UNIVERSITÀ DI TORINO

INFN

EOB@Work25

“A novel Lagrange-multiplier approach to the effective-one-body dynamics of binary systems in post-Minkowskian gravity”

Based on [T. Damour, A. Nagar, AP, P. Rettegno; **2503.05487**]

Andrea Placidi 02/09/2025

MOTIVATION

Why should we give up our beloved Hamiltonian paradigm?

EOB with post-Minkowskian information

PM results

$$\chi_{\rm PM}(j,\gamma,\tilde{a}_i) = \sum_n 2\frac{\chi_n(\gamma,\tilde{a}_i,\nu)}{j^n}$$

$\gamma =$ Lorentz factor

EOB mass-shell constraint

$$\mathscr{C} = g^{\mu\nu}_{\rm eff}(x^\rho,\gamma,\tilde{a}_i)p_\mu p_\nu + 1 = 0$$

$$\gamma = E_{\rm eff}/\mu$$

$$\boxed{\chi_n(\gamma,\tilde{a}_i,\nu) \longrightarrow g^{\mu\nu}_{\rm eff}(x^\rho,\gamma,\tilde{a}_i)}$$

EOB with post-Minkowskian information

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$$\boxed{\chi_n(\gamma,\tilde{a}_i,\nu) \longrightarrow g_{\mathrm{eff}}^{\mu\nu}(x^\rho,\gamma,\tilde{a}_i)}$$

In particular, for spin-aligned binaries:

$$\mathscr{C} = -\frac{[\gamma - \mathscr{G}(r,\gamma,\tilde{a}_i)p_\varphi]^2}{A(r,\gamma,\tilde{a}_i)} + \frac{p_r^2}{B(r,\gamma,\tilde{a}_i)} + \frac{p_\varphi^2}{r_c(r,\gamma,\tilde{a}_i)^2} + 1$$

$$\mathscr{G}(r,\gamma,\tilde{a}_i) = G_S(r,\gamma,\tilde{a}_i)\hat{S} + G_{S_*}(r,\gamma,\tilde{a}_i)\hat{S}_*$$

$$\boxed{\chi_n^{a_i-\mathrm{even}}\text{+gauge fixing} \longrightarrow A, B, r_c}$$

$$\boxed{\chi_n^{a_i-\mathrm{odd}} \longrightarrow G_S, G_{S_*}}$$

EOB Hamiltonian in PM gravity

[T. Damour; 2018] [A. Antonelli et al.; 2019]
[M. Khalil et al.; 2022] [A. Buonanno et al.; 02/2024] [A. Buonanno et al.; 05/2024]

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$\gamma \to \hat{H}_{\text{eff}}$ here ●, and the constraint is solved for $\hat{H}_{\text{eff}}$

$$\longrightarrow \boxed{\hat{H}_{\text{eff}} = \hat{H}_{\text{eff}}(r, p_r, p_\varphi, \gamma, \tilde{a}_i)}$$

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$$\boxed{\hat{H}_{\text{eff}} = \hat{H}_{\text{eff}}(r, p_r, p_\varphi, \underline{\gamma,} \tilde{a}_i)}$$

This is a **recursive definition**: at each PM order every γ must be replaced by the ordinary Hamiltonian $\hat{H}_{\text{eff}}(r, p_r, p_\varphi, \tilde{a}_i)$ obtained at the previous orders, starting from $\hat{H}^{\text{1PM}}_{\text{eff}} = \hat{H}_{\text{Kerr}}(r, p_r, p_\varphi, \tilde{a}_i)$

First issue:

$\mathscr{C}$ has an intricate dependence on γ

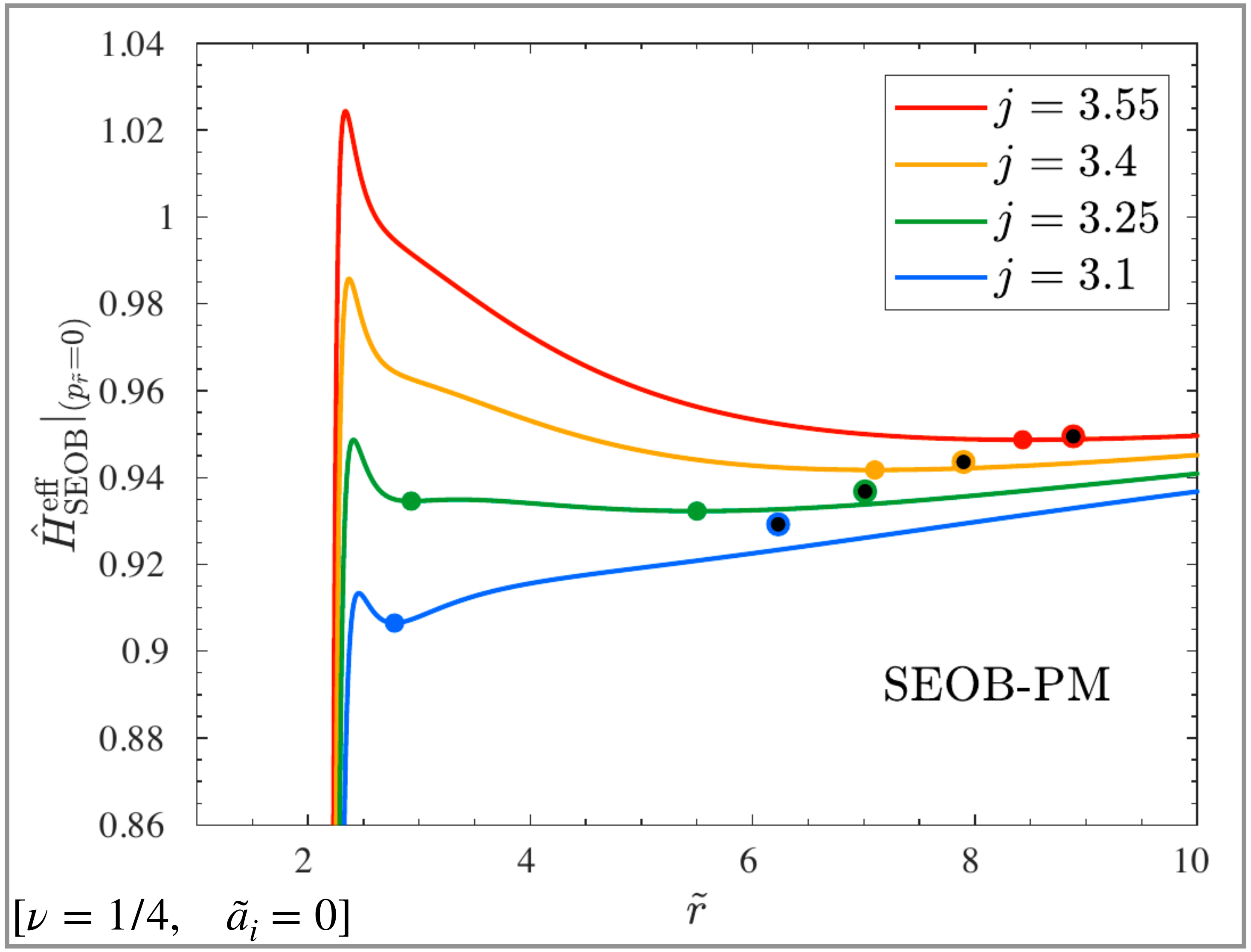
- 3PM → hyperbolic functions
- 4PM → elliptic integrals, polylogs

