25 Years of the Effective One Body (EOB) framework

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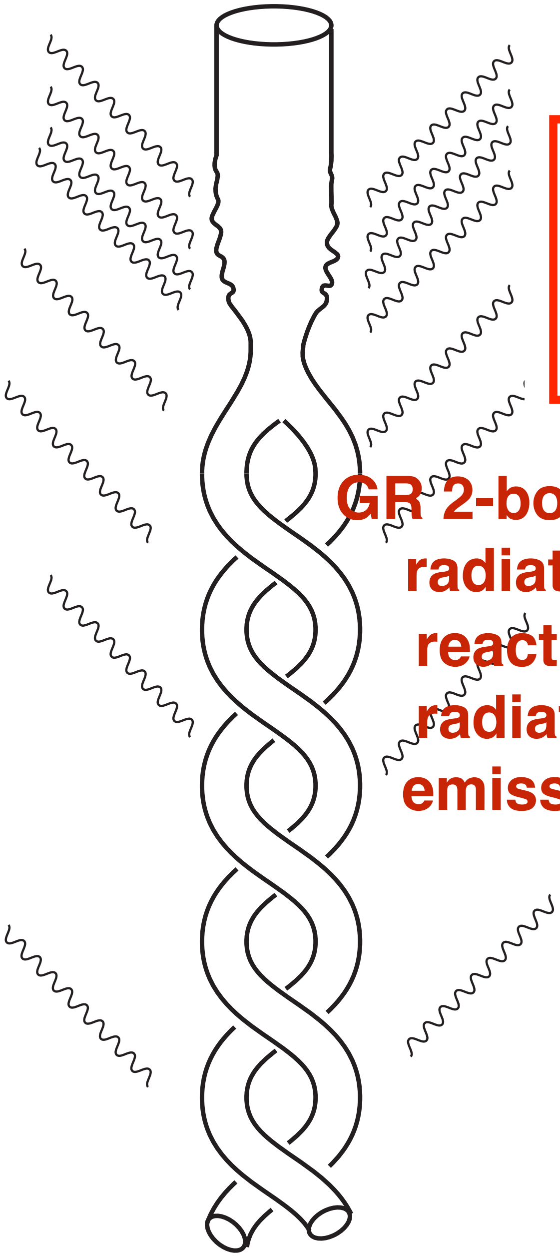
$$R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \qquad R_{\mu\nu} = 0$$

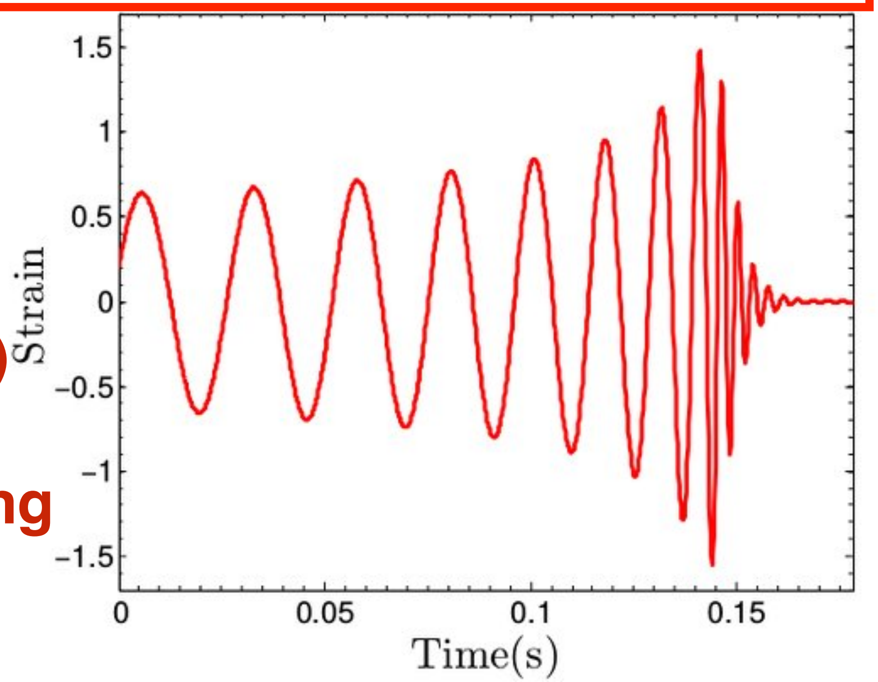
$$ds^2 = g_{\mu\nu}(x^\lambda)\, dx^\mu dx^\nu$$

$$-g^{\mu\nu} g_{\alpha\beta,\,\mu\nu} + g^{\mu\nu} g^{\rho\sigma} \Big(g_{\alpha\mu,\,\rho}\, g_{\beta\nu,\,\sigma} - g_{\alpha\mu,\,\rho}\, g_{\beta\sigma,\,\nu} + g_{\alpha\mu,\,\rho}\, g_{\nu\sigma,\,\beta} + g_{\beta\mu,\,\rho}\, g_{\nu\sigma,\,\alpha} - \frac{1}{2} g_{\mu\rho,\,\alpha}\, g_{\nu\sigma,\,\beta} \Big) = 0$$



GR 2-body pb, radiation-reaction, radiation emission.

waveform h(t) used for matched filtering



needed with ever-increasing faithfulness:

Tools used for the GR 2-body pb

Post-Newtonian (PN) approximation (**expansion in 1/c; ie v^2/c^2 and GM/(c^2r)**)

Post-Minkowskian (PM) approximation (**expansion in G; ie in GM/(c^2b)**) and its recent **Worldline EFT avatars**

Multipolar post-Minkowskian (MPM) approximation theory to the GW emission of binary systems

Matched Asymptotic Expansions useful both for the motion of strongly self-gravitating bodies, and for the nearzone-wavezone matching

Gravitational Self-Force (SF): expansion in m1/m2, with « first law of BH mechanics » (LeTiec-Blanchet-Whiting'12,…)

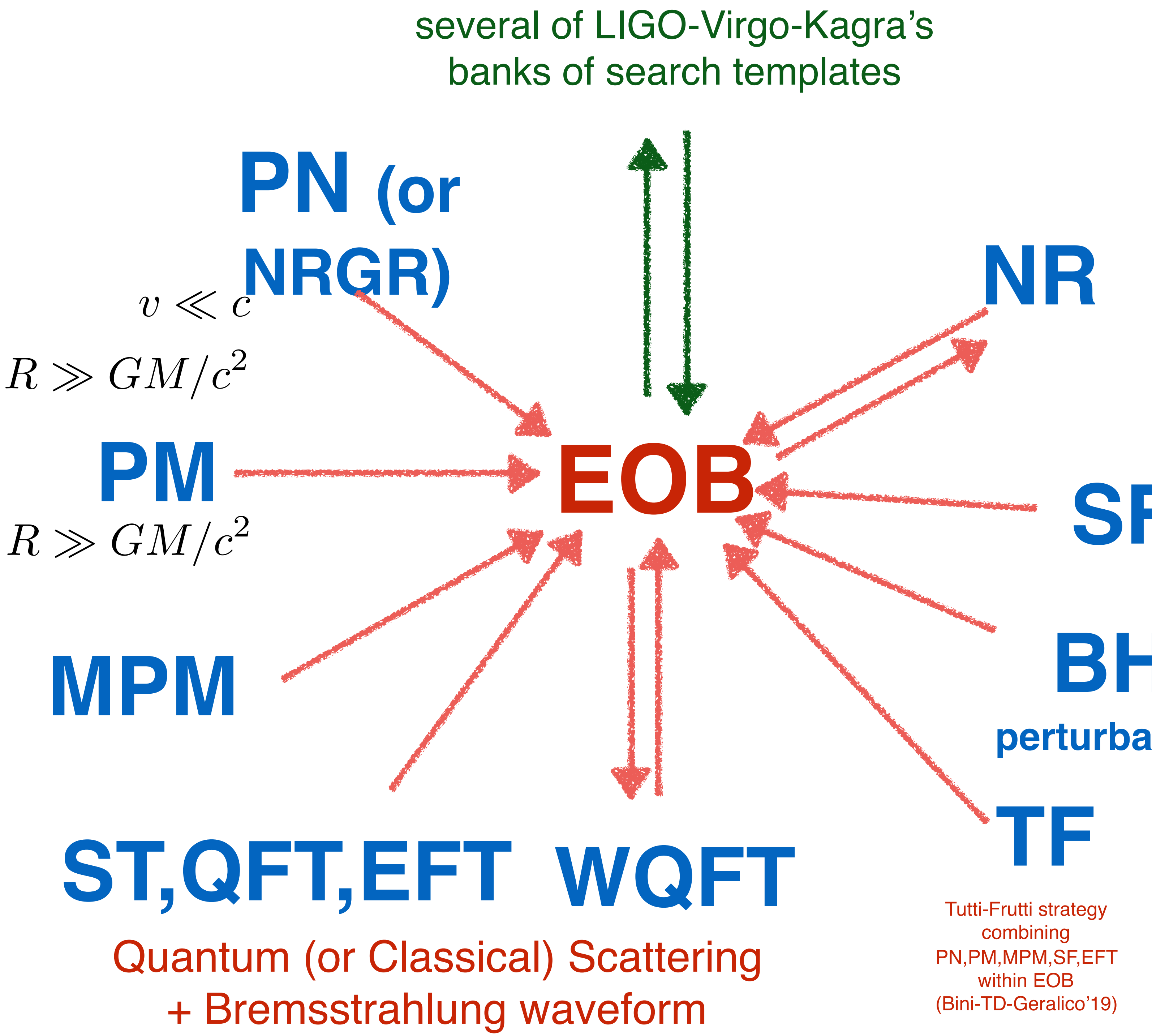
Effective One-Body (EOB) Approach

Numerical Relativity (NR)

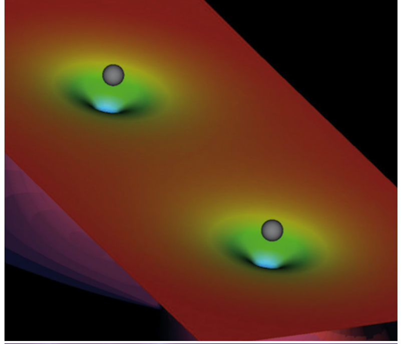
NRGR Effective Field Theory (EFT) à la Goldberger-Rothstein

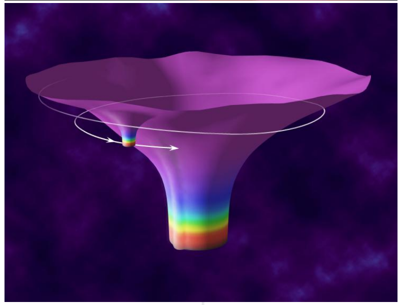
Tutti Frutti method

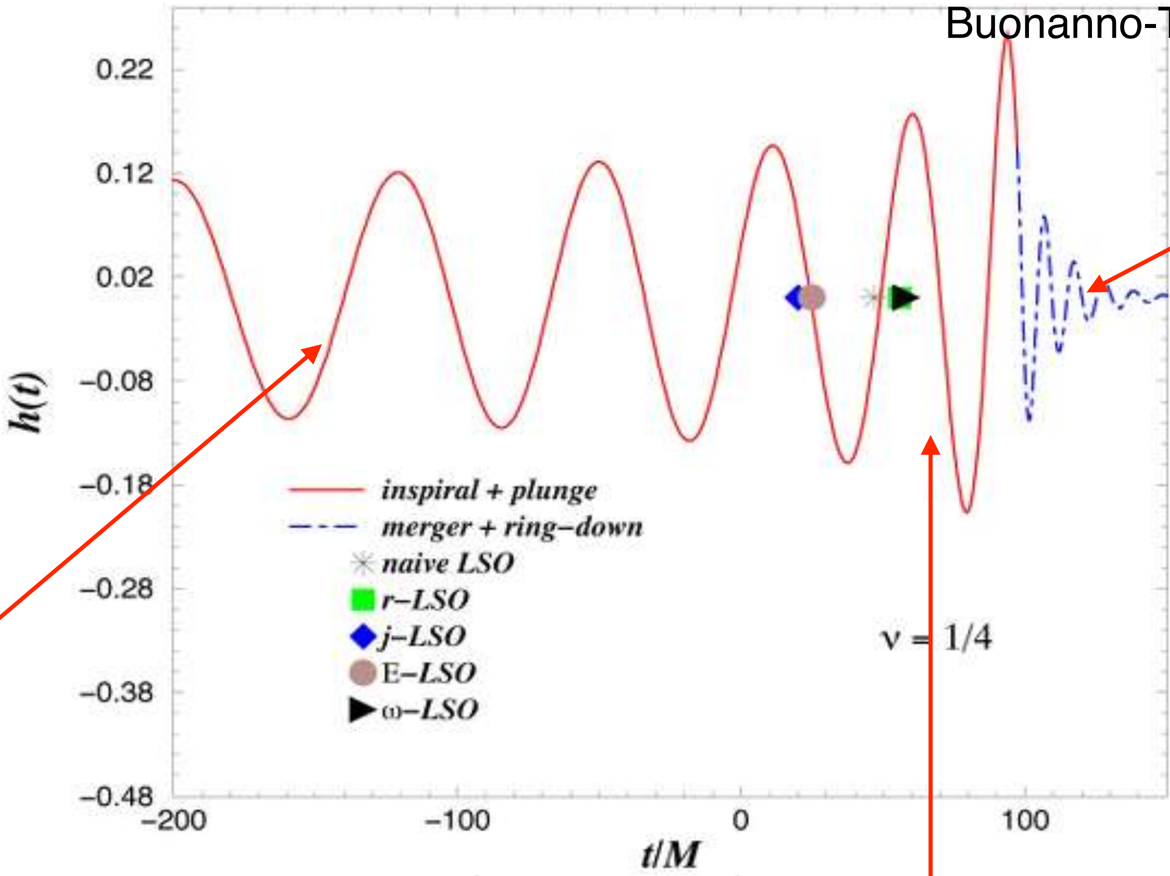
Scattering :quantum amplitude or Eikonal or various Worldline approaches aided by Double-Copy, Generalized Unitarity,« Feynman-integral Calculus » (IBP, DE, regions, reverse unitarity,…),
2 to 3 amplitude for GW generation (aided by Kosower-Maybee-O'Connell)

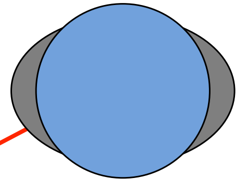

several of LIGO-Virgo-Kagra's
banks of search templates
PN (or NRGR)
$v \ll c$
$R \gg GM/c^2$
PM
$R \gg GM/c^2$
MPM
EOB
NR
$v \sim c$
$R \sim GM/c^2$
SF
BH perturbation
$m_1 \ll m_2$
TF
ST,QFT,EFT
WQFT
Quantum (or Classical) Scattering
+ Bremsstrahlung waveform
Tutti-Frutti strategy
combining
PN,PM,MPM,SF,EFT
within EOB
(Bini-TD-Geralico'19)

The Effective One-Body (EOB) approach to the GW signal emitted by the Merger of two Black Holes

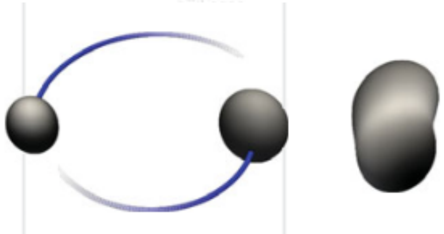








Ringdown (BBH): « vibration modes » of final BH (QNM); perturbation of BHs à la Regge-Wheeler-Zerilli-Teukolsky +Vishveshwara



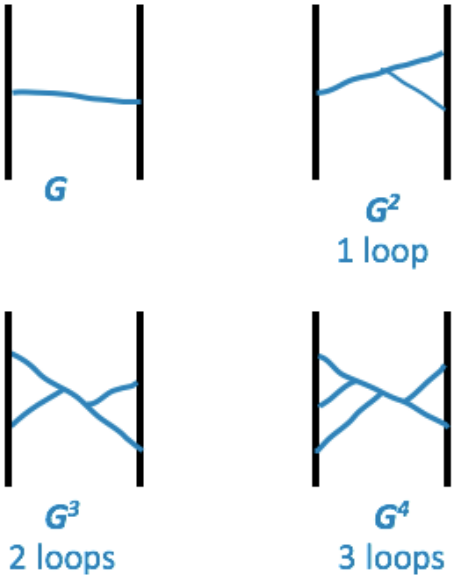
Inspiral: perturbative computation of higher-order contributions to E=H and F (expansion in v^2/c^2 + tidal polarizability of NS)

Late inspiral, « plunge » and merger:
first estimated by the Effective One-Body method (AB-TD 2000) later confirmed and improved by using numerical simulations (Pretorius…2005)

Effective One-Body (EOB) approach: H + Rad-Reac Force

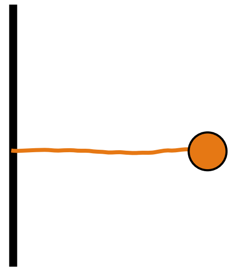
Historically rooted in QM: Brezin-Itzykson-ZinnJustin'70
eikonal scattering amplitude+ Wheeler's:`Think quantum mechanically'

Real 2-body system
(in the c.o.m. frame)



