EPFL

# QFT as a set of coupled ODEs

## Seminar Università di Genova / INFN

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# Motivation

- QFT is fantastic in statistical physics, particle physics, cosmology
- Many phenomenologically relevant QFTs are *strongly coupled*

**How do we formulate predictions in strongly coupled QFTs?**

| Method | Downsides |
|---|---|
| Conformal bootstrap | Only CFT |
| S-matrix bootstrap | Only 2-to-2 |
| Hamiltonian truncation | Cutoff breaks Lorentz, many divergences |
| Lattice | Expensive, sign problem, no dynamics |
| Integrability | Only 2D and extra conserved charges |
| Holography | Only large $N$ and SUSY |

# Basic idea

**A framework in the continuum, with no divergences and (*maybe??)* cheap**

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- In CFT, local observables are fixed by $\{\Delta_i, C_{ijk}\}$

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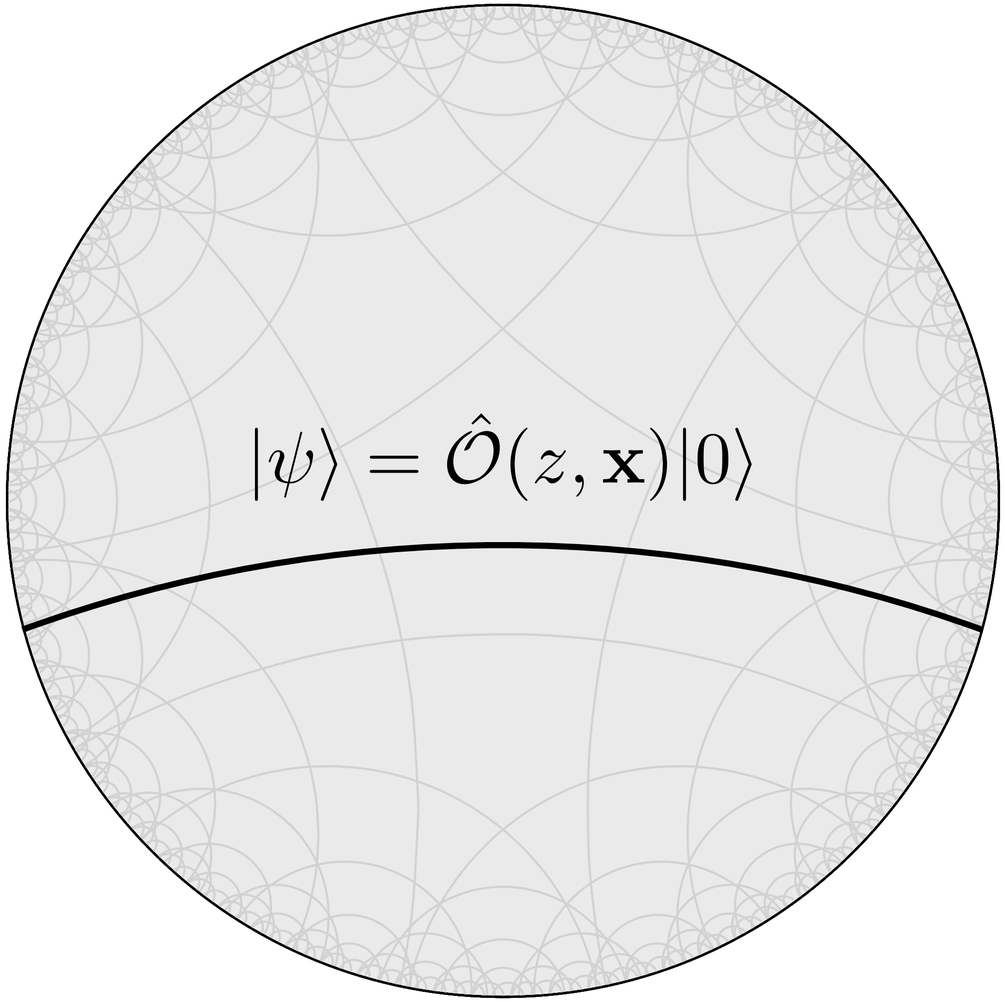
- QFT in AdS defines a family of CFTs (without stress tensor)
- Observables in QFT in AdS fixed by $\{\Delta_i(\lambda),\ b_i^{\hat{\mathcal{O}}}(\lambda)\ , C_{ijk}(\lambda)\}$

# The Boundary Operator Expansion (BOE)



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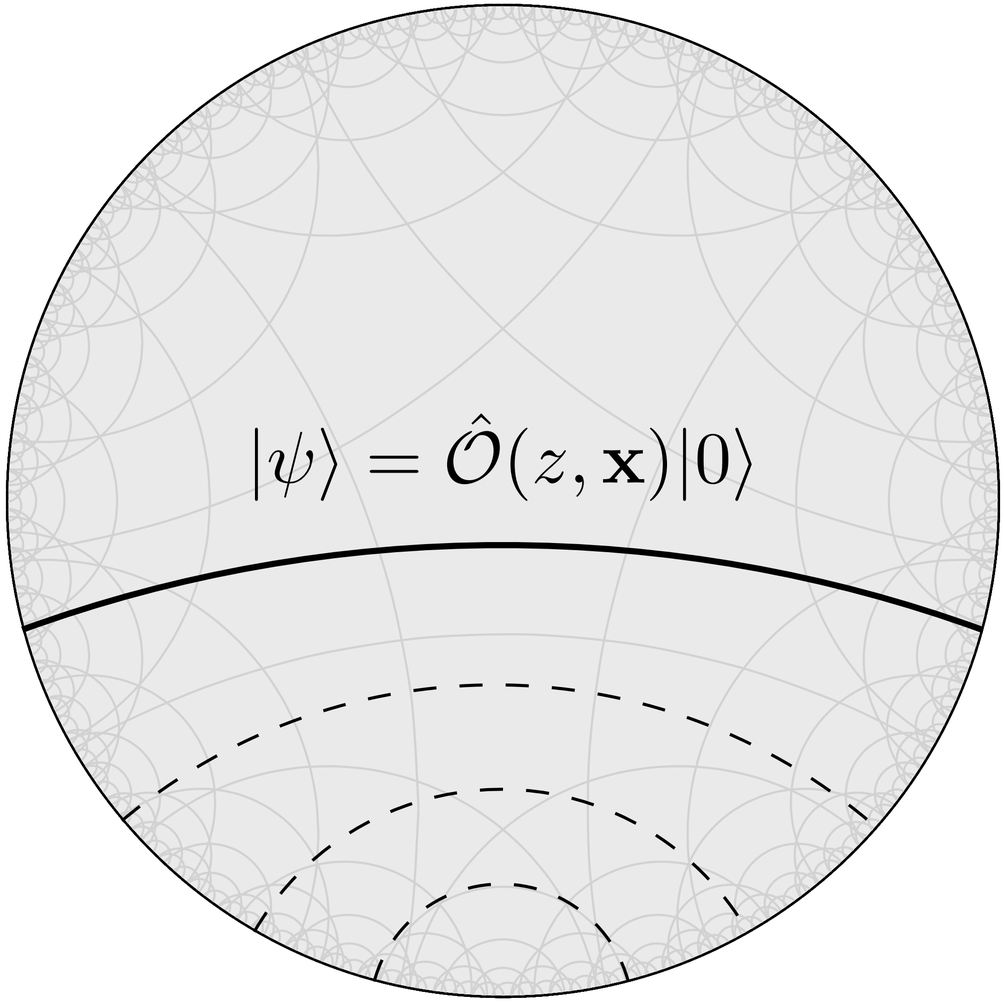

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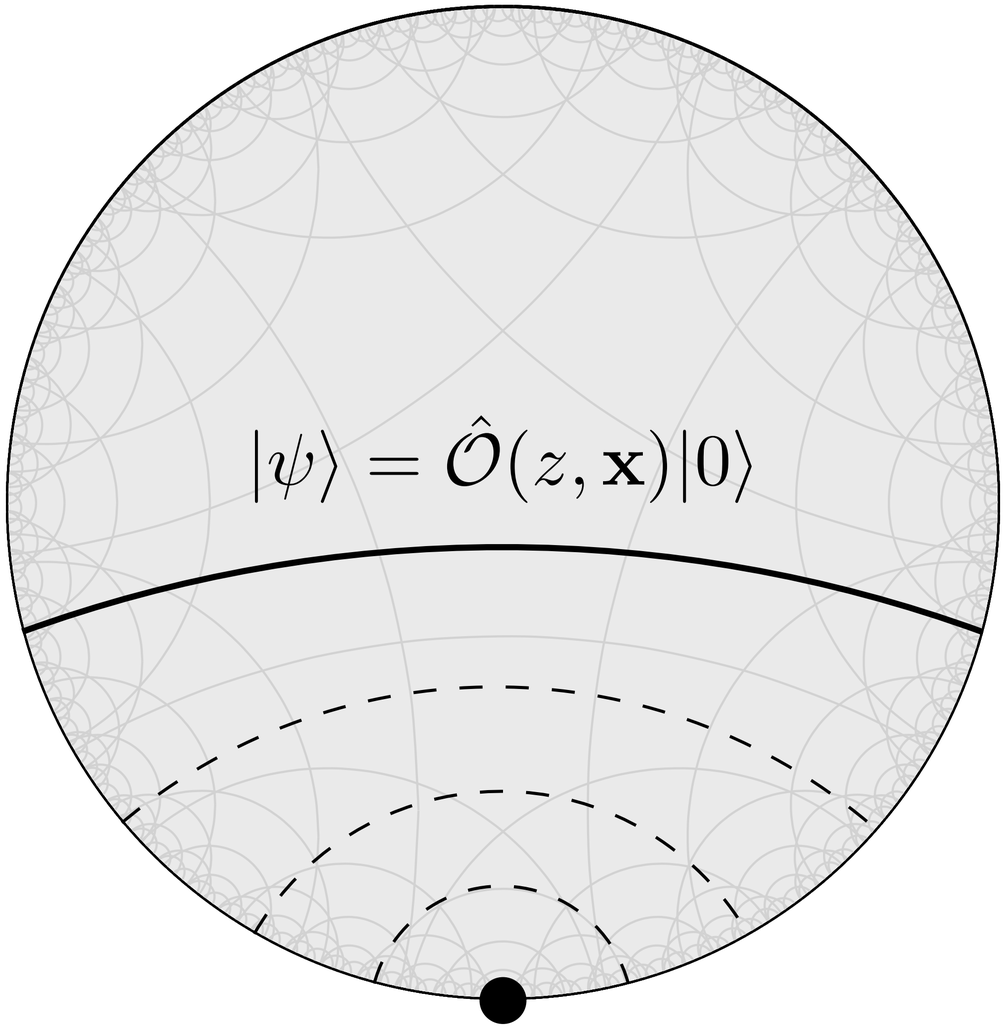


$$|\psi\rangle = \sum_i z^{\Delta_i} b_i^{\hat{\mathcal{O}}} \left(\mathcal{O}_i(\mathbf{x}) + \text{desc.}\right)$$

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- For arbitrary $\lambda$ , we can compute <u>universally</u> the derivatives of the data

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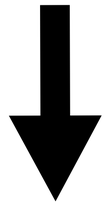
- Observables in QFT in AdS fixed by $\{\Delta_i(\lambda),\ b_i^{\hat{\mathcal{O}}}(\lambda)\ , C_{ijk}(\lambda)\}$
- For arbitrary $\lambda$ , we can compute universally the derivatives of the data
- Truncated system of ODEs can in principle be solved iteratively, flowing the data to strong coupling and to flat space

# Basic idea

What are local correlation functions in AdS?