\+ required recursive steps

EOB Hamiltonian in PM gravity

Second issue:

Radial potential of the corresponding conservative dynamics



Using the results of [A. Buonanno et al.; 05/2024]

- Colored dots → local minima, stable circular orbits
- Black dots → position of the effective particle as it moves along the radiation-reacted dynamics

EOB Hamiltonian in PM gravity

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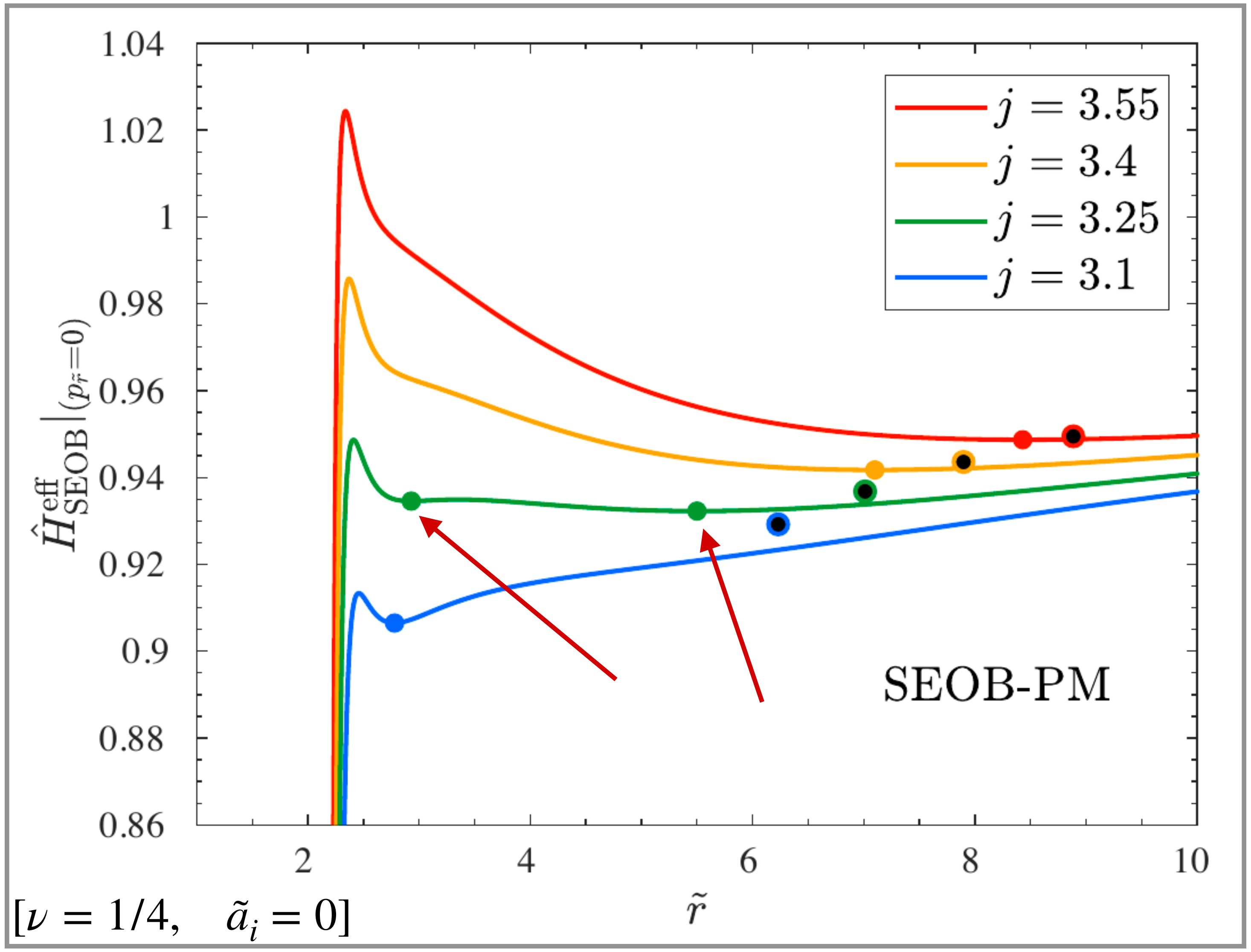

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→ 2 local minima (and maxima)

