Sudakov resummation of the thrust distribution in e^+e^- annihilation and the determination of the strong coupling

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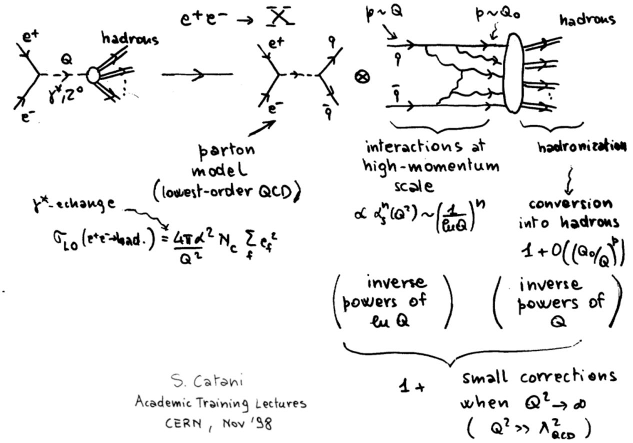
Based on:

U. Aglietti, G.F., W.-L. Ju, J. Miao, 2502.01570, PRL 134 (2025) 25

U. Aglietti, G.F., W.-L. Ju, 2506.18707

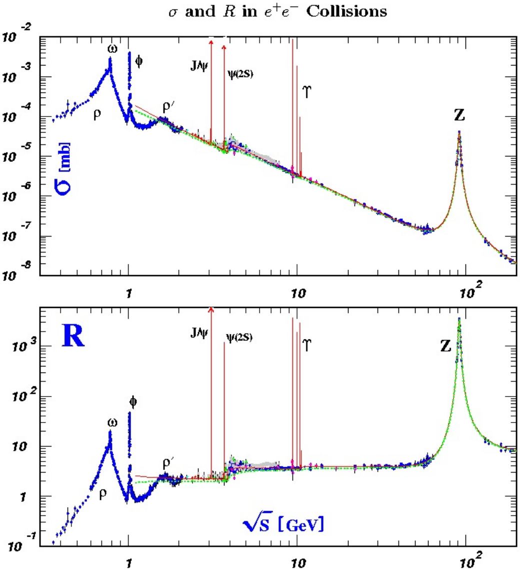
Genoa – 1/10/2025

e^+e^- annihilation into hadrons



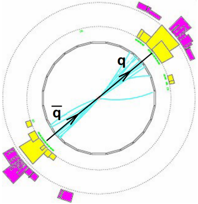
$$R(Q) \equiv \frac{\sigma(e^+e^- \to hadrons)}{\sigma(e^+e^- \to \mu^+\mu^-)} = R_0(Q)\left(1 + \sum_{n=1}^{\infty} C_n \left(\frac{\alpha_S(Q^2)}{\pi}\right)^n\right) + \mathcal{O}\left(\frac{\Lambda^4_{QCD}}{Q^4}\right)$$

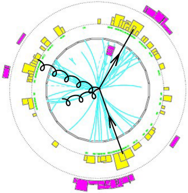
e^+e^- annihilation into hadrons



Event shape variables in e^+e^- annihilation

- The idea: define a quantity X characterizing the type of "shape" of an event (pencil-like, planar, spherical,...).
- Key point: to be (reliably) calculable in pQCD, X has to be **infrared** and **collinear** (IRC) safe, i.e. insensitive to soft and/or collinear emissions.
- Sterman-Weinberg (1977) safety criteria: Non Perturbative (NP) effects are power suppressed if X is invariant under the branching: $\mathbf{p_i} \to \mathbf{p_j} + \mathbf{p_k}$ when $\mathbf{p_j} // \mathbf{p_k}$ (collinear emission) or $\mathbf{p_j} \to \mathbf{0}$ or $\mathbf{p_k} \to 0$ (soft emission).





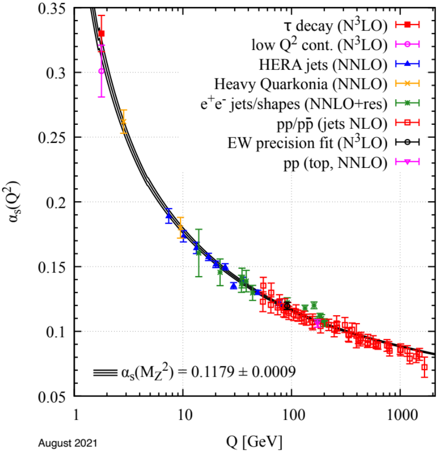
Examples of IRC safe shape variables

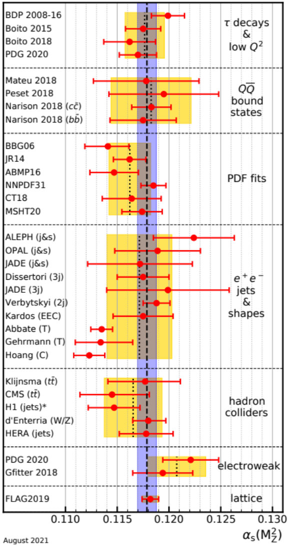
- Thrust [Farhi ('77)]:
  $T = \max_{\mathbf{n}} \frac{\sum_i |\mathbf{p_i}\cdot\mathbf{n}|}{\sum_i |\mathbf{p_i}|}$
- Spherocity [Georgi,Machacek ('77)]:
  $S = \frac{4}{\pi} \min_{\mathbf{n}} \left(\frac{\sum_i |\mathbf{p_i}\times\mathbf{n}|}{\sum_i |\mathbf{p_i}|} \right)^2$
- Momentum tensor and C- and D-param.
  [Parisi ('78)]: $\Theta^{\alpha\beta} = \frac{\sum_i p_i{}^{\alpha} p_i{}^{\beta}}{\sum_j |\mathbf{p_j}|}$
- Heavy jet mass [Clavelli ('79)]:
  $\rho = \frac{1}{Q^2} \max\{M_L^2, M_R^2\}$, with
  $M_{L/R}^2 = \left(\sum_{i\in H_{L/R}} p_i\right)^2$
- Total and wide jet broadening
  [Rakow,Webber ('81)]:
  $B_T = B_L + B_R, \quad B_W = \max\{B_L, B_R\}$,
  $B_{L/R} = \frac{\sum_{i\in H_{L/R}} |\mathbf{p_i}\times\mathbf{n_T}|}{2\sum_i |\mathbf{p_i}|}$