And how do we compute them?

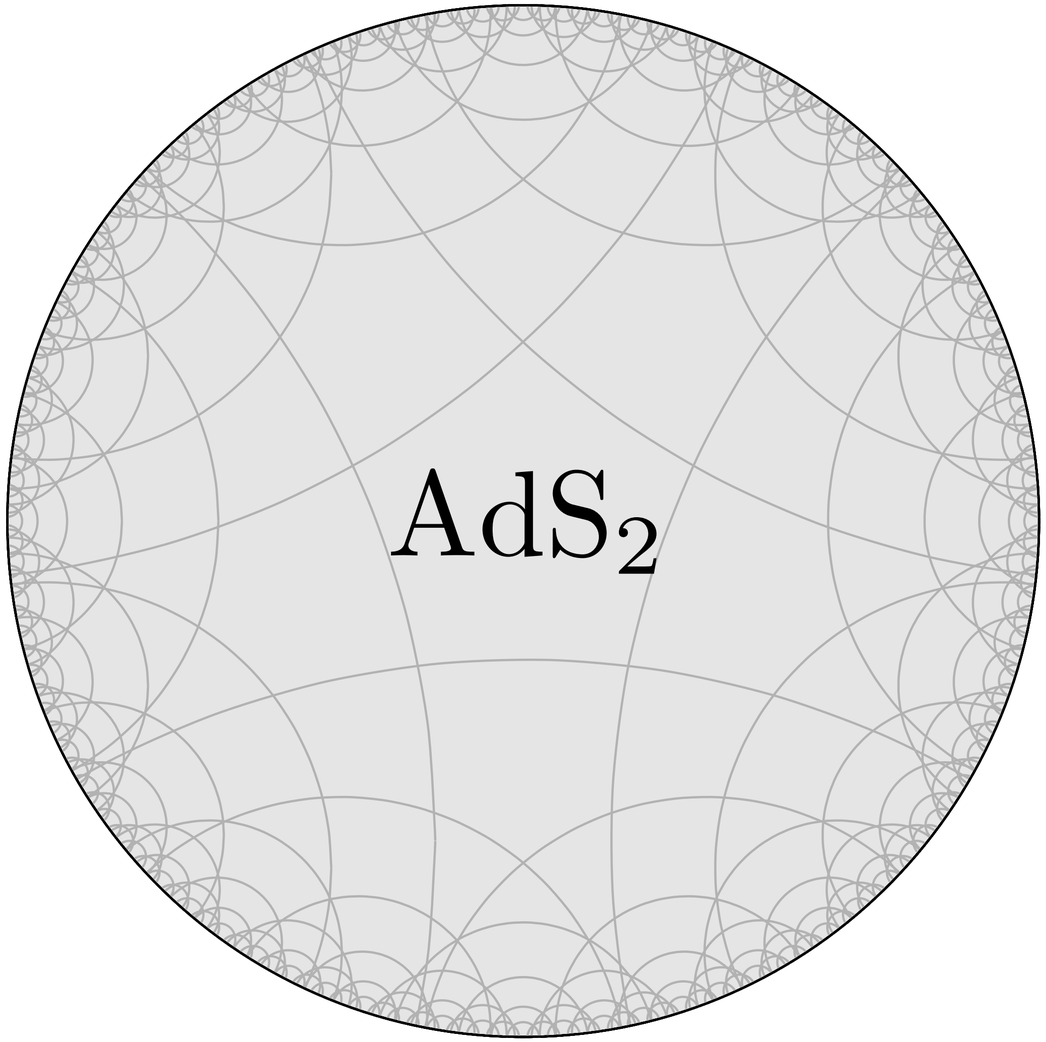


What is the S-matrix in QFT in flat space?

And how do we compute it?

[Paulos, Penedones, Toledo, van Rees, Vieira, ’16]

# Basic idea




$$S = S_{\mathrm{QFT}_2} + \lambda \int_{\mathrm{AdS}_2} \hat{\Phi}$$

"QFT data" fully fixes all local observables

$$\Delta_i(\lambda)\,, \quad C_{ijk}(\lambda)\,, \quad b_i^{\hat{\mathcal{O}}_a}(\lambda)$$

Taking $\lambda \to \lambda + \delta\lambda$ , with PT we compute

$$\frac{\Delta_i}{d\lambda}(\lambda)\,, \quad \frac{C_{ijk}}{d\lambda}(\lambda)\,, \quad \frac{db_i^{\hat{\mathcal{O}}_a}}{d\lambda}(\lambda)$$

Truncated system of ODEs can be iteratively solved

Inspiration: [Hollands, Wald, '23]

1D conformal manifolds: [Behan, '17]

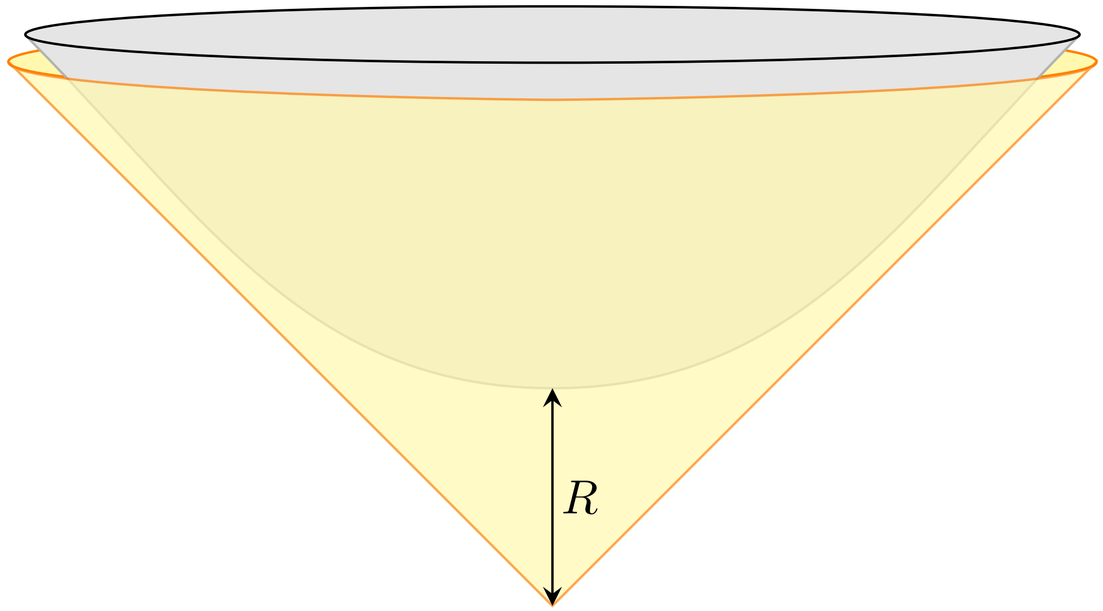
# Outline

- Correlation functions from QFT data
- Perturbation theory, once and for all
- Examples and checks
- Further properties of the flow equations

# Observables from QFT data

# Geometry

## Embedding space formalism




- Embed $\mathrm{AdS}_2$ in $\mathbb{R}^{1,2}$

$$X^A X^B \eta_{AB} = -R^2$$

- Boundary $\equiv$ space of light rays

$$P^A P^B \eta_{AB} = 0\,, \qquad P^A \sim \lambda P^A$$

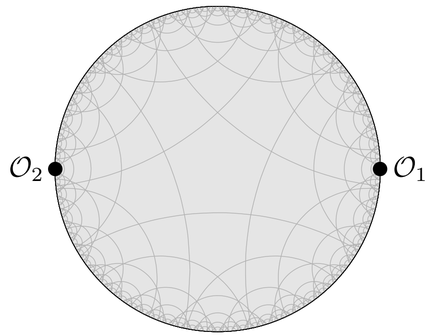
- Poincaré metric $\quad ds^2 = \dfrac{\mathrm{d}z^2 + \mathrm{d}\tau^2}{z^2}$

# Correlation functions

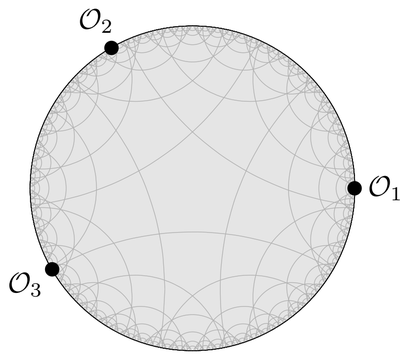
$$P_{12} \equiv -2P_1 \cdot P_2$$

$$\Delta_{123} \equiv \Delta_1 + \Delta_2 - \Delta_3$$

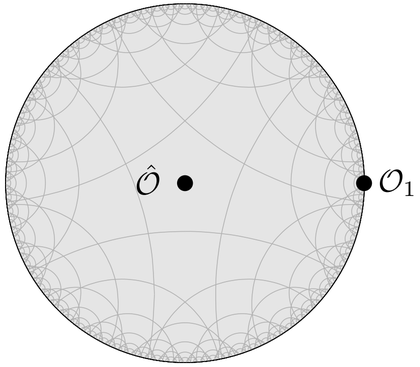
- At any point in the RG flow




$$= \frac{\delta_{\Delta_1 \Delta_2}}{P_{12}^{\Delta_1(\lambda)}}$$




$$= \frac{C_{123}(\lambda)}{P_{12}^{\Delta_{123}(\lambda)} P_{23}^{\Delta_{231}(\lambda)} P_{13}^{\Delta_{132}(\lambda)}}$$




$$= \frac{b_1^{\hat{\mathcal{O}}}(\lambda)}{(-2P \cdot X)^{\Delta_1(\lambda)}}$$

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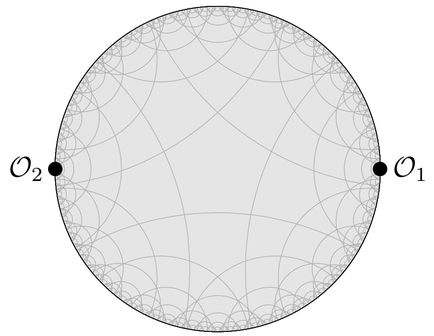


