RELATIVE ENTROPY IN QFT

State of the Art, Open Problems and Future Directions

Leonardo Sangaletti

Università degli Studi di Genova

08th October 2025

Outline

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In QFT (infinitely many degrees of freedom) the algebras of **local systems** do not admit trace class operators (**Type III von Neumann factors**) $\Longrightarrow$ **Divergent traces** $\Longrightarrow$ **Relative** entropy (rather then *entanglement entropy*) is suitable for generalization to **QFTs** (on the continuum). It subtracts the **vacuum UV divergences** common to every state.

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$$\mathcal{S}(\rho|\sigma) := \operatorname{Tr} \rho_1(\ln \rho_1 - \ln \sigma_1).$$

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- We identify ρ_1, σ_1 with the **reduced density matrices** of two vectors Ψ, Φ in an enlarged Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$ (**purification** (Witten 2018)). It follows

$$\mathcal{S}(\rho|\sigma) = -(\Psi, \ln \Delta_{\Psi|\Phi} \Psi), \tag{1}$$

where

$$\ln \Delta_{\Psi|\Phi} = \ln \sigma_1 \otimes \mathbb{1} - \mathbb{1} \otimes \ln \rho_2 .$$

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- Eq. (1) generalizes to the QFT setting (Araki 1975; Araki 1976; Uhlmann 1977), with the positive operator $\Delta_{\Psi,\Phi}$ the **relative modular operator** between Ψ, Φ (Araki and Masuda 1982).
It satisfies "good" entropic properties (E.g. **Positivity, Monotonicity w.r.t. localization**)

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States ω are **positive, normalised linear functionals** acting on $\mathcal{A}$:

$$\omega \in \mathcal{A}^* \quad \text{s.t.} \quad \omega(A^*A) \geq 0,\ \omega(\mathbb{1}) = 1.$$

We recover net of **operator algebras** via **GNS construction** (Bratteli and Robinson 1987) (*not all equivalent!*). Given ω on $\mathcal{A}(\mathbb{M})$ (*quasi-local algebra*), we obtain a representation π^ω on an **Hilbert space** $\mathcal{H}^\omega$ containing a **vector** $|\Omega\rangle$ that implements ω:

$$\omega(A) = \langle\Omega|\, \pi^\omega(A)\, |\Omega\rangle\,, \quad \forall A \in \mathcal{A}(\mathbb{M}).$$

We obtain a net $\mathfrak{A}$ of **von Neumann algebras** by defining:

$$\mathfrak{A}(\mathcal{O}) := \pi^\omega(\mathcal{A}(\mathcal{O}))''.$$

It coincides with the closure in the **weak topology** (*useful and makes physical sense!*)

Example: Scalar Field

Massive real scalar field equation:

$$(\Box - m^2)\phi(x) = 0.$$

Admits a unique **causal propagator** E (retarded minus advanced Green functions):

$$E(x,y) = \frac{i}{(2\pi)^3}\int \mathrm{sign}(p_0)\delta(p_0^2 - \omega_{\mathbf{p}}^2)e^{ip(x-y)}\mathrm{d}^4p, \quad \omega_{\mathbf{p}} = \sqrt{\|\mathbf{p}\|^2 + m^2}.$$

E defines a **symplectic form** $E(f,g)$ on the space of real valued test functions $f,g \in \mathcal{C}_c^\infty(\mathbb{M}) \implies$ Unique associated C^*-algebra $\mathcal{W}(\mathbb{M})$, so called **Weyl algebra** or **CCR algebra** (D. Petz 1990; Gérard 2023) .