EOB Hamiltonian in PM gravity

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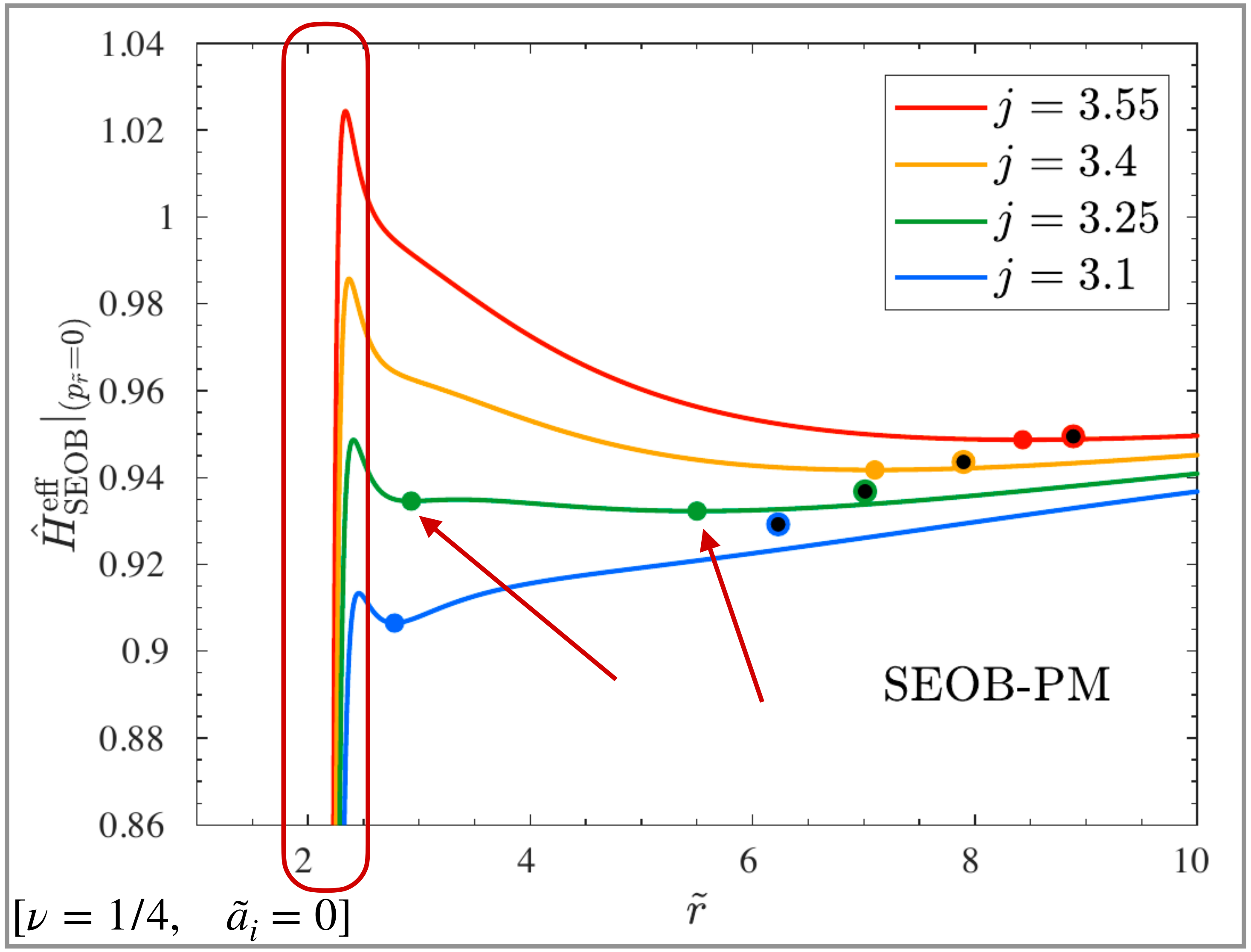

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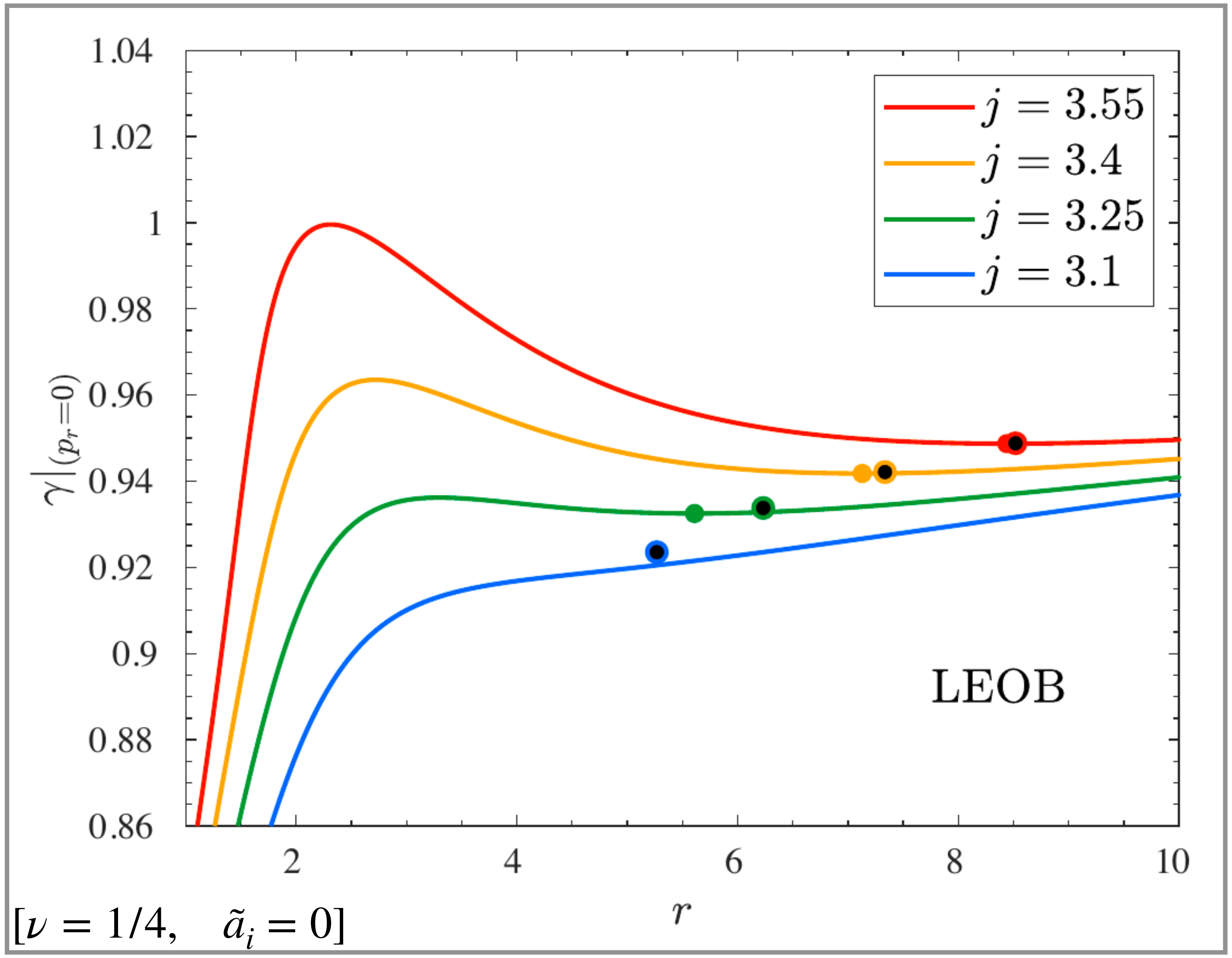
→ 2 local minima (and maxima)

→ Singular behaviour around $\tilde{r} = 2$

Why a Lagrangian EOB?

Both the issues are solved!

- No need to recursively solve for an Hamiltonian
- Radial potential closer to the test-mass counterpart



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HOW DOES IT WORK?