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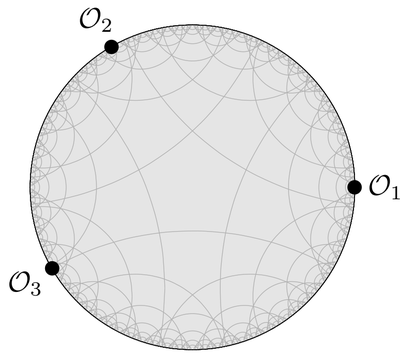


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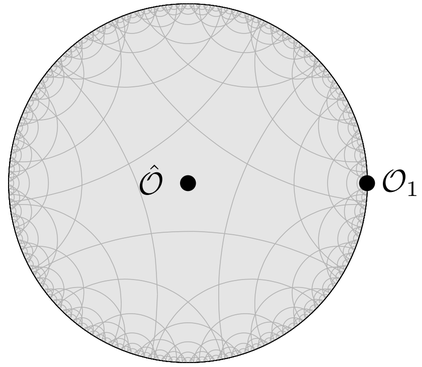


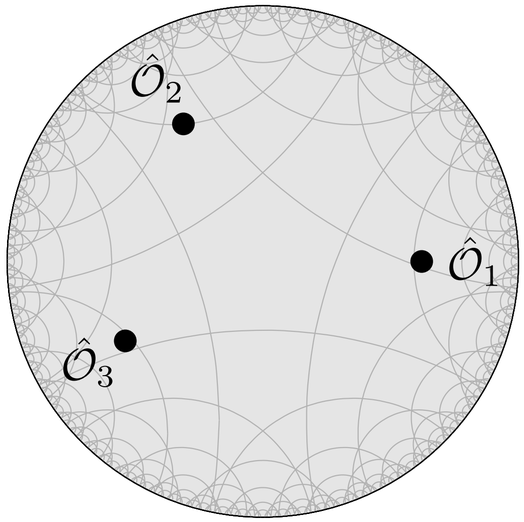

$$= \frac{b_1^{\hat{\mathcal{O}}}(\lambda)}{(-2P \cdot X)^{\Delta_1(\lambda)}}$$

These are the coefficients in the BOE

$$\hat{\mathcal{O}}(z,\tau) = \sum_i z^{\Delta_i} b_i^{\hat{\mathcal{O}}} \left(\mathcal{O}_i(\tau) + \text{descendants}\right)$$

# Correlation functions

- State-operator map: any other correlation function is built from QFT data




$$= \sum_{i,j,k} b_i^{\hat{\mathcal{O}}_1} b_j^{\hat{\mathcal{O}}_2} b_k^{\hat{\mathcal{O}}_3} C_{ijk} \left(\text{kinematic block}\right)_{ijk}$$

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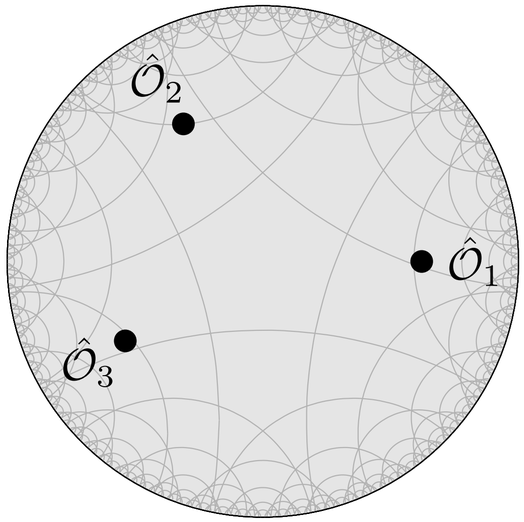


Dynamics Theory-independent

$$= \sum_{i,j,k} b_i^{\hat{\mathcal{O}}_1} b_j^{\hat{\mathcal{O}}_2} b_k^{\hat{\mathcal{O}}_3} C_{ijk} \, (\text{kinematic block})_{ijk}$$

# The QFT data

- QFT observables are thus fully fixed by these functions

$$\Delta_i(\lambda)\,, \quad C_{ijk}(\lambda)\,, \quad b_i^{\hat{\mathcal{O}}_a}(\lambda)$$

- For boundary observables, we only need

$$\Delta_i(\lambda)\,, \quad C_{ijk}(\lambda)\,, \quad b_i^{\hat{\Phi}}(\lambda)$$

- ODEs for these functions come by solving perturbation theory once and for all

# Perturbation theory, once and for all

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## Shifting data

- Change the coupling infinitesimally

$$S = S_{\mathrm{QFT}_2} + (\lambda + \delta\lambda) \int_X \hat{\Phi}$$

- At leading order, the only effect is

$$\delta\Delta_i(\lambda) = \delta\lambda \; \Delta_i'(\lambda)$$
$$\delta C_{ijk}(\lambda) = \delta\lambda \; C_{ijk}'(\lambda)$$
$$\delta b_{\hat{\mathcal{O}}i}(\lambda) = \delta\lambda \; b_{\hat{\mathcal{O}}i}'(\lambda)$$

# Perturbation theory, once and for all

## 2-pt function

$$S = S_{\mathrm{QFT}_2} + (\lambda + \delta\lambda) \int_X \hat{\Phi}$$

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# Perturbation theory, once and for all

## 2-pt function

$$S = S_{\mathrm{QFT}_2} + (\lambda + \delta\lambda) \int_X \hat{\Phi} \qquad\qquad \delta\Delta_i(\lambda) = \delta\lambda \; \Delta_i'(\lambda)$$

- At the same time,

$$\langle \mathcal{O}(P_1)\mathcal{O}(P_2) \rangle_{(\lambda+\delta\lambda)} = \langle \mathcal{O}(P_1)\mathcal{O}(P_2) \rangle_{(\lambda)} \; - \; \delta\lambda \int_X \langle \mathcal{O}(P_1)\mathcal{O}(P_2)\hat{\Phi}(X) \rangle_{(\lambda)}$$

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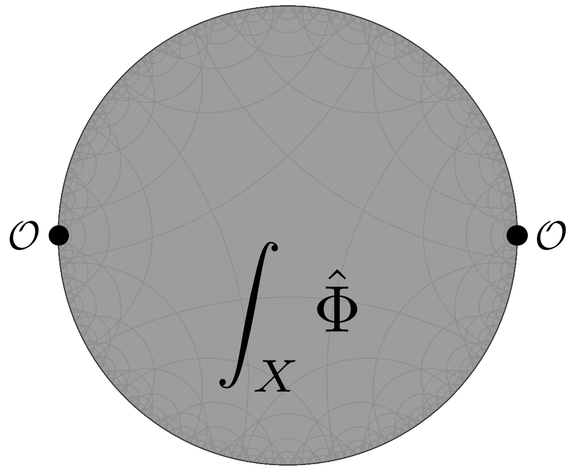

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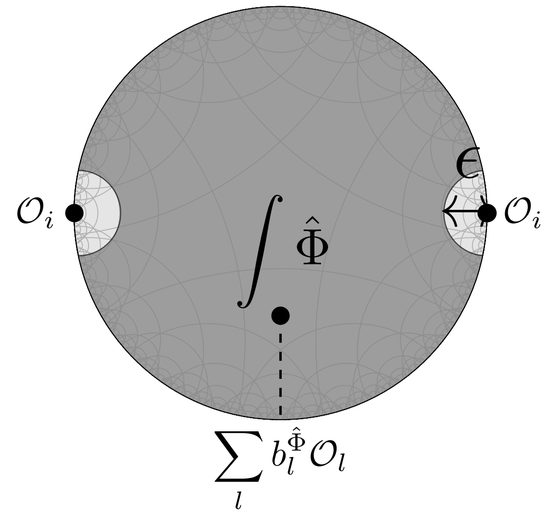
- Renormalization conditions

$$\mathcal{O}_{(\lambda+\delta\lambda)} = (1 + Z(\epsilon))\, \mathcal{O}^R_{(\lambda+\delta\lambda)}$$

$$\langle \mathcal{O}_i^R(P_1)\mathcal{O}_j^R(P_2)\rangle_{(\lambda+\delta\lambda)} \equiv \frac{\delta_{ij}}{P_{12}^{\Delta_i+\delta\Delta_i}}$$

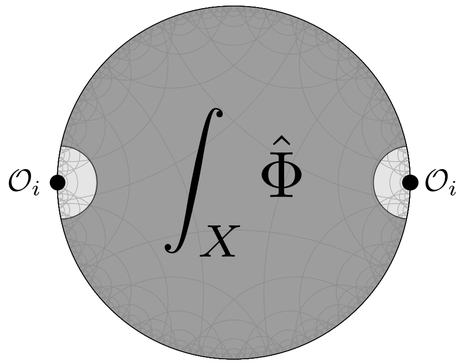


# Perturbation theory, once and for all

## Conformal blocks

- We want to carry out these integrals universally

$$\int_{X,\epsilon} \langle 0|\mathcal{O}_i(P_1)\mathcal{O}_i(P_2)\hat{\Phi}(X)|0\rangle$$



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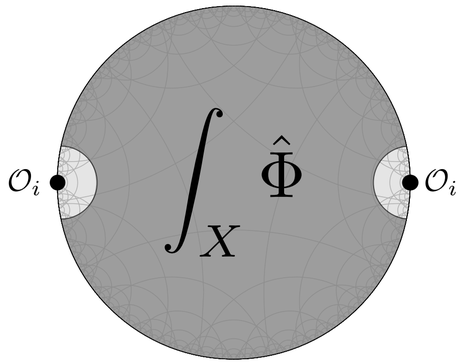
$$\int_{X,\epsilon} \langle 0|\mathcal{O}_i(P_1)\mathcal{O}_i(P_2)\hat{\Phi}(X)|0\rangle \;=\; \int_{X,\epsilon} \sum_j \int_P \langle 0|\mathcal{O}_i(P_1)\mathcal{O}_i(P_2)|\Delta_j, P\rangle\langle \Delta_j, P|\hat{\Phi}(X)|0\rangle$$



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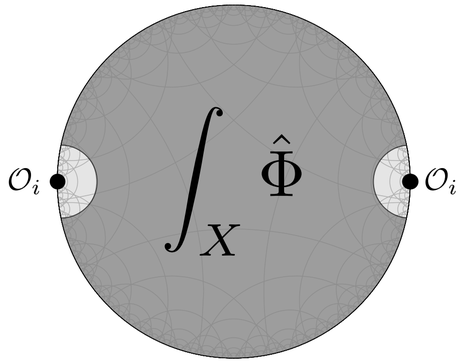
$\cup$ $\cup$