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Let ω^0 be the **vacuum state** $\stackrel{\text{GNS}}{\Longrightarrow}$ Recover the usual representation on the **symmetric Fock space**. $\mathfrak{A}(\mathcal{O})$ is generated by polynomials in:

$$W(f) = e^{i\phi(f)}, \quad \mathrm{supp}\, f \subset \mathcal{O}$$

where (*formally*):

$$\phi(f) = \int \mathrm{d}^4 x\, f(x)\phi(x) = \int \mathrm{d}^4 x\, f(x) \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3 \omega_{\mathbf{p}}} \left[a_{\mathbf{p}} e^{-i\omega_{\mathbf{p}} x^0 + i\mathbf{p}\mathbf{x}} + a_{\mathbf{p}}^\dagger e^{i\omega_{\mathbf{p}} x^0 - i\mathbf{p}\mathbf{x}} \right].$$

Outline

Modular Theory in a Nutshell

Let $\mathcal{H}$ be a **Hilbert space**, $\mathfrak{A}$ a **von Neumann algebra** and $|\Omega\rangle$ be a **cyclic** ($\overline{\mathfrak{A}\,|\Omega\rangle} = \mathcal{H}$) and **separating** ($A\,|\Omega\rangle = 0,\ A \in \mathfrak{A} \implies A = 0$) vector. $\implies$ **modular theory** can be constructed Review: (Borchers 2000):

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- **Tomita operator** S is (*well*) densely defined on $\mathfrak{A}\,|\Omega\rangle$ by:

$$SA\,|\Omega\rangle = A^\dagger\,|\Omega\rangle \ \forall A \in \mathfrak{A}.$$

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- Let $S = J\Delta^{1/2}$ be the **polar decomposition** of S. Δ (**modular operator**) defines an **automorphism** for $\mathfrak{A}$ (*time evolution!* (Longo 2020)) via the unitary group $\Delta^{it}, t \in \mathbb{R}$:

$$\mathrm{Ad}\Delta^{it}\mathfrak{A} = \mathfrak{A}, \forall t \in \mathbb{R}.$$

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- The state defined by $|\Omega\rangle$ (via $\omega(\cdot) = \langle\Omega|\cdot|\Omega\rangle$) is **KMS** (*thermal*) with respect to the **modular evolution**.

Relative Modular Operator

Let (*for simplicity*) $|\Omega\rangle, |\Psi\rangle$ be two cyclic and separating vectors. We define $S_{\Omega|\Psi}$ on the dense domain $\mathfrak{A}|\Omega\rangle$ by

$$S_{\Omega|\Psi} A |\Omega\rangle = A^{\dagger} |\Psi\rangle, \quad \forall A \in \mathfrak{A}.$$

$S_{\Omega|\Psi}$ is again antilinear,unbounded and closable.

The polar decomposition is:

$$S_{\Omega|\Psi} = J_{\Omega|\Psi} \Delta^{1/2}_{\Omega|\Psi}.$$

with $\Delta_{\Omega|\Psi}$ the **relative modular operator**.

Obviously $S = S_{\Omega|\Omega}$, $J = J_{\Omega|\Omega}$ and $\Delta^{1/2} = \Delta^{1/2}_{\Omega|\Omega}$.
In addition, $\Delta_{\Omega|\Psi}$ depends on $|\Psi\rangle$ only through the state $\langle\Psi| \cdot |\Psi\rangle$ that it implements on $\mathfrak{A}$ (Araki and Masuda 1982) (*important for entropy!*).

Modular Operator in QFT

In QFT, we assign von Neumann algebras $\mathfrak{A}$ to spacetime regions $\mathcal{O}$:

$$\mathfrak{A}(\mathcal{O}) = \pi^{\omega}(\mathcal{A}(\mathcal{O}))''. \tag{2}$$

What about **cyclic and separating vectors for local algebras**?

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Reeh-Schlieder Theorem (Reeh and Schlieder 1961)

The vacuum vector $|\Omega\rangle$ for a scalar field theory on $\mathbb{M}$ is separating (*not too surprising*) and cyclic (*surprising*) for every local algebra $\mathfrak{A}(\mathcal{O})$.

It follows from **analytic properties** of correlation functions. Can be generalised to KMS states (Jäkel 2000) and to more generic (globally hyperbolic) spacetimes (Strohmaier, Verch, and Wollenberg 2002; Sanders 2009).

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We can construct modular theory for local algebras, but...**can we compute modular operators?**

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Known Modular Operators

Explicit expression for Δ only known in some cases, mostly for **geometric modular flow** $\mathrm{Ad}\,\Delta^{it}$, i.e. $\ln \Delta$ (**modular hamiltonian**) generates a (combination of) **spacetime symmetries**.