Key point: *linearity*
$\mathbf{p_j} + \mathbf{p_k} \to z\,\mathbf{p} + (1-z)\,\mathbf{p} = \mathbf{p}$
ensure **infrared** and **collinear** safety.

Name of Observable	Definition	Typical Value for: 			QCD calculation
Thrust	$T = \max_{\vec{n}} \left(\frac{\sum_i \lvert\vec{p}_i \vec{n}\rvert}{\sum_i \lvert\vec{p}_i\rvert} \right)$	1	≥2/3	≥1/2	(resummed) $O(\alpha_s^2)$
Thrust major	Like T, however T_{maj} and $\vec{n}_{maj}$ in plane $\perp \vec{n}_T$	0	≤1/3	≤1/√2	$O(\alpha_s^2)$
Thrust minor	Like T, however T_{min} and $\vec{n}_{min}$ in direction $\perp$ to $\vec{n}_T$ and $\vec{n}_{maj}$	0	0	≤1/2	$O(\alpha_s^2)$
Oblateness	$O = T_{maj} - T_{min}$	0	≤1/3	0	$O(\alpha_s^2)$
Sphericity	$S = 1.5\,(Q_1 + Q_2)$; $Q_1 \le \ldots \le Q_3$ are Eigenvalues of $S^{\alpha\beta} = \frac{\sum_i p_i^{\alpha} p_i^{\beta}}{\sum_i p_i^2}$	0	≤3/4	≤1	none (not infrared safe)
Aplanarity	$A = 1.5\, Q_1$	0	0	≤1/2	none (not infrared safe)
Jet (Hemisphere) masses	$M_{\pm}^2 = \left(\sum_i E_i^2 - \sum_i \vec{p}_i^{\,2}\right)_{i\in S_{\pm}}$ ($S_{\pm}$: Hemispheres $\perp$ to $\vec{n}_T$) $M_H^2 = \max(M_+^2, M_-^2)$ $M_D^2 = \lvert M_+^2 - M_-^2 \rvert$	0 0	≤1/3 ≤1/3	≤1/2 0	(resummed) $O(\alpha_s^2)$
Jet broadening	$B_{\pm} = \frac{\sum_{i\in S_{\pm}} \lvert\vec{p}_i \times \vec{n}_T\rvert}{2\sum_i \lvert\vec{p}_i\rvert}$; $B_T = B_+ + B_-$ $B_w = \max(B_+, B_-)$	0 0	≤1/(2√3) ≤1/(2√3)	≤1/(2√2) ≤1/(2√3)	(resummed) $O(\alpha_s^2)$
Energy-Energy Correlations	$EEC(\chi) = \sum_{events} \sum_{i,j} \frac{E_i E_j}{E_{vis}^2} \int_{\chi-\frac{\Delta\chi}{2}}^{\chi+\frac{\Delta\chi}{2}} \delta(\chi - \chi_{ij})$				(resummed) $O(\alpha_s^2)$
Asymmetry of EEC	$AEEC(\chi) = EEC(\pi-\chi) - EEC(\chi)$				$O(\alpha_s^2)$

Determinations of α_S

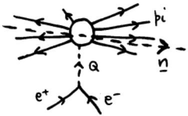




[Caola et al.('21,'22)],[Nason,Zanderighi('23,'25)],[PDG('23)]

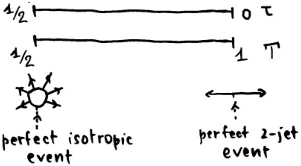
The Thrust in e^+e^- annihilation

$$T \equiv 1 - \tau = \max_{\mathbf{n}} \frac{\sum_i |\mathbf{p_i} \cdot \mathbf{n}|}{\sum_i |\mathbf{p_i}|}$$



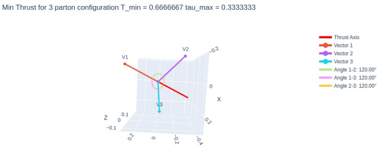
- The sum is over all final state particles i with three-momentum $\mathbf{p_i}$.
- The maximum is taken with respect to the direction of the unit three-vector $\mathbf{n}$.
- T maximizes the longitudinal momentum along the vector $\mathbf{n}$.
- The vector which realizes the maximum is called thrust axis: $\mathbf{n}_T$.

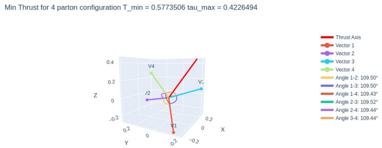
The allowed kinematical range for T is: $1/2 \leq T \leq 1$ $(0 \leq \tau \leq 1/2)$



Upper limit for τ, $\tau_{max}^{(N)}$, depends on the number N of final-state particle. Only in the (formal) limit $N \to \infty$ it approaches $\tau_{max}^{(N\to\infty)} \to 1/2$.