$C_{iij}$ $b_j^{\hat{\Phi}}$



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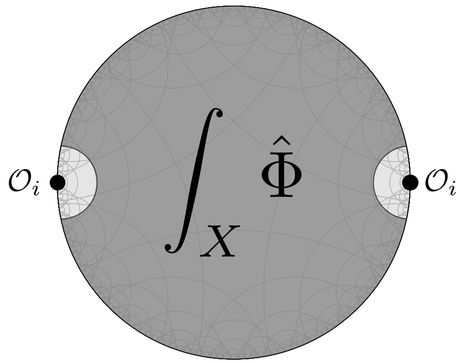
$$= \frac{1}{P_{12}^{\Delta_i}} \int_{X,\epsilon} \sum_j C_{iij}\, b_j^{\hat{\Phi}}\, G_{\Delta_i,\Delta_j}(\chi)$$

Conformal block




$$\chi \equiv \frac{P_{12}}{(P_1 \cdot X)(P_2 \cdot X)}$$

# Perturbation theory, once and for all

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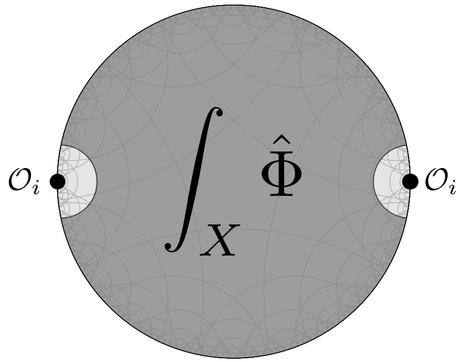



Can we swap?

$$\chi \equiv \frac{P_{12}}{(P_1 \cdot X)(P_2 \cdot X)}$$

# Local blocks

- Big technical parenthesis: must use *local blocks* [Levine, Paulos ’23], [Meineri, Penedones, Spirig ’23]
- Naive conformal blocks present unphysical cuts
- Local blocks form a basis with physical analytic structure
- Endowed with an extra free parameter $\alpha$
- There exists an integral transform

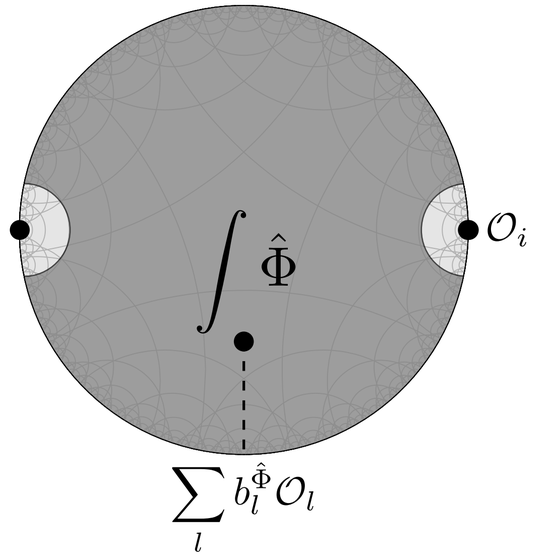
$$F^{(\alpha)}[G] = G^{(\alpha)}$$

$$F^{(\alpha)}[I] = I^{(\alpha)}$$

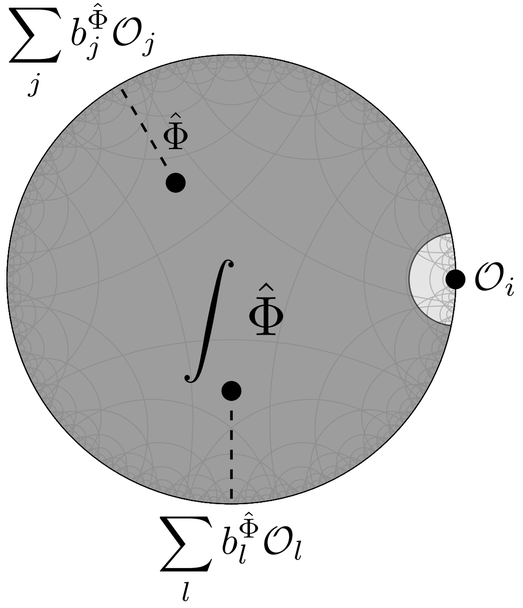
# Perturbation theory, once and for all

- Expanding for small $\delta\lambda$ and matching,

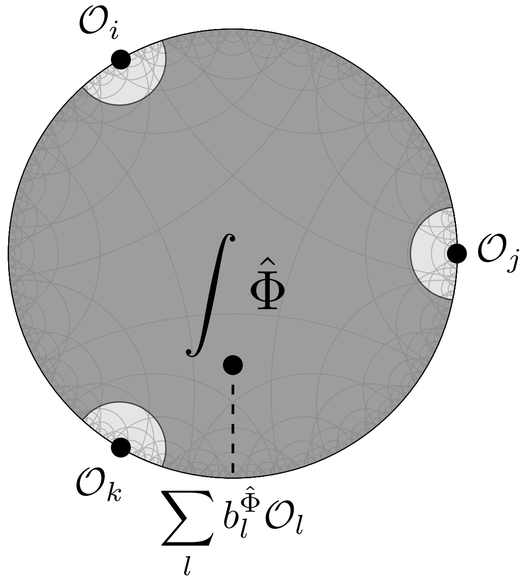
$$\frac{d\Delta_i}{d\lambda} =$$




$$\frac{db_i^{\hat{\Phi}}}{d\lambda} =$$



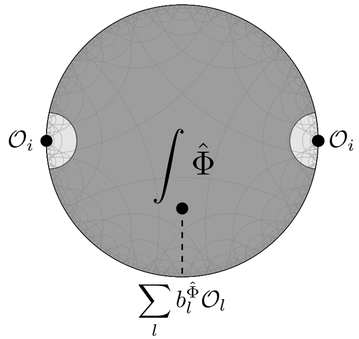

$$\frac{dC_{ijk}}{d\lambda} =$$

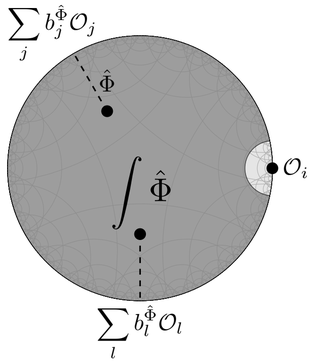


# The flow equations

- The ODEs:

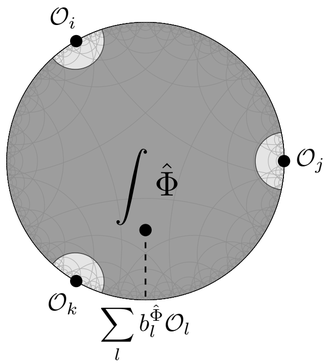
$$\frac{d\Delta_i}{d\lambda} = \sum_l b_l^{\hat{\Phi}} C_{iil} \mathcal{I}^{(\alpha)}(\Delta_l)$$




$$\frac{db_i^{\hat{\Phi}}}{d\lambda} = \sum_{j,l} b_l^{\hat{\Phi}} b_j^{\hat{\Phi}} C_{ijl} \mathcal{J}_{\Delta_i}^{(\alpha)}(\Delta_l, \Delta_j)$$

$$\frac{dC_{ijk}}{d\lambda} = \sum_{l,m} b_l^{\hat{\Phi}} C_{lim} C_{mjk} \mathcal{K}^{(\alpha)}_{\Delta_i \Delta_j \Delta_k}(\Delta_l, \Delta_m) + (ijk) \text{ perms.}$$



# Examples and Checks

# Example
## Scalar GFF

$$S = -\frac{1}{2}\int \sqrt{g}\; d^2x \left(\partial_\mu \hat{\phi} \partial^\mu \hat{\phi} + m^2 \hat{\phi}^2\right)$$

- In this example $\hat{\Phi} = \hat{\phi}^2$ and $\lambda = \frac{m^2}{2}$ .

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- The theory is solvable. It has, for example, primaries $\phi$ and $[\phi^2]_n$

$$\Delta_\phi(m^2) = \frac{1}{2} \pm \sqrt{\frac{1}{4} + m^2} \qquad \Delta_{[\phi^2]_n}(m^2) = 2\Delta_\phi(m^2) + 2n$$

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$$b^{\hat{\phi}}_{\phi}(m^2) = \sqrt{\frac{\Gamma(\Delta)}{2\sqrt{\pi}\Gamma(\Delta+\frac{1}{2})}} \qquad b^{\hat{\phi}^2}_{[\phi^2]_n}(m^2) = \frac{\Gamma(\Delta)}{\sqrt{2\pi}\Gamma\left(\Delta+\frac{1}{2}\right)}\sqrt{\frac{(\frac{1}{2})_n(\Delta)_n^3(2\Delta-\frac{1}{2})_n}{n!\left(\Delta-\frac{1}{4}\right)_n(2\Delta)_n\left(\Delta+\frac{1}{4}\right)_n\left(\Delta+\frac{1}{2}\right)_n}}$$

# Example

## Scalar GFF

$$\frac{1}{2\Delta_\phi - 1} = \frac{d\Delta_\phi}{dm^2} = \frac{1}{2}\sum_{n=0}^{\infty} b^{\hat{\phi}^2}_{[\phi^2]_n} C_{[\phi^2]_n \phi\phi} \mathcal{I}^{(\alpha)}(2\Delta_\phi + 2n)$$

# Example

## Scalar GFF

From boundary expansion of $\langle \hat{\phi}^2 \hat{\phi}^2 \rangle$

$$\frac{1}{2\Delta_\phi - 1} = \frac{d\Delta_\phi}{dm^2} = \frac{1}{2} \sum_{n=0}^{\infty} b^{\hat{\phi}^2}_{[\phi^2]_n} C_{[\phi^2]_n \phi\phi} \mathcal{I}^{(\alpha)}(2\Delta_\phi + 2n)$$