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- **Rindler wedge**: **Scalar field** (also massive, possibly interacting), **right wedge** $\mathscr{W}_{\mathrm{r}}$ of $\mathbb{M}^{d+1}$:

$$\mathscr{W}_{\mathrm{r}} := \{x \in \mathbb{M}^{d+1} | x^1 > |x^0|\},$$

  vacuum vector $|\Omega\rangle$

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vacuum vector $|\Omega\rangle \implies \mathrm{Ad}\,\Delta^{it}$ coincides with **Rindler time** evolution, $\ln\Delta = K$ with the **boost generator** in x^1 direction and J with the x^0, x^1 **reflection** (Bisognano and Wichmann 1975; Bisognano and Wichmann 1976):

$$\mathrm{Ad}\,J\mathfrak{A}(\mathscr{W}_\mathrm{r}) = \mathfrak{A}(\mathscr{W}_\mathrm{l}).$$

The vacuum state is KMS (*looks thermal*) for **uniformly accelerated observers**.

Modular Operator in QFT

- **Massless free scalar field** on $\mathbb{M}$, **vacuum vector** $|\Omega\rangle$, algebra of the **future light cone** (Buchholz 1977) and algebra of the **double cone** (Hislop and Longo 1982).

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Open problems:

- For a **massive theory**, **double cone** (*equivalent to* $\mathcal{O}$) we have no exact results (even for free scalar theory and vacuum state on $\mathbb{M}$). Some **numerical results** for scalar theories (Bostelmann, Cadamuro, and Minz 2023), **perturbative results** for fermionic theories (Cadamuro, Fröb, and Minz 2024) and numerical results on the **lattice** (Eisler et al. 2020; Javerzat and Tonni 2022).
- **Interacting theories** (in more than $1+1$ dimensions).

Relative Modular Operator

Is the situation better for the **relative** modular operator?

Relative Modular Operator

Is the situation better for the **relative** modular operator? No, **much worse**.

Relative Modular Operator

Explicit expressions for $\Delta_{\Psi|\Phi}$ are generally unknown.
Exception: if $\Psi = UU'\Omega$ and $\Phi = VV'\Omega$ (*both cyclic and separating*), with $U, V \in \mathfrak{M}$; $U', V' \in \mathfrak{M}'$ all unitary:

- Invertibility implies
$$S_{\Psi|\Phi} = (U^{-1})^{\dagger} V' S (U')^{-1} V^{\dagger}.$$
- Unitarity and uniqueness of polar decomposition imply
$$\Delta_{\Psi|\Phi} = VU'\Delta(U')^{\dagger} V^{\dagger}.$$
- In conclusion, the relative entropy is:
$$\mathcal{S}_{\Psi|\Phi} = -\langle \Psi | \ln \Delta_{\Psi|\Phi} | \Psi \rangle = -\langle V^{\dagger} U \Omega | \ln \Delta | V^{\dagger} U \Omega \rangle$$
independent of U', V' (*as it should!*).

Knowing $\mathrm{Ad}\,\Delta^{it}$, the relative entropy between $|\Omega\rangle$ and a **coherent excitaion** $W(f)\,|\Omega\rangle = e^{i\phi(f)}\,|\Omega\rangle$ can be computed for a scalar theory. If $\mathrm{Ad}\,\Delta^{it}$ acts geometrically:

$$\boxed{\mathcal{S}_{\Omega|W(f)\Omega} = \frac{1}{2} E\left(\frac{\mathrm{d}}{\mathrm{d}\,t}\alpha_t(f)\Big|_{t=0}, f\right), \quad \textit{with} \quad \mathrm{Ad}\,\Delta^{it} W(f) = W(\alpha_t(f)).}$$

Only the **“classical” causal propagator** enters!

Relative Entropy for Coherent States

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Example: Rindler wedge
By explicit computation, for $|\Omega\rangle$ the vacuum vector and $\Psi = W(f)\,|\Omega\rangle$, $\mathrm{supp}\, f \subset \mathscr{W}_{\mathrm{r}}$ (Casini, Grillo, and Pontello 2019; Ciolli, Longo, and Ruzzi 2020):

$$\mathcal{S}_{\Omega|\Psi} = \int 2\pi x^1 [T_{00}(Ef)]_{x^0=0}\mathrm{d}^d\mathbf{x} \geq 0,$$

$$\text{where } T_{00}(\phi) = \frac{1}{2}m^2\phi^2 + \frac{1}{2}\sum_{i=0}^{d}\partial_{x^i}\phi,$$

i.e. the **classical Noether charge** associated to the boosts $\Longrightarrow$ Recover the **classical entropy of a wave packet**.