Euler-Lagrange EOB equations

LEOB: a novel Lagrange-multiplier approach

$$S[X^\mu, P_\mu] = \int [P_\mu dX^\mu]^{\text{on-shell}} = \int P_i dX^i - H_{\text{eff}}\left(X^i, P_i\right) dT_{\text{eff}}$$

replaced by

Lagrange multiplier

Evolution parameter associated to e_L

$$S[X^\mu, P_\mu, e_{\text{L}}] = \int P_\mu dX^\mu - e_{\text{L}}\, \mathscr{C}\left(X^\mu, P_\mu\right) d\tau$$

EOB mass-shell constraint

From the variational principle:

$$\delta S[X^\mu, P_\mu, e_{\text{L}}] = 0 \quad \rightarrow \quad \frac{dX^\mu}{d\tau} = e_{\text{L}} \frac{\partial \mathscr{C}}{\partial P_\mu}\,, \qquad \frac{dP_\mu}{d\tau} = -\, e_{\text{L}} \frac{\partial \mathscr{C}}{\partial X^\mu}\,, \qquad \mathscr{C} = 0$$

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Fixing $\tau = T_{\text{real}}$ while considering $\dfrac{dT_{\text{eff}}}{dT_{\text{real}}} = \dfrac{dE_{\text{real}}}{dE_{\text{eff}}} = \dfrac{M}{E_{\text{real}}}$ and $X^0 = T_{\text{eff}}$

$$\frac{M}{E_{\text{real}}} = e_{\text{L}} \frac{\partial \mathscr{C}}{\partial P_0} = -\, e_{\text{L}} \frac{\partial \mathscr{C}}{\partial E_{\text{eff}}} \longrightarrow \boxed{e_L = -\frac{M}{E_{\text{real}}} \left(\frac{d\mathscr{C}}{dE_{\text{eff}}}\right)^{-1}}$$

LEOB equations of motion

In terms of mass-rescaled quantities, the resulting **Euler-Lagrange equations** are:

$$\boxed{\frac{dx^i}{dt_{\text{real}}} = -\frac{1}{h}\left(\frac{\partial \mathscr{C}}{\partial \gamma}\right)^{-1}\frac{\partial \mathscr{C}}{\partial p_i}, \qquad \frac{dp_i}{dt_{\text{real}}} = \frac{1}{h}\left(\frac{\partial \mathscr{C}}{\partial \gamma}\right)^{-1}\frac{\partial \mathscr{C}}{\partial x^i}, \qquad \frac{d\gamma}{dt_{\text{real}}} = 0}$$

$$\boxed{h = \frac{E_{\text{real}}}{M} = \sqrt{1 + 2\nu(\gamma - 1)}}$$

The extra equation $\mathscr{C} = 0$ is only relevant for the initial conditions

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Adding **dissipative effects**:

+ radiation-reaction force $\mathscr{F}_\mu$ in the evolution equation of p_μ , i.e.

$$\boxed{\frac{dx^i}{dt_{\mathrm{real}}} = -\frac{1}{h}\left(\frac{\partial \mathscr{C}}{\partial \gamma}\right)^{-1}\frac{\partial \mathscr{C}}{\partial p_i}, \qquad \frac{dp_i}{dt_{\mathrm{real}}} = \frac{1}{h}\left(\frac{\partial \mathscr{C}}{\partial \gamma}\right)^{-1}\frac{\partial \mathscr{C}}{\partial x^i} + \mathscr{F}_i, \qquad \frac{d\gamma}{dt_{\mathrm{real}}} = -\mathscr{F}_0}$$

with the condition $\dfrac{dx^\mu}{dt_{\mathrm{real}}}\mathscr{F}_\mu = 0$ ensuring that $\mathscr{C} = 0$ holds along the whole radiation-reacted evolution

APPLICATION

A PM-informed LEOB waveform model for quasi-circular spin-aligned binaries

Our choices for the model

Gauge fixing:

Lagrange-Just-Boyer-Lindquist gauge

- $A(r, \gamma, \tilde{a}_i)B(r, \gamma, \tilde{a}_i) = \dfrac{r^2}{r_c^2(r, \gamma, \tilde{a}_i)}$ (as in a Kerr metric in BL coordinates)
- $r_c(r, \gamma, \tilde{a}_i) = r_c^{\mathrm{Kerr}}(r, \tilde{a}_i)$

Analytical information:

- $\tilde{a}_i^0 \rightarrow$ local 4PM contributions + 4PN completion (non-local part up to e^6) in A
- $\tilde{a}_i^1 \rightarrow$ 4PM + static 5PM-4PN contribution in G_S, G_{S_*}
- $\tilde{a}_i^2 \rightarrow r_c^{\mathrm{Kerr}}$ and 4PM spin-spin term in A
- $\tilde{a}_i^3 \rightarrow$ 5PM spin-spin term in G_S, G_{S_*}
- $\tilde{a}_i^4 \rightarrow$ 5PM spin^4 term in A

We consider here the *physical* PM counting: +1 order for each power of $1/r$ *and* $\tilde{a}_i$

Our choices for the model

Waveform and radiation reaction:

Standard PN-based prescription of TEOBResumS-Dalí [A. Nagar et al.; 07/2024]

$$h_{\ell m} = h^{N}_{\ell m} \hat{h}_{\ell m} \hat{h}^{\rm NQC}_{\ell m}$$

$$\hat{\mathscr{F}}_{\varphi} = -\frac{32}{5} \nu r_{\Omega}^{4} \Omega^{5} \hat{f}(x;\nu) + \hat{\mathscr{F}}^{\rm H}_{\varphi}$$

