Hamiltonian Truncation Effective Theory

Improving convergence in quantum field theoretical calculations

Andrea Maestri

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Hamiltonian Truncation

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- *Truncated Conformal Space Approach:* $H = H_0 + V$

  Yurov, Zamolodchikov (1990) [1]

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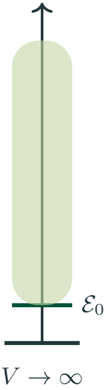

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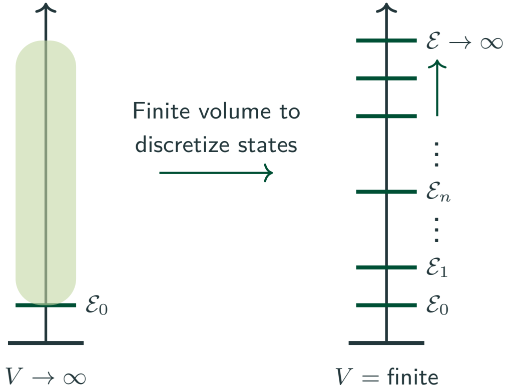

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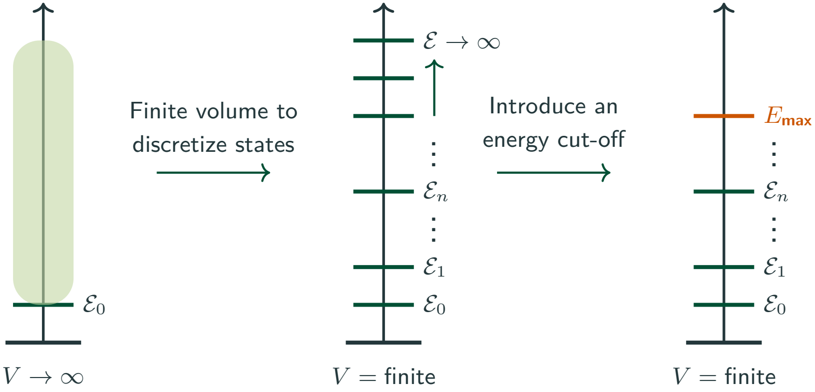

Diagonalising H in the Truncated Subspace

- *Truncated subspace:* $\mathcal{H} \longrightarrow \mathcal{H}_{\text{eff}} \doteq \text{span}\{|\mathcal{E}_i\rangle \big| \mathcal{E}_i \leq E_{\max}\}$
- *Diagonalise H on $\mathcal{H}_{\text{eff}}$:* compute matrix elements $H_{ij} \doteq \langle \mathcal{E}_i | H | \mathcal{E}_j \rangle_{\text{eff}}$
- *Working on the "cylinder":* $\mathbb{R} \times \mathbb{S}_R^{d-1}$ Hogervorst, Rychkov, Rees (2015) [2]

Accuracy

$$A \sim E_{\max}^{-\nu}$$

- $\nu = d - 2\Delta_{\mathcal{V}}$

Computational cost

$$T \sim \exp\left(E_{\max}^{1-1/d}\right)$$

- number of states in $\mathcal{H}_{\text{eff}}$

Logarithmic behaviour:

$$A(T) \sim \ln(T) \implies$$

Improving accuracy

Mirò *et al.* (2017) [3]

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Application to $\lambda\phi^4$ theory in two dimensions

We work in $\mathbb{R} \times \mathbb{S}^1_R$, with periodic condition $\phi(x + 2\pi R, t) = \phi(x, t)$

$$H = \sum_k \omega_k \, a_k^\dagger a_k + \frac{\lambda}{4!} \int_0^{2\pi R} \mathrm{d}x \, {:}\phi^4{:} \qquad \longleftarrow \qquad \omega_k = \sqrt{(k/R)^2 + m^2}$$

$$\mathcal{H}_{\text{eff}} = \text{span}\Big\{ |\bar{n}_i\rangle = |n_0, \ldots, n_i, \ldots\rangle \;\Big|\; \sum_k \omega_k \, n_k \leq E_{\max} \Big\}$$

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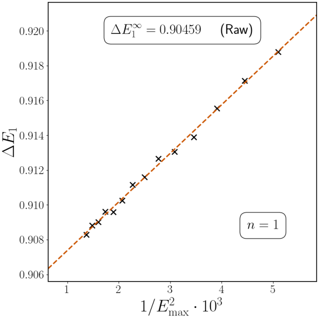