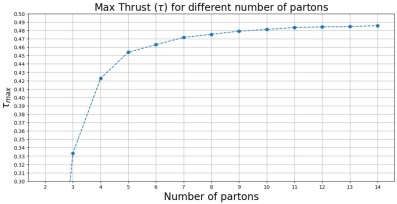
- $\tau_{max}^{(N)}$ is important to correctly normalize the thrust cross section.
- In massless approx.: $\tau_{max}^{(3)} = 1/3 = 0.3333\cdots$, $\tau_{max}^{(4)} = 1 - 1/\sqrt{3} = 0.4226\cdots$





- Finding $\tau_{max}^{(N)}$ is a non-trivial (double optimization) kinematical problem: given N particle momenta one needs to find $\mathbf{n_T}$ and then find the maximum value of τ by varying the particle momenta finding the new thrust axis.
- We used stochastic optimization algorithms (Genetic Algorithm and Particle Swarm Optimization) perturbing the initially randomly generated momenta.

N	3	4	5	6
$\tau_{\rm max}^{(N)}$	0.3333	0.4226	0.4539	0.4629
N	7	8	9	10
$\tau_{\rm max}^{(N)}$	0.4716	0.4753	0.4790	0.4811
N	11	12	13	14
$\tau_{\rm max}^{(N)}$	0.4834	0.4842	0.4845	0.4857



Fixed-order QCD expansion

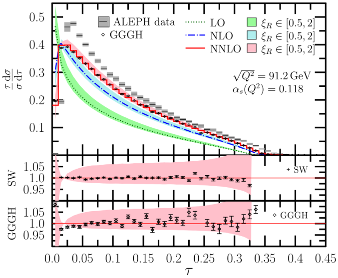
Thrust distribution can be systematically calculated in pQCD as a fixed-order expansion in $\alpha_S = \alpha_S(\mu^2)$

$$\frac{1}{\sigma_{\rm tot}}\frac{d\sigma}{d\tau} = \delta(\tau) + \frac{\alpha_S}{\pi}\frac{d\mathcal{A}}{d\tau} + \left(\frac{\alpha_S}{\pi}\right)^2\frac{d\mathcal{B}}{d\tau} + \left(\frac{\alpha_S}{\pi}\right)^3\frac{d\mathcal{C}}{d\tau} + \mathcal{O}(\alpha_S^4)\,,$$

The LO function ($\tau > 0$) is

$$\frac{d\mathcal{A}}{d\tau} = 4 + 6\tau - \frac{2}{\tau} + \left(-4 + \frac{8}{3(1-\tau)\tau}\right)\ln\left(\frac{1-2\tau}{\tau}\right) \xrightarrow{\tau\to 0} -\frac{8}{3}\frac{\ln\tau}{\tau} - \frac{2}{\tau}\,.$$

QCD corrections up to NNLO known [Gehrmann-De Ridder et al. ('07)],[Weinzierl ('09)],[Del Duca et al. ('16)]. Calculations based on a numerical integration of the matrix elements. NNLO parton-level event generator public available EERAD3 [Gehrmann-De Ridder et al. ('14)].



Sudakov resummation

- Bulk of events in the two-jet limit $\tau \to 0$ (semi-inclusive region).
- In the fixed-order expansion large Sudakov logarithms appear due to incomplete cancellation between real radiation (constrained by kinematics) and virtual (unconstrained) emissions:

$$\frac{1}{\sigma_{tot}} \frac{d\sigma}{d\tau} \overset{\tau \to 0}{\sim} \sum_{n=1}^{\infty} \sum_{k=1}^{2n-1} \alpha_S^n \frac{1}{\tau} \ln^k \frac{1}{\tau} \,.$$

- *Cumulative* cross section:

$$R_T(\tau) \equiv \frac{1}{\sigma_{\rm tot}} \int_0^\tau d\tau' \frac{d\sigma}{d\tau'} \qquad \text{thus} \qquad \frac{1}{\sigma_{\rm tot}} \frac{d\sigma}{d\tau} = \frac{dR_T(\tau)}{d\tau} \,.$$

$$R_T(\tau) \overset{\tau \to 0}{\sim} \sum_{n=1}^{\infty} \sum_{k=1}^{2n} \alpha_S^n \ln^k \frac{1}{\tau} \,.$$

- Fixed-order expansion unreliable in the low τ region where $\alpha_S(Q) \ln^2 1/\tau \sim 1$.
- To obtain reliable predictions in the two-jet region, resummation of Sudakov logarithms is mandatory.

Idea of (analytic) resummation

Idea of large logs (Sudakov) resummation: reorganize the perturbative expansion by all-order summation (L is a large log).

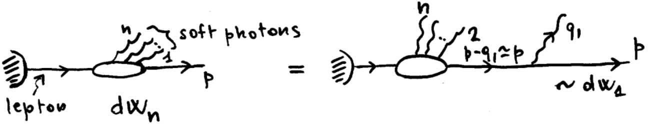
$\alpha_S L^2$	$\alpha_S L$	$\cdots$	$\cdots$	$\cdots$	$\mathcal{O}(\alpha_S)$
$\alpha_S^2 L^4$	$\alpha_S^2 L^3$	$\alpha_S^2 L^2$	$\alpha_S^2 L$	$\cdots$	$\mathcal{O}(\alpha_S^2)$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\alpha_S^n L^{2n}$	$\alpha_S^n L^{2n-1}$	$\alpha_S^n L^{2n-2}$	$\cdots$	$\cdots$	$\mathcal{O}(\alpha_S^n)$
dominant logs	next-to-dominant logs	$\cdots$	$\cdots$	$\cdots$	$\cdots$

- Ratio of two successive rows $\mathcal{O}(\alpha_S L^2)$: fixed order expansion valid when $\alpha_S L^2 \ll 1$.
- Ratio of two successive columns $\mathcal{O}(1/L)$: resummed expansion valid when $1/L \ll 1$.

Soft gluon exponentiation

Analytic resummation feasible if:
dynamics AND kinematics factorize ⇒ exponentiation.