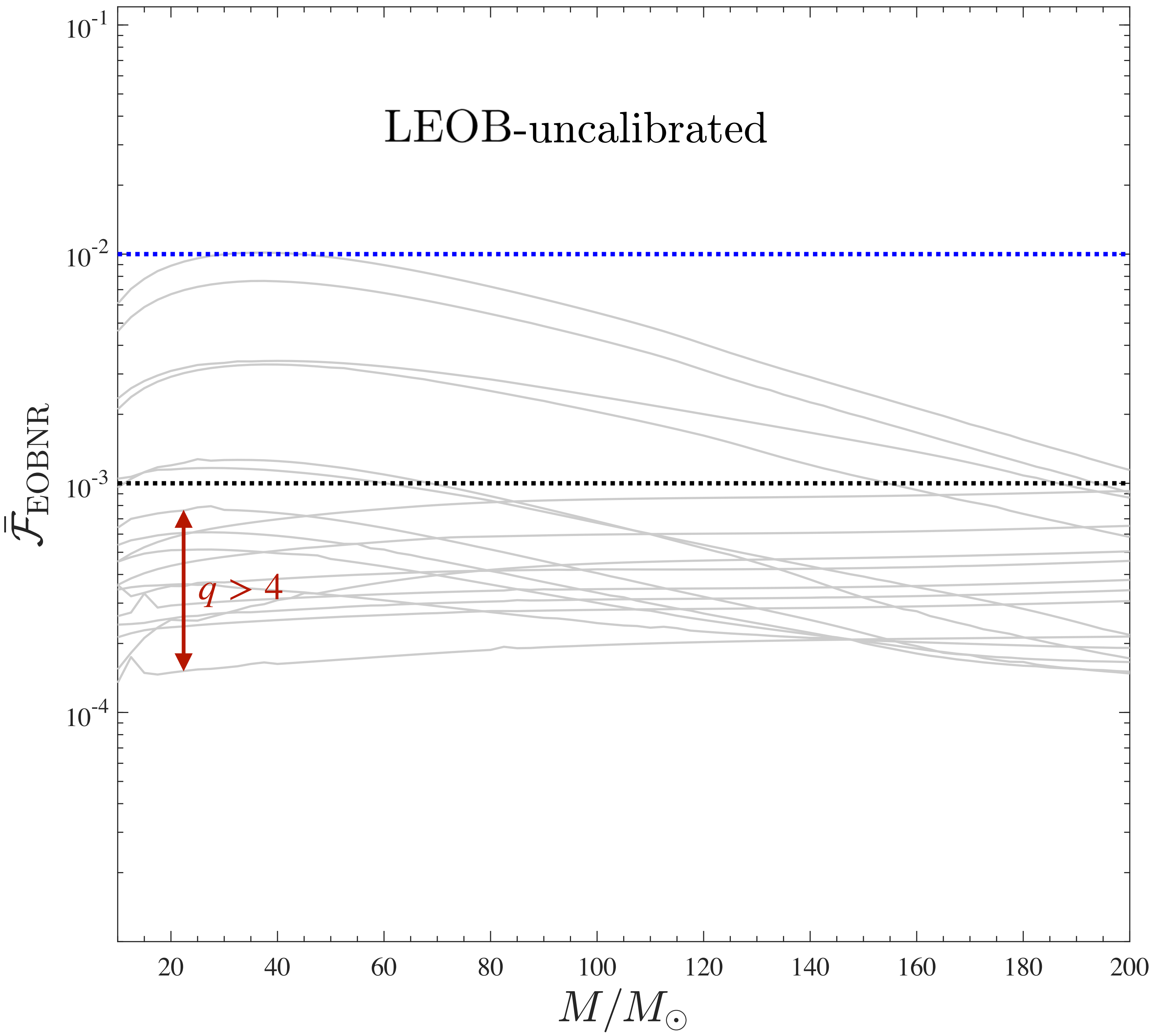
$$\hat{\mathscr{F}}_{r} = 0$$

NR calibration of the dynamics:

- $A \to$ 5PM-5PN parameter $a^{\rm NR}_{52}(\nu)$ in the orbital part
- $G_S, G_{S_*} \to$ 5PM-4PN parameter $\hat{g}^{\rm NR}_{32}(\nu, \tilde{a}_i)$ (the same in both functions)

Nonspinning case - uncalibrated

Unfaithfulness on a sample of 18 SXS nonspinning simulations with $1 \leq q \leq 15$