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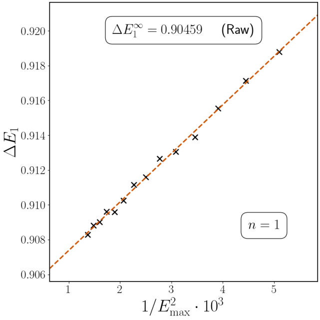

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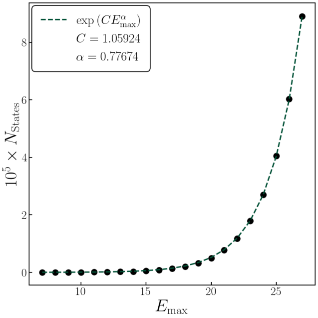
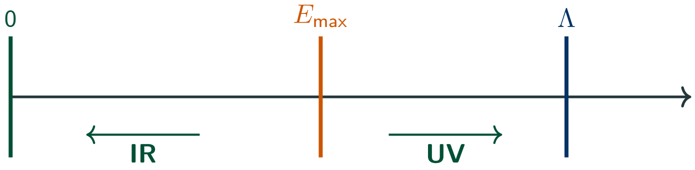


Effective Field Theory

Effective Field Theory applied to Hamiltonian Truncation

- *Effective Field Theory:* applies with clear scale hierarchy

  Houtz, Cohen, Farnsworth, Luty (2022) [4]



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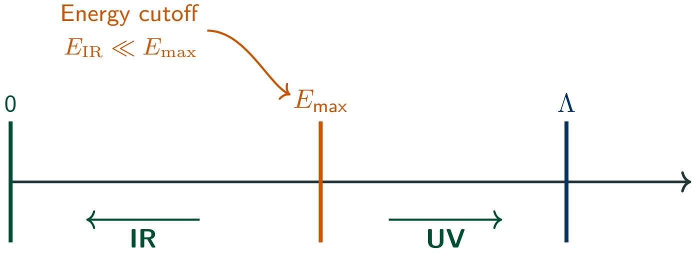

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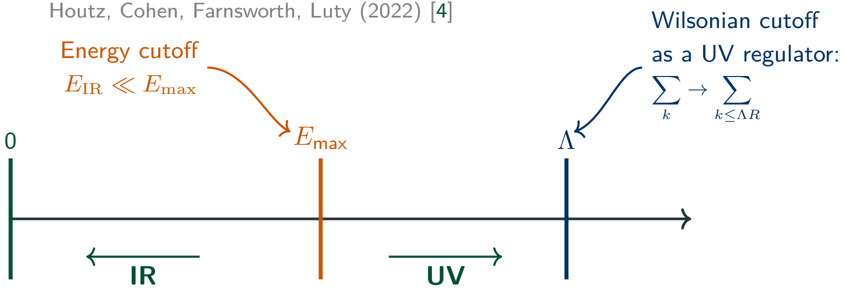

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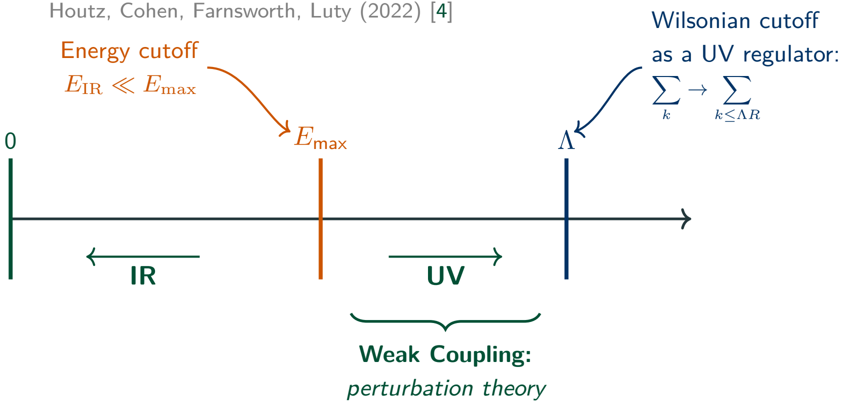