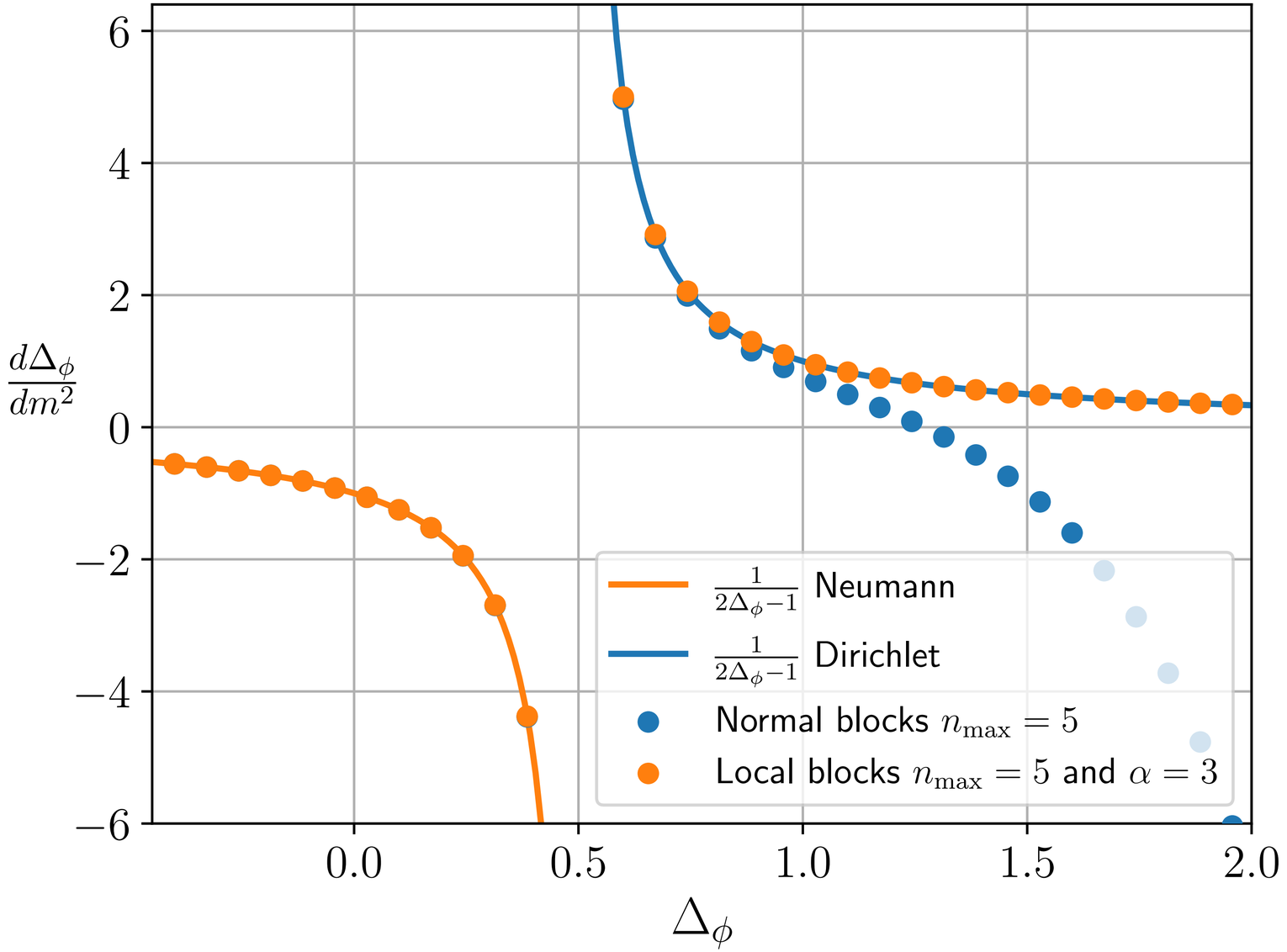
From harmonic analysis on $\langle \phi\phi\phi\phi \rangle$

[Simmons-Duffin, Kravchuk, Karateev, ’19]

$$\mathcal{I}^{(\alpha)}(\Delta) = \frac{\pi \Gamma(\Delta + \frac{1}{2}) \Gamma(2\alpha - 1)}{2^{2\alpha-2}(\Delta - 1)\Gamma(\frac{\Delta}{2})\Gamma(\frac{\Delta}{2} + 1)\Gamma(\frac{\Delta - 1}{2} + \alpha)\Gamma(\alpha - \frac{\Delta}{2})}$$

# Flow of scaling dimensions

$$\frac{d\Delta_\phi}{dm^2} = \frac{1}{2}\sum_{n=0}^{n_{\max}} b^{\hat{\phi}^2}_{[\phi^2]_n} C_{[\phi^2]_n \phi\phi} \mathcal{I}^{(\alpha)}(2\Delta_\phi + 2n)$$




Sum over normal blocks converges only if

$$\Delta_i < \frac{3}{4}$$

# Flow of scaling dimensions

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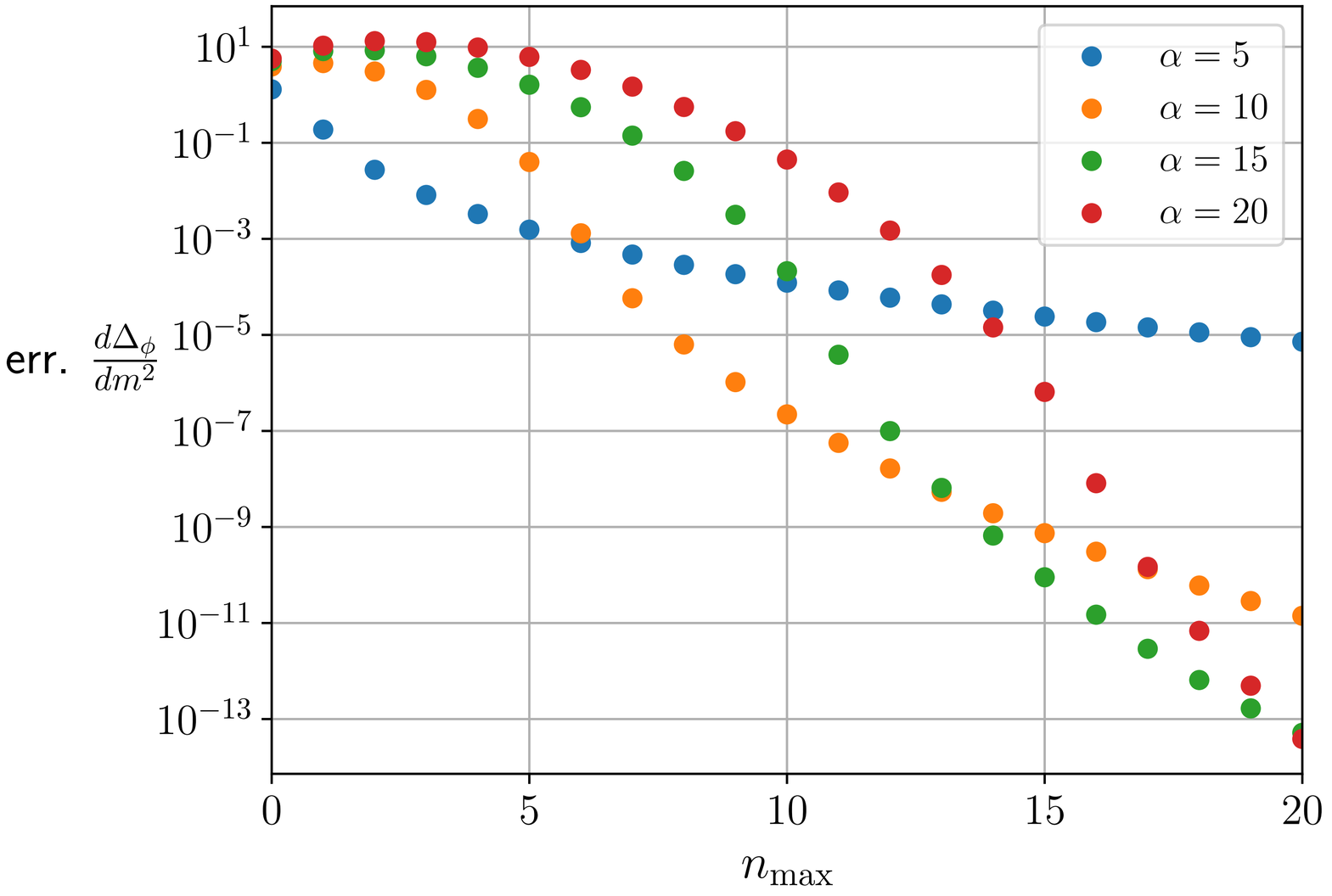


$$\Delta_\phi = \frac{3\sqrt{3}}{2}$$

Universal error estimate:

$$\Delta_{\max}^{2(\Delta_i - \alpha) + \Delta_{\hat{\Phi}}}$$

# Flow of BOE coefficient

From harmonic analysis on $\langle\phi\phi\phi\phi\phi\phi\rangle$

[Antunes, Costa, Goncalves, Boas '22]

[Antunes, Harris, Kaviraj, Schomerus '23]

$$\frac{db^{\hat{\phi}^2}_{\phi^2}}{dm^2} = \frac{1}{2}\sum_{l,j}^{\infty} b^{\hat{\phi}^2}_{[\phi^2]_l} b^{\hat{\phi}^2}_{[\phi^2]_j} C_{\phi^2[\phi^2]_l[\phi^2]_j} \mathcal{J}^{(\alpha)}_{2\Delta_\phi}(2\Delta_\phi + 2l, 2\Delta_\phi + 2j)$$