An effective particle of mass mu in some effective metric

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$



$$g_{\rm eff}^{\mu\nu}(X)$$

1:1 map

mass-shell constraint

$$0 = g_{\rm eff}^{\mu\nu}(X) P_\mu P_\nu + \mu^2 + Q(X, P)$$

Level correspondence
in the semi-classical limit:
Bohr-Sommerfeld ->
identification of
quantized action variables

$$J = \ell\hbar = \frac{1}{2\pi}\oint p_\varphi d\varphi$$
$$N = n\hbar = I_r + J$$
$$I_r = \frac{1}{2\pi}\oint p_r dr$$

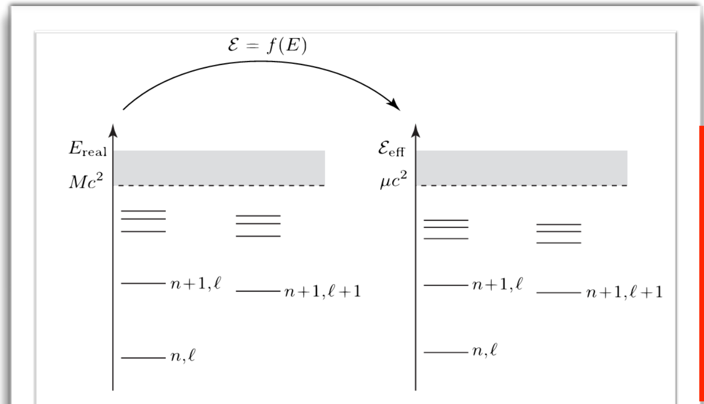


Figure 1: Sketch of the correspondence between the quantized energy levels of the real and effective conservative dynamics. n denotes the 'principal quantum

Crucial energy map

$$\mathcal{E}_{\rm eff} = \frac{(\mathcal{E}_{\rm real})^2 - m_1^2 - m_2^2}{2(m_1 + m_2)}$$

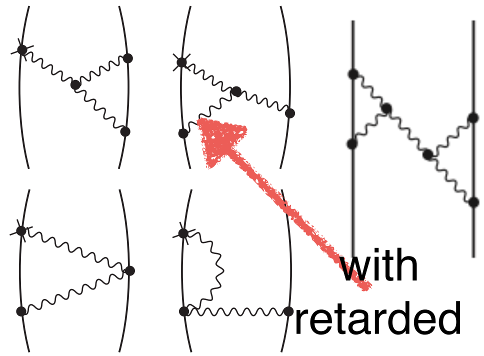
as functions of I_r and I_phi=J

State of the art for PN dynamics

- 1PN (including v^2/c^2) [Lorentz-Droste '17], Einstein-Infeld-Hoffmann '38
- 2PN (inc. v^4/c^4) Ohta-Okamura-Kimura-Hiida '74, Damour-Deruelle '81
  Damour '82, Schäfer '85, Kopeikin '85
- 2.5 PN (inc. v^5/c^5) Damour-Deruelle '81, Damour '82, Schäfer '85,
  LO-radiation-reaction Kopeikin '85

First complete 2PN and 2.5PN dynamics obtained by using 2PM (G^2) EOM of Bel et al.'81

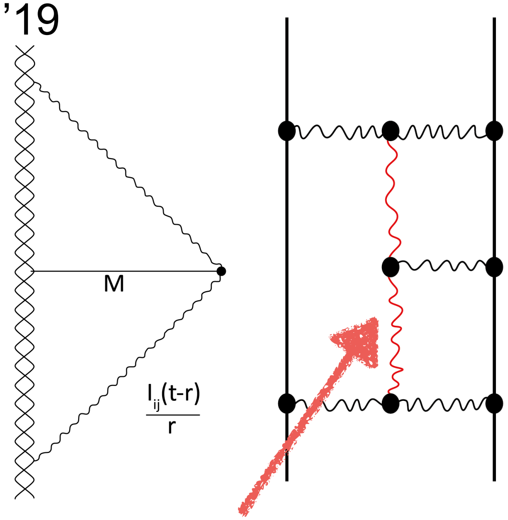
- 3 PN (inc. v^6/c^6) Jaranowski-Schäfer '98, Blanchet-Faye '00,
  Damour-Jaranowski-Schäfer '01, Itoh-Futamase '03,
  Blanchet-Damour-Esposito-Farèse' 04, Foffa-Sturani '11
- 3.5 PN (inc. v^7/c^7) Iyer-Will '93, Jaranowski-Schäfer '97, Pati-Will '02,
  Königsdörffer-Faye-Schäfer '03, Nissanke-Blanchet '05, Itoh '09



with retarded propagator

- **4PN** (inc. v^8/c^8) Jaranowski-Schäfer '13, Foffa-Sturani '13,'16
  Bini-Damour '13, Damour-Jaranowski-Schäfer '14, Marchand+'18, Foffa+'19

New feature at G^4/c^8 (4PN and 4PM) : **non-locality in time** (linked to IR divergences of formal PN-expansion) (Blanchet,TD '88)



- **5PN** (inc. v^{10}/c^{10} and **G^6**) Bini-Damour-Geralico'19: complete **modulo two**
- **numerical** parameters; Bluemlein et al'21: potential-graviton contrib. and
- partial determination of radiation-graviton contrib.

- **6PN** (inc. v^{12}/c^{12} and **G^7**) Bini-Damour-Geralico'20: complete **modulo four**
- additional parameters

soft (radiation) gravitons

Inclusion of **spin-dependent effects**: Barker-O' Connell'75, Faye-Blanchet-Buonanno'06, Damour-Jaranowski-Schaefer'08, Porto-Rothstein '06, Levi '10, Steinhoff-Hergt-Schaefer '10, Steinhoff'11, Levi-Steinhoff'15-18, Bini-TD, Vines , Guevara-Ochirov-Vines,….

2-body Taylor-expanded 4PN Hamiltonian [DJS, 2014, JS 2015]

$$
\begin{aligned}
c^8 H_{4\mathrm{PN}}^{\mathrm{local}}(\mathbf{x}_a,\mathbf{p}_a) = {}& \frac{7(\mathbf{p}_1^2)^5}{256m_1^9} + \frac{Gm_1m_2}{r_{12}} H_{48}(\mathbf{x}_a,\mathbf{p}_a) + \frac{G^2m_1m_2}{r_{12}^2} m_1 H_{46}(\mathbf{x}_a,\mathbf{p}_a) \\
&+ \frac{G^3m_1m_2}{r_{12}^3}\left(m_1^2 H_{441}(\mathbf{x}_a,\mathbf{p}_a) + m_1m_2 H_{442}(\mathbf{x}_a,\mathbf{p}_a)\right) \\
&+ \frac{G^4m_1m_2}{r_{12}^4}\left(m_1^3 H_{421}(\mathbf{x}_a,\mathbf{p}_a) + m_1^2m_2 H_{422}(\mathbf{x}_a,\mathbf{p}_a)\right) \\
&+ \frac{G^5m_1m_2}{r_{12}^5} H_{40}(\mathbf{x}_a,\mathbf{p}_a) + (1 \leftrightarrow 2), \qquad \text{(A3)}
\end{aligned}
$$

$$
\begin{aligned}
H_{48}(\mathbf{x}_a,\mathbf{p}_a) = {}& \frac{45(\mathbf{p}_1^2)^4}{128m_1^8} - \frac{9(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1^2)^2}{64m_1^6m_2^2} + \frac{15(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1^2)^3}{64m_1^6m_2^2} - \frac{9(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1^2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{16m_1^6m_2^2} \\
&- \frac{3(\mathbf{p}_1^2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{32m_1^6m_2^2} + \frac{15(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1^2)^2\mathbf{p}_2^2}{64m_1^6m_2^2} - \frac{21(\mathbf{p}_1^2)^3\mathbf{p}_2^2}{64m_1^6m_2^2} - \frac{35(\mathbf{n}_{12}\cdot\mathbf{p}_1)^5(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3}{256m_1^5m_2^3} \\
&+ \frac{25(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3\mathbf{p}_1^2}{128m_1^5m_2^3} + \frac{33(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3(\mathbf{p}_1^2)^2}{256m_1^5m_2^3} - \frac{85(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{256m_1^5m_2^3} \\
&- \frac{45(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{128m_1^5m_2^3} - \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1^2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{256m_1^5m_2^3} + \frac{25(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{64m_1^5m_2^3} \\
&+ \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{64m_1^5m_2^3} - \frac{3(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1\cdot\mathbf{p}_2)^3}{64m_1^5m_2^3} + \frac{3\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)^3}{64m_1^5m_2^3} + \frac{55(\mathbf{n}_{12}\cdot\mathbf{p}_1)^5(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_2^2}{256m_1^5m_2^3} \\
&- \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2\mathbf{p}_2^2}{128m_1^5m_2^3} - \frac{25(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1^2)^2\mathbf{p}_2^2}{256m_1^5m_2^3} - \frac{23(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{256m_1^5m_2^3} \\
&+ \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{128m_1^5m_2^3} - \frac{7(\mathbf{p}_1^2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{256m_1^5m_2^3} - \frac{5(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^4\mathbf{p}_1^2}{64m_1^4m_2^4} + \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_2)^4(\mathbf{p}_1^2)^2}{64m_1^4m_2^4} \\
&- \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{4m_1^4m_2^4} + \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{16m_1^4m_2^4} - \frac{5(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_2^2}{64m_1^4m_2^4} + \frac{21(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2\mathbf{p}_2^2}{64m_1^4m_2^4} \\
&- \frac{3(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1^2)^2\mathbf{p}_2^2}{32m_1^4m_2^4} - \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{4m_1^4m_2^4} + \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{16m_1^4m_2^4} + \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2\mathbf{p}_2^2}{16m_1^4m_2^4} \\
&- \frac{\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2\mathbf{p}_2^2}{32m_1^4m_2^4} + \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4(\mathbf{p}_2^2)^2}{64m_1^4m_2^4} - \frac{3(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\mathbf{p}_1^2(\mathbf{p}_2^2)^2}{32m_1^4m_2^4} - \frac{7(\mathbf{p}_1^2)^2(\mathbf{p}_2^2)^2}{128m_1^4m_2^4}, \qquad \text{(A4a)}
\end{aligned}
$$

$$
\begin{aligned}
H_{46}(\mathbf{x}_a,\mathbf{p}_a) = {}& \frac{369(\mathbf{n}_{12}\cdot\mathbf{p}_1)^6}{160m_1^6} - \frac{889(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4\mathbf{p}_1^2}{192m_1^6} + \frac{49(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1^2)^2}{16m_1^6} - \frac{63(\mathbf{p}_1^2)^3}{64m_1^6} - \frac{549(\mathbf{n}_{12}\cdot\mathbf{p}_1)^5(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{128m_1^5m_2} \\
&+ \frac{67(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2}{16m_1^5m_2} - \frac{167(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1^2)^2}{128m_1^5m_2} + \frac{1547(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4(\mathbf{p}_1\cdot\mathbf{p}_2)}{256m_1^5m_2} - \frac{851(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{128m_1^5m_2} \\
&+ \frac{1099(\mathbf{p}_1^2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{256m_1^5m_2} + \frac{3263(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{1280m_1^4m_2^2} + \frac{1067(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2}{480m_1^4m_2^2} - \frac{4567(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1^2)^2}{3840m_1^4m_2^2} \\
&- \frac{3571(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)}{320m_1^4m_2^2} + \frac{3073(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{480m_1^4m_2^2} + \frac{4349(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{1280m_1^4m_2^2} \\
&- \frac{3461\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{3840m_1^4m_2^2} + \frac{1673(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4\mathbf{p}_2^2}{1920m_1^4m_2^2} - \frac{1999(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\mathbf{p}_1^2\mathbf{p}_2^2}{3840m_1^4m_2^2} + \frac{2081(\mathbf{p}_1^2)^2\mathbf{p}_2^2}{3840m_1^4m_2^2} - \frac{13(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3}{8m_1^3m_2^3} \\
&+ \frac{191(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3\mathbf{p}_1^2}{192m_1^3m_2^3} - \frac{19(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{384m_1^3m_2^3} - \frac{5(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{384m_1^3m_2^3} \\
&+ \frac{11(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{192m_1^3m_2^3} + \frac{77(\mathbf{p}_1\cdot\mathbf{p}_2)^3}{96m_1^3m_2^3} + \frac{233(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_2^2}{96m_1^3m_2^3} - \frac{47(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2\mathbf{p}_2^2}{32m_1^3m_2^3} \\
&+ \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{384m_1^3m_2^3} - \frac{185\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{384m_1^3m_2^3} - \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^4}{4m_1^2m_2^4} + \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_2)^4\mathbf{p}_1^2}{4m_1^2m_2^4} \\
&- \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3(\mathbf{p}_1\cdot\mathbf{p}_2)}{2m_1^2m_2^4} + \frac{21(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{16m_1^2m_2^4} + \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_2^2}{6m_1^2m_2^4} + \frac{49(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2\mathbf{p}_2^2}{48m_1^2m_2^4} \\
&- \frac{133(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{24m_1^2m_2^4} - \frac{77(\mathbf{p}_1\cdot\mathbf{p}_2)^2\mathbf{p}_2^2}{96m_1^2m_2^4} + \frac{197(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_2^2)^2}{96m_1^2m_2^4} - \frac{173\mathbf{p}_1^2(\mathbf{p}_2^2)^2}{48m_1^2m_2^4} + \frac{13(\mathbf{p}_2^2)^3}{8m_2^6}, \qquad \text{(A4b)}
\end{aligned}
$$

$$
\begin{aligned}
H_{441}(\mathbf{x}_a,\mathbf{p}_a) = {}& \frac{5027(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4}{384m_1^4} - \frac{22993(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\mathbf{p}_1^2}{960m_1^4} - \frac{6695(\mathbf{p}_1^2)^2}{1152m_1^4} - \frac{3191(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{640m_1^3m_2} \\
&+ \frac{28561(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2}{1920m_1^3m_2} + \frac{8777(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{384m_1^3m_2} + \frac{752969\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{28800m_1^3m_2} \\
&- \frac{16481(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{960m_1^2m_2^2} + \frac{94433(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2}{4800m_1^2m_2^2} - \frac{103957(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)}{2400m_1^2m_2^2} \\
&+ \frac{791(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{400m_1^2m_2^2} + \frac{26627(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\mathbf{p}_2^2}{1600m_1^2m_2^2} - \frac{118261\mathbf{p}_1^2\mathbf{p}_2^2}{4800m_1^2m_2^2} + \frac{105(\mathbf{p}_2^2)^2}{32m_2^4}, \qquad \text{(A4c)}
\end{aligned}
$$

$$
\begin{aligned}
H_{442}(\mathbf{x}_a,\mathbf{p}_a) = {}& \left(\frac{2749\pi^2}{8192} - \frac{211189}{19200}\right)\frac{(\mathbf{p}_1^2)^2}{m_1^4} + \left(\frac{63347}{1600} - \frac{1059\pi^2}{1024}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\mathbf{p}_1^2}{m_1^4} + \left(\frac{375\pi^2}{8192} - \frac{23533}{1280}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4}{m_1^4} \\
&+ \left(\frac{10631\pi^2}{8192} - \frac{1918349}{57600}\right)\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{m_1^2m_2^2} + \left(\frac{13723\pi^2}{16384} - \frac{2492417}{57600}\right)\frac{\mathbf{p}_1^2\mathbf{p}_2^2}{m_1^2m_2^2} \\
&+ \left(\frac{1411429}{19200} - \frac{1059\pi^2}{512}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2}{m_1^2m_2^2} + \left(\frac{248991}{6400} - \frac{6153\pi^2}{2048}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1^2m_2^2} \\
&- \left(\frac{30383}{960} + \frac{36405\pi^2}{16384}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^2m_2^2} + \left(\frac{1243717}{14400} - \frac{40483\pi^2}{16384}\right)\frac{\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1^3m_2} \\
&+ \left(\frac{2369}{60} + \frac{35655\pi^2}{16384}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^3m_2} + \left(\frac{43101\pi^2}{16384} - \frac{391711}{6400}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2}{m_1^3m_2} \\
&+ \left(\frac{56955\pi^2}{16384} - \frac{1646983}{19200}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1^3m_2}, \qquad \text{(A4d)}
\end{aligned}
$$

$$
H_{421}(\mathbf{x}_a,\mathbf{p}_a) = \frac{64861\mathbf{p}_1^2}{4800m_1^2} - \frac{91(\mathbf{p}_1\cdot\mathbf{p}_2)}{8m_1m_2} + \frac{105\mathbf{p}_2^2}{32m_2^2} - \frac{9841(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2}{1600m_1^2} - \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{2m_1m_2}, \qquad \text{(A4e)}
$$

$$
\begin{aligned}
H_{422}(\mathbf{x}_a,\mathbf{p}_a) = {}& \left(\frac{1937033}{57600} - \frac{199177\pi^2}{49152}\right)\frac{\mathbf{p}_1^2}{m_1^2} + \left(\frac{176033\pi^2}{24576} - \frac{2864917}{57600}\right)\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1m_2} + \left(\frac{282361}{19200} - \frac{21837\pi^2}{8192}\right)\frac{\mathbf{p}_2^2}{m_2^2} \\
&+ \left(\frac{698723}{19200} + \frac{21745\pi^2}{16384}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2}{m_1^2} + \left(\frac{63641\pi^2}{24576} - \frac{2712013}{19200}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1m_2} \\
&+ \left(\frac{3200179}{57600} - \frac{28691\pi^2}{24576}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_2^2}, \qquad \text{(A4f)}
\end{aligned}
$$

$$
H_{40}(\mathbf{x}_a,\mathbf{p}_a) = -\frac{m_1^4}{16} + \left(\frac{6237\pi^2}{1024} - \frac{169799}{2400}\right)m_1^3m_2 + \left(\frac{44825\pi^2}{6144} - \frac{609427}{7200}\right)m_1^2m_2^2. \qquad \text{(A4g)}
$$

$$
\begin{aligned}
H_{4\mathrm{PN}}^{\mathrm{nonloc}}(t) = {}& -\frac{1}{5}\frac{G^2M}{c^8} I_{ij}^{(3)}(t) \\
&\times \mathrm{Pf}_{2r_{12}/c}\int_{-\infty}^{+\infty}\frac{\mathrm{d}v}{|v|} I_{ij}^{(3)}(t+v),
\end{aligned}
$$

nonlocal in time

Explicit 4PN EOB (non-spinning) dynamics (Damour-Jaranowski-Schaefer '14)

