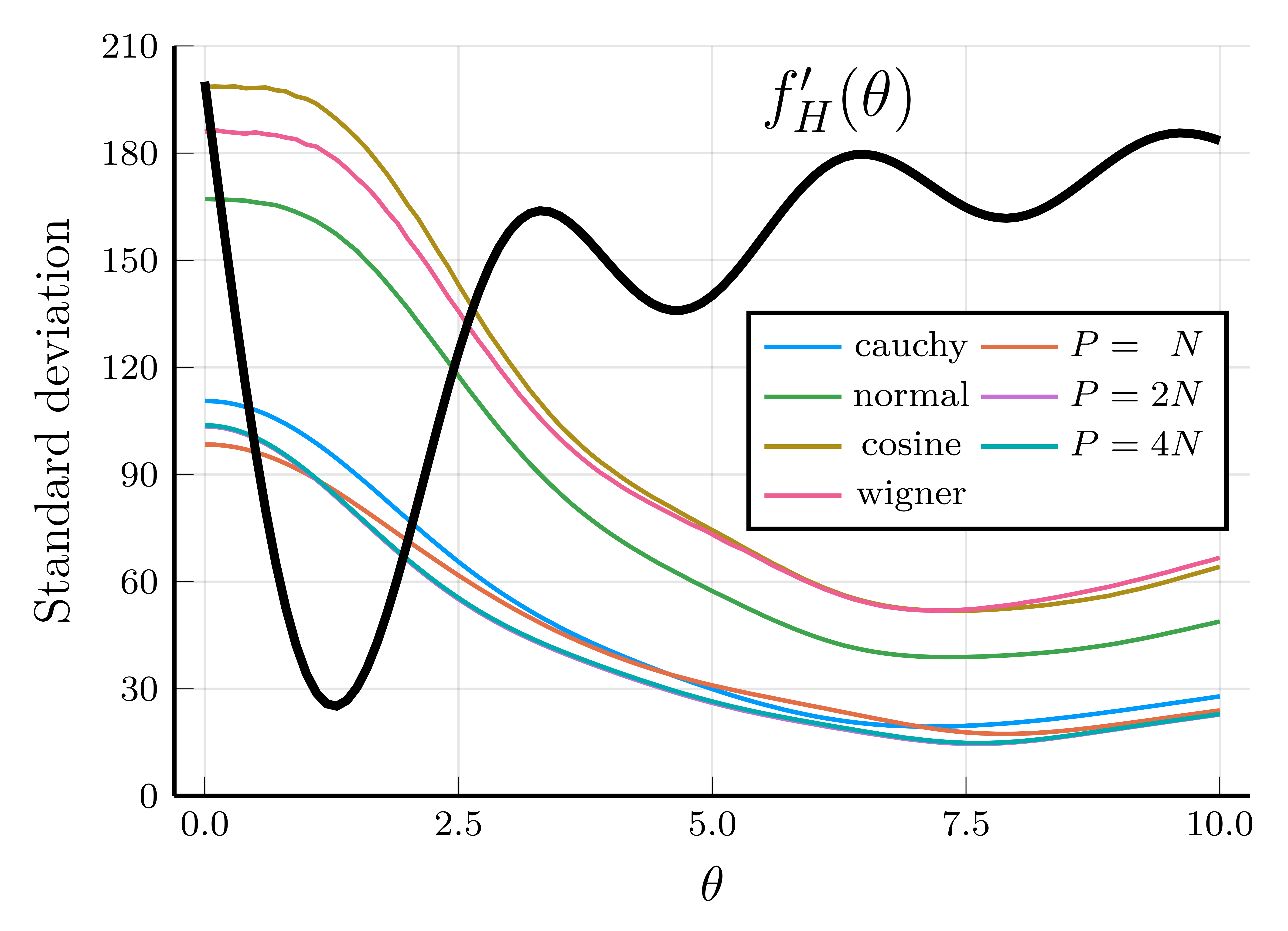
Probability of detecting five photons as (3,1,1) in a three-mode interferometer with squeezed vacuum input as a function of a single phase in the first mode.