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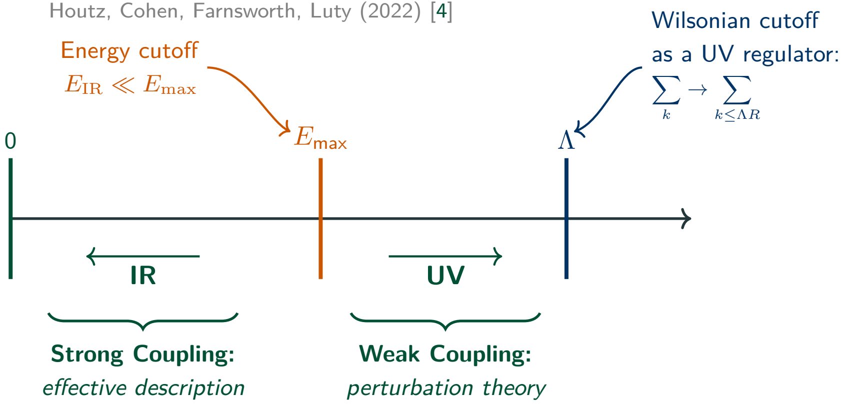

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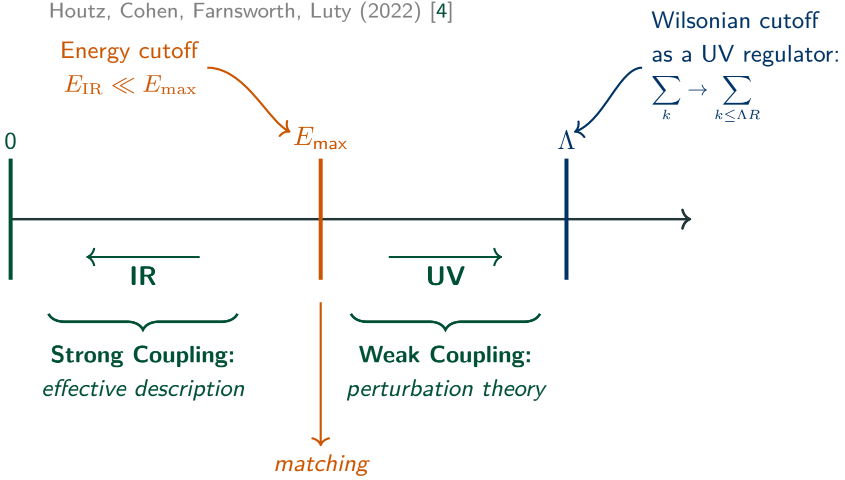

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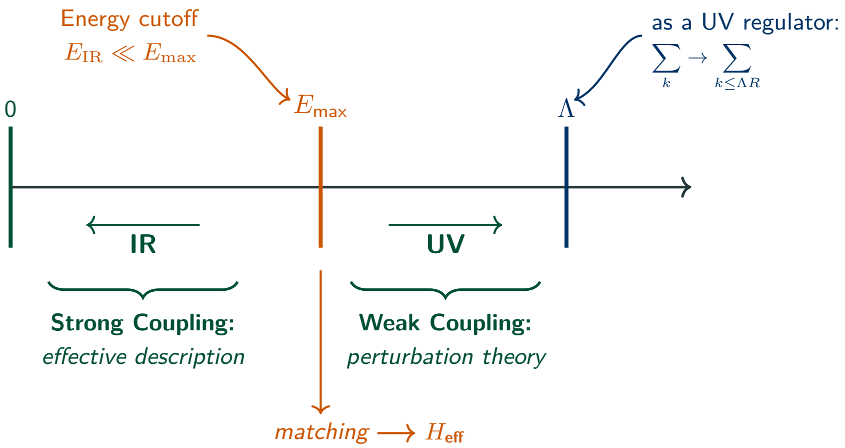

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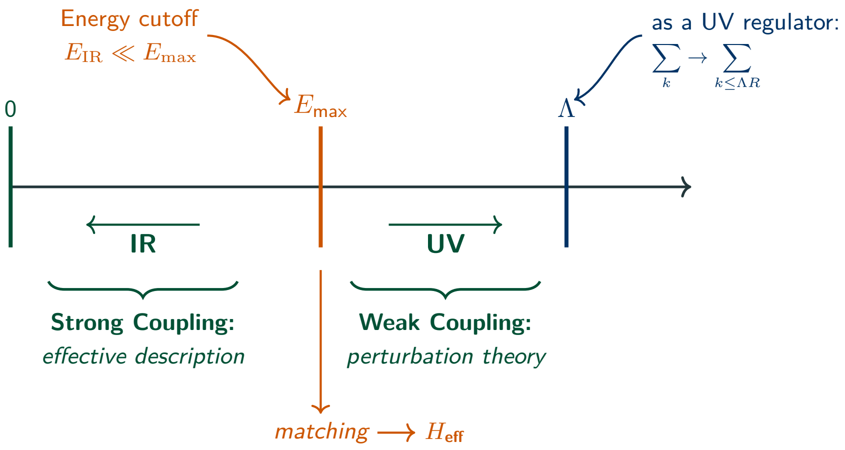

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Matching Procedure

Effective Hamiltonian $H_{\text{eff}} = H_0 + H_1 + H_2 + \cdots\,, \quad H_n \sim \mathcal{O}(V^n)$

Key idea: Compare observables in the full and effective theories

Requirements:
1. Fix H_{eff} uniquely,
2. Allow systematic expansion in IR scales / E_{max},
3. Ensure separation of scales at each order.