A **simple**, but crucial transformation between the real energy and the effective one:

$$\mathcal{E}_{\rm eff} = \frac{(\mathcal{E}_{\rm real})^2 - m_1^2 - m_2^2}{2(m_1+m_2)} .$$

A **simple(gauge-fixed) post-geodesic** effective

$$g_{\rm eff}^{\mu\nu} P'_\mu P'_\nu + \mu^2 c^2 + Q(P'_\mu) = 0 ,$$

$$M = m_1 + m_2 , \qquad \mu = \frac{m_1 m_2}{m_1+m_2} , \qquad \nu = \frac{m_1 m_2}{(m_1+m_2)^2}$$

$$ds^2_{\rm eff} = -A(R;\nu)dt^2 + B(R;\nu)dR^2 + R^2(d\theta^2 + \sin^2\theta d\phi^2)$$

Padé resummed

$$a_4 = \frac{94}{3} - \frac{41\pi^2}{32} \,;\; a_5^c = \frac{2275\pi^2}{512} + \ldots \,;\; a_5' = \ldots$$

$$u \equiv \frac{GM}{R c^2}$$

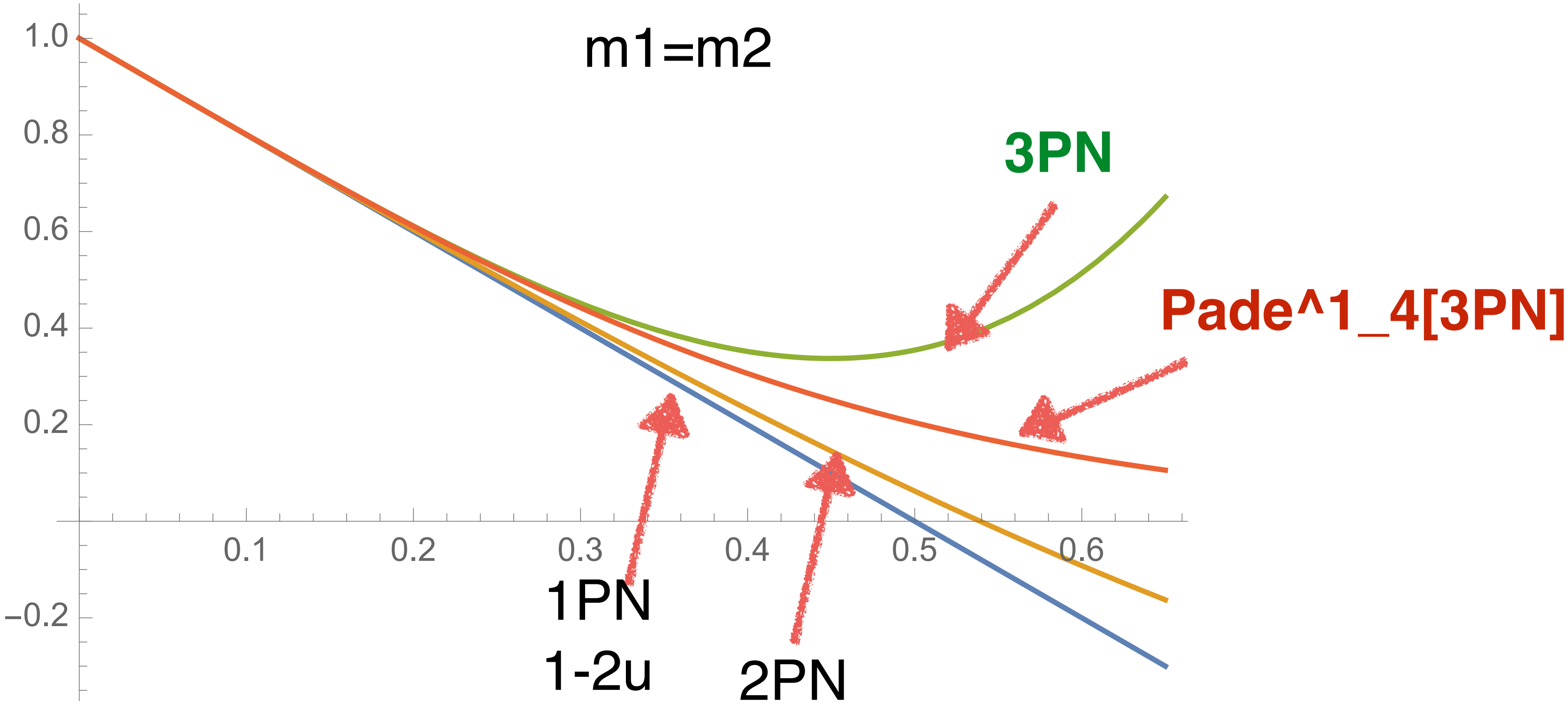
$$A^{\rm PN}(u;\nu) = 1 - 2u + 2\nu u^3 + \nu a_4 u^4 + (\nu a_5^c + \nu^2 a_5' + \frac{64}{5}\nu \ln u) u^5$$

$$(AB)^{-1} = \bar{D}(u) = 1 + 6\nu u^2 + (52\nu - 6\nu^2)u^3 + \left(\left(-\frac{533}{45} - \frac{23761\pi^2}{1536} + \frac{1184}{15}\gamma_E - \frac{6496}{15}\ln 2 + \frac{2916}{5}\ln 3\right)\nu + \left(\frac{123\pi^2}{16} - 260\right)\nu^2 + \frac{592}{15}\nu \ln u\right)u^4,$$

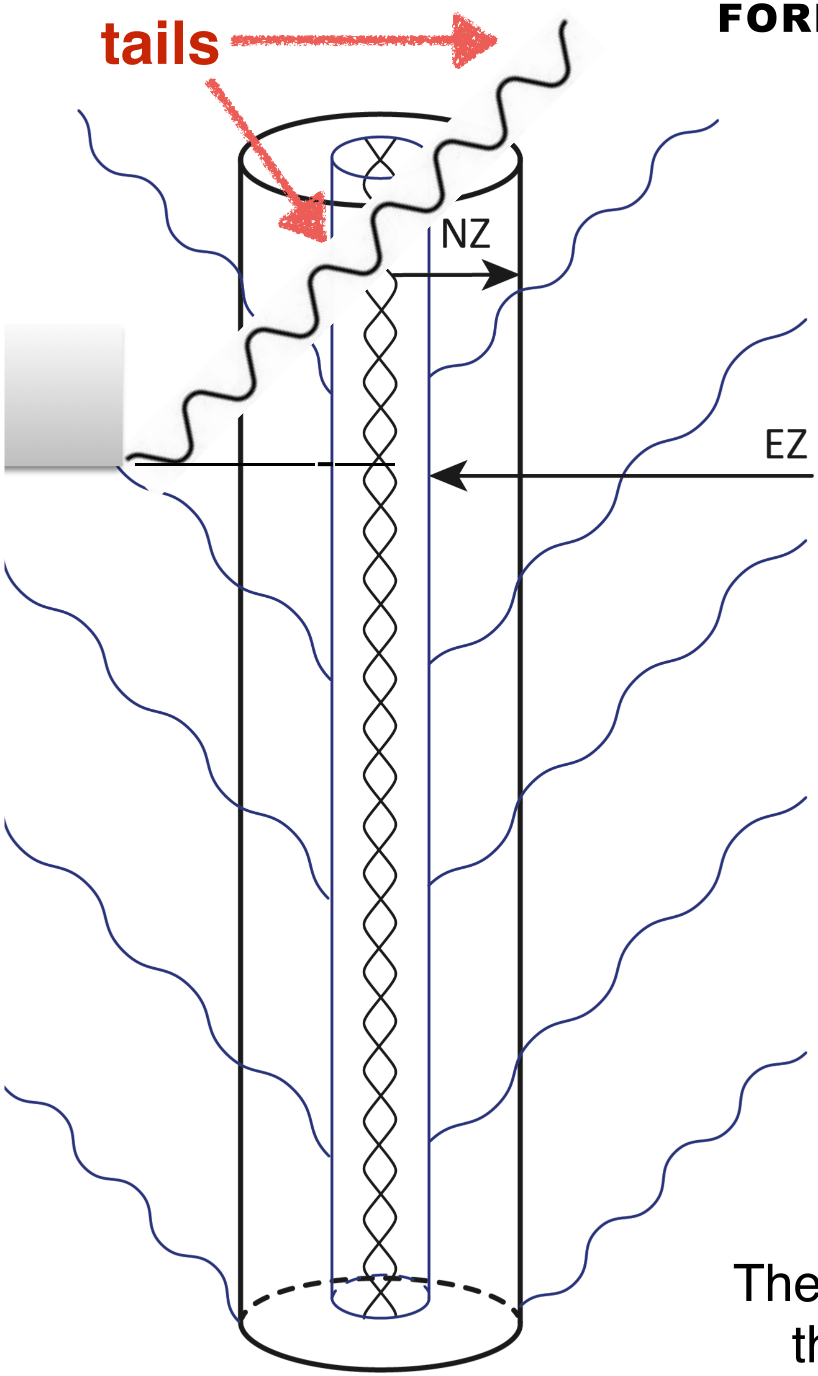
$$\hat{Q}(\mathbf{r}',\mathbf{p}') = \left(2(4-3\nu)\nu u^2 + \left(\left(-\frac{5308}{15} + \frac{496256}{45}\ln 2 - \frac{33048}{5}\ln 3\right)\nu - 83\nu^2 + 10\nu^3\right)u^3\right)(\mathbf{n}'\cdot\mathbf{p}')^4 + \left(\left(-\frac{827}{3} - \frac{2358912}{25}\ln 2 + \frac{1399437}{50}\ln 3 + \frac{390625}{18}\ln 5\right)\nu - \frac{27}{5}\nu^2 + 6\nu^3\right)u^2(\mathbf{n}'\cdot\mathbf{p}')^6 + \mathcal{O}[\nu u(\mathbf{n}'\cdot\mathbf{p}')^8].$$

only gauge-invariant information

Resummed A(u) potential



GRAVITATIONAL WAVE GENERATION: MULTIPOLAR POST-MINKOWSKIAN FORMALISM (BLANCHET-DAMOUR-IYER)



Decomposition of space-time in various overlapping regions:

1. **near-zone: r << lambda : PN**
2. **exterior zone: r >> r_source: MPM**
3. **far wave-zone: Bondi-type expansion**

then matching between the zones

in exterior zone, iterative solution of Einstein's vacuum field equations by means of a double expansion in non-linearity and in multipoles, with crucial use of analytic continuation (complex B) for dealing with formal UV divergences at r=0

$$g = \eta + G h_1 + G^2 h_2 + G^3 h_3 + ...,$$
$$\Box h_1 = 0,$$
$$\Box h_2 = \partial\partial h_1 h_1,$$
$$\Box h_3 = \partial\partial h_1 h_1 h_1 + \partial\partial h_1 h_2,$$
$$h_1 = \sum_\ell \partial_{i_1 i_2 \dots i_\ell} \left(\frac{M_{i_1 i_2 \dots i_\ell}(t - r/c)}{r} \right) + \partial\partial\dots\partial \left(\frac{\epsilon_{j_1 j_2 k} S_{k j_3 \dots j_\ell}(t - r/c)}{r} \right),$$
$$h_2 = FP_B \Box^{-1}_{\rm ret} \left(\left(\frac{r}{r_0} \right)^B \partial\partial h_1 h_1 \right) + ...,$$
$$h_3 = FP_B \Box^{-1}_{\rm ret} \dots$$

STF tensors encoding multipole moments

mass-type and spin-type multipole moments

The PN-matched MPM formalism has allowed to compute the GW emission to very high accuracy (Blanchet et al)

Perturbative computation of GW flux from binary system

- lowest order : Einstein 1918 Peters-Mathews 63
- 1 + (v^2/c^2) : Wagoner-Will 76
- … + (v^3/c^3) : Blanchet-Damour 92, Wiseman 93
- … + (v^4/c^4) : Blanchet-Damour-Iyer Will-Wiseman 95
- … + (v^5/c^5) : Blanchet 96
- … + (v^6/c^6) : Blanchet-Damour-Esposito-Farèse-Iyer 2004
- … + (v^7/c^7) : Blanchet
- … + (v^8/c^8) + (v^9/c^9) : Blanchet et al 2023

$$\nu = \frac{m_1 m_2}{(m_1+m_2)^2}$$

$$x = \left(\frac{v}{c}\right)^2 = \left(\frac{G(m_1+m_2)\Omega}{c^3}\right)^{\frac{2}{3}} = \left(\frac{\pi G(m_1+m_2)f}{c^3}\right)^{\frac{2}{3}}$$

$$\begin{aligned}
\mathcal{F} = \frac{32c^5}{5G}\nu^2 x^5 \Bigg\{ & 1 + \left(-\frac{1247}{336} - \frac{35}{12}\nu\right)x + 4\pi x^{3/2} + \left(-\frac{44711}{9072} + \frac{9271}{504}\nu + \frac{65}{18}\nu^2\right)x^2 + \left(-\frac{8191}{672} - \frac{583}{24}\nu\right)\pi x^{5/2} \\
& + \left[\frac{6643739519}{69854400} + \frac{16}{3}\pi^2 - \frac{1712}{105}\gamma_\mathrm{E} - \frac{856}{105}\ln(16\,x) + \left(-\frac{134543}{7776} + \frac{41}{48}\pi^2\right)\nu - \frac{94403}{3024}\nu^2 - \frac{775}{324}\nu^3\right]x^3 \\
& + \left(-\frac{16285}{504} + \frac{214745}{1728}\nu + \frac{193385}{3024}\nu^2\right)\pi x^{7/2} \\
& + \Bigg[-\frac{323105549467}{3178375200} + \frac{232597}{4410}\gamma_\mathrm{E} - \frac{1369}{126}\pi^2 + \frac{39931}{294}\ln 2 - \frac{47385}{1568}\ln 3 + \frac{232597}{8820}\ln x \\
& \quad + \left(-\frac{1452202403629}{1466942400} + \frac{41478}{245}\gamma_\mathrm{E} - \frac{267127}{4608}\pi^2 + \frac{479062}{2205}\ln 2 + \frac{47385}{392}\ln 3 + \frac{20739}{245}\ln x\right)\nu \\
& \quad + \left(\frac{1607125}{6804} - \frac{3157}{384}\pi^2\right)\nu^2 + \frac{6875}{504}\nu^3 + \frac{5}{6}\nu^4\Bigg]x^4 \\
& + \Bigg[\frac{265978667519}{745113600} - \frac{6848}{105}\gamma_\mathrm{E} - \frac{3424}{105}\ln(16\,x) + \left(\frac{2062241}{22176} + \frac{41}{12}\pi^2\right)\nu \\
& \quad - \frac{133112905}{290304}\nu^2 - \frac{3719141}{38016}\nu^3\Bigg]\pi x^{9/2} + \mathcal{O}(x^5)\Bigg\}. \qquad (4)
\end{aligned}$$

LO quadrupole radiation

4PN

4.5PN

Resummed EOB waveform

Damour-Nagar 2007, Damour-Iyer-Nagar 2008

$$h_{\ell m} \equiv h^{(N,\epsilon)}_{\ell m} \hat{h}^{(\epsilon)}_{\ell m} \hat{h}^{\rm NQC}_{\ell m}$$

$$\hat{h}^{(\epsilon)}_{\ell m} = \hat{S}^{(\epsilon)}_{\rm eff} T_{\ell m} e^{i\delta_{\ell m}} \rho^{\ell}_{\ell m}$$

$$T_{\ell m} = \frac{\Gamma(\ell + 1 - 2i\hat{\hat{k}})}{\Gamma(\ell+1)} e^{\pi \hat{\hat{k}}} e^{2i\hat{\hat{k}} \ln(2kr_0)}$$

resums an infinite # of leading logs

$$\begin{aligned}\rho_{22}(x;\nu) = 1 &+ \left(\frac{55\nu}{84} - \frac{43}{42}\right) x + \left(\frac{19583\nu^2}{42336} - \frac{33025\nu}{21168} - \frac{20555}{10584}\right) x^2 \\ &+ \left(\frac{10620745\nu^3}{39118464} - \frac{6292061\nu^2}{3259872} + \frac{41\pi^2\nu}{192} - \frac{48993925\nu}{9779616} - \frac{428}{105}\mathrm{eulerlog}_2(x) + \frac{1556919113}{122245200}\right) x^3 \\ &+ \left(\frac{9202}{2205}\mathrm{eulerlog}_2(x) - \frac{387216563023}{160190110080}\right) x^4 + \left(\frac{439877}{55566}\mathrm{eulerlog}_2(x) - \frac{16094530514677}{533967033600}\right) x^5 + \mathcal{O}(x^6),\end{aligned}$$

$$\mathcal{F}_{\varphi} \equiv -\frac{1}{8\pi\Omega} \sum_{\ell=2}^{\ell_{\max}} \sum_{m=1}^{\ell} (m\Omega)^2 |R h^{(\epsilon)}_{\ell m}|^2$$