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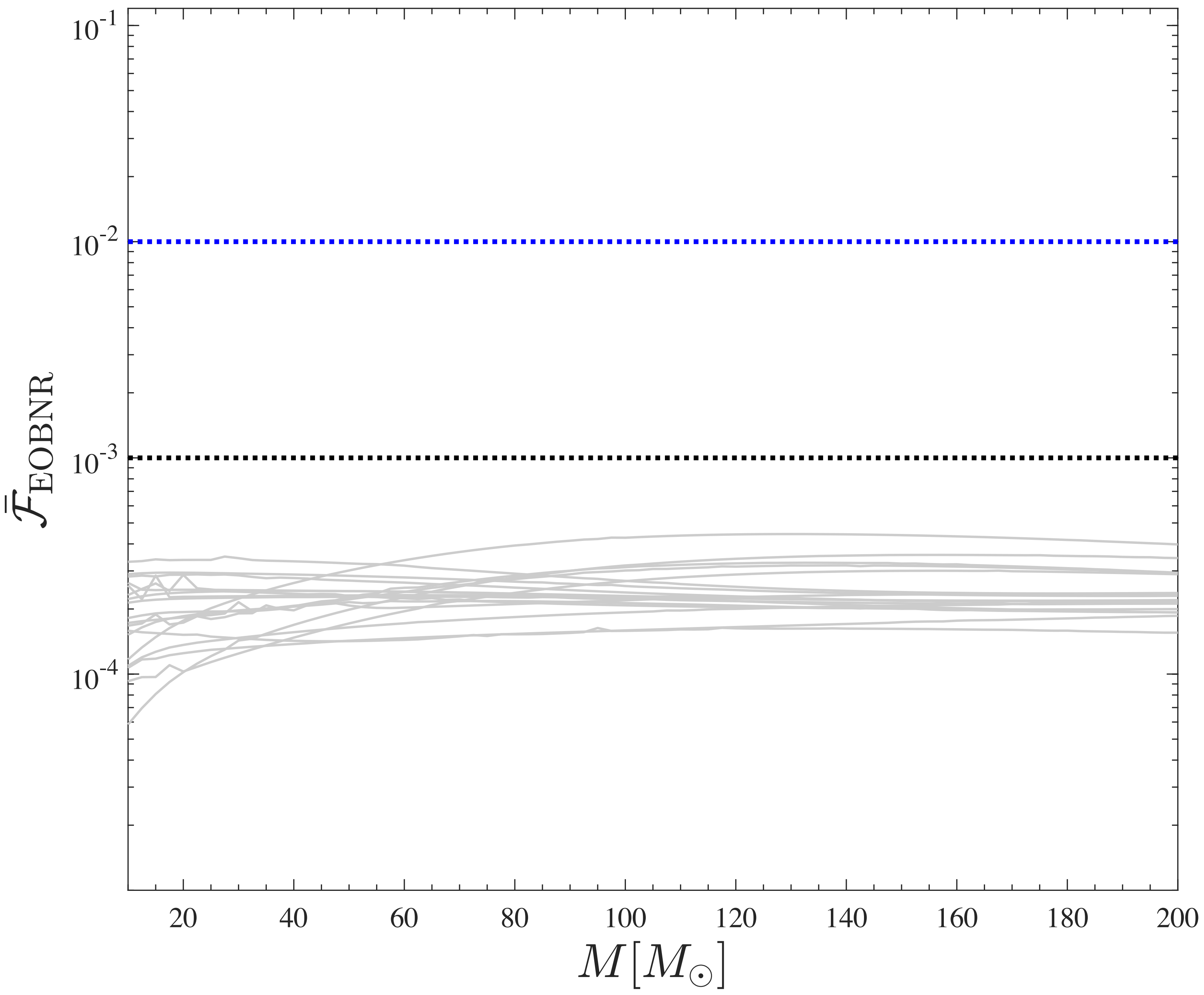
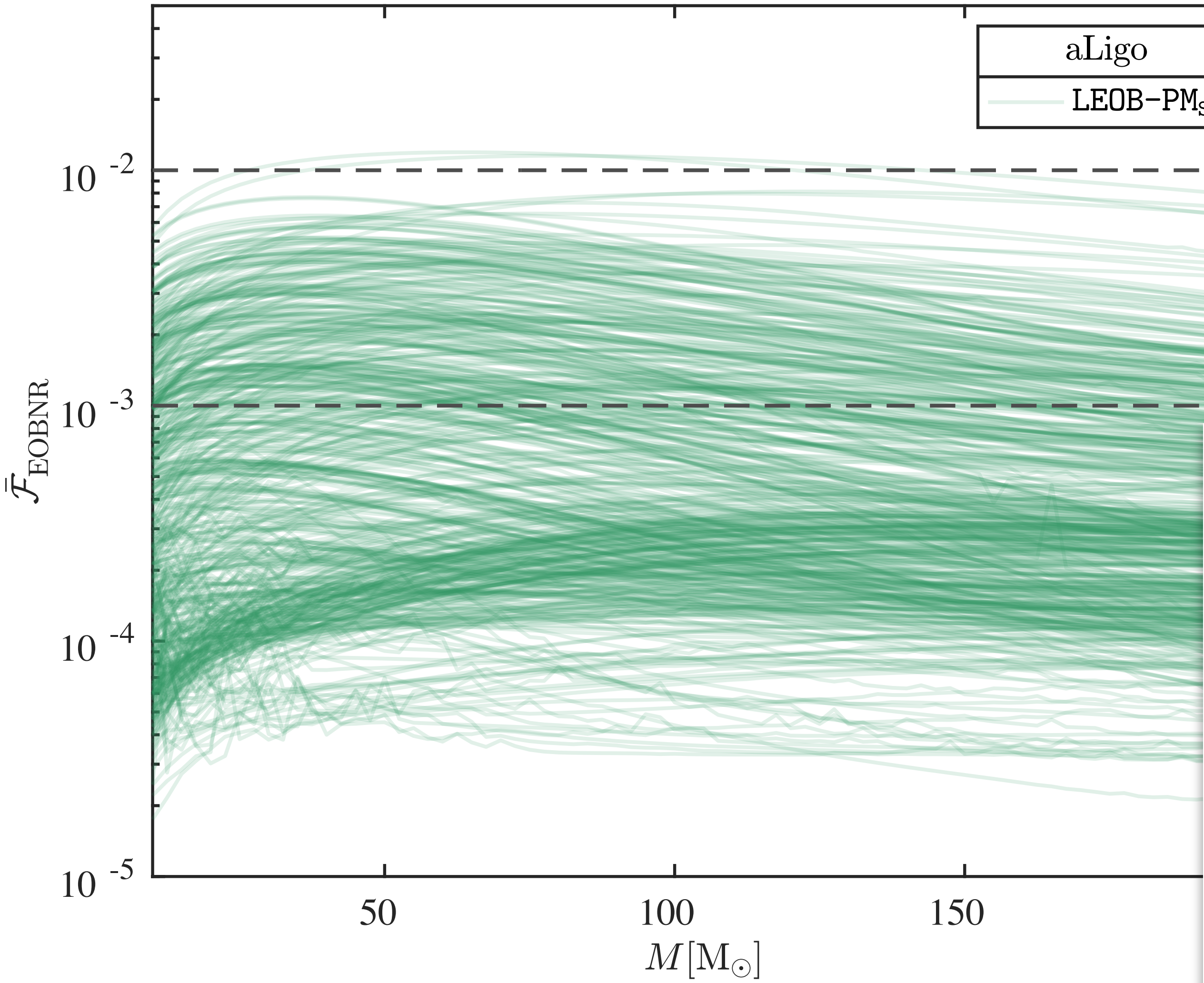


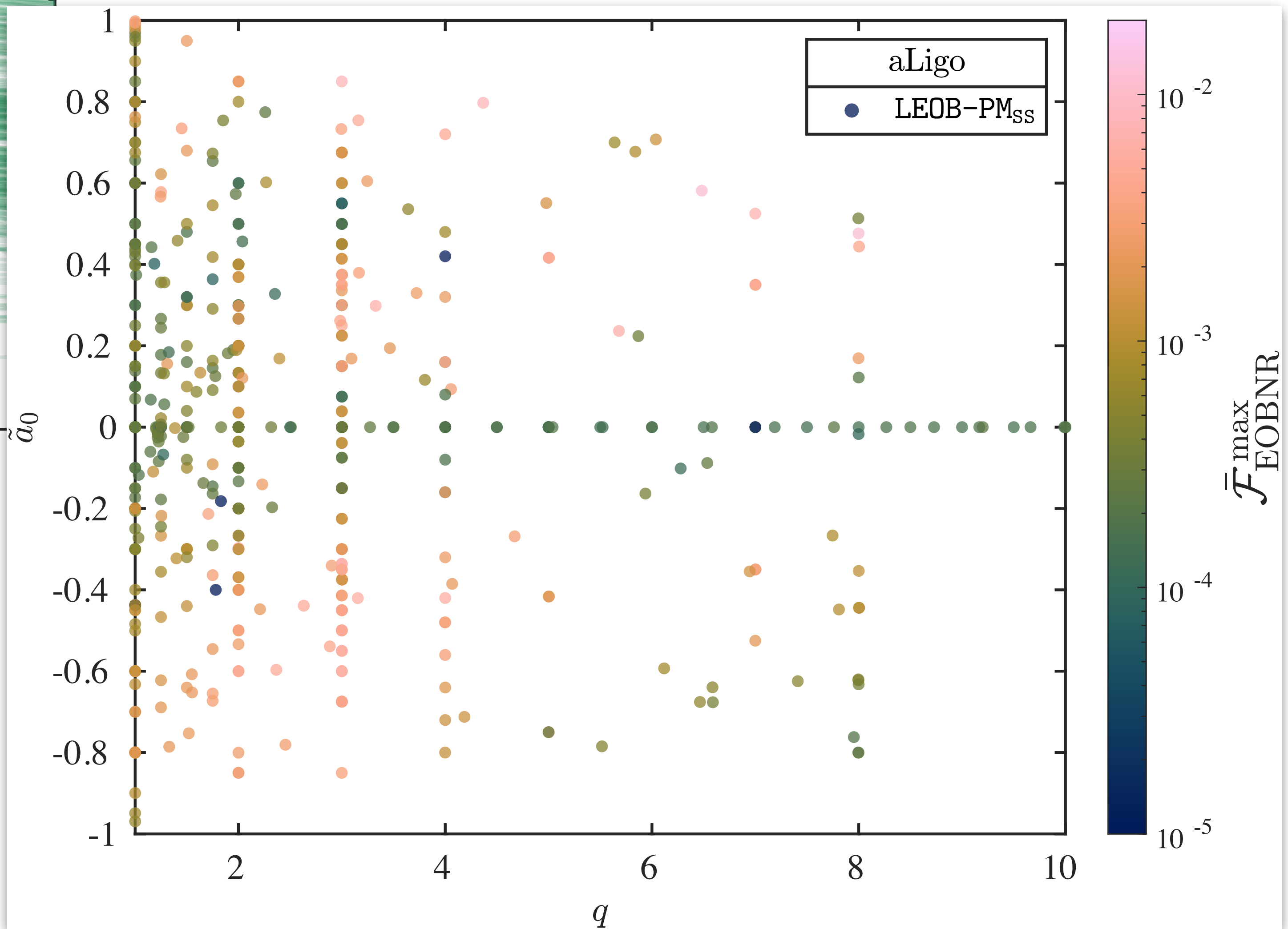
For nearly equal-mass configurations the NR-calibration yields an improvement of almost two orders of magnitude.