Issue: Matching the spectrum fails 1.

Approach: Matching the **transition matrix**:

$$\lim_{t_f \to \infty} \langle f|U_{IP}(t_f, 0)|i\rangle = \delta_{ij} + \frac{\langle \mathbf{f}|\mathbf{T}|\mathbf{i}\rangle}{E_{fi} + i\epsilon}$$

$$\Longrightarrow \boxed{\langle f|T|i\rangle_{\text{fund}} = \langle f|T|i\rangle_{\text{eff}}}$$

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Matching conditions up to order $\mathcal{O}(V^2)$

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- We will perform the matching using a **diagrammatic approach**
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Diagrammatic Matching

Diagrammatic Matching order $\mathcal{O}(V)$

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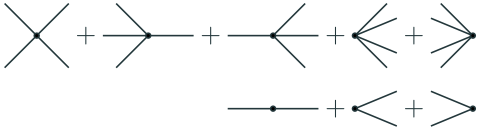

$$= \frac{\lambda}{4!}\int dx \langle f|:\phi^4:|i\rangle$$

$$= \frac{1}{2}\, m_V^2 \int dx \langle f|:\phi^2:|i\rangle$$

- Same diagrams in effective and full theory
- Defines the parameters of $H_1 = \int dx \left[\frac{1}{2}m_{V1}^2 :\phi^2: + \frac{\lambda_1}{4!}:\phi^4:\right]$

Matching conditions $\mathcal{O}(V)$

$$\lambda_1 = \lambda, \qquad m_{V1}^2 = m_V^2$$

At this order, effective and "raw" Hamiltonian truncation coincide

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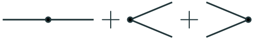

$$= \frac{1}{2}\, m_V^2 \int dx \langle f|:\phi^2:|i\rangle$$

- Same diagrams in effective and full theory
- Defines the parameters of $H_1 = \int dx \left[\frac{1}{2} m_{V1}^2 :\phi^2: + \frac{\lambda_1}{4!} :\phi^4:\right]$

Matching conditions $\mathcal{O}(V)$

$$\lambda_1 = \lambda, \qquad m_{V1}^2 = m_V^2$$

At this order, effective and "raw" Hamiltonian truncation coincide

Diagrammatic Matching order $\mathcal{O}(V^2)$

- Matching at second order: $\langle f|H_2|i\rangle_{\text{eff}} = \sum_{\alpha}^{>} \frac{\langle f|V|\alpha\rangle\langle\alpha|V|i\rangle}{E_{f\alpha}}$
- Diagrams contributing with four external legs $T_2^{(4)}$:

- Diagrams contributing with two external legs $T_2^{(2)}$:

- Diagram evaluation example:

$$= \frac{1}{8}\left(\frac{\lambda}{2\pi R}\right)^2 \sum_{1,\dots,4} \delta_{12,34}\, \langle f|\, \phi_4^{(-)}\, \phi_3^{(-)}\, \phi_2^{(+)}\, \phi_1^{(+)}\, |i\rangle$$

$$\times \sum_{5,6} \delta_{34,56}\, \frac{\Theta(-E_{\max} + \mathcal{E}_f - \omega_3 - \omega_4 + \omega_5 + \omega_6)}{4\omega_5\omega_6(\omega_3 + \omega_4 - \omega_5 - \omega_6)}$$

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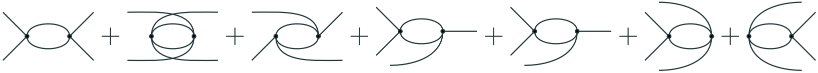

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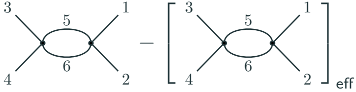

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Diagrammatic Matching order $\mathcal{O}(V^2)$