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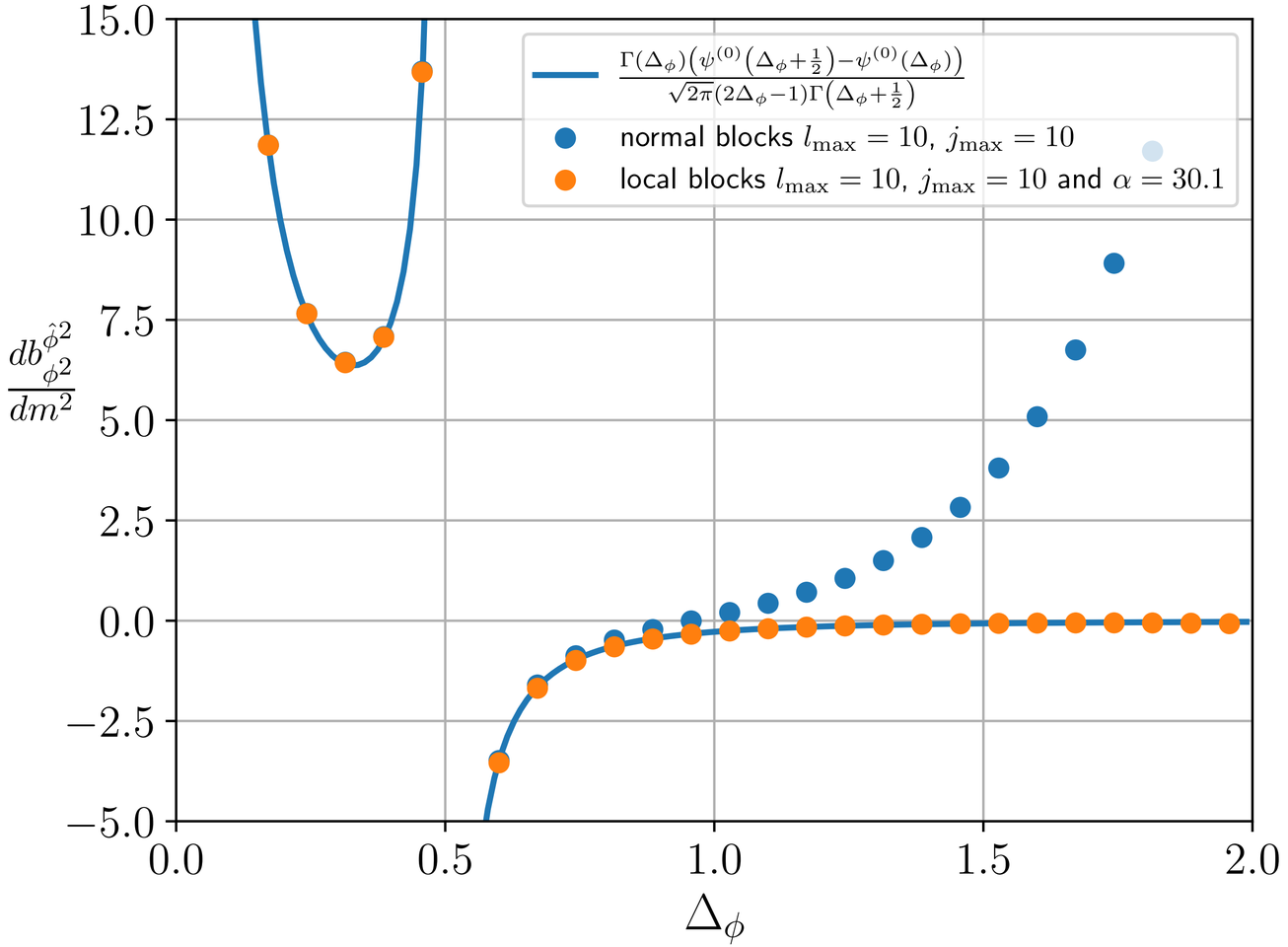
$$\mathcal{J}^{(\alpha)}_{\Delta_3}(\Delta_1,\Delta_2) = \frac{2\sqrt{\pi}\Gamma\left(\frac{\Delta_1+1}{2}\right)\Gamma\left(\Delta_2+\frac{1}{2}\right)\Gamma\left(\alpha-\frac{1}{2}\right)\Gamma\left(\alpha-\frac{\Delta_{23}}{2}\right)}{(\Delta_1-1)\Delta_{23}\Gamma\left(\frac{\Delta_{123}}{2}\right)\Gamma\left(\alpha-\frac{\Delta_1}{2}\right)\Gamma\left(\alpha+\frac{\Delta_{132}-1}{2}\right)\Gamma\left(\frac{\Delta_2+\Delta_3+1}{2}\right)}$$
$$\times\ {}_3F_2\left(\begin{matrix}\frac{\Delta_{132}}{2}, & \frac{\Delta_1-1}{2}, & \alpha+\frac{\Delta_1-1}{2}\\ \Delta_1+\frac{1}{2}, & \alpha-\frac{\Delta_{23}}{2}+\frac{\Delta_1-1}{2} & \end{matrix};1\right)$$

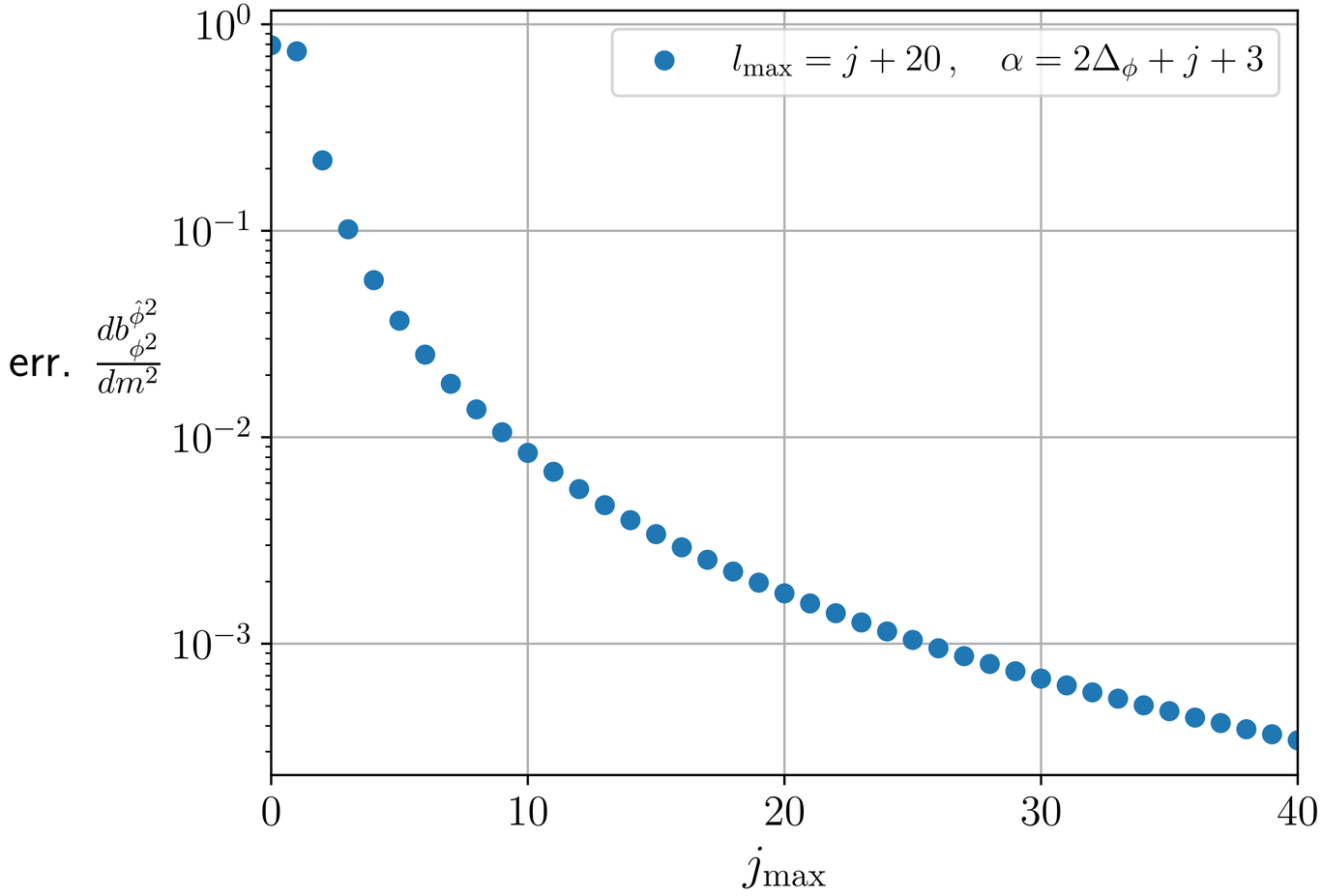
# Flow of BOE coefficient

$$\frac{db^{\hat{\phi}^2}_{\phi^2}}{dm^2} = \frac{1}{2}\sum_{j=0}^{j_{\max}}\sum_{l=0}^{l_{\max}} b^{\hat{\phi}^2}_{[\phi^2]_l} b^{\hat{\phi}^2}_{[\phi^2]_j} C_{\phi^2[\phi^2]_l[\phi^2]_j} \mathcal{J}^{(\alpha)}_{2\Delta_\phi}(2\Delta_\phi + 2l, 2\Delta_\phi + 2j)$$



# Flow of BOE coefficient

$$\frac{db^{\hat{\phi}^2}_{\phi^2}}{dm^2} = \frac{1}{2}\sum_{j=0}^{j_{\max}}\sum_{l=0}^{l_{\max}} b^{\hat{\phi}^2}_{[\phi^2]_l} b^{\hat{\phi}^2}_{[\phi^2]_j} C_{\phi^2[\phi^2]_l[\phi^2]_j} \mathcal{J}^{(\alpha)}_{2\Delta_\phi}(2\Delta_\phi + 2l, 2\Delta_\phi + 2j)$$




$\Delta_\phi = 2.1$

# Flow of OPE coefficient

$$0 = \frac{dC_{\phi\phi 1}}{dm^2} = \sum_{l=0}^{\infty} b^{\hat{\phi}^2}_{[\phi^2]_l} C_{\phi\phi[\phi^2]_l} \Big( \mathcal{K}^{(\alpha)}_{\Delta_\phi,\Delta_\phi,0}(2\Delta_\phi+2l,\Delta_\phi) + \mathcal{K}^{(\alpha)}_{\Delta_\phi,0,\Delta_\phi}(2\Delta_\phi+2l,\Delta_\phi) + \mathcal{K}^{(\alpha)}_{0,\Delta_\phi,\Delta_\phi}(2\Delta_\phi+2l,2\Delta_\phi+2l) \Big)$$

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This is the numerical bottleneck for now:

$$\mathcal{K}^{(\alpha)}_{\Delta_i\Delta_j\Delta_k}(\Delta_l,\Delta_m) = \int \frac{dc_1 dc_2 dc_3}{(2\pi i)^3} f^{\Delta_l,\Delta_m}_{\Delta_i\Delta_j\Delta_k}(c_1,c_2,c_3)$$