- Dynamics factorization: general propriety of QCD for soft emissions [Gatheral('83)],[Frenkel,Taylor('84)], [Catani,Ciafaloni('84,'85)] analogous of eikonal approximation in QED [Yennie,Frautschi,Suura('61)]



$$dw_n(q_1,\dots,q_n) \simeq \frac{1}{n!}\prod_{i=1}^{n} dw_1(q_i)$$

- Thrust kinematics factorize in Laplace space [Catani,Trentadue,Turnock,Webber('91)]

$$\Theta(\tau - \sum_{j=1}^{n}\frac{k_i^2}{Q^2}) = \frac{1}{2\pi i}\int_C \frac{dN}{N} e^{N\tau}\prod_{j=1}^{n} e^{-Nk^2/Q^2}$$

- Exponentiation holds in Laplace space (results then transformed into physical space):

$$\tau \ll 1 \quad \Leftrightarrow \quad N \gg 1\,, \qquad \ln 1/\tau \gg 1 \quad \Leftrightarrow \quad \ln N \gg 1\,.$$

Sudakov resummation for Thrust

- We closely follow the CTTW formalism [Catani,Trentadue,Turnock,Webber('91,'93)]
- Resummation applied also by [Gardi et al.('99)],[Banfi et al.('01)], [Davison,Webber('09)], [Gehrmann et al.('08)], [Dissertori et al.('09)]
- Resummation also reformulated in the framework of SCET [Schwartz('08)], [Becher,Schwartz('08)],[Hornig et al.('09)],[Almeida et al.('14)],[Abbate et al.('11,'12)],[Benitez et al.('24)]

Cumulative cross section can be written as:

$$R_T(\tau) = C(\alpha_S(Q^2))\,\Sigma(\tau,\alpha_S(Q^2)) + D(\tau,\alpha_S(Q^2))\,;$$

$C(\alpha_S)$ is a hard-virtual factor and $D(\alpha_S)$ is a *remainder* function vanishing at small τ:

$$C(\alpha_S) = 1 + \sum_{n=1}^{\infty}\left(\frac{\alpha_S}{\pi}\right)^n C_n\,, \qquad D(\tau,\alpha_S) = \sum_{n=1}^{\infty}\left(\frac{\alpha_S}{\pi}\right)^n D_n(\tau)\,.$$

$\Sigma\,(\tau,\alpha_s)$ is a long-distance form factor (resums the large Sudakov logarithms).

In the Laplace-conjugated space:

$$\Sigma(\tau,\alpha_S) = \frac{1}{2\pi i}\int_C \frac{dN}{N}\,e^{N\tau}\,e^{\mathcal{F}(\alpha_S,L)}\,,$$

where the contour C runs parallel to the imaginary axis and lies to the right of all singularities.

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Sudakov resummation for Thrust

[Catani,Trentadue,Turnock,Webber('91,'93)]

The exponent $\mathcal{F}(\alpha_S L)$ reads:

$$\mathcal{F}(\alpha_S, L) \quad = \quad \int_0^1 \frac{du}{u} \left(e^{-uN} - 1\right) \left[\int_{u^2Q^2}^{uQ^2} \frac{dq^2}{q^2} 2A(\alpha_S(q^2)) + B(\alpha_S(uQ^2))\right] ,$$

with

$$A(\alpha_S) = \sum_{n=1}^{\infty} A_n \, \alpha_S^n, \quad B(\alpha_S) = \sum_{n=1}^{\infty} B_n \, \alpha_S^n \; .$$

Performing the integrals ($e^{-uN} - 1 \simeq -\Theta(u - N_0/N)$, $N_0 = e^{-\gamma_E}$, $\gamma_E = 0.5772\cdots$):

$$\mathcal{F}(\alpha_S, L) \quad = \quad L\, f_1(\lambda) + f_2(\lambda) + \frac{\alpha_S}{\pi} f_3(\lambda) + \sum_{n=4}^{\infty} \left(\frac{\alpha_S}{\pi}\right)^{n-2} f_n(\lambda) \, ,$$

with $\lambda \equiv \beta_0 \alpha_S L/\pi$, $L = \ln N$ and $\alpha_S L \sim 1$.

$\mathcal{F}(\alpha_S, L)$ resums to all order classes of $\ln N$ (large for $N \gg 1$):
LL ($\sim\alpha_S^n L^{n+1}$): $f^{(1)}$; NLL ($\sim\alpha_S^n L^n$): $f^{(2)}$; $\cdots$ N^{k}LL ($\sim\alpha_S^n L^{n+k-1}$): $f^{(k+1)}$;

Sudakov resummation for Thrust

$$
\begin{aligned}
f_1(\lambda) &= -\frac{A_1}{\beta_0 \lambda}\left[(1-2\lambda)\ln(1-2\lambda) - 2(1-\lambda)\ln(1-\lambda)\right], \\
f_2(\lambda) &= \frac{A_2}{\beta_0^2}\ln\frac{1-2\lambda}{(1-\lambda)^2} + \frac{2\gamma_E A_1}{\beta_0}\ln\frac{1-2\lambda}{1-\lambda} \\
&+ \frac{A_1\beta_1}{\beta_0^3}\left[\ln^2(1-\lambda) - \frac{\ln^2(1-2\lambda)}{2} - \ln\frac{1-2\lambda}{(1-\lambda)^2}\right] + \frac{B_1}{\beta_0}\ln(1-\lambda), \\
f_3(\lambda) &= -\frac{A_3\lambda^2}{\beta_0^2(1-\lambda)(1-2\lambda)} + \cdots, \\
f_4(\lambda) &= -\frac{A_4\lambda^2\left(2\lambda^2 - 6\lambda + 3\right)}{3\beta_0^2(1-\lambda)^2(1-2\lambda)^2} + \cdots, \\
f_5(\lambda) &= \cdots ,
\end{aligned}
$$

LL coefficients: A_1, β_0 , NLL coefficients: A_2, B_1,β_1, C_1
NNLL coefficients: A_3,B_2, β_2, C_2, N^3LL coefficients: A_4,B_3, β_4, C_3,
N^4LL coefficients: A_5, B_4, β_5, C_4,

- $f_n(\lambda)$ are singular at $\lambda = 1/2$ (i.e. $N \sim N_L = \exp 1/[2\beta_0\alpha_S(Q^2)]$) and and $\lambda = 1$ (i.e. $N \sim N'_L = \exp 1/[\beta_0\alpha_S(Q^2)]$). These singularities have to be regularized with a NP prescription/model *outside* perturbation theory.