Application: Hamiltonian System

Gates from many-body quantum dynamics

$$f_H(\theta) = \langle\psi| e^{i\hat{H}\theta} \hat{O} e^{-i\hat{H}\theta} |\psi\rangle$$

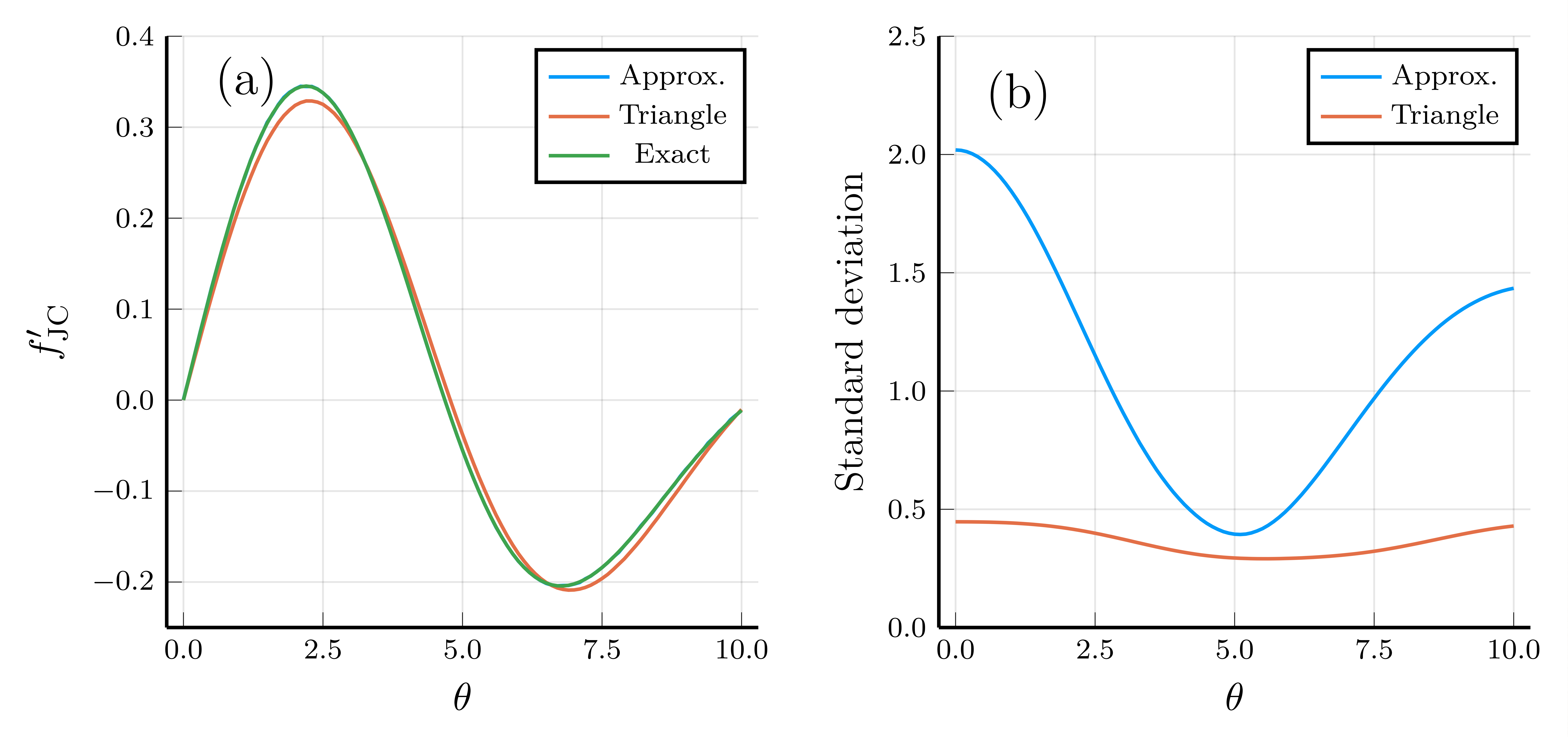
$$\hat{H} = \frac{1}{4} \sum_{i=1}^{L-1} \left(\hat{X}_i \hat{X}_{i+1} + \hat{Y}_i \hat{Y}_{i+1} \right)$$



Application: all degrees of freedom in cavity QED

$$\hat{H}_{\mathrm{JC}} = \frac{\delta}{2}\hat{\sigma}_Z + \frac{\lambda}{2}(\hat{a}^\dagger\hat{\sigma}_- + \hat{a}\hat{\sigma}_+)$$

$$f_{\mathrm{JC}}(\theta) = \langle\psi(\alpha)\,|\,e^{i\hat{H}_{\mathrm{JC}}\theta}\hat{Z}e^{-i\hat{H}_{\mathrm{JC}}\theta}\,|\,\psi(\alpha)\rangle$$



Conclusions

- We have generalised parameter shift rules to systems with possibly arbitrary and unknown spectra
- Approximately works even for unbounded (infinite dimensional) systems, for low energy inputs
- Applicable to systems with exponentially many frequencies
- Different regularisations can enforce different constraints
  - Minimise the total number of measurements
  - Enforce stability against parameter fluctuations

[Submitted on 6 Oct 2025]

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