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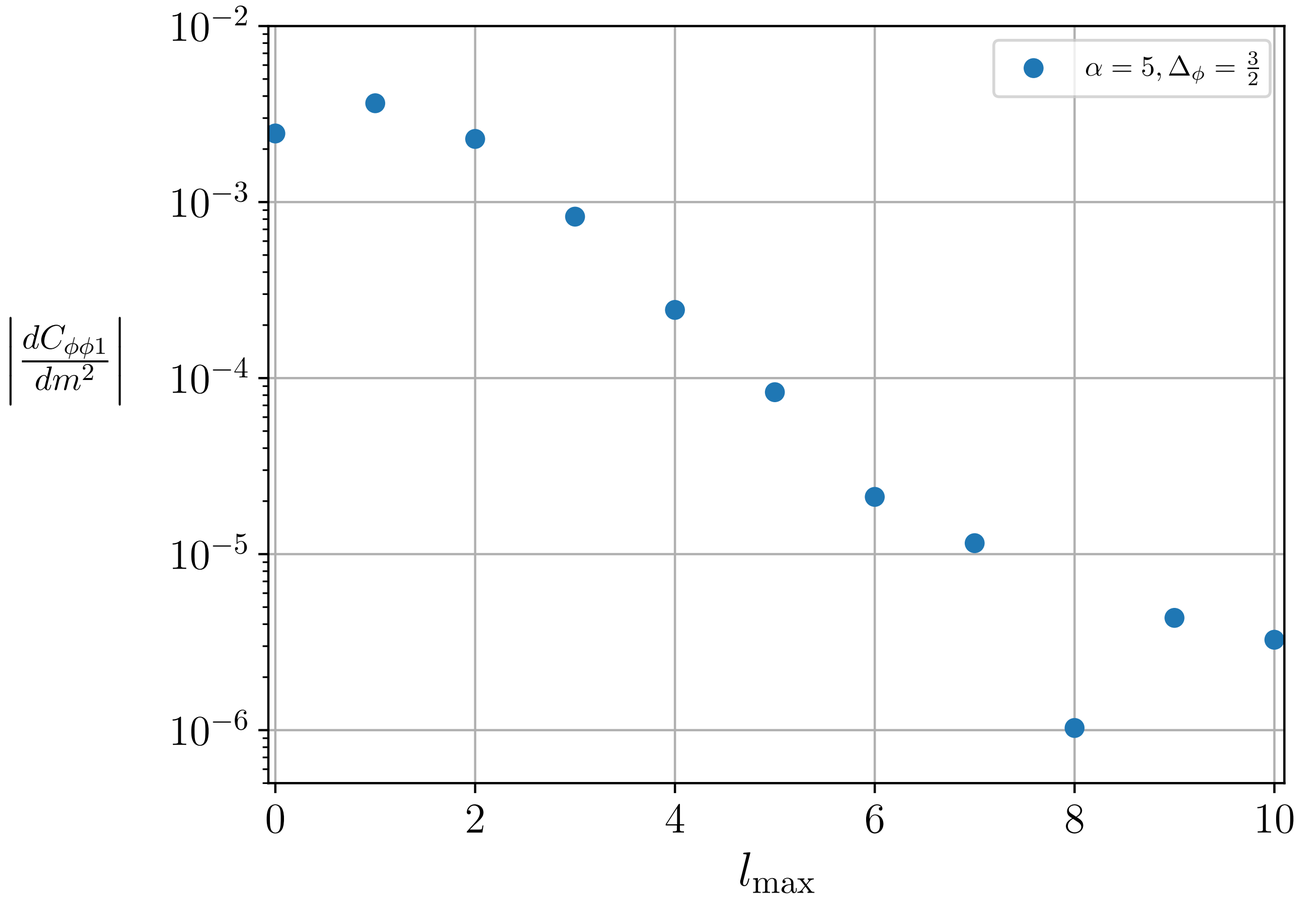


# Further properties of the flow equations

# The flat space limit and the S-matrix

- The coupling $\lambda$ is in units of $R$
- Flow can be seen as happening from heavily curved AdS to flat space
- In principle, spectrum and S-matrix can be extracted

$$\frac{m_i}{m_0} = \lim_{\lambda\to\infty} \frac{\Delta_i(\lambda)}{\Delta_0(\lambda)}$$

$$S(s) \sim \lim_{\lambda\to\infty} \sum_{\mathcal{O}_\Delta} f(\Delta) C^2_{\phi\phi\mathcal{O}_\Delta}(\lambda) \delta\left(\frac{\sqrt{s}}{m} - \frac{\Delta(\lambda)}{\Delta_\phi(\lambda)}\right)$$

[Meineri, Penedones, Spirig, ’23]

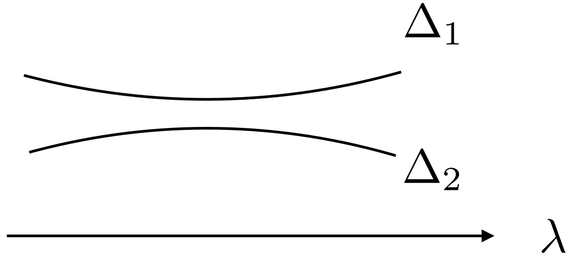
[Paulos, Penedones, Toledo, van Rees, Vieira, ’16]

More: [van Rees, Zhao, ’22], [Polchinski, ’99], [Cordova, He, Paulos ’22]

# Further properties

## Level repulsion

- Eigenvalues of hamiltonians which depend on continuous parameters don’t cross
- Proven in RMT and finite dimensional QM
- Expected to hold generically in QFT
- We can prove it with our equations




$$0 < \Delta_1 - \Delta_2 \ll 1 :$$

$$\frac{d^2(\Delta_1 - \Delta_2)}{d\lambda^2} = \frac{c}{(\Delta_1 - \Delta_2)^3}$$

# Further properties

## Mergers and annihilations of theories

- Expected when operators hit marginality

[Kaplan et al 2009, Gorbenko et al 2018]

- From the first flow equation,

$$\Delta_* \gtrsim 1\,, \qquad \frac{d\Delta_i}{d\lambda} \sim \frac{c}{\Delta_* - 1}$$

$$\Delta_i(\lambda) \sim \Delta_i(\lambda_c) \pm \sqrt{2c(\lambda - \lambda_c)} \qquad c = \frac{C_{ii*}^2 b_1^{\hat{\Phi}}}{C_{***}}$$

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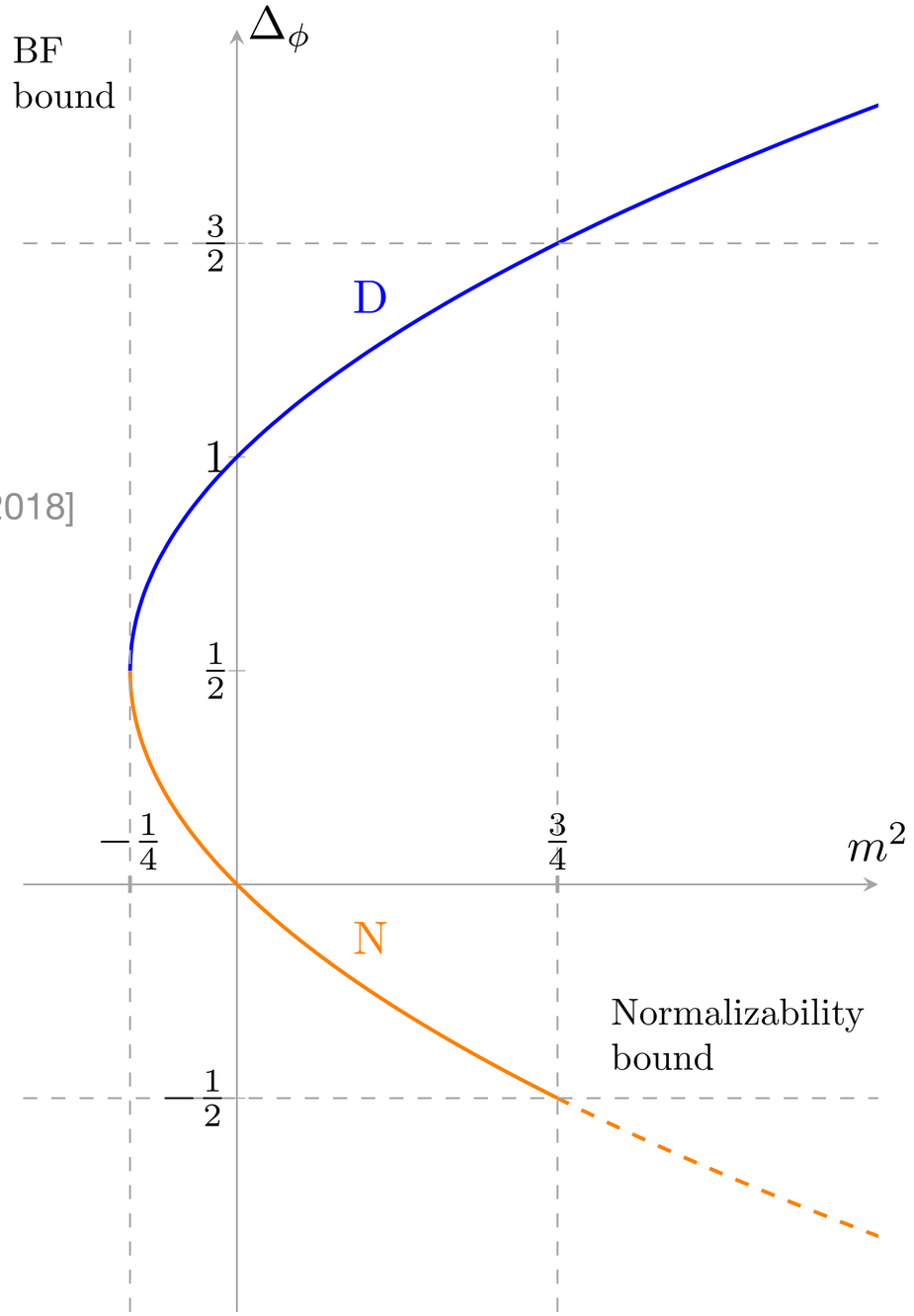
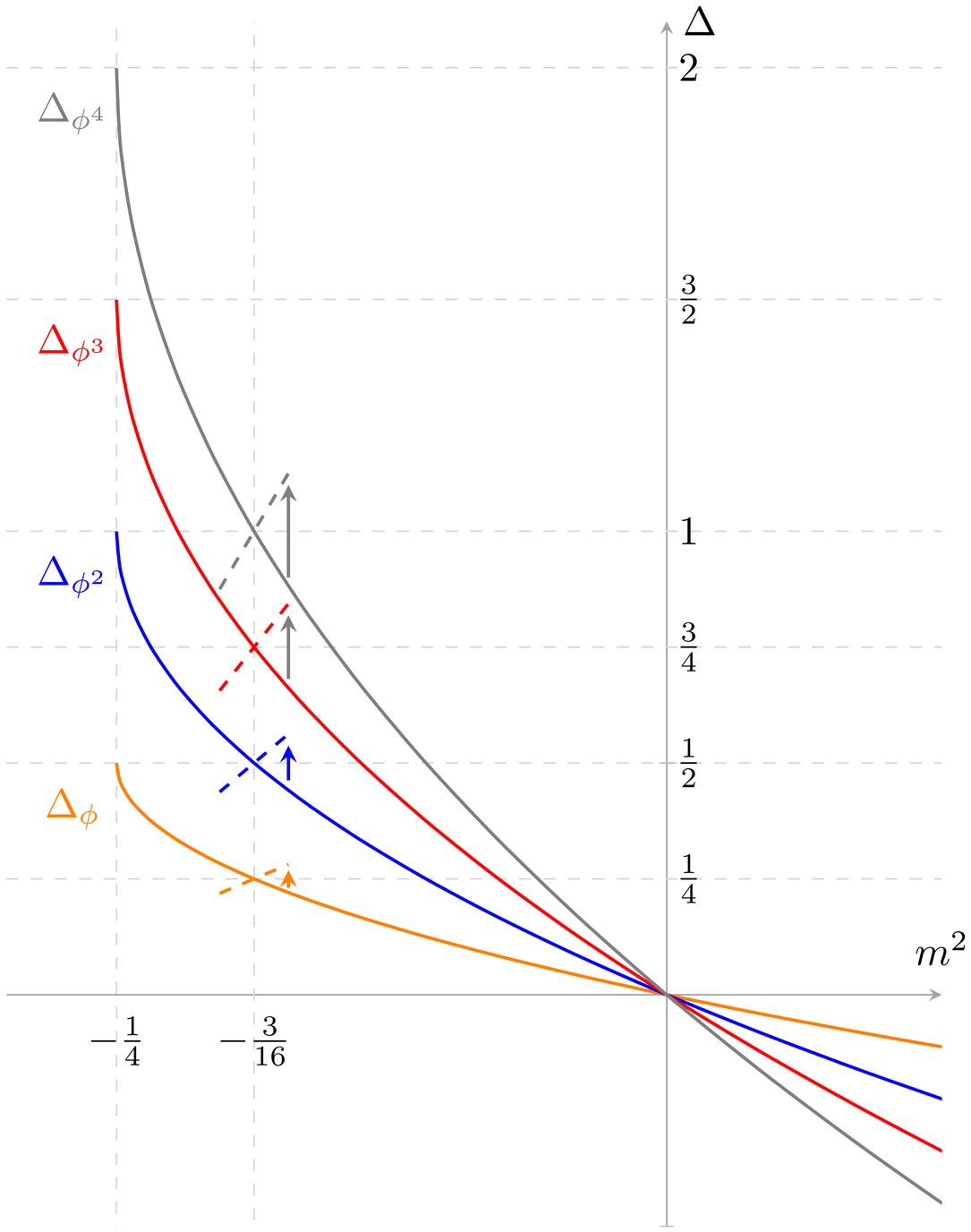
- Example in free massive theory:



# Further properties

## Free bulk and interacting boundary

- There exists a family of 1D CFTs called *Long Range Ising*
- Critical point of Ising model with decaying interactions
- It is equivalent to the following construction
  - Take free massive scalar in AdS2
  - Turn on $\phi^4$ only on the boundary
  - Flow to IR fixed point on boundary
- Fits in our flow equations framework



# Conclusion

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## Summary

- QFT in AdS can be phrased in terms of $\Delta_i(\lambda)\,, \quad C_{ijk}(\lambda)\,, \quad b_i^{\hat{\mathcal{O}}}(\lambda)$
- Their evolution under RG flows is governed by coupled ODEs
- No more divergences to deal with
- Fully non-perturbative
- Nontrivial checks for scalar and fermion GFF
- They know a lot about QFT (level repulsion, merger-annihilation, etc..)
- What next?

# Conclusion

## Short term wishes

- Efficiency
- Solve simple interacting cases (ex: LRI family)
- More concise form? No $b_i^{\hat{\Phi}}$ , no $\lambda$

# Conclusion

## Longer term wishes

- Generalization to multiple deformations (maybe trivial)
- Ising Field Theory and flows between minimal models [Lauria, Milam, van Rees ’24]
- $O(N)$ nonlinear sigma model and $O(N)$ GN model [Copetti, Di Pietro, Ji, Komatsu ’24]
- Generalization to higher dimensions

  ⋮
- $\mathrm{YM}_4$ [Ciccone, De Cesare , Di Pietro, Serone ’24]
- $\mathrm{QCD}_4$!

# What is QFT?

- This framework inherits all axioms from CFT (except bdy stress tensor)
- Existence of bulk stress tensor implies [Meineri, Penedones, Spirig '23]

$$c^{\mathrm{UV}} = \sum_i \frac{24\Gamma(2\Delta_i)}{4^{\Delta_i}\Gamma^2(\Delta_i+2)} \left(b_i^{\hat{\Theta}}\right)^2 \qquad \Delta_i = \sum_j b_j^{\hat{\Theta}} C_{iij} \kappa(\Delta_j, \alpha)$$

- Possibly exciting for axiomatic (and computable) approach to QFT

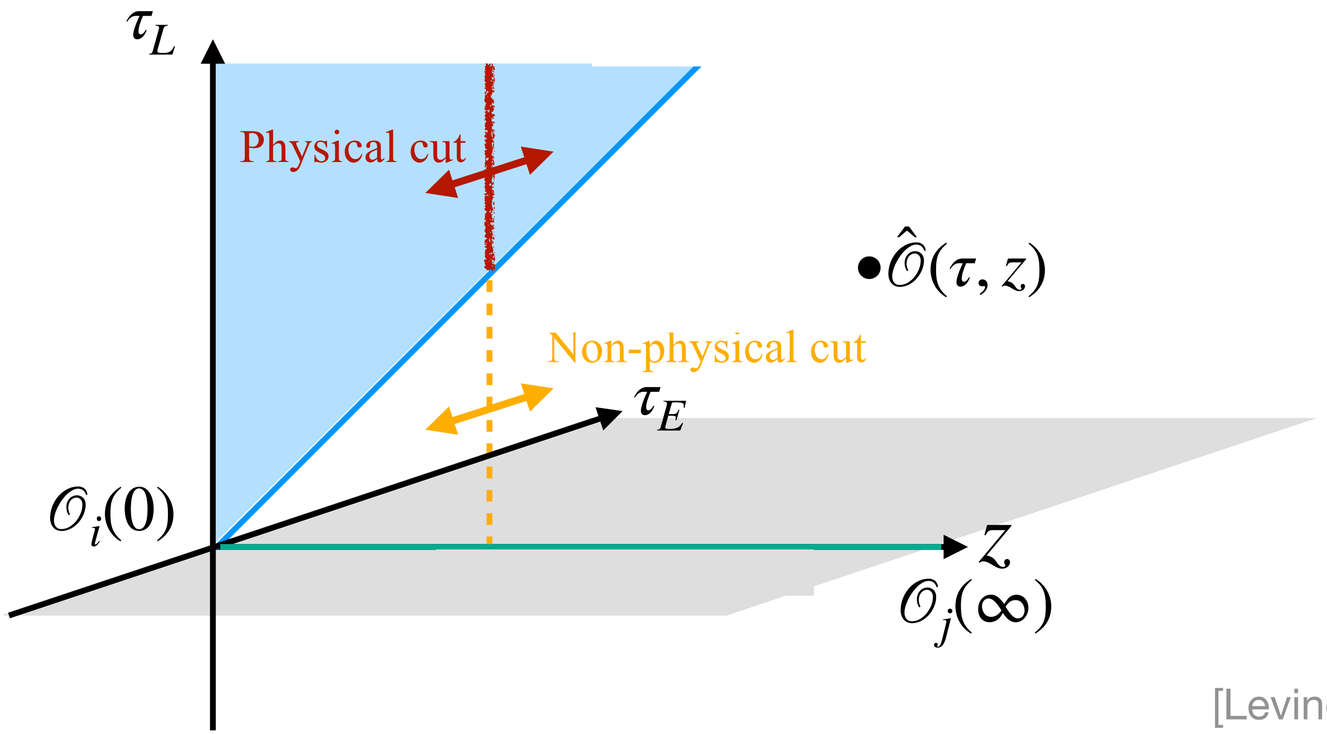
**Thank you**

# Backup

# Local blocks

## Analytic structure

$\langle \hat{\mathcal{O}} \mathcal{O}_i \mathcal{O}_j \rangle$




$G_{\Delta_i \Delta_j}(\chi)$

[Levine, Paulos '23]

[Meineri, Penedones, Spirig '23]

# Local blocks

## The dispersive integral


Consider the following trivial rewriting

$$\langle \mathcal{O}_i(P_1)\mathcal{O}_i(P_2)\hat{\mathcal{O}}(X)\rangle = \frac{1}{P_{12}^{\Delta_i}} f(\chi) \qquad f(\chi) = \chi^\alpha \oint_\chi \frac{d\chi'}{2\pi i}\frac{1}{\chi'-\chi}\frac{f(\chi')}{\chi'^\alpha}$$

Imposing a physical analytic structure for the correlator,

$$f(\chi) = \chi^\alpha \int_{-\infty}^{0} \frac{d\chi'}{2\pi i}\frac{1}{\chi'-\chi}\mathrm{Disc}_{\chi'\le 0}\left(\frac{f(\chi')}{\chi'^\alpha}\right)$$

Now plug in $f(\chi) = \sum_j b_{\hat{\mathcal{O}}_j} C_{iij} G_{\Delta_i,\Delta_j}(\chi)$ . Swapping (valid for $\alpha > \Delta_i + \frac{\Delta_{\hat{\mathcal{O}}}}{2}$ )

$$\langle \mathcal{O}_i(P_1)\mathcal{O}_i(P_2)\hat{\mathcal{O}}(X)\rangle = \frac{1}{P_{12}^{\Delta_i}} \sum_j b_{\hat{\mathcal{O}}_j} C_{iij} G^{(\alpha)}_{\Delta_i,\Delta_j}(\chi) \qquad G^{(\alpha)}_{\Delta_i,\Delta_j}(\chi) \equiv \chi^\alpha \int_{-\infty}^{0} \frac{d\chi'}{2\pi i}\frac{1}{\chi'-\chi}\mathrm{Disc}_{\chi'\le 0}\left(\frac{G_{\Delta_i,\Delta_j}(\chi')}{\chi'^\alpha}\right)$$

# Renormalization condition

- Explicitly, we choose

$$\langle \mathcal{O}_i^R(P_1)\mathcal{O}_j^R(P_2)\rangle_{(\lambda+\delta\lambda)} \equiv \frac{\delta_{ij}}{P_{12}^{\Delta_i+\delta\Delta_i}}$$

$$\langle \mathcal{O}_i^R(P_1)\mathcal{O}_j^R(P_2)\mathcal{O}_k^R(P_3)\rangle_{(\lambda+\delta\lambda)} \equiv \frac{C_{ijk}+\delta C_{ijk}}{P_{12}^{\Delta_{ijk}+\delta\Delta_{ijk}} P_{23}^{\Delta_{jki}+\delta\Delta_{jki}} P_{13}^{\Delta_{ikj}+\delta\Delta_{ikj}}}$$

$$\mathcal{O}_i(P)\Big|_{(\lambda+\delta\lambda)} = \left(\delta_i^j + \delta Z_i^j \sum_{n=0}^{\infty} a_n D_P^n\right) \mathcal{O}_j^R(P)\Big|_{(\lambda+\delta\lambda)}$$