Inversion of Laplace transform

Formal inversion from N space to τ space is:

$$\Sigma(\tau, \alpha_S) = \frac{1}{2\pi i} \int_{C-i\infty}^{C+i\infty} \frac{dN}{N} e^{N\tau} e^{\mathcal{F}(\alpha_S, L)} ,$$

- This formula involves (formally non-integrable) Landau singularity of α_S
- Exact analytic Laplace inversion cannot be computed.

CTTW solution: Taylor expand $\mathcal{F}(\alpha_S, L)$ around the point (not the saddle point)

$$\ln N = \ln(1/\tau) \equiv \ell ,$$

$$\Sigma(\tau, \alpha_S) = \frac{1}{2\pi i} \int_C \frac{dN}{N} e^{N\tau} \exp\left[\sum_{k=0}^{\infty} \frac{\partial^k \mathcal{F}(\alpha_S, \ell)}{\partial \ell^k} \frac{\ln^k(\tau N)}{k!}\right] ,$$

not possible to evaluate the series exactly: a new hierarchy is defined in $\tau-$space. N^{n}LL in τ space defined by keeping the terms $\alpha_S^{n-1}(\alpha_S \ell)^k$, (for all k).

Crucial point: the correspondence $\ln N$ to $\ln(1/\tau)$ is not exact.Kinematics factorization and exponentiation are valid only in N space.

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Approximated analytic inversion (I)

The approximated analytic form factor in τ space reads up to N^{4}LL:

$$\Sigma(\tau,\alpha_S) = \frac{1}{\Gamma[\phi]} e^{\ell\, f_1(\lambda_\tau)+f_2(\lambda_\tau)+\frac{\alpha_S}{\pi} f_3(\lambda_\tau)+\left(\frac{\alpha_S}{\pi}\right)^2 f_4(\lambda_\tau)+\left(\frac{\alpha_S}{\pi}\right)^3 f_5(\lambda_\tau)}$$
$$\times \Bigg[1+\mathcal{F}^{(1)}_{res}(\alpha_S,\ell)\psi_0(\phi)+\frac{1}{2}(\mathcal{F}^{(2)}(\alpha_S,\ell)+(\mathcal{F}^{(1)}_{res}(\alpha_S,\ell))^2)(\psi_0^2(\phi)-\psi_1(\phi))$$
$$+\frac{1}{6}(\mathcal{F}^{(3)}(\alpha_S,\ell)+3\mathcal{F}^{(2)}(\alpha_S,\ell)\mathcal{F}^{(1)}_{res}(\alpha_S,\ell)+(\mathcal{F}^{(1)}_{res})^3(\alpha_S,\ell)(\psi_0^3(\phi)-3\psi_0(\phi)\psi_1(\phi)+\psi_2(\phi))$$
$$+\frac{1}{24}(\mathcal{F}^{(4)}(\alpha_S,\ell)+3(\mathcal{F}^{(2)}(\alpha_S,\ell))^2+4\mathcal{F}^{(3)}(\alpha_S,\ell)\mathcal{F}^{(1)}_{res}(\alpha_S,\ell)+6\mathcal{F}^{(2)}(\alpha_S,\ell)(\mathcal{F}^{(1)}_{res}(\alpha_S,\ell))^2+(\mathcal{F}^{(1)}_{res}(\alpha_S,\ell))^4)$$
$$\times(\psi_0^4(\phi)-6\psi_1(\phi)+3\psi_1^3(\phi)+4\psi_0(\phi)\psi_2(\phi)-\psi_3(\phi))\Bigg],$$

where $\Gamma(x)$ is the Euler Γ function, $\psi_n(x) \equiv \frac{d^{n+1}\ln\Gamma(x)}{dx^{n+1}}$, $\phi \equiv 1 - f_1(\lambda_\tau) - \lambda f'(\lambda_\tau)$, $\mathcal{F}^{(1)}_{res}(\alpha_S,\ell) \equiv \mathcal{F}^{(1)}(\alpha_S,\ell) - f_1(\lambda_\tau) - \lambda f'(\lambda_\tau)$, $\lambda_\tau = \alpha_S\beta_0\,\ell/\pi$.
The analytic form factor in τ space can then be re-written in an "exponentiated form"

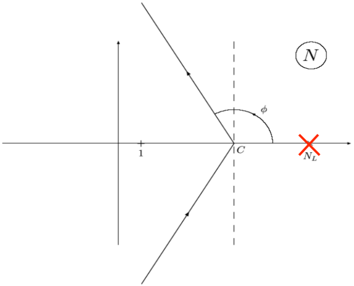
$$\Sigma(\tau,\alpha_S) = e^{\ell\, g_1(\lambda_\tau)+g_2(\lambda_\tau)+\frac{\alpha_S}{\pi} g_3(\lambda_\tau)+\sum_{n=4}^{\infty}\left(\frac{\alpha_S}{\pi}\right)^{n-2} g_n(\lambda_\tau)},$$

Exact numerical inversion (II)

$$\Sigma(\tau, \alpha_S) = \frac{1}{2\pi i} \int_{C_{MP}} \frac{dN}{N} e^{N\tau} e^{\mathcal{F}(\alpha_S, L)} ,$$

where the contour C runs parallel to the imaginary axis and lies to the right of all singularities of the integrand.