- Complicated IR dependence negligible assuming scale separation:

$$\underbrace{m_Q \lesssim \omega_{1234} \lesssim \mathcal{E}_{i,f}}_{\textbf{IR scales}} \ll E_{\mathsf{max}} \quad \Longrightarrow \quad \underbrace{\omega_{5,6} \gtrsim E_{\max}}_{\textbf{From step function}}$$

- **Local approximation**: set all external energies to zero

$$\simeq -\frac{\lambda^2}{128\pi R}\int dx\, \langle f|\left[\phi^{(-)}\right]^2\left[\phi^{(+)}\right]^2|i\rangle$$

$$\times \sum_k \frac{\Theta(-E_{\max}+2\omega_k)}{\omega_k^3}\left[1+\mathcal{O}\left(\frac{\mathcal{E}_{i,f}}{E_{\max}}\right)\right]$$

- $\mathcal{O}\left(\frac{\mathcal{E}_{i,f}}{E_{\max}}\right) \longrightarrow T$ satisfies requirement 2. IR expansion / $E_{\max}$

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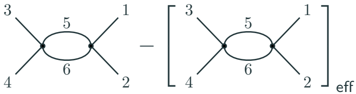

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$$\lambda_2 = \frac{3\lambda^2}{16\pi R} \sum_k \frac{\Theta(2\omega_k - E_{\text{max}})}{\omega_k^3}$$

$$m_{V2}^2 = \frac{\lambda}{16\pi R} \left[\frac{\lambda}{6\pi R} \sum_{3,4,5} \delta_{345,0} \frac{\Theta(\omega_3 + \omega_4 + \omega_5 - E_{\text{max}})}{\omega_3\omega_4\omega_5(-\omega_3 - \omega_4 - \omega_5)} - m_V^2 \sum_k \frac{\Theta(2\omega_k - E_{\text{max}})}{2\omega_k^3}\right]$$

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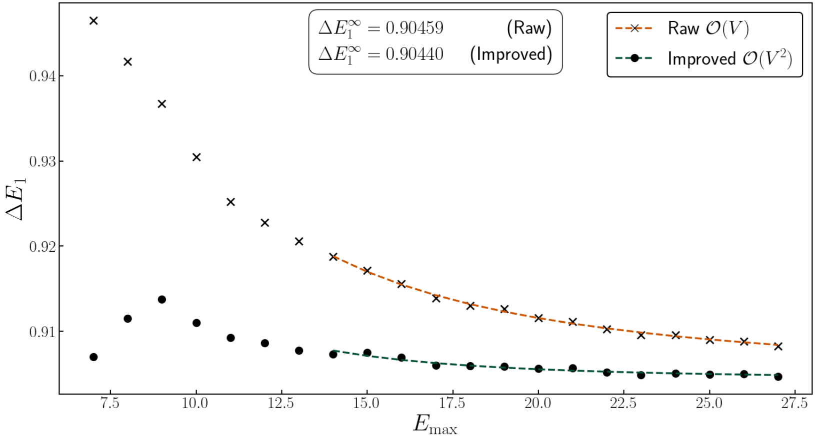
Numerical Results

Improved Scaling of the Accuracy



Behaviour of the energy difference $\Delta E_1 \doteq E_1 - E_0$ between the ground state and the first excited state, shown as a function of the energy cut-off $E_{\max}$

Two-point correlation function

Two-point correlation function:

$$G(x) \doteq \langle\psi_0|{:}\phi(x)\phi(0){:}|\psi_0\rangle \sim e^{-|x|/\xi}$$

Correlation length:

$$\xi \sim 1/m_{ph} \sim C\,|\lambda-\lambda_c|^{-1}$$

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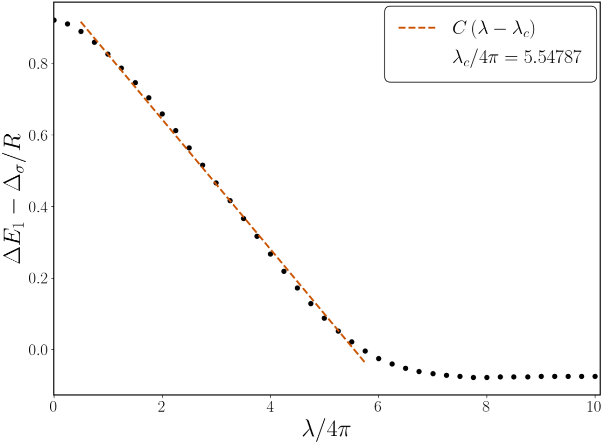