Recent developments: Cipriani+25, Ivanov+25

EOB

$$\frac{dr}{dt} = \left(\frac{A}{B}\right)^{1/2} \frac{\partial \hat{H}_{\rm EOB}}{\partial p_{r_*}},$$

$$\frac{dp_{r_*}}{dt} = -\left(\frac{A}{B}\right)^{1/2} \frac{\partial \hat{H}_{\rm EOB}}{\partial r},$$

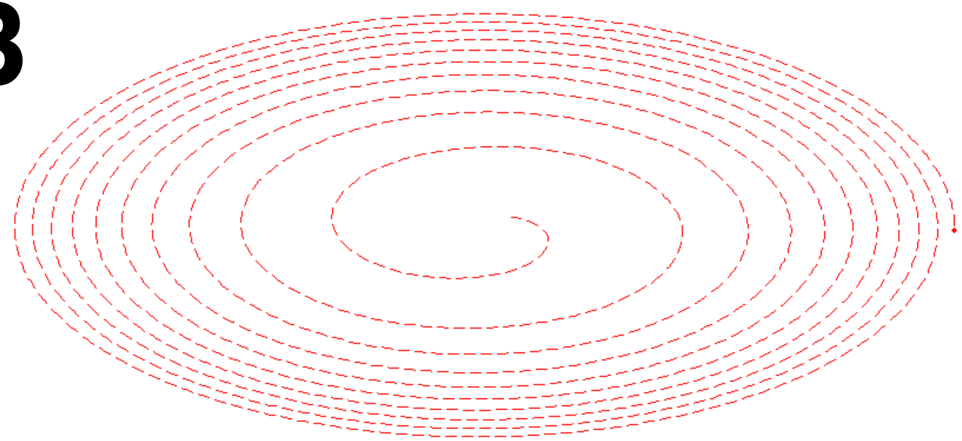
$$\Omega \equiv \frac{d\varphi}{dt} = \frac{\partial \hat{H}_{\rm EOB}}{\partial p_\varphi},$$

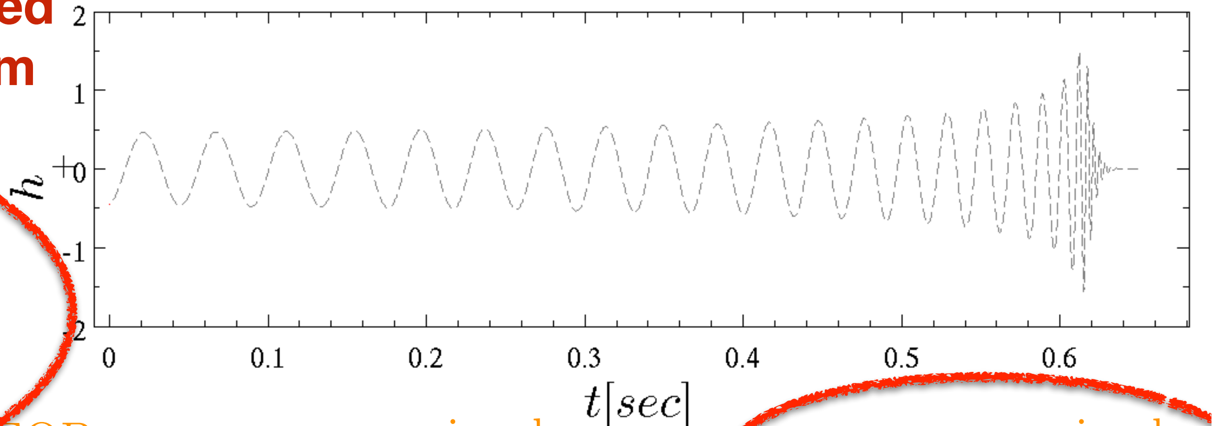
$$\frac{dp_\varphi}{dt} = \hat{\mathcal{F}}_\varphi \, .$$

Hamiltonian: conservative dynamics

Rad Reac Force

Resummed waveform





$$h_{\ell m} \equiv h_{\ell m}^{(N,\epsilon)} \hat{h}_{\ell m}^{(\epsilon)} \hat{h}_{\ell m}^{\rm NQC}$$

$$\hat{h}_{\ell m}^{(\epsilon)} = \hat{S}_{\rm eff}^{(\epsilon)} T_{\ell m} e^{i\delta_{\ell m}} \rho_{\ell m}^{\ell}$$

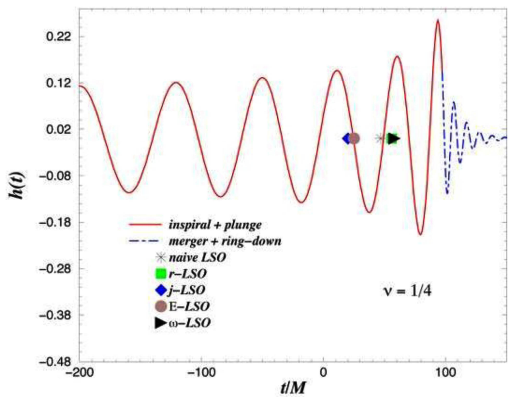
$$\mathcal{F}_\varphi \equiv -\frac{1}{8\pi\Omega} \sum_{\ell=2}^{\ell_{\max}} \sum_{m=1}^{\ell} (m\Omega)^2 |R h_{\ell m}^{(\epsilon)}|^2$$

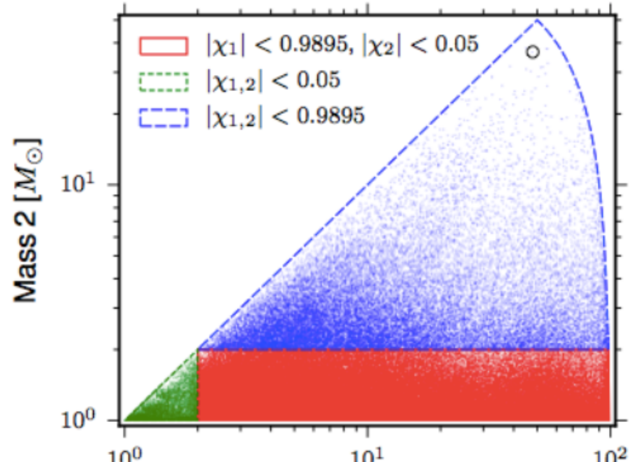
$$h_{\ell m}^{\rm EOB} = \theta(t_m - t) h_{\ell m}^{\rm insplunge}(t) + \theta(t - t_m) h_{\ell m}^{\rm ringdown}$$

$$h_{\ell m}^{\rm ringdown}(t) = \sum_N C_N^+ e^{-\sigma_N^+ (t - t_m)}$$

$$T_{\ell m} = \frac{\Gamma(\ell + 1 - 2i\hat{\hat{k}})}{\Gamma(\ell + 1)} e^{\pi \hat{\hat{k}}} e^{2i\hat{\hat{k}} \log(2kr_0)},$$

Complete waveforms for BBH coalescences





Spinning EOB effective Hamiltonian

Damour'01,Damour-Jaranowski-Schaefer'08,Barausse-Buonanno'11,Taracchini etal'12,Damour-Nagar'14,………

$$H_{\rm eff} = H_{\rm orb} + H_{\rm so} \quad \to H_{\rm EOB} = Mc^2\sqrt{1+2\nu\left(\frac{H_{\rm eff}}{\mu c^2}-1\right)}$$

$$\hat{H}^{\rm eff}_{\rm orb} = \sqrt{A\left(1 + B_p \boldsymbol{p}^2 + B_{np}(\boldsymbol{n}\cdot\boldsymbol{p})^2 - \frac{1}{1+\frac{(\boldsymbol{n}\cdot\boldsymbol{\chi}_0)^2}{r^2}}\frac{(r^2+2r+(\boldsymbol{n}\cdot\boldsymbol{\chi}_0)^2)}{\mathcal{R}^4+\Delta(\boldsymbol{n}\cdot\boldsymbol{\chi}_0)^2}((\boldsymbol{n}\times\boldsymbol{p})\cdot\boldsymbol{\chi}_0)^2 + Q_4\right)}.$$

$$H_{\rm so} = G_S \boldsymbol{L}\cdot\boldsymbol{S} + G_{S^*}\boldsymbol{L}\cdot\boldsymbol{S}^*,$$

$$\mathbf{S} = \mathbf{S_1} + \mathbf{S_2}\,;\ \mathbf{S}_* = \frac{m_2}{m_1}\mathbf{S_1} + \frac{m_1}{m_2}\mathbf{S_2}\,,$$

Gyrogravitomagnetic ratios (when neglecting spin^2 effects)

$$r^3 G_S^{\rm PN} = 2 - \frac{5}{8}\nu u - \frac{27}{8}\nu p_r^2 + \nu\left(-\frac{51}{4}u^2 - \frac{21}{2}up_r^2 + \frac{5}{8}p_r^4\right) + \nu^2\left(-\frac{1}{8}u^2 + \frac{23}{8}up_r^2 + \frac{35}{8}p_r^4\right)$$

$$r^3 G_{S_*}^{\rm PN} = \frac{3}{2} - \frac{9}{8}u - \frac{15}{8}p_r^2 + \nu\left(-\frac{3}{4}u - \frac{9}{4}p_r^2\right) - \frac{27}{16}u^2 + \frac{69}{16}up_r^2 + \frac{35}{16}p_r^4 + \nu\left(-\frac{39}{4}u^2 - \frac{9}{4}up_r^2 + \frac{5}{2}p_r^4\right) + \nu^2\left(-\frac{3}{16}u^2 + \frac{57}{16}up_r^2 + \frac{45}{16}p_r^4\right)$$

SPIN-EOB TO BE REEXAMINED ?

From EOB vs NR to EOB-NR waveforms

Buonanno-Cook-Pretorius 2007

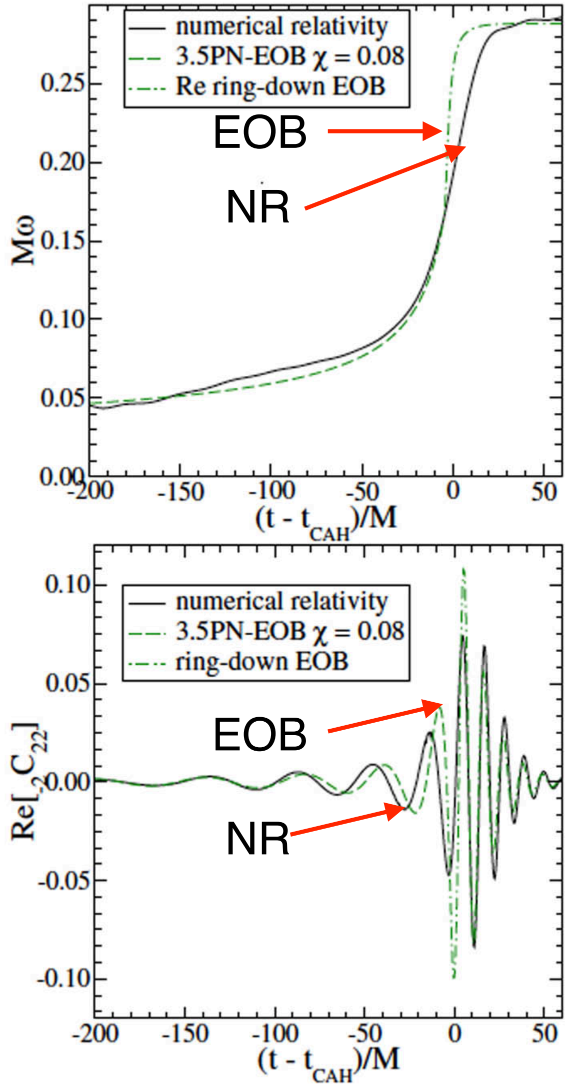
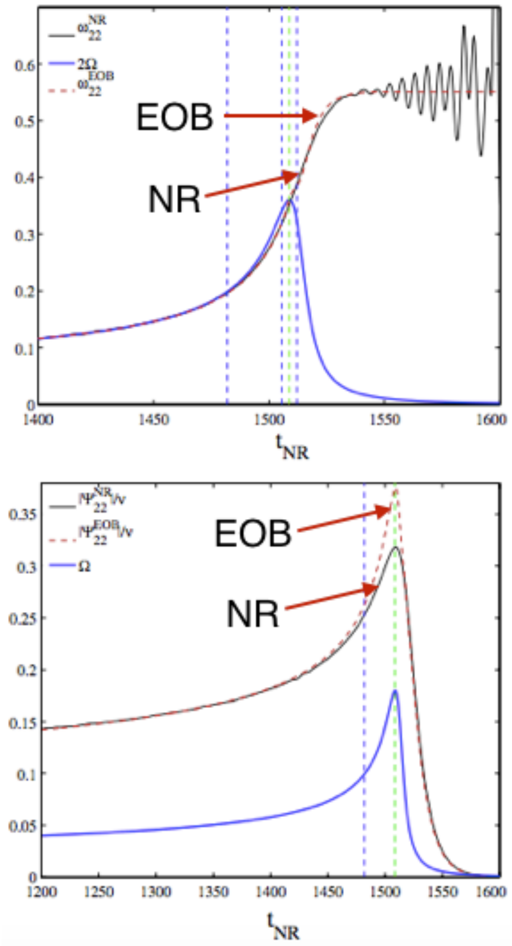
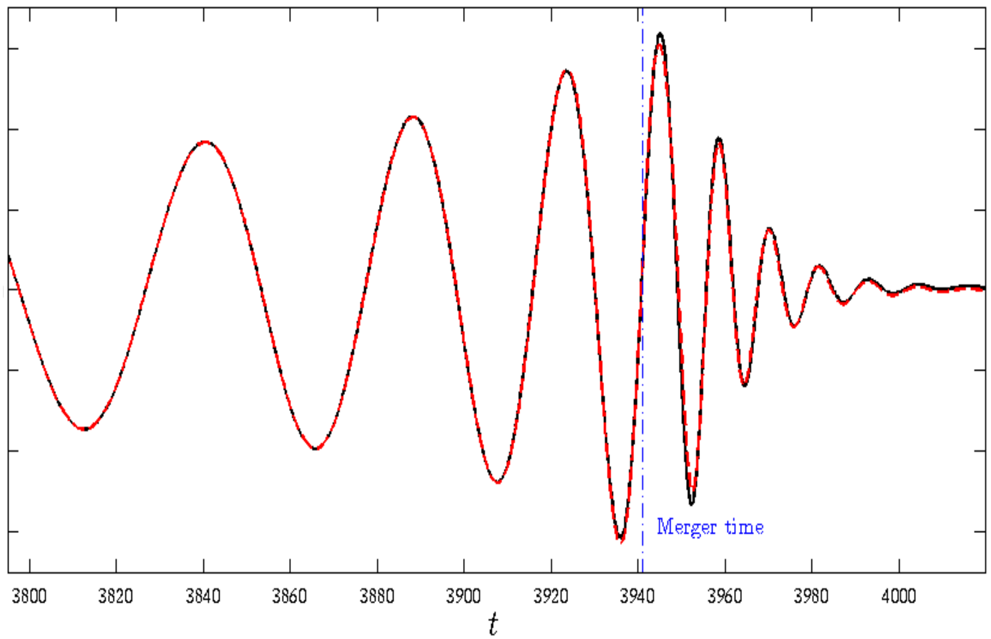


FIG. 21 (color online). We compare the NR and EOB frequency and Re[$_{-2}C_{22}$] waveforms throughout the entire inspiral–merger–ring-down evolution. The data refers to the $d = 16$ run.