Exact numerical inversion can be performed with a prescription to avoid the Landau Pole. **Minimal Prescription** [Catani,Mangano,Nason,Trentadue('96)]: the contour of integration C_{MP} lies to the right of all physical singularities but to the left of the (unphysical) Landau pole. The results obtained by using this prescription converge asymptotically to the perturbative series and do not include any power correction.



Remainder function and unitarity

Outside $\tau \ll 1$ resummation is not justified. Matching with F.O. results is necessary

$$D(\tau, \alpha_S) = R_T(\tau, \alpha_S)|_{f.o.} - [C(\alpha_S(Q^2))\, \Sigma(\tau, \alpha_S(Q^2))]_{f.o.}\,;$$

However resummation ambiguity at large τ only partially solved by matching. Impose the *physical* constraint:

$$R_T(\tau_{\max}) = 1\,,$$

which is violated by higher-order terms beyond the nominal fixed-order accuracy of the calculation (e.g. by $\mathcal{O}(\alpha_S^4)$ terms at N^{3}LL+NNLO). It can be fulfilled in τ space using CTTW['93]

$$\ell \equiv \ln(1/\tau) \quad \mapsto \quad \tilde{\ell} \equiv \ln(1/\tau - 1/\tau_{\max} + 1) \overset{\tau \lll 1}{=} \ell + \mathcal{O}(\tau)\,,$$

which acts as a perturbative unitarity constraint: $\Sigma(\tau = \tau_{\max}, \alpha_S)|_{\ell \to \tilde{\ell}} = 1\,.$

An analogous constraint can be imposed in N space:

$$L \equiv \ln N \quad \mapsto \quad \tilde{L} \equiv \ln(N - N_c) \overset{N \ggg 1}{=} L + \mathcal{O}(1/N)\,,$$

with N_c a constant such that $\Sigma(\tau = \tau_{\max}, \alpha_S)|_{L \to \tilde{L}} = 1\,.$ These replacements, besides affecting $\Sigma(\tau, \alpha_S)$, have also an impact on the remainder function $D(\tau, \alpha_S)$.

In the large-τ region ($\tau \lesssim \tau_{\max}$), thrust distribution is affected by instabilities from enhanced soft/collinear corrections near the kinematic fixed-order boundaries (*Sudakov shoulder* [Catani,Webber['97]).

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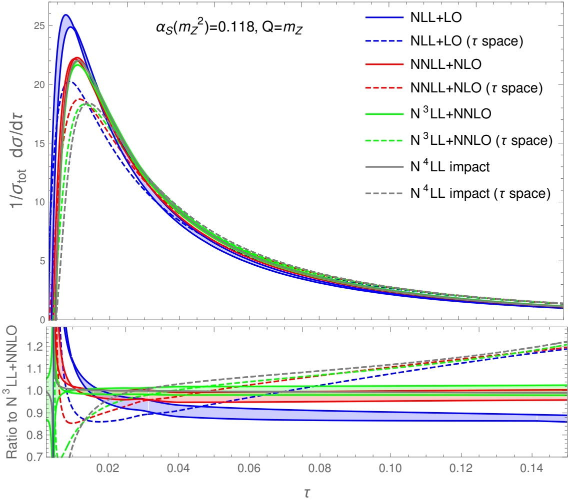
Numerical results: perturbative effects



Thrust distribution at $Q = 91.1876$ GeV in pQCD. Results from resummation in Laplace-conjugated space (solid bands), including renormalization scale variations $Q/2 \le \mu_R \le 2Q$, compared with physical τ-space approximated results (dashed lines).

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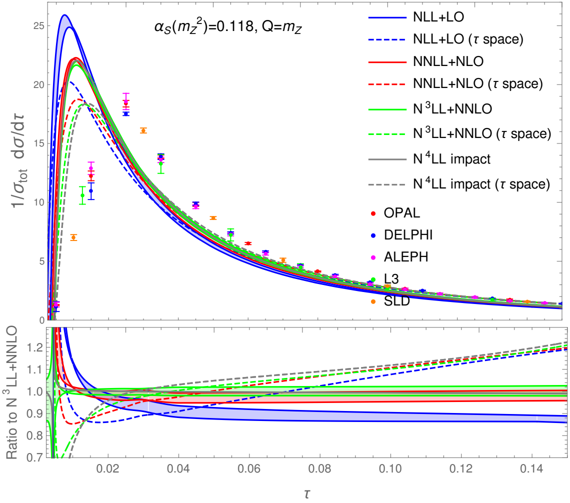

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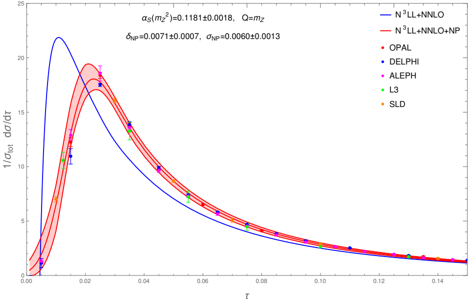
Numerical results: non perturbative effects

NP effects included using an analytic model based on a correlation [Catani at al.('91)] or shape function [Korchemsky,Sterman('99)] $f_{NP}(\tau_h, \tau)$ depending on 2 parameters:

$$\frac{d\sigma_h}{d\tau_h} = \int d\tau \frac{d\sigma}{d\tau} f_{NP}(\tau, \tau_h)\,,$$

$$f_{NP}(\tau_h, \tau) = \frac{1}{\sqrt{2\pi}\sigma_{NP}} \exp\left[-\frac{(\tau_h - \tau - \delta_{NP})^2}{2\,\sigma_{NP}^2}\right],$$



The thrust distribution at $Q = 91.1876$ GeV at N^{3}LL+NNLO in QCD without (blue solid line) and with (red band) the inclusion of NP effects.