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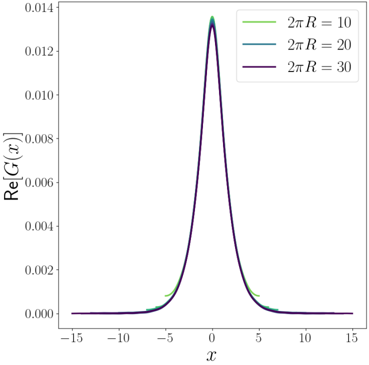

- $\lambda/\pi = 1$

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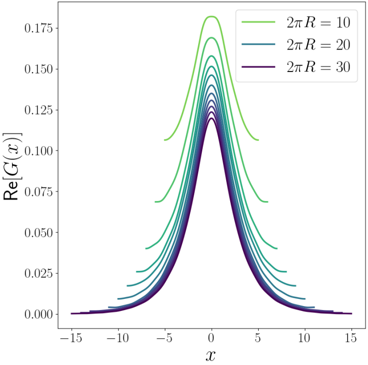

- $\lambda/\pi = 5$

Extension of HTET beyond the rest frame

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Problem: *Energy cut-off is not Lorentz-invariant*

Key idea: Imposing the cut-off on the invariant mass μ_0
Chen *et al.* (2022) [5]

Step:
1. Fix total momentum P ($[H, P] = 0$)
2. Compute $\mu_0 \doteq \sqrt{\mathcal{E}^2 - P^2}$,
3. Impose cut-off $\mu_0 \ \leq \ E_{\max}$

Truncated Subspace

$$\mathcal{H}_{\text{eff}} \doteq \text{span}\{|\bar{n}_i\rangle \,\big|\, \mathcal{E} = \sum_k \omega_k n_k \leq \sqrt{E_{\max}^2 + P^2}\}$$

Vacuum "Effective" vacuum energy in the boosted frame:

$$E_{\text{vac}}^p \doteq E_1^p - \sqrt{m_{\text{ph}}^2 + P^2}$$

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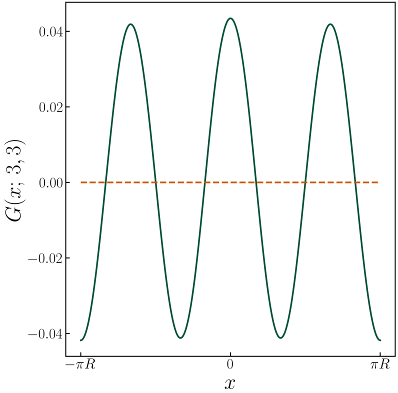
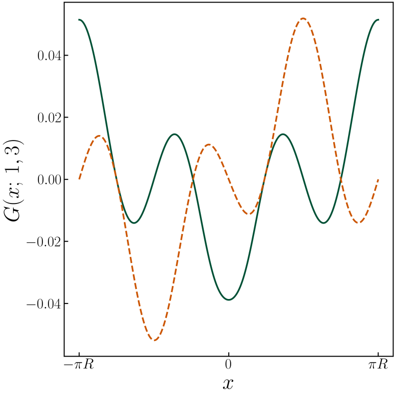
- **Two–point correlation functions** between states of non–zero momentum

$$G(x;\,\mathbf{p},\mathbf{q}) \doteq \langle\psi_0^q|{:}\phi(x)\phi(0){:}|\psi_0^p\rangle$$



- $p = q = 3$



- $p = 1 \quad q = 3$

Connection to PDF and TMD

From qPDFs/qTMDs on the Lattice to PDFs/TMDs

- **qPDFs/qTMDs** are computed on the lattice
- Tend to **PDFs/TMDs** for $P_z \to \infty$, but $P_z \sim \pi/a \sim$ 2-3 GeV
- **Large Momentum Effective Theory** $\longrightarrow \mathcal{O}\left(\frac{\Lambda^2_{\text{QCD}}}{P_z^2}, \frac{M^2}{P_z^2}\right) \sim 1\text{-}20\ \%$

Limitations:

- Statistical noise
- Complicated renormalisation
- Unstable numerical reconstruction
- Limited access to endpoints $x \to 0, 1$

Lattice (Euclidean)

qPDF $\tilde{f}_i(x, P_z)$
qTMD $\tilde{f}_i(x, \boldsymbol{k}_T, P_z)$

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$\xrightarrow{P_z \to \infty}$

Light-Cone

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LaMET $\longrightarrow$

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$\xrightarrow{\text{LaMET}}$

Light-Cone

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Exploiting the HTET approach to improve current methods?