Spinning case - calibrated

Unfaithfulness on a sample of 530 SXS spin-aligned simulations



Performance consistent with the PN-based TEOB models



$\tilde{a}_0 = \tilde{a}_1 + \tilde{a}_2$

Conclusions

With **LEOB**, our novel Lagrange-multiplier approach, we are able to provide a complete description, within the EOB framework and in the form of Euler-Lagrange equations, of the dynamical evolution of black hole binaries.

Crucially, LEOB avoids the need to solve the EOB mass-shell constraint for an effective Hamiltonian, at the cost of having one additional evolution equation for the energy γ.

The simplification and flexibility brought about by the LEOB approach have a notable impact in the development of PM-based EOB models, and we expect even more benefits when higher-order PM results will be released.

When applied to a complete waveform model for quasi-circular spin-aligned binaries, the LEOB approach yields good results already before any NR tuning of the dynamics. Moreover, the LEOB dynamics is flexible enough to allow for a successful NR calibration, which pushes the performance of the model at the level of the state-of-the-art PN-based EOB models

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Thanks for your attention!

Backup Slides

NR-tuned parameters

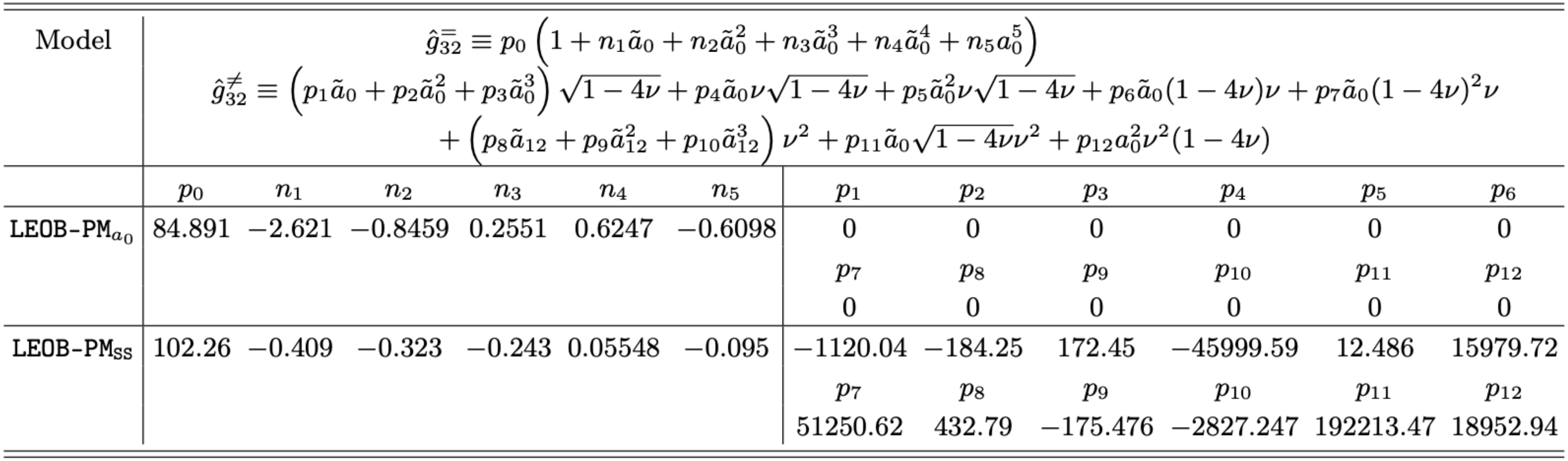
Orbital part:

$$a_{52}^{\mathrm{NR}}(\nu) = 263.55\nu - 0.171$$

Spin part:

$$\hat{g}_{32}^{\mathrm{NR}}(\nu, \tilde{a}_i) = \hat{g}_{32}^{=} + \hat{g}_{32}^{\neq}$$

$$\tilde{a}_0 = \tilde{a}_1 + \tilde{a}_2, \quad \tilde{a}_{12} = \tilde{a}_1 - \tilde{a}_2$$



Model	$\hat{g}_{32}^{=} \equiv p_0 \left(1 + n_1\tilde{a}_0 + n_2\tilde{a}_0^2 + n_3\tilde{a}_0^3 + n_4\tilde{a}_0^4 + n_5 a_0^5\right)$ $\hat{g}_{32}^{\neq} \equiv \left(p_1\tilde{a}_0 + p_2\tilde{a}_0^2 + p_3\tilde{a}_0^3\right)\sqrt{1-4\nu} + p_4\tilde{a}_0\nu\sqrt{1-4\nu} + p_5\tilde{a}_0^2\nu\sqrt{1-4\nu} + p_6\tilde{a}_0(1-4\nu)\nu + p_7\tilde{a}_0(1-4\nu)^2\nu$ $+ \left(p_8\tilde{a}_{12} + p_9\tilde{a}_{12}^2 + p_{10}\tilde{a}_{12}^3\right)\nu^2 + p_{11}\tilde{a}_0\sqrt{1-4\nu}\nu^2 + p_{12}a_0^2\nu^2(1-4\nu)$											
	p_0	n_1	n_2	n_3	n_4	n_5	p_1	p_2	p_3	p_4	p_5	p_6
`LEOB-PM`$_{a_0}$	84.891	−2.621	−0.8459	0.2551	0.6247	−0.6098	0	0	0	0	0	0
							p_7	p_8	p_9	p_{10}	p_{11}	p_{12}
							0	0	0	0	0	0
`LEOB-PM`$_{\mathrm{SS}}$	102.26	−0.409	−0.323	−0.243	0.05548	−0.095	−1120.04	−184.25	172.45	−45999.59	12.486	15979.72
							p_7	p_8	p_9	p_{10}	p_{11}	p_{12}
							51250.62	432.79	−175.476	−2827.247	192213.47	18952.94