TD-Nagar-Dorband-Pollney-Rezzolla 2008



EOB-NR vs NR



EOB-NR is obtained by tuning some yet unknown theoretical EOB parameter to a sample of NR simulations

NR-completed resummed 5PN EOB radial A potential

« We think, however, that a suitable "numerically fitted" and, if possible, "analytically extended" EOB Hamiltonian should be able to fit the needs of upcoming GW detectors. » (TD 2001)

here Damour-Nagar-Bernuzzi '13, Nagar-et al '16; alternative: Taracchini et al '14, Bohe et al '17

4PN analytically complete + 5 PN logarithmic term in the A(u, nu) function,
With u = GM/R and nu = m1 m2 / (m1 + m2)^2
[Damour 09, Blanchet et al 10, Barack-Damour-Sago 10, Le Tiec et al 11, Barausse et al 11, Akcay et al 12, Bini-Damour 13, Damour-Jaranowski-Schäfer 14, Nagar-Damour-Reisswig-Pollney 15]

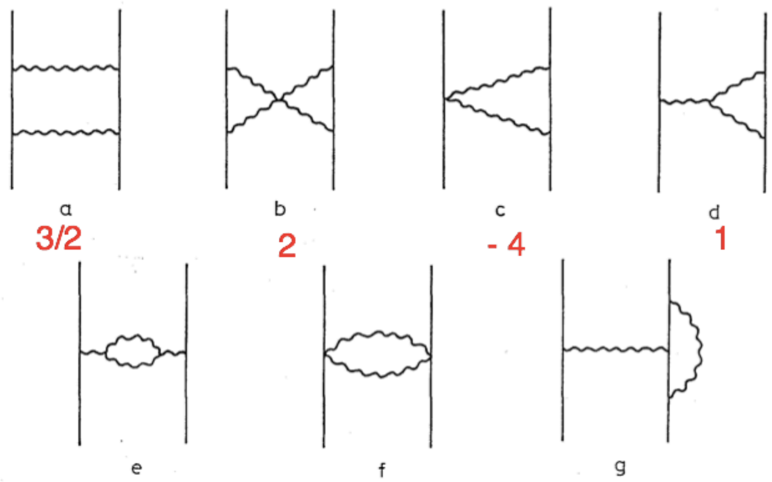
$$A(u;\nu,a_6^c) = P_5^1\Big[1 - 2u + 2\nu u^3 + \nu\left(\frac{94}{3} - \frac{41}{32}\pi^2\right) u^4 + \nu\left[-\frac{4237}{60} + \frac{2275}{512}\pi^2 + \left(-\frac{221}{6} + \frac{41}{32}\pi^2\right)\nu + \frac{64}{5}\ln(16e^{2\gamma}u)\right] u^5 + \nu\left[a_6^c(\nu) - \left(\frac{7004}{105} + \frac{144}{5}\nu\right)\ln u\right] u^6\Big]$$

$$u = \frac{GM}{c^2 R}$$

$$\nu = \frac{m_1 m_2}{(m_1 + m_2)^2}$$

$$a_6^{c\,\text{NR-tuned}}(\nu) = 81.38 - 1330.6\,\nu + 3097.3\,\nu^2$$

Quantum Scattering Amplitudes and 2-body Dynamics



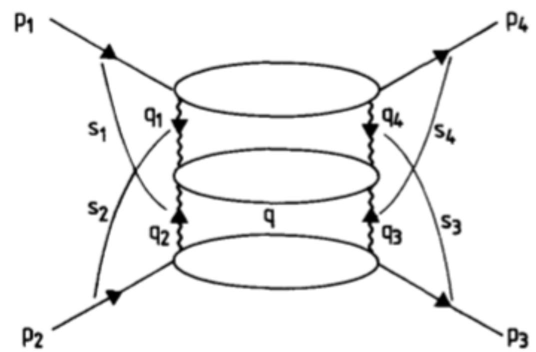


Fig. 3. The "H" diagram that provides the leading correction to the eikonal.

• Quantum Scattering Amplitudes —> Potential
one-graviton exchange :
Corinaldesi '56 '71,
Barker-Gupta-Haracz 66,
Barker-O'Connell 70, Hiida-Okamura72

Nonlinear: Iwasaki 71 [1PN],
Okamura-Ohta-Kimura-Hiida 73[2 PN]

Using modern amplitude techniques: Bjerrum-Bohr+..2003-

Amati-Ciafaloni-Veneziano 1987-2008

Ultra-High-Energy (s >> M_Planck^2)
Four-graviton Scattering at 2 loops

Eikonal phase \delta in D=4
with one- and two-loop corrections using the Regge-Gribov approach

confirmed by DiVecchia+'19

$$\delta = \frac{Gs}{\hbar}\left(\log\left(\frac{L_{IR}}{b}\right) + \frac{6\ell_s^2}{\pi b^2} + \frac{2G^2 s}{b^2}(1 + \frac{2i}{\pi}\log(\cdots))\right)$$

Personally becoming aware of the ACV results in Parma 2008, plus discussions at IHES with Donoghue and Vanhove
—> GSF and EOB (TD 2010): scattering and zero-binding zoom-whirl orbit (Barack et al'19)

Having so computed $\mathcal{E}$ and J one might then, for instance, compare the EOB prediction for the scattering angle $\theta(\mathcal{E}, J)$ (which follows from the EOB Hamiltonian) with GSF computations of θ for a sample of values of $\mathcal{E}$ and J. We see that, in principle, we have access here to one function of *two* real variables, which is ample information for determining the functions entering the EOB formalism.

Reviving the PM Two-Body Dynamics

(pioneered by Bertotti'56, Havas-Goldberg'62, Rosenblum'78, Westpfahl'79, Portilla'80, Bel et al.81)

using Classical and/or Quantum Two-Body Scattering

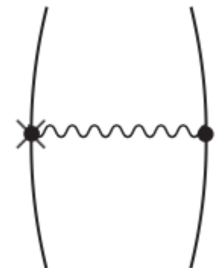
TD 2016, 2017:

Gravitational scattering, post-Minkowskian approximation, and effective-one-body theory

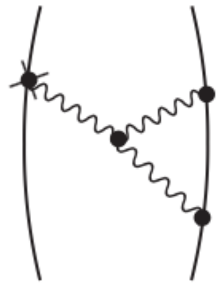
High-energy gravitational scattering and the general relativistic two-body problem

A technique for translating the classical scattering function of two gravitationally interacting bodies into a corresponding (effective one-body) Hamiltonian description has been recently introduced [Phys. Rev. D **94**, 104015 (2016)]. Using this technique, we derive, for the first time, to second-order in Newton's constant (i.e. one classical loop) the Hamiltonian of two point masses having an arbitrary (possibly relativistic) relative velocity. The resulting (second post-Minkowskian) Hamiltonian is found to have a tame high-energy structure which we relate both to gravitational self-force studies of large mass-ratio binary systems, and to the ultra high-energy quantum scattering results of Amati, Ciafaloni and Veneziano. We derive several consequences of our second post-Minkowskian Hamiltonian: (i) the need to use special phase-space gauges to get a tame high-energy limit; and (ii) predictions about a (rest-mass independent) linear Regge trajectory behavior of high-angular-momenta, high-energy circular orbits. Ways of testing these predictions by dedicated numerical simulations are indicated. We finally indicate a way to connect our classical results to the quantum gravitational scattering amplitude of two particles, and we urge amplitude experts to use their novel techniques to compute the two-loop scattering amplitude of scalar masses, from which one could deduce the third post-Minkowskian effective one-body Hamiltonian.

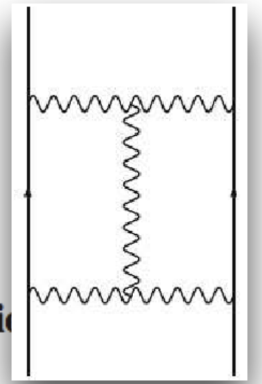
tree-level G^1



one-loop G^2



two-loop G^3+G^4



Cheung-Rothstein-Solon 2018

From Scattering Amplitudes to Classical Potentials in the Post-Minkowskian Expansi

We combine tools from effective field theory and generalized unitarity to construct a map between on-shell scattering amplitudes and the classical potential for interacting spinless particles. For general relativity, we obtain analytic expressions for the classical potential of a binary black hole system at second order in the gravitational constant and all orders in velocity. Our results exactly match all known results up to fourth post-Newtonian order, and offer a simple check of future higher order calculations. By design, these methods should extend to higher orders in perturbation theory.

one-loop G^2

Two equivalent gauge-invariant routes to derive the EOB dynamics

Delaunay Hamiltonian BuonannoTD '99

$$E = H_D(I_r, I_\theta, I_\phi)$$

Scattering angle TD'16-18

$$\frac{1}{2}\chi = \Phi(E_{\rm real}, J; m_1, m_2, G)$$

$$0 = g_{\rm eff}^{\mu\nu} P_\mu P_\nu + \mu^2 + Q$$

$$\mathcal{E}_{\rm eff} = \frac{(\mathcal{E}_{\rm real})^2 - m_1^2 - m_2^2}{2(m_1+m_2)}$$

$$\frac{\mathcal{E}_{\rm eff}}{\mu} = \gamma = -\frac{p_1 \cdot p_2}{m_1 m_2}$$

$$\chi(\gamma, j) = 2\frac{\chi_1(\gamma)}{j} + 2\frac{\chi_2(\gamma)}{j^2} + 2\frac{\chi_3(\gamma)}{j^3} + 2\frac{\chi_4(\gamma)}{j^4} + O\left[\frac{1}{j^5}\right]$$

$$\frac{1}{j} = \frac{G m_1 m_2}{J}$$

$g_{\rm eff}^{\mu\nu}$ **= Schwarzschild metric M=m1+m2**

$$Q = \left(\frac{GM}{R}\right)^2 q_2(E) + \left(\frac{GM}{R}\right)^3 q_3(E) + O(G^4)$$

$$q_2 = -\frac{4}{\pi}(\chi_2 - \chi_2^{\rm Schw})$$

$$q_3 = \frac{4}{\pi}\frac{2\gamma^2-1}{\gamma^2-1}(\chi_2 - \chi_2^{\rm Schw}) - \frac{\chi_3 - \chi_3^{\rm Schw}}{\gamma^2-1}$$

PM scattering results (here without spin)

ultrarelativistic $\gamma \to \infty$ Amati-Ciafaloni-Veneziano'90

3PM=G^3 **Bern-Cheung-Roiban-Shen-Solon-Zeng'19**

inclusion of radiative effects TD'21,DiVecchia+'21,Hermann+21,..

4PM=G^4 conservative: Bern+'22, Dlapa+'22

including radiation-reaction: Dlapa+'23, Damgaard+'23

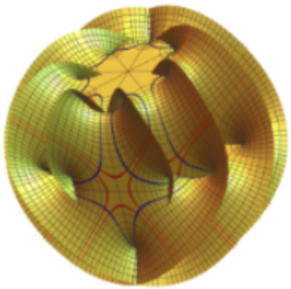
$$\mathcal{F}_{\rm rad-reac} = O(G^2/c^5) \implies \exists\, \mathcal{F}^2_{\rm rad-reac} = O(G^4) \text{ effects}$$

(Bini-TD-Geralico'21)

5PM=G^5

$$(\Delta p_a^\mu)^{\rm 5PM} \sim \frac{G^5}{b^5} m_1 m_2 (m_2^4 + m_1 m_2^3 + m_1^2 m_2^2 + m_1^3 m_2 + m_1^4)$$

probe 1SF **2SF ?** 1SF probe

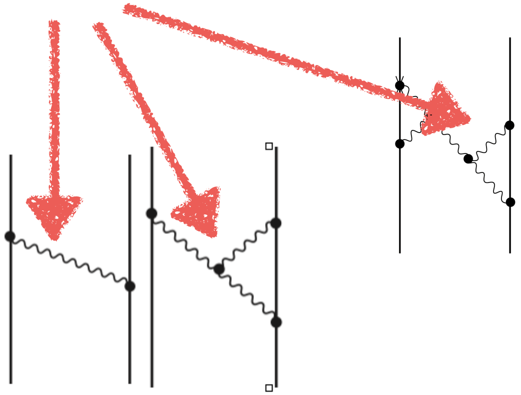


CY3 $(n=3)$

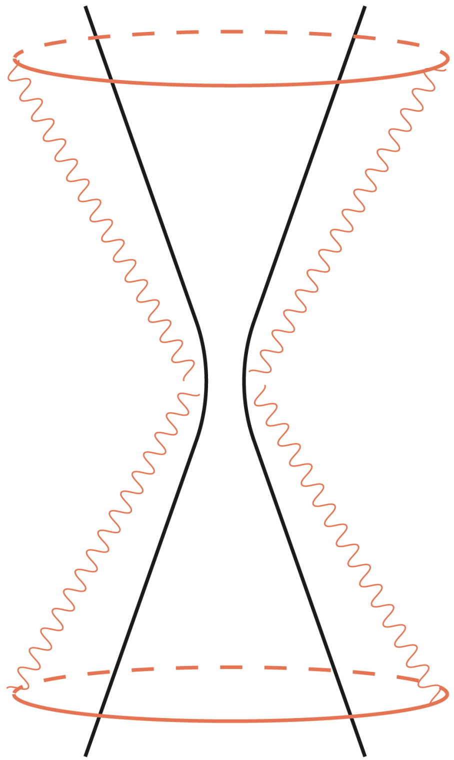
Driesse+'24

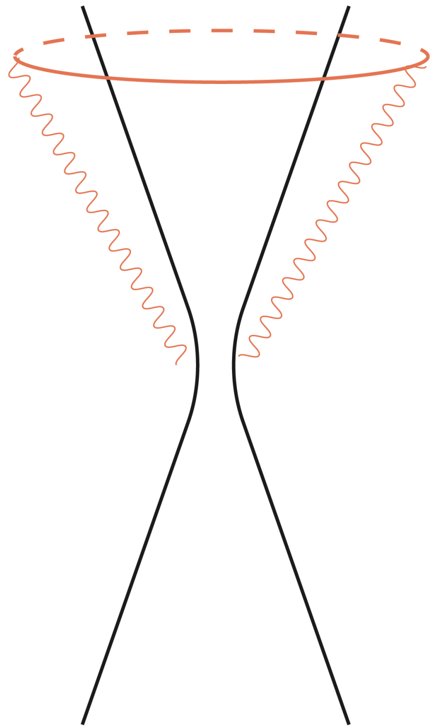
Conservative vs Radiation-reacted
Classical Gravitational Scattering

Fokker-Wheeler-Feynman
conservative action using
G_sym=1/2(G_ret+Gadv)
=Re[G_F]=PV(1/p^2)



Subtleties arise at G^4 when iterating several PV(1/p^2)





Radiation-reaction effects enter scattering at G^3/c^5 (Bini-TD'12)

$$\frac{1}{2}\chi^{\rm rad} = +\frac{8G^3}{5c^5}\frac{m_1^3 m_2^3}{J^3}\nu v^2 + \cdots$$

chi^rad linked to radiated E and J

Radiation-reaction effects in scattering play a crucial role at **high-energy**

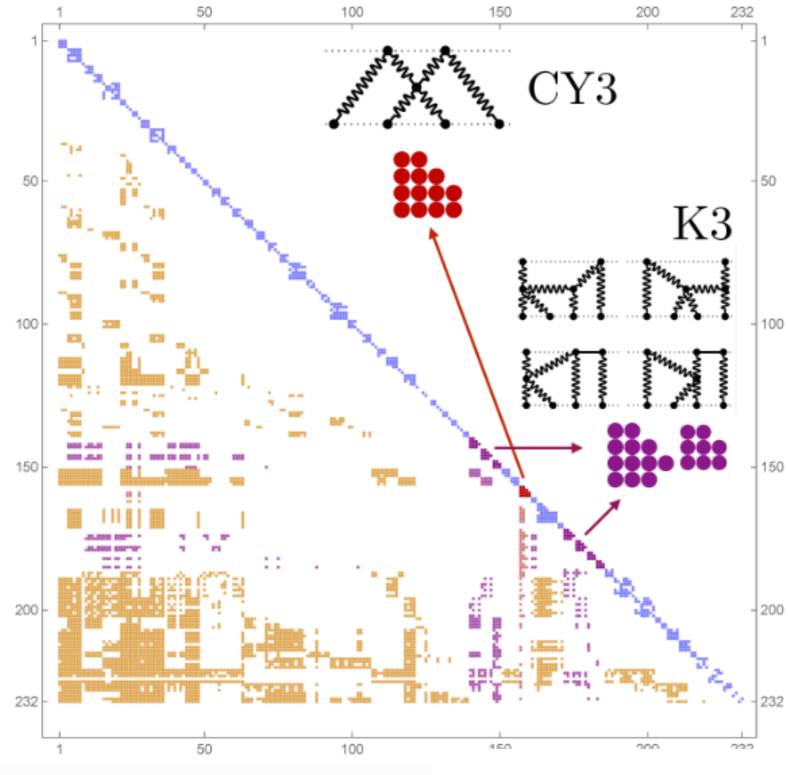
(DiVecchia-Heissenberg-Russo-Veneziano'20, TD'21, Hermann-Parra-Martinez-Ruf-Zeng'21,….)

they resolve the O(G^3) puzzle of the discrepancy between the HE limit of Amati-Ciafaloni-Veneziano'90(+ Ciafaloni-Colferai'14), and the G^3 result of Bern et al'19,20

5PM=G^5=4-loop; currently at «1 SF» level

Emergence of Calabi-Yau manifolds in high-precision black hole scattering

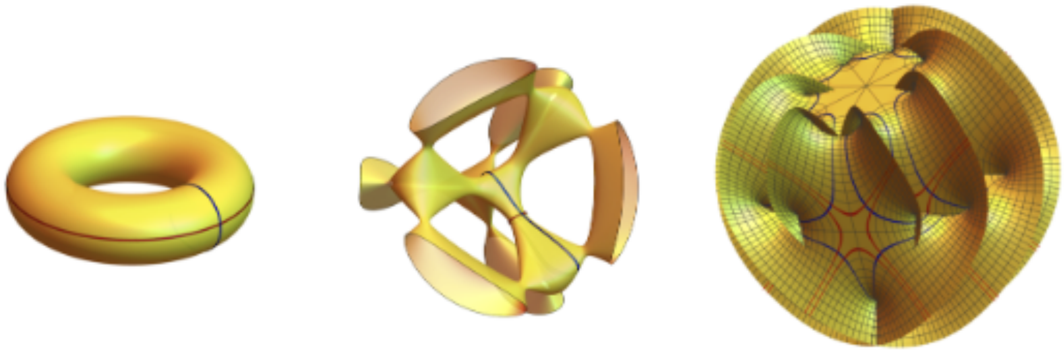
Mathias Driesse[1] Gustav Uhre Jakobsen[1,2] Albrecht Klemm[3,4] Gustav Mogull[1,2,5]
Christoph Nega[6] Jan Plefka[1] Benjamin Sauer[1] and Johann Usovitsch[1] 2024



calculation of the impulse calls for the evaluation of millions of Feynman integrals, which may have at most 13 propagators of the kinds seen in Fig. 6. To evaluate them we generate linear integration-by-parts (IBP) identities which reduces the problem to one solving a large system of linear equations. The task was nevertheless enormous, and consumed around 300,000 core hours on HPC clusters.