Key Idea: Work directly in light-cone coordinates

Basis Light-Front Quantization

Basis: $\mathcal{H}_{BLFQ}(K, N_{\max})$

- Longitudinal momentum: DLCQ
- Transverse d.o.f.: 2D-HO

Application:

- QED, QCD, and nuclear systems

  Form factors, PFD/TMD

Light-cone Conformal Truncation

Basis: $\mathcal{H}_{LCT}(\Delta_{\max})$

- States of a CFT

Application:

- $1+1$D: ϕ^4, Gross–Neveu, QCD
- $2+1$D: ϕ^4

  Spectral densities, form factors, correlators... PDFs?

Outlook Implement the HTET approach to improve accuracy with systematic diagrammatic matching

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References

[1] Yurov, V. P. and Zamolodchikov, A. B.“ ‘Truncated conformal space approach to scaling Lee-Yang model’” (1990).

[2] Hogervorst, M., Rychkov, S., and Rees, B. C. van“ ‘Truncated conformal space approach in d dimensions: A cheap alternative to lattice field theory?’” (2015).

[3] Elias-Miró, J., Rychkov, S., and Vitale, L. G.“ ‘NLO renormalization in the Hamiltonian truncation’” (2017).

[4] Cohen, T. et al.“ ‘Hamiltonian Truncation Effective Theory’” (2022).

[5] Chen, H. et al.“ *Giving Hamiltonian Truncation a Boost*” (2022).

Backup Slides

Definition of the Transition Matrix

- Time evolution operator in the interaction picture:

$$U_{IP}(t_f, t_i) = T\left[e^{-i\int_{t_i}^{t_f} dt\; V_{IP}(t)}\right], \text{ with } V_{IP}(t) = e^{iH_0 t}\left(V\, e^{-\epsilon t}\right)e^{-iH_0 t}$$

- Definition of the matching operator:

$$\langle f|\Sigma|i\rangle \doteq \lim_{t_f\to\infty}\langle f|U_{IP}(t_f,0)|i\rangle$$

- Implicit relation can be evaluated iteratively, order by order:

$$U_{IP}(t_f, t_i) = \mathbb{I} - i\int_{t_i}^{t_f} dt\, U_{IP}(t_f, t)\, V_{IP}(t)$$

- This leads to the expansion ($E_{\alpha\beta} \doteq \mathcal{E}_\alpha - \mathcal{E}_\beta$):

$$\langle f|\Sigma|i\rangle = \delta_{fi} + \frac{\langle f|V|i\rangle}{E_{fi}+i\epsilon} + \sum_\alpha \frac{\langle f|V|\alpha\rangle\langle\alpha|V|i\rangle}{(E_{f\alpha}+i\epsilon)(E_{fi}+i\epsilon)} + \mathcal{O}(V^3)$$

- The common denominator $E_{fi}+i\epsilon$ can be factored out:

$$\langle f|\Sigma|i\rangle \doteq \delta_{ij} + \frac{\langle \mathbf{f}|\mathbf{T}|\mathbf{i}\rangle}{E_{fi}+i\epsilon}$$

Wilsonian Renormalization and Scale Separation

- By introducing the Wilsonian cut-off Λ, we obtain:

$$m_V^2 = m_0^2 - m_Q^2 + \frac{\lambda}{8\pi R} \sum_{|k| \leq \Lambda R} \frac{1}{\omega_k}$$

- Renormalized mass m_R^2 depending on an arbitrary renormalization scale μ:

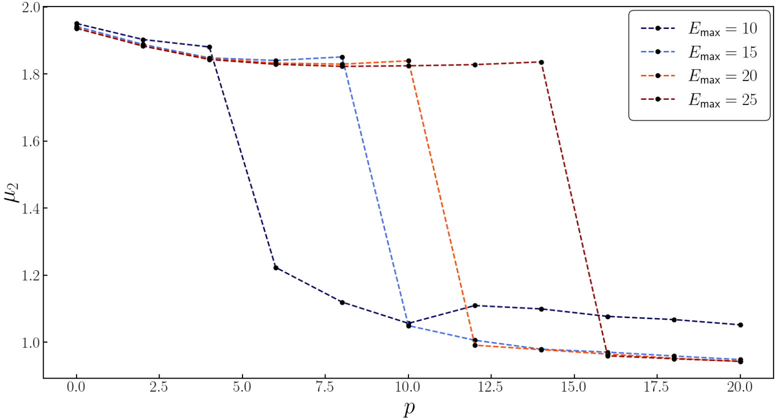
$$m_R^2(\mu) \doteq m_0^2 + \frac{\lambda}{8\pi R} \sum_{\mu R < |k| \leq \Lambda R} \frac{1}{\omega_k}$$