Outline

Generalizations of the relative entropy exist. In (Fröb and S. 2025) we consider the **Petz-Rényi relative entropy** (Rényi 1961; Dénes Petz 1985; Dénes Petz 1986) of order $\alpha \in [0,1)$:

$$\mathcal{S}_\alpha(\Omega|\Psi) := \frac{1}{\alpha - 1} \ln \langle\Omega| \Delta_{\Omega|\Psi}^{1-\alpha} |\Omega\rangle ,$$

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- We prove that $\mathcal{S}_\alpha(\Omega|\Psi)$ can be computed by **analytic continuation** of relative modular flow $\mathrm{Ad}\, \Delta^{it}_{\Omega|\Psi}$.
- For coherent excitations of $|\Omega\rangle$ and geometric modular flow we prove:

$$\boxed{\mathcal{S}_\alpha(W(f)\Omega|\Omega) = \frac{1}{\alpha - 1} F(i(\alpha - 1)), \quad F(t) = \omega_2(f_{-t/2}, f_{t/2}) - \omega_2(f, f).}$$

It is **genuinely quantum!** (while for $\alpha \to 1$ only E contributes).

Explicit Computation of Petz-Rényi Entropy

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- **Massive scalar field** on $\mathscr{W}_r$ in the **vacuum**

$$\mathcal{S}_\alpha(W(f)\Omega|\Omega) = \frac{1}{\alpha-1}\iint \left[\omega_2(\Lambda_{-\frac{i(\alpha-1)}{2}}x, \Lambda_{\frac{i(\alpha-1)}{2}}y) - \omega_2(x,y)\right] f(x)f(y)\mathrm{d}^{d+1}x\mathrm{d}^{d+1}y,$$

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  with Λ boost in x^1 direction.

- **Chiral current** of scalar field on **light ray** in a **thermal equilibrium KMS state**:

$$\mathcal{S}_\alpha(W(f)\Omega\|\Omega) = -\frac{1}{4\pi(\alpha-1)}\lim_{\epsilon\to0^+}\iint_0^\infty \times\left[\ln\left[-\cos(\pi\alpha)\left(e^{\frac{2\pi u}{\beta}} - e^{\frac{2\pi v}{\beta}}\right) - i\sin(\pi\alpha)\left(e^{\frac{2\pi u}{\beta}} + e^{\frac{2\pi v}{\beta}} - 2\right) - i\epsilon\right] - \ln\left(e^{\frac{2\pi u}{\beta}} - e^{\frac{2\pi v}{\beta}} - i\epsilon\right)\right] f'(u)f'(v)\mathrm{d}u\mathrm{d}v.$$

Non Unitary Excitations

We are recently working on the case of **non unitary excitations**. Using known results about **Rényi divergences** (Berta, Scholz, and Tomamichel 2018) we obtain an **explicit upper bound** on $\mathcal{S}_{\Omega|\Psi}$ in terms of Δ, for $\Psi = A|\Omega\rangle$ and A not necessarily unitary.

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Pros:

- First result for non unitary excitations.
- The bound applies to a **dense set of state**.
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Pros:

- First result for non unitary excitations.
- The bound applies to a **dense set of state**.
- The bound works also for certain **unbounded excitations** (*E.g. the field!*)

Cons:

- It is only a bound, not equality.
- Does not apply to every excitation.

Conclusions and Outlook

Entanglement entropy is not well defined in QFT, but **relative entropy** is.

The formula for the relative entropy in QFT is given by Tomita-Takesaki modular theory in terms of the **relative modular operator**.

Explicit expressions are available only in few cases (E.g. **Rindler wedge**). In particular, for **unitary excitations** and only when the **modular flow is known**.

Main open problems: **non unitary** excitations? More results for the modular flow, in particular **massive scalar field in a double cone**?

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