Picard-Fuchs equation for the CY3 periods

$$\left[\left(x\frac{\mathrm{d}}{\mathrm{d}x}-1\right)^4 - x^4\left(x\frac{\mathrm{d}}{\mathrm{d}x}+1\right)^4\right]\varpi(x) = 0\,.$$



Torus ($n=1$) K3 ($n=2$) CY3 ($n=3$)

Various ways of formulating the EOB mass-shell condition

$$g^{\mu\nu}(X^\lambda)P_\mu P_\nu + \mu^2 + Q(X^\mu, P_\mu) = 0$$

-P_0= gamma=H_eff

expressed as a function
of R and P_R^2 or **P**^2 or E

Traditionally DJS gauge: Q(R,P_R) ~ (GM/R)^2 P_r^4+…
convenient both for solving E=H(X,P) and for mildly-eccentric dynamics

in PM gravity: TD 1710.10599 introduced various gauges:

post-Schwarzschild: $g_{\text{eff}}^{\mu\nu}$ **= Schwarzschild metric M=m1+m2** $Q = \left(\frac{GM}{R}\right)^2 q_2(E) + \left(\frac{GM}{R}\right)^3 q_3(E) + O(G^4)$

absorbing Q in A(g,R): $A_{S^*}(R,\gamma,\nu) = \frac{A_S(R)}{1 - \frac{1}{\gamma^2} A_S(R) Q(R,\gamma,\nu)}$

Newtonianlike potential:

$$p_{\bar{r}}^2 + \frac{j^2}{\bar{r}^2} = p_\infty^2 + w(\bar{r},\gamma).$$

$$w(\bar{r},\gamma) = \frac{w_1(\gamma)}{\bar{r}} + \frac{w_2(\gamma)}{\bar{r}^2} + \frac{w_3(\gamma)}{\bar{r}^3} + \frac{w_4(\gamma)}{\bar{r}^4} + O\left[\frac{1}{\bar{r}^5}\right]$$

A novel Lagrange-multiplier approach to the effective-one-body dynamics of binary systems in post-Minkowskian gravity

Damour-Nagar-Placidi-Rettegno, March 2025

a geodesic-like mass-shell condition involving an energy-dependent effective metric:

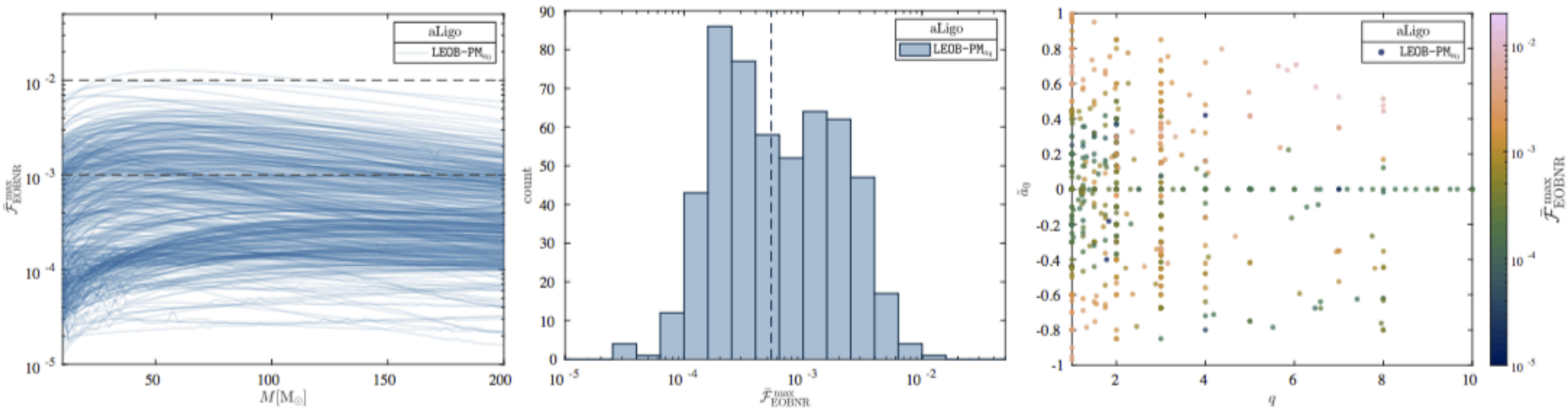
$$g_*^{\mu\nu}(X^\mu,\gamma)P_\mu P_\nu + \mu^2 = 0$$

up to now solved perturbatively
wrt the energy gamma=-P_0/mu;
this led to several drawbacks

Lagrange multiplier approach:

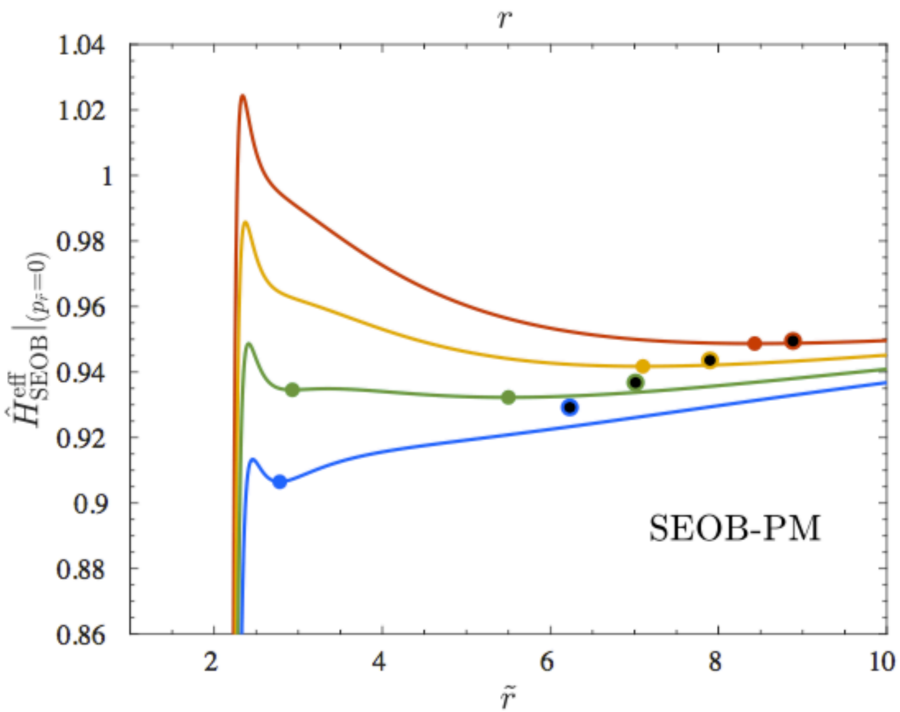
$$\mathcal{C} = g_*^{\mu\nu}(x^\lambda, p_0)p_\mu p_\nu + 1$$

$$S[x^\mu, p_\mu, e_{\rm L}] = \int \left[p_\mu \frac{dx^\mu}{d\tau} - e_{\rm L}\, \mathcal{C}\,(x^\mu, p_\mu) \right] d\tau$$

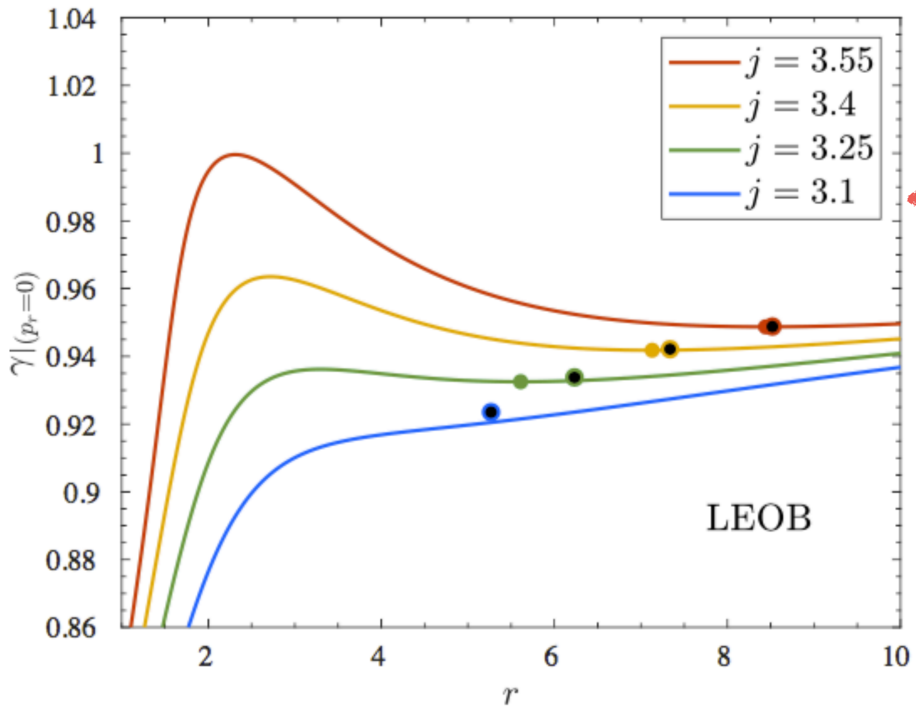


median unfaithfulness= 5.39×10^{-4}

Radial potentials for inspiralling orbits in two versions of EOB-PM



when iteratively solving the mass-shell condition to get H_eff(R,J)
SEOB-PM, Buonanno+24



when exactly solving the mass-shell condition to get H_eff(R,J)
LEOB, Damour+25

Mass polynomiality structure in scattering (TD'20)

$$\frac{dx_a^\mu}{ds_a} = g^{\mu\nu}(x_a)u_{a\nu},$$

$$\frac{du_{a\mu}}{ds_a} = -\frac{1}{2}\partial_\mu g^{\alpha\beta}(x_a)u_{a\alpha}u_{a\beta},$$

$$R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu} = 8\pi G T^{\mu\nu},$$

$$T^{\mu\nu}(x) = \sum_{a=1,2} m_a \int ds_a u_a^\mu u_a^\nu \frac{\delta^4(x - x_a(s_a))}{\sqrt{g}}$$

$$\Delta p_{a\mu} = -\frac{m_a}{2}\int_{-\infty}^{+\infty} ds_a \partial_\mu g^{\alpha\beta}(x_a)u_{a\alpha}u_{a\beta}$$

$$\Delta p_{1\mu} = -2Gm_1m_2\frac{2(u_{10}\cdot u_{20})^2 - 1}{\sqrt{(u_{10}\cdot u_{20})^2 - 1}}\frac{b_\mu}{b^2} + \frac{Gm_1m_2}{b}\Delta_\mu.$$

polynomial in Gm1/b and Gm2/b

conservative scattering case

$$\frac{1}{2}\chi(E_{\text{real}}, J) = \frac{\chi_1(\gamma,\nu)}{j} + \frac{\chi_2(\gamma,\nu)}{j^2} + \frac{\chi_3(\gamma,\nu)}{j^3} + \frac{\chi_4(\gamma,\nu)}{j^4} + \cdots,$$

at G^n

$$h^{n-1}(\gamma,\nu)\chi_n(\gamma,\nu) = P^\gamma_{d(n)}(\nu),$$

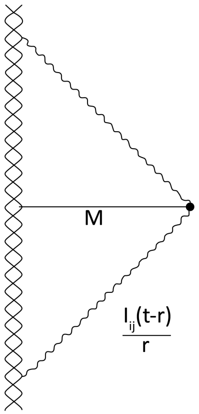
polynomial in nu of degree [(n-1)/2]

0SF scattering gives access to full G^2 dynamics !
1SF scattering gives access to full G^3 and G^4 conservative dynamics !
2SF scattering gives access to full G^5 and G^6 conservative dynamics

Tutti-Frutti method (Bini-TD-Geralico,'19,20,21)

with PN local Hamiltonian

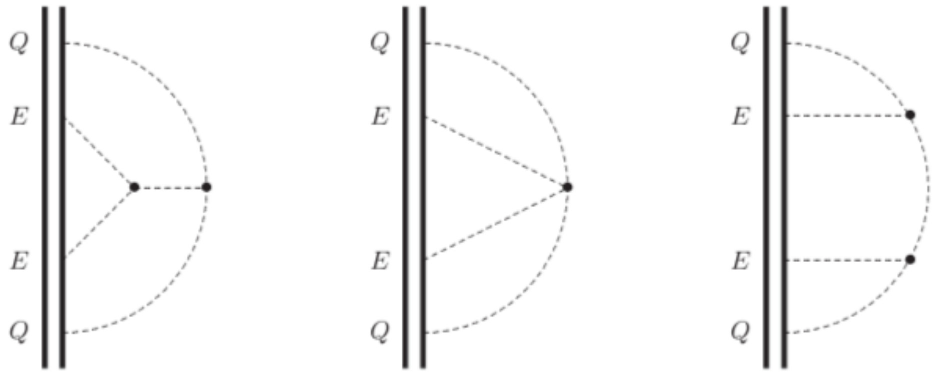
$$S^{\leq n\mathrm{PN}}_{\mathrm{tot}}[x_1(s_1), x_2(s_2)] = S^{\leq n\mathrm{PN}}_{\mathrm{loc}}[x_1(s_1), x_2(s_2)] + S^{\leq n\mathrm{PN}}_{\mathrm{nonloc}}[x_1(s_1), x_2(s_2)].$$



$$S^{4+5\mathrm{PN}}_{\mathrm{nonloc}}[x_1(s_1), x_2(s_2)] = \frac{G^2\mathcal{M}}{c^3}\int dt \mathrm{PF}_{2r^h_{12}(t)/c} \times \int \frac{dt'}{|t-t'|}\mathcal{F}^{\mathrm{split}}_{\mathrm{1PN}}(t,t'),$$

$$\mathcal{F}^{\mathrm{split}}_{\mathrm{1PN}}(t,t') = \frac{G}{c^5}\left(\frac{1}{5}I^{(3)}_{ab}(t)I^{(3)}_{ab}(t') + \frac{1}{189c^2}I^{(4)}_{abc}(t)I^{(4)}_{abc}(t') + \frac{16}{45c^2}J^{(3)}_{ab}(t)J^{(3)}_{ab}(t')\right).$$

starting at 5.5PN, G^5



(a) (b) (c)

Then combining:
1GSF computations
polynomiality in nu
Delaunay averaging, one determines H_loc modulo a few 2SF parameters

Tutti-Frutti method

(**Bini-TD-Geralico** '19,'20'21)

combines PN, MPM, EOB, Delaunay, Self-Force, mass-polynomiality of scattering angle

SIXTH POST-NEWTONIAN LOCAL-IN-TIME DYNAMIC

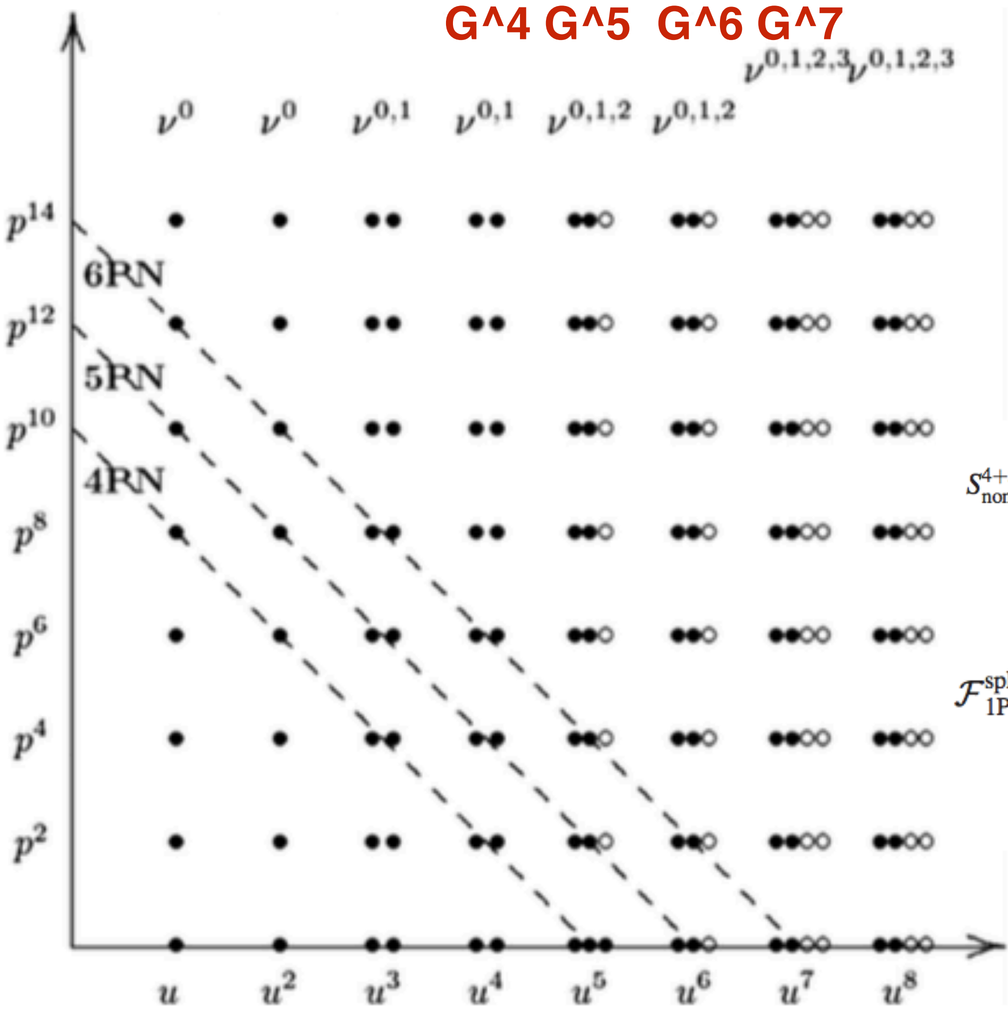


FIG. 1. Schematic representation of the irreducible information contained, at each post-Minkowskian level (keyed by a power of $u = GM/r$), in the local dynamics. Each vertical column of dots describes the post-Newtonian expansion (keyed by powers of p^2) of an energy-dependent function parametrizing the scattering angle. The various columns at a given post-Minkowskian level correspond to increasing powers of the symmetric mass-ratio ν. See text for details.