$\alpha_S(m_Z)$ extraction from Thrust

We performed a three parameter ($\alpha_S(m_Z^2)$, δ_{NP}, σ_{NP}) fit in the small/intermediate τ region ($0 < \tau < 0.15$) using LEP and SLD data at the Z boson peak ($Q = m_Z$) [SLD('94),Wicke('99),ALEPH('03),OPAL('04),L3('04)].
At N^{3}LL+NNLO accuracy we get:

$$\alpha_S(m_Z^2) = 0.1181 \pm 0.0018\,, \qquad \delta_{NP} = 0.0071 \pm 0.0007\,, \qquad \sigma_{NP} = 0.0060 \pm 0.0013\,,$$

where the uncertainties include experimental and theoretical (perturbative) errors (the latter estimated by means of a renormalization scale variation of a factor two).

Inclusion of N^{4}LL correction modify the result negligibly.

Same fit at NNLL+NLO accuracy gives:

$$\alpha_S(m_Z^2) = 0.1194 \pm 0.0020\,,\ \delta_{NP} = 0.0071 \pm 0.0007\,,\ \sigma_{NP} = 0.0062 \pm 0.0014$$

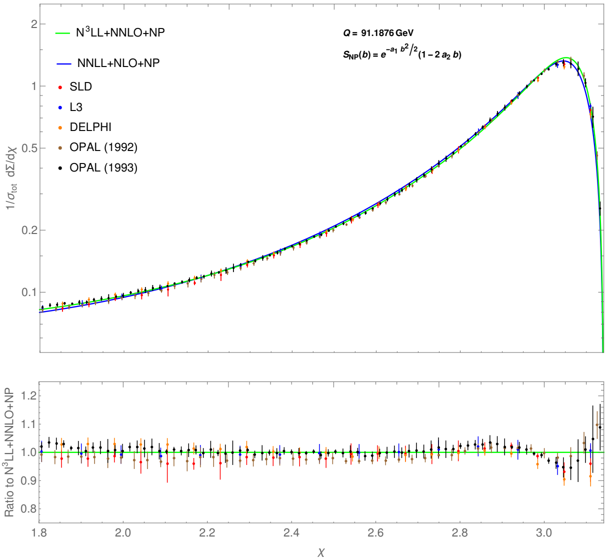
Same fit using the τ-space resummation formalism at N^{3}LL+NNLO accuracy:

$$\alpha_S(m_Z^2) = 0.1120 \pm 0.0019\,,\ \delta_{NP} = 0.0083 \pm 0.0010\,,\ \sigma_{NP} = 0.0055 \pm 0.0020$$

Key point: Approximations used to perform the resummation in τ space have to be properly included in order to obtain a reliable determination of $\alpha_S(m_Z^2)$.

Energy Energy Correlation function

[Aglietti,G.F.('24)]



Comparison at $N^3LL+NNLO$ and $NNLL+NLO$ with NP effects parameterized by a form factor

$$S(Q,b) \to S(Q,b)S_{NP}(b)$$

$$S_{NP} = \exp\{-a_1 b^2\}(1-a_2 b)$$

[Dokshitzer,Marchesini, Webber('99)]

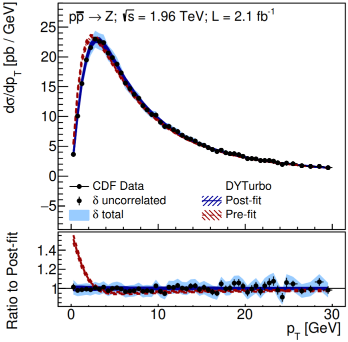
Fit results

NNLL+NLO:
$\alpha_S(m_Z) = 0.121 \pm 0.002$,
$a_1 = 1.9 \pm 1.4\,GeV^2$,
$a_2 = 0.4 \pm 0.1\,GeV$

$N^3LL+NNLO$:
$\alpha_S(m_Z) = 0.120 \pm 0.002$,
$a_1 = 1.8 \pm 1.4\,GeV^2$,
$a_2 = 0.3 \pm 0.1\,GeV$

Drell–Yan transverse momentum at the Tevatron

[Camarda,G.F.,Schött('22)]



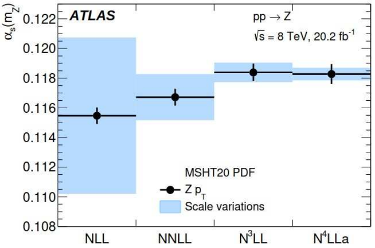
Statistical uncertainty
Experimental systematic uncertainty
PDF uncertainty (NNPDF4.0)
PDF uncertainty (envelope of PDFs)
Scale variations uncertainties
Matching at $\mathcal{O}(\alpha_S^3)$
Non-perturbative model
Flavour model
QED ISR
Lower limit of fit range
Total

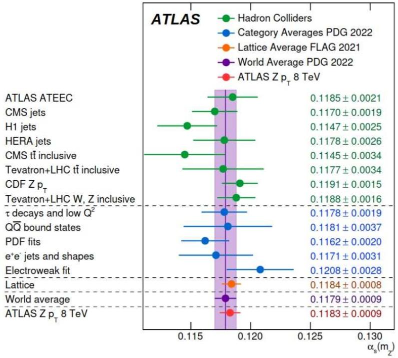
Simultaneous fit of $\alpha_S(m_Z)$ and g at N^{3}LL+$\mathcal{O}(\alpha_S^3)$ (N^{3}LL+N^{3}LO):

$$\alpha_S(m_Z) = 0.1191^{+0.0013}_{-0.0016}$$

Drell–Yan transverse momentum at the LHC

[Cieri,G.F.,ATLAS Coll.('23)]