Link between LEOB and the H-based EOB

The traditional Hamiltonian EOB dynamics is regained when using a mass-shell constraint in the explicitly solved form

$$\hat{\mathscr{C}}_H \equiv \hat{H}_{\text{eff}}(x^i, p_i) - \gamma$$

which implies

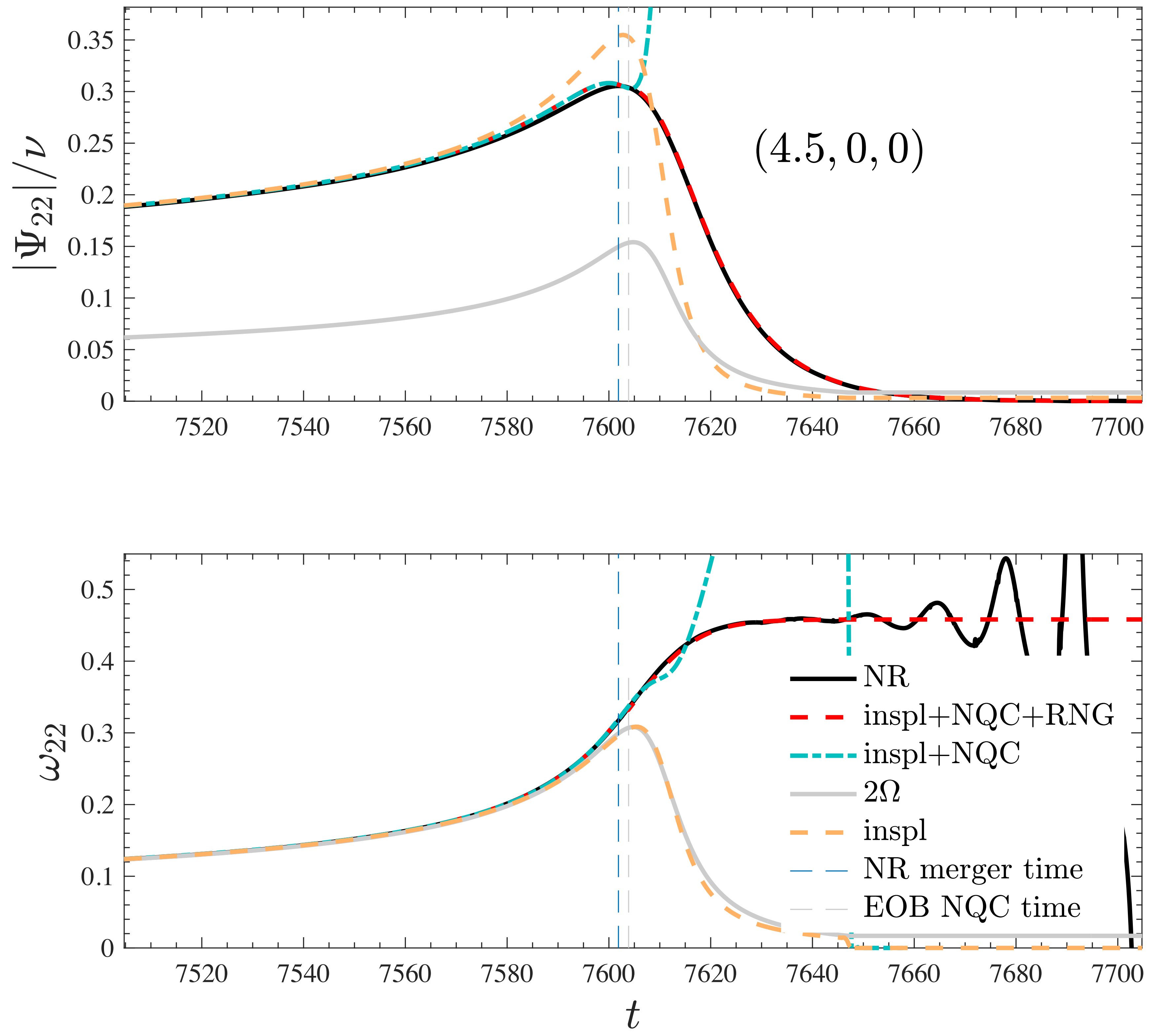
$$\frac{\partial \hat{\mathscr{C}}_H}{\partial \gamma} = -1\,, \qquad \frac{\partial \hat{\mathscr{C}}_H}{\partial p_i} = \frac{\partial \hat{H}_{\text{eff}}}{\partial p_i}\,, \qquad \frac{\partial \hat{\mathscr{C}}_H}{\partial x^i} = \frac{\partial \hat{H}_{\text{eff}}}{\partial x^i}$$

Under this conditions in fact:

$$\boxed{\frac{dx^i}{dt_{\text{real}}} = -\frac{1}{h}\left(\frac{\partial \hat{\mathscr{C}}_H}{\partial \gamma}\right)^{-1}\frac{\partial \hat{\mathscr{C}}_H}{\partial p_i}, \qquad \frac{dp_i}{dt_{\text{real}}} = \frac{1}{h}\left(\frac{\partial \hat{\mathscr{C}}_H}{\partial \gamma}\right)^{-1}\frac{\partial \hat{\mathscr{C}}_H}{\partial x^i} + \mathscr{F}_i, \qquad \frac{d\gamma}{dt_{\text{real}}} = -\mathscr{F}_0}$$

$$\boxed{\frac{dx^i}{dt_{\text{real}}} = \frac{1}{h}\frac{\partial \hat{H}_{\text{eff}}}{\partial p_i}, \qquad \frac{dp_i}{dt_{\text{real}}} = -\frac{1}{h}\frac{\partial \hat{H}_{\text{eff}}}{\partial x^i} + \mathscr{F}_i}$$

Effect of NQC corrections and ringdown



Kepler-preserving radius in LEOB

$$r_\Omega = \left\{ \frac{\left(r_c^3 \psi_c\right)^{-1/2} + \mathcal{G}}{h\left(1 - \frac{\gamma_{\rm orb}}{2}\partial_\gamma \ln A - p_\varphi \partial_\gamma \mathcal{G}\right)} \right\}^{-2/3} ,$$

with

$$\psi_c = -\frac{2}{\partial_r A}\left(\partial_r u_c + \frac{\partial_r \mathcal{G}}{u_c A}\frac{\gamma_{\rm orb}}{p_\varphi}\right) .$$