6PN conservative dynamics complete at 3PM and 4PM

$$S^{4+5\mathrm{PN}}_{\mathrm{nonloc}}[x_1(s_1), x_2(s_2)] = \frac{G^2\mathcal{M}}{c^3}\int dt\,\mathrm{PF}_{2r^h_{12}(t)/c} \times \int \frac{dt'}{|t-t'|}\mathcal{F}^{\mathrm{split}}_{1\mathrm{PN}}(t,t').$$

$$\mathcal{F}^{\mathrm{split}}_{1\mathrm{PN}}(t,t') = \frac{G}{c^5}\left(\frac{1}{5}I^{(3)}_{ab}(t)I^{(3)}_{ab}(t') + \frac{1}{189c^2} \dots + \frac{16}{45c^2}J^{(3)}_{ab}(t)J^{(3)}_{ab}(t')\right).$$

5PN conservative complete at 5PM and 6PM modulo d5 and a6

Tutti Frutti vs Worldline Effective Field Theory (Bini-TD'25)

$$\Delta p_a^{\mu} = \Delta p_a^{\mathrm{cons}\,\mu} + \Delta p_a^{\mathrm{rrlin}J_{\mathrm{rad}}\,\mu} + \Delta p_a^{\mathrm{rrlin}P_{\mathrm{rad}}\,\mu} + \Delta p_1^{\mathrm{rr\,remain\,sup}\,\mu}$$

uniquely determined at G^4 by mass polynomiality and rad-reac structure

at G^4

$$c^{\mathrm{4diss}}_{\mathrm{1b,1rad}} = \frac{G^4}{b^4} m_1^2 m_2^2 \left\{ (m_1+m_2)\left[\mathcal{E}(\gamma)\frac{\gamma(6\gamma^2-5)}{(\gamma^2-1)^{3/2}} - \pi\frac{3}{4}\hat{J}_2(\gamma)\frac{(5\gamma^2-1)}{(\gamma^2-1)^{3/2}} - \hat{J}_3(\gamma)\frac{(2\gamma^2-1)}{(\gamma^2-1)^2}\right] - m_1\mathcal{E}(\gamma)\frac{2\gamma^2-1}{(\gamma+1)\sqrt{\gamma^2-1}} \right\}. \quad (3.2$$

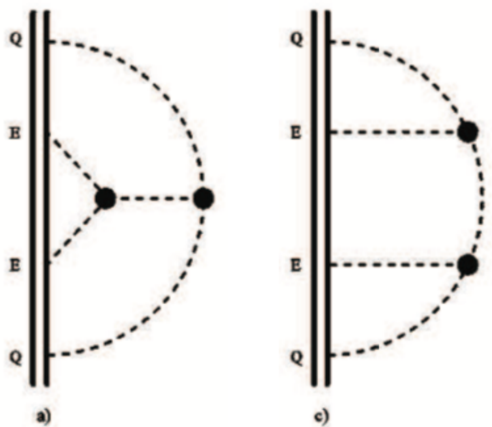
determines J3 from cb1rad

$$c^{\mathrm{4diss}}_{\mathrm{1b,2rad}} = \frac{m_1}{m_2-m_1} P^{\mathrm{rad}}_{xG^4}$$

direct link between cb2rad and Px^rad

at G^5 1SF

$$f_b^{G^5,\mathrm{1SF}} = -\frac{2\tilde{\chi}_5^{\mathrm{1cons}}}{(\gamma^2-1)^2} - \frac{\chi_1 \hat{J}_4^0}{b^5(\gamma^2-1)^2} + K$$

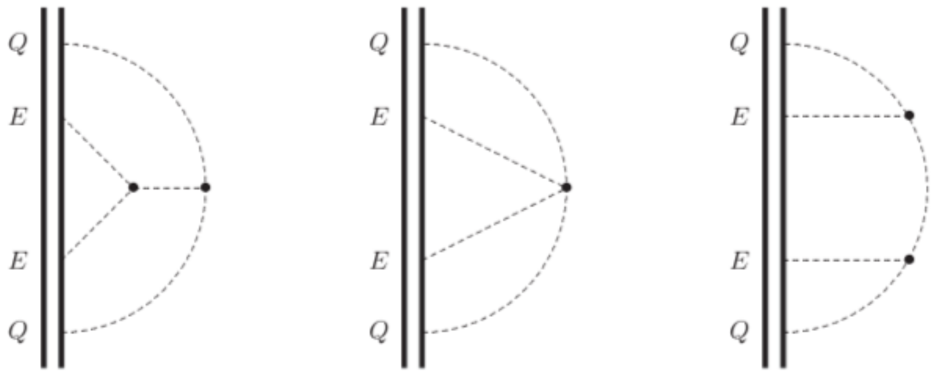


TF determined to 6PN ıcluding a **tail-of-tail** term

determined to 6PN by TF + Heissenberg'24

TF determined P_mu^rad to 5.5 PN

High-post-Newtonian-order dynamical effects induced by tail-of-tail interactions in a two body system (BDG'25)



(a) (b) (c)

tail-of-tail conservative action

(DJS'14,BD'25,using Blanchet'05,Goldberger-Ross'10,…)

$$S^{\rm tail-of-tail}_{\rm time-sym} = \frac{1}{2}\left(\frac{G\mathcal{M}}{c^3}\right)^2 G\sum_{l\geq 2}\frac{1}{c^{2l+1}}a_l\beta_l^{\rm even}\int dt\times \int_{-\infty}^{\infty} dt' I_L^{(l+2)}(t) I_L^{(l+2)}(t')\ln\frac{c|t-t'|}{2r_0} + \frac{1}{2}\left(\frac{G\mathcal{M}}{c^3}\right)^2 G\sum_{l\geq 2}\frac{1}{c^{2l+3}}b_l\beta_l^{\rm odd}\int dt\times dt' J_L^{(l+2)}(t) J_L^{(l+2)}(t')\ln\frac{c|t-t'|}{2r_0},$$

1SF confirmations and 2SF new results at the 6.5PN level

$$A(u,\nu) = 1-2u+\sum_{n\geq 3}a_n(\nu,\ln u)u^n,$$
$$\bar{D}(u,\nu) = 1+\sum_{n\geq 2}\bar{d}_n(\nu,\ln u)u^n.$$

$$a_{6.5} = \frac{13696}{525}\nu\pi,$$
$$a_{7.5} = -\frac{10052}{225}\nu^2\pi - \frac{512501}{3675}\nu\pi;$$

agreement at 6.5PN with 1SF Driesse et al (using a recent result of Geralico'25)

prediction of the conservative G^5 scattering at 2SF in terms of TF undetermined parameters

Strong-field scattering of two black holes: Numerical relativity meets post-Minkowskian gravity

Thibault Damour[1] and Piero Rettegno[2,3] (PRD March 2023)

w(r) introduced in TD 1710.10599

$$\chi_{n\mathrm{PM}}(\gamma, j) \equiv \sum_{i=1}^{n} 2\frac{\chi_i(\gamma)}{j^i}$$

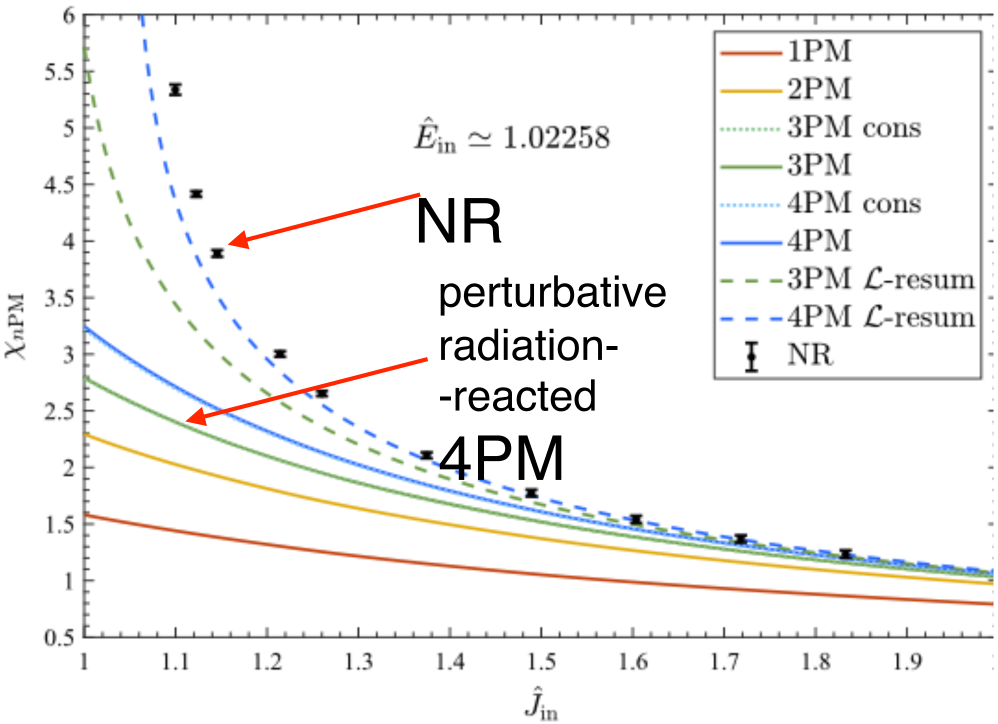
$$\mu^2 + g^{\mu\nu}_{\mathrm{eff}} P_\mu P_\nu + Q(X^\mu, P_\mu) = 0,$$

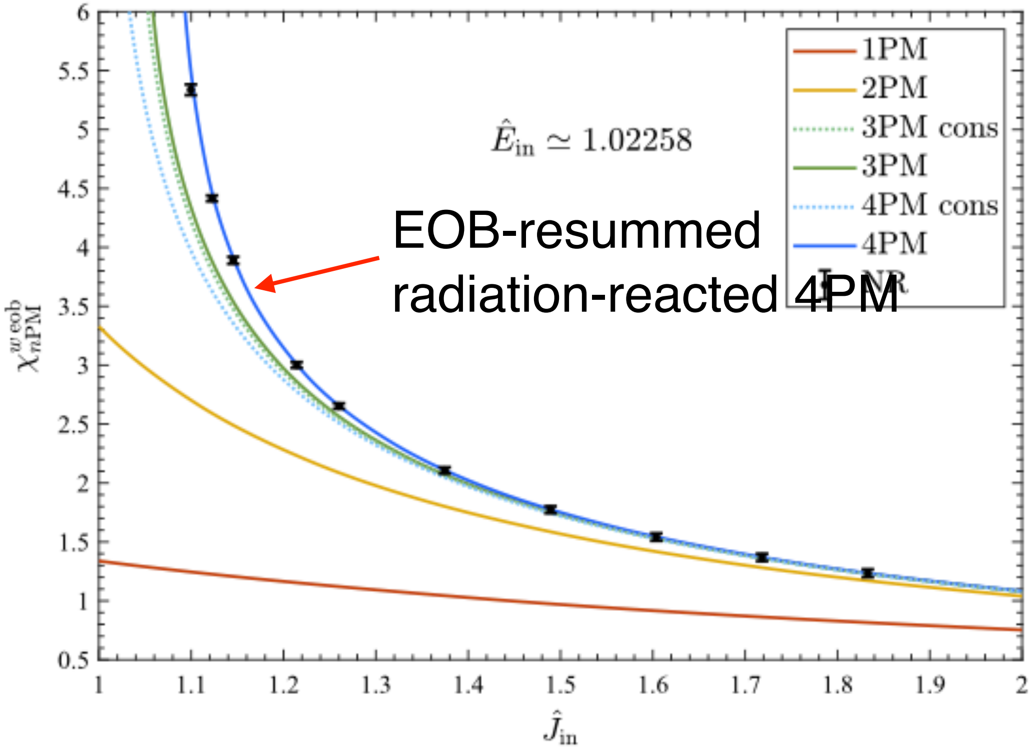
$$p_{\bar{r}}^2 + \frac{j^2}{\bar{r}^2} = p_\infty^2 + w(\bar{r}, \gamma).$$

$$w(\bar{r}, \gamma) = \frac{w_1(\gamma)}{\bar{r}} + \frac{w_2(\gamma)}{\bar{r}^2} + \frac{w_3(\gamma)}{\bar{r}^3} + \frac{w_4(\gamma)}{\bar{r}^4} + O\left[\frac{1}{\bar{r}^5}\right]$$

Newtonianlike EOB radial potential

$$\chi^{w\,\mathrm{eob}}_{n\mathrm{PM}}(\gamma, j) \equiv 2j \int_0^{\bar{u}_{\max}(\gamma,j)} \frac{d\bar{u}}{\sqrt{p_\infty^2 + w_{n\mathrm{PM}}(\bar{u}, \gamma) - j^2\bar{u}^2}} - \pi.$$

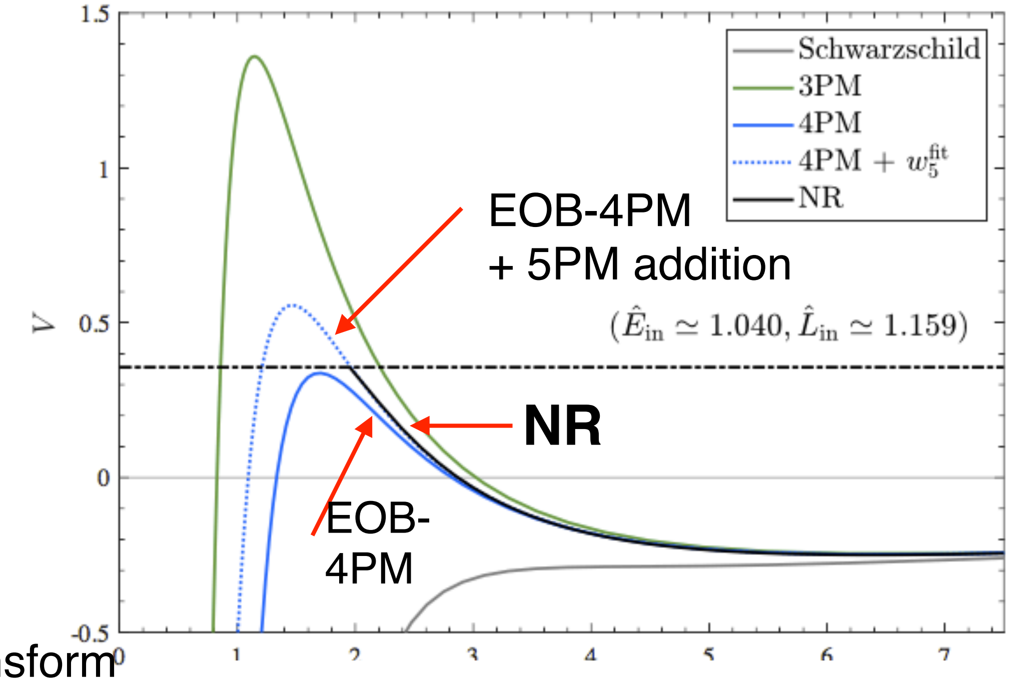




Rettegno-Pratten+23, Buonanno-Jakobsen+..24, Swain-Pratten-Schmidt'25, Long-Pfeiffer+25

Strong-field scattering of two spinning black holes: Numerical Relativity versus post-Minkowskian gravity (Rettegno et al.'23)

Higher-energy non-spinning: Comparison between the effective potential V=L^2/r^2-w(r) extracted from NR simulations and its EOB-PM equivalent