Simultaneous fit of $\alpha_S(m_Z)$ and NP parameters at N^{4}LL+$\mathcal{O}(\alpha_S^3)$:

$$\alpha_S(m_Z) = 0.11828^{+0.00084}_{-0.00088}$$

Analytic inversion with the Saddle-Point method (III)

$$\Sigma(\tau, \alpha_S) = \frac{1}{2\pi i} \int_C dN\, e^{\,g_\tau(N)} = \frac{1}{2\pi i} \int_C \frac{dN}{N} e^{N\tau} e^{\mathcal{F}(\alpha_S, L)} \,,$$

the saddle point $\overline{N}$ is the stationary point of the exponent: $g'_\tau(\overline{N}) = 0$, i.e.

$$\overline{N}\,\tau = 1 - \frac{d}{d\lambda} \left[\lambda f_1(\lambda) + \beta_0 \sum_{n=1}^{\infty} \left(\frac{\alpha_S}{\pi} \right)^n f_{n+1}(\lambda) \right]_{\lambda \to \overline{\lambda}}$$

where $\overline{\lambda} = \beta_0 \alpha_S \ln(\overline{N})/\pi$. The (usual) Taylor expansion is around $N = 1/\tau$: the *free* theory ($\alpha_S \to 0$) saddle point. Already at LL the term $\lambda f_1(\lambda) \sim 1$ can't be neglected. The saddle-point method then gives:

$$\Sigma(\tau, \alpha_S) = \frac{e^{\,g_\tau(\overline{N})}}{\sqrt{2\pi\, |g''_\tau(\overline{N})|}} \int_{-\infty}^{+\infty} \frac{dx}{\sqrt{\pi}}\, e^{-x^2}\, K(x) \simeq \frac{e^{\,g_\tau(\overline{N})}}{\sqrt{2\pi\, |g''_\tau(\overline{N})|}} \,,$$

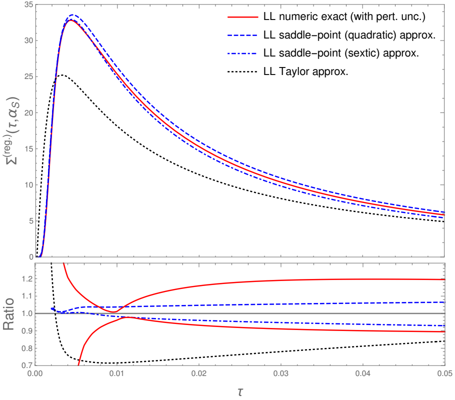
where $K(x) \simeq 1 + \cdots$ contains the anharmonic corrections and $\overline{N}$ can be obtained analytically (by a recursion method) or numerically.
Same techniques applied in the similar case of threshold resummation by [Bonvini,Forte,Ridolfi('12,'15)

Saddle-Point method: numerical results

The application of the SP method is complicated by the Landau singularity which has to be regularized. We used an analytic (or dispersive) approach [Shirkov, Solovtsov('96)]],[Aglietti,Ricciardi('04)],[Aglietti,G.F.,Ricciardi('07)

$$\widetilde{\alpha}_S(q^2) = \frac{1}{\beta_0} \arctan\left(\beta_0 \alpha_S(q^2)\right) = \alpha_S(q^2) - \frac{\beta_0^2}{3} \alpha_S(q^2)^3 + \cdots .$$

$$f_1(\lambda) \to \widetilde{f}_1(\lambda) = \frac{A_1}{2\beta_0 \lambda} \left\{ 2(1-\lambda) \ln[(1-\lambda)^2 + (\beta_0 \alpha_S)^2] - (1-2\lambda) \ln[(1-2\lambda)^2 + (\beta_0 \alpha_S)^2] \cdots \right\}$$



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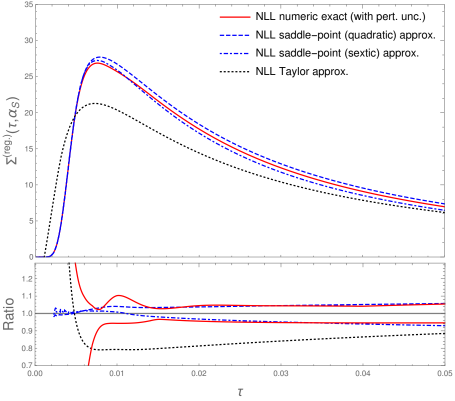

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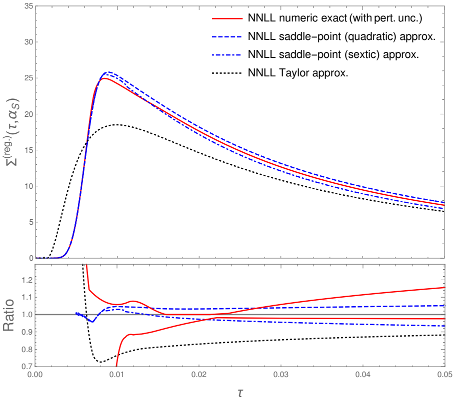

Conclusions

- Presented resummed calculation for thrust distribution in e^+e^- to full $N^3LL+NNLO$ accuracy (including also the N^4LL terms) in pQCD.
- Resummation performed in the Laplace-conjugated space. Results inverted *exactly* (in numerical way).
- pQCD with NP effects compared with LEP and SLD data at the Z-boson peak.
- Extract value of the QCD coupling $\alpha_S(m_Z^2) = 0.1181 \pm 0.0018$, fully consistent with the world average.
- Commonly used (approximate) analytic inversion gives quite different results with lower determinations of $\alpha_S(m_Z^2)$.
- Proposed an analytic inversion using the saddle-point method which nicely agrees with the exact numeric inversion.