- Ref. [4]: separation of scales is manifest in terms of $m_R^2(\mu \sim E_{\max})$
- We then define the *normal ordered mass* as

$$m_{NO} = m_R(\mu = 0) \Longrightarrow m_Q = m_{NO}, \quad m_V^2 = 0.$$

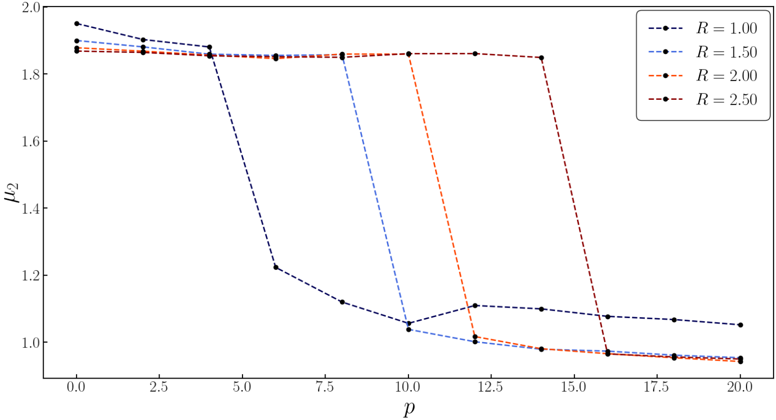
Range di validità HTET boosted-frame $\mathrm{E}_{\max}$

- If the cut-off $E_{\max}$ or the radius R is too small, the boost is ill-defined.
- Analysis by invariant-mass: $\mu_i(p) = \sqrt{\left(E_i^p - E_{\mathrm{vac}}^p\right)^2 + \left(\frac{p}{R}\right)^2}$



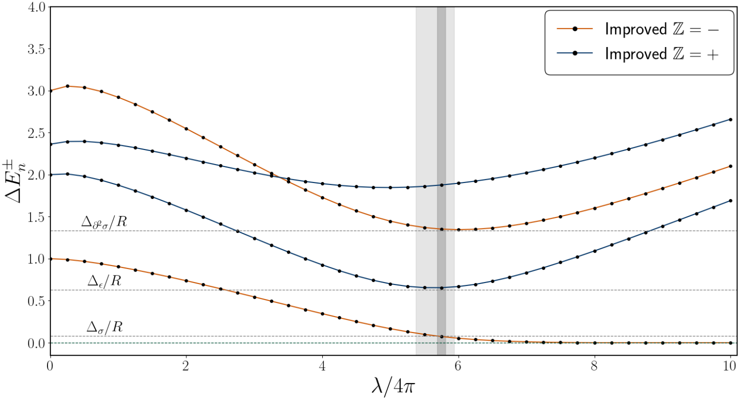
Range di validità HTET boosted-frame $\mathbb{R}$

- If the cut-off $E_{\max}$ or the radius R is too small, the boost is ill-defined.
- Analysis by invariant-mass: $\mu_i(p) = \sqrt{\left(E_i^p - E_{\text{vac}}^p\right)^2 + \left(\frac{p}{R}\right)^2}$



$\mathbb{Z}_2$ Symmetry Breaking

- For $\lambda \lesssim \lambda_c$, $E_1 \approx E_0$, indicating that the *mass gap* is closing, signal of spontaneous symmetry breaking
- At $\lambda = \lambda_c$, a second-order phase transition occurs and the gap vanishes completely
- For $\lambda > \lambda_c$, the theory enters has two degenerate vacua



LaMET Limitations

$$q(x,\mu) \;=\; \int_{-1}^{1} \frac{dy}{|y|}\, C^{-1}\!\left(\tfrac{x}{y}, \tfrac{\mu}{P_z}\right)\, \tilde{q}(y, P_z, \mu)$$

Statistical noise:

Bilocal correlators exhibit noise that increases exponentially with spatial separation (required for P_z), making precise extraction difficult.

Complicated renormalisation:

The infinite Wilson line introduces additional linear divergences, requiring a scheme (e.g. non-perturbative RI/MOM) followed by conversion to $\overline{\text{MS}}$.

Unstable numerical reconstruction:

The deconvolution is an ill-posed problem, highly sensitive to numerical and statistical errors.

Limited access to endpoints $x \to 0, 1$:

Lattice data are more reliable in the intermediate region.