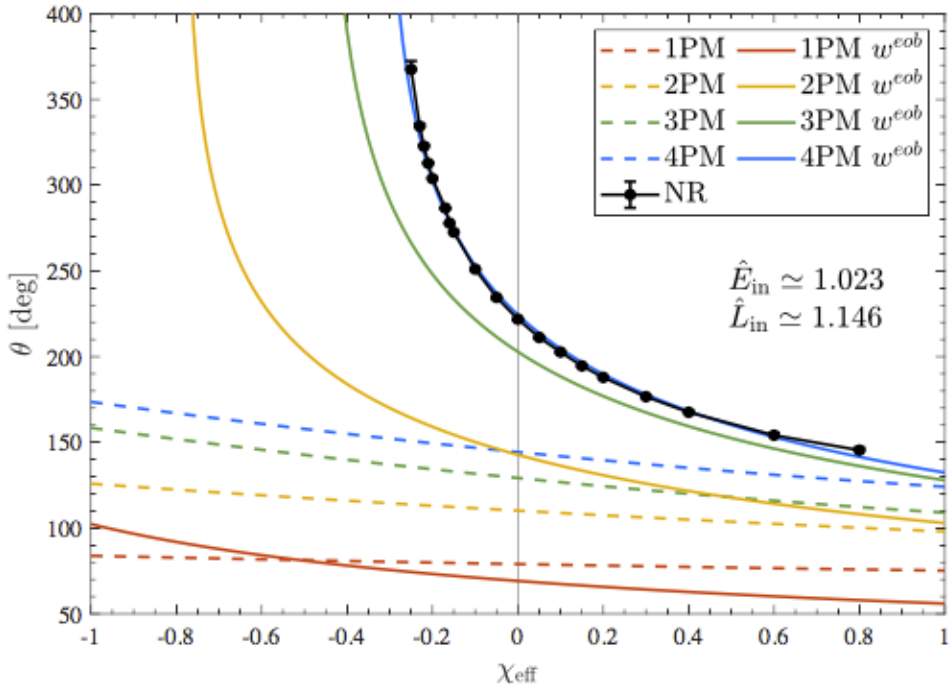


In the spin-aligned case, one can transform the PM-expanded scattering angle

$$\theta_{n\mathrm{PM}}(\gamma, \ell, S_i) \equiv \sum_{k=1}^{n} 2\frac{\theta_k(\gamma, \ell, S_i)}{\ell^k}$$

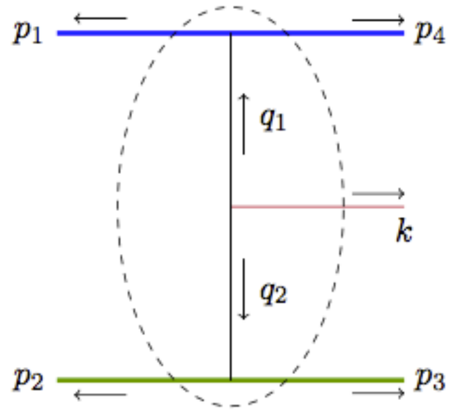
into an equivalent spin-dependent EOB potential

$$w_{n\mathrm{PM}}(\bar{r}, \gamma, \ell, S_i) = w^{\mathrm{orb}}(\bar{r}, \gamma) + \frac{\ell\, w^{\mathrm{S}}_{n\mathrm{PM}}(\bar{r}, \gamma)}{\bar{r}^2} + \frac{w^{\mathrm{S}^2}_{n\mathrm{PM}}(\bar{r}, \gamma)}{\bar{r}^2} + \frac{\ell\, w^{\mathrm{S}^3}_{n\mathrm{PM}}(\bar{r}, \gamma)}{\bar{r}^4} + \frac{w^{\mathrm{S}^4}_{n\mathrm{PM}}(\bar{r}, \gamma)}{\bar{r}^4}.$$



PM waveform computation $W(k^\mu) = \epsilon^\mu \epsilon^\nu h_{\mu\nu}(\omega, \theta, \phi)$

G^1=1PM (linearized,Einstein 1918) stationary $\propto \delta(\omega)$



LO (tree level) waveform

G^2=2PM: classical time-domain W(t,n): Kovacs-Thorne 1977

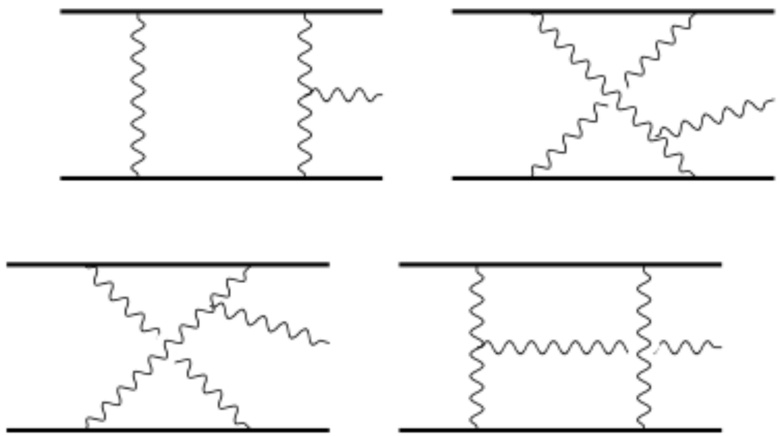
quantum-based: yields W(k,p1,p2,p3,p4)=W(k,p1,p2,q1)

Johansson-Ochirov'15, GoldbergerRidgway'17 Luna-Nicholson-OConnellWhite'18
Mougiakakos-Riva-Vernizzi'21,Bautista-Siemonsen'22, De Angelis-Gonzo-Novichkov'23

Recent NLO (one-loop) waveform

G^3=3PM

Brandhuber+'23, Herderschee+'23, Georgoudis+'23,
Bohnenblust+'24



5-point HEFT one-loop amplitude
—-> O(G^3) waveform via KMOC

5-point amplitude: 2 —> 3

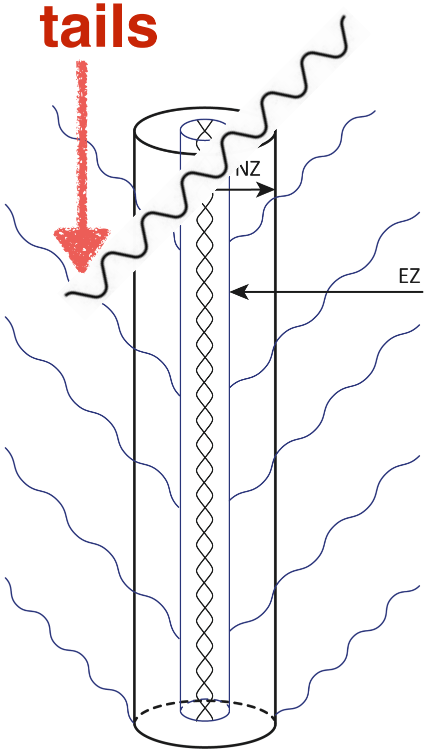
$$\begin{aligned}\mathscr{M}(\varepsilon, k, p_1, p_2, q_1, q_2)\\ &\equiv i\langle p_3 p_4 | \hat{a}(k)\mathbb{T}|p_1 p_2\rangle + \langle p_3 p_4|\mathbb{T}^\dagger \hat{a}(k)\mathbb{T}|p_1 p_2\rangle \\ &= i\langle p_3 p_4 k|\mathbb{T}|p_1 p_2\rangle + \langle p_3 p_4|\mathbb{T}^\dagger \hat{a}(k)\mathbb{T}|p_1 p_2\rangle\,,\end{aligned}$$

« cut term » important (Caron-Huot+'23)

Comparing one-loop amplitude to MPM waveform

(Bini-TD-Geralico'23)

tails



algorithmic STF tensors encoding multipole moments (related to the source moments I_L,J_L)

$$
\begin{aligned}
g &= \eta + G h_1 + G^2 h_2 + G^3 h_3 + \ldots,\\
\Box h_1 &= 0,\\
\Box h_2 &= \partial\partial h_1 h_1,\\
\Box h_3 &= \partial\partial h_1 h_1 h_1 + \partial\partial h_1 h_2,\\
h_1 &= \sum_\ell \partial_{i_1 i_2 \ldots i_\ell}\left(\frac{M_{i_1 i_2 \ldots i_\ell}(t-r/c)}{r}\right) + \partial\partial \ldots \partial\left(\frac{\epsilon_{j_1 j_2 k} S_{k j_3 \ldots j_\ell}(t-r/c)}{r}\right),\\
h_2 &= FP_B \Box^{-1}_{\rm ret}\left(\left(\frac{r}{r_0}\right)^B \partial\partial h_1 h_1\right) + \ldots,\\
h_3 &= FP_B \Box^{-1}_{\rm ret} \ldots.
\end{aligned}
$$

radiative multipole moments (observable at infinity) U_L, V_L

$$
r h^{\rm TT}_{ij} = \frac{4G}{c^2} P(n)_{ijab} \sum_{l=2}^{\infty} \frac{1}{c^l}\frac{1}{l!}\left(U_{abL-2} n_{L-2} - \frac{2l}{c(l+1)} n_{cL-2} \epsilon_{cd(a} V_{b)dL-2}\right)
$$

$$
\mathcal{M}^{\rm MPM}(k,b,u_1,u_2,m_1,m_2) = -i\frac{\kappa}{2}\epsilon^\mu \epsilon^\nu h^{\rm MPM}_{\mu\nu}(\omega,\theta,\phi) = -i\frac{\kappa}{2}\int dt e^{i\omega t}\epsilon^\mu\epsilon^\nu h^{\rm MPM}_{\mu\nu}(t,\theta,\phi)
$$

$$
\mathcal{M}^{\rm HEFT}(k,b,u_1,u_2,m_1,m_2) =
$$

$$
e^{i\frac{b_1+b_2}{2}\cdot k}\int \frac{d^D q}{(2\pi)^{D-2}}\,\delta\!\left(2p_1\cdot\left(q+\frac{k}{2}\right)\right)\delta\!\left(2p_2\cdot\left(-q+\frac{k}{2}\right)\right)\, e^{iq\cdot(b_1-b_2)}\mathcal{M}^{(1)}_{5,\rm HEFT}\left(q+\frac{k}{2}, -q+\frac{k}{2}; h\right)
$$

Comparison one-loop amplitude vs MPM waveform

$$W(t,\theta,\phi) \sim \frac{1}{c^4}\left(G\,(\text{stationary}) + G^2(1+\frac{1}{c^1}+\frac{1}{c^2}+\frac{1}{c^3}+\cdots) + G^3(1+\frac{1}{c^1}+\frac{1}{c^2}+\frac{1}{c^3}+\cdots)+O(G^4)\right)$$

tree-level

one-loop

Aim: accuracy up to radiation-reaction effects: O(1/c^5) beyond LO quadrupole

$$U_{ij}(\omega) \sim \left(G(1+\frac{1}{c^2}+\frac{1}{c^4}) + G^2(1+\frac{1}{c^2}+\frac{1}{c^3}+\frac{1}{c^4}+\frac{1}{c^5}) + O(G^4)\right) + O(\frac{1}{c^6})$$

Newtonian G^2 **LO tail** **rad-reac plus similar effects**

$$U_{ij}^{\text{tail}}(t) = \frac{2G\mathcal{M}}{c^3}\int_0^\infty d\tau I_{ij}^{(4)}(t-\tau)\left(\ln\left(\frac{\tau}{2b_0}\right)+\frac{11}{12}\right)$$

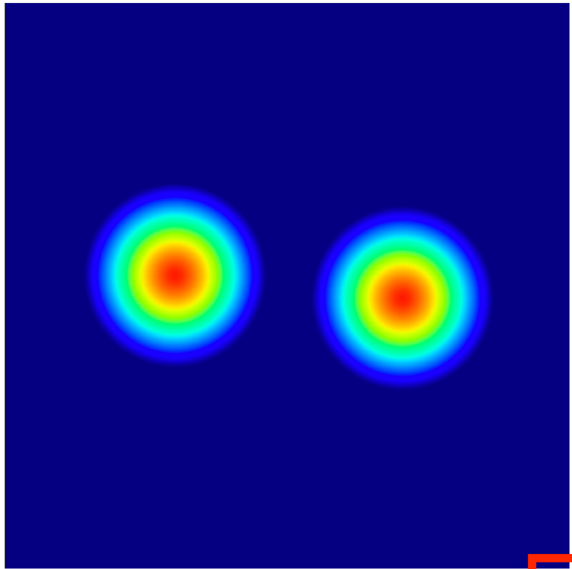
Main results of the initial EFT-MPM comparison (Bini-TD-Geralico, 2023):
mismatch at the Newtonian level, except if one refers the one-loop amp. to classical averaged momenta, rather than incoming momenta; then the terms linked to time-even PN corrections to multipoles agree but there are many mismatches at the G^2/c^5 level

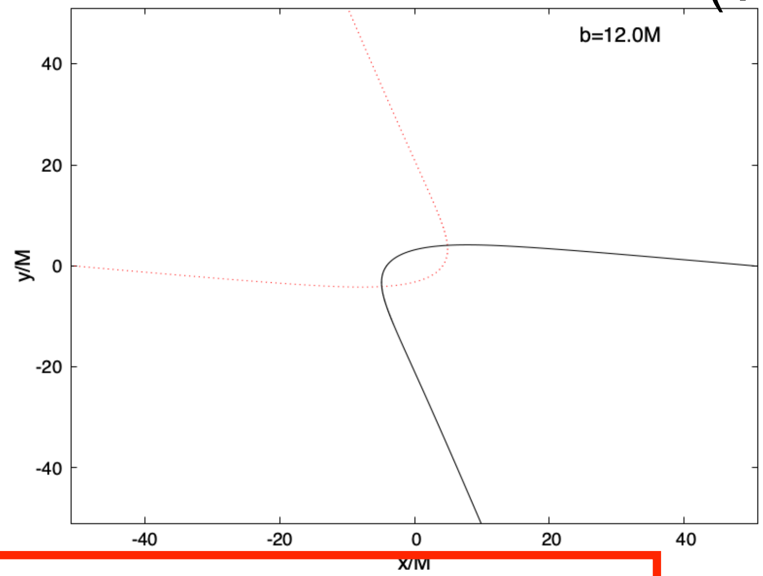
Updated comparisons (Georgoudis et al.'23,'24, Bini et al. '24) lead to perfect agreement after taking into account three subtle effects:

(1) the bilinear-in-amplitude KMOC term generates the needed rotation
(2) IR divergences generate an additional (D-4)/(D-4) contribution
(3) zero-frequency gravitons contribute additional terms at h~G and h~G^3
(4) interesting links beween zero-freq gravitons and BMS frame (Veneziano-Vilkovisky)

Gravitational scattering of solitonic boson stars

(TD-Jain-Sperhake, wip)





complex scalar field $\varphi(t,r) = A(r)e^{\mathrm{i}(\epsilon\omega t+\delta\Phi)}$

$$S = \int \frac{\sqrt{-g}}{2}\left\{\frac{R}{8\pi G} - [g^{\mu\nu}\nabla_\mu\bar{\varphi}\nabla_\nu\varphi + V(|\varphi|)]\right\}\mathrm{d}^4x, \qquad V(|\varphi|) = \mu^2|\varphi|^2\left(1-2\frac{|\varphi|^2}{\sigma_0^2}\right)^2$$

NR results for various systems: BS-BS, BS-antiBS, BS-BSpi/2,BS-BSpi

$$68\ \mathrm{deg} \le \chi \le 259\ \mathrm{deg}$$

$\frac{b}{GM}$	$\frac{J_{\mathrm{in}}^{\mathrm{NR}}}{GM^2}$	$\chi_{\mathrm{NR}}^{\mathrm{BS-BS}}$	$\chi_{\mathrm{NR}}^{\mathrm{BS-\overline{BS}}}$	$\chi_{\mathrm{NR}}^{\mathrm{BS-BS^{\frac{\pi}{2}}}}$	$\chi_{\mathrm{NR}}^{\mathrm{BS-BS^{\pi}}}$
9.9	1.15315	258.79(95)	216.41(1.41)	214.18(1.40)	198.41(1.26)
10.5	1.22360	170.51(89)	165.42(81)	165.28(81)	161.23(75)
11.0	1.28186	143.91(62)	142.32(60)	142.28(60)	140.83(58)
12.0	1.39839	114.43(64)	114.20(63)	114.33(53)	113.96(65)
13.0	1.51492	96.[illegible]	96.89(84)	96.75(60)	96.81(82)
14.1	1.64982	83.39(1.14)	83.38(1.14)	83.32(1.10)	83.32(1.11)
15.0	1.74798	75.51(1.23)	75.73(1.37)	75.62(1.30)	75.62(1.30)
16.0	1.86450	68.16(1.27)	68.44(1.44)	67.97(1.14)	68.50(1.47)

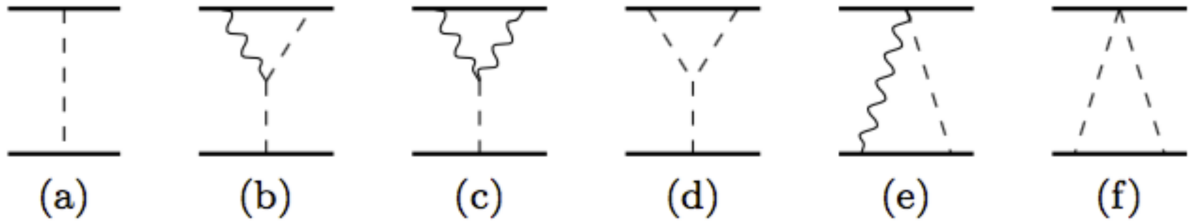
preliminary

BS scattering: Analytics vs Numerics

$$\chi(\gamma, j) = \chi^{\rm BH}(\gamma, j) + \chi^{\rm tidal}(\gamma, j) + \chi^{\rm scalar}(\gamma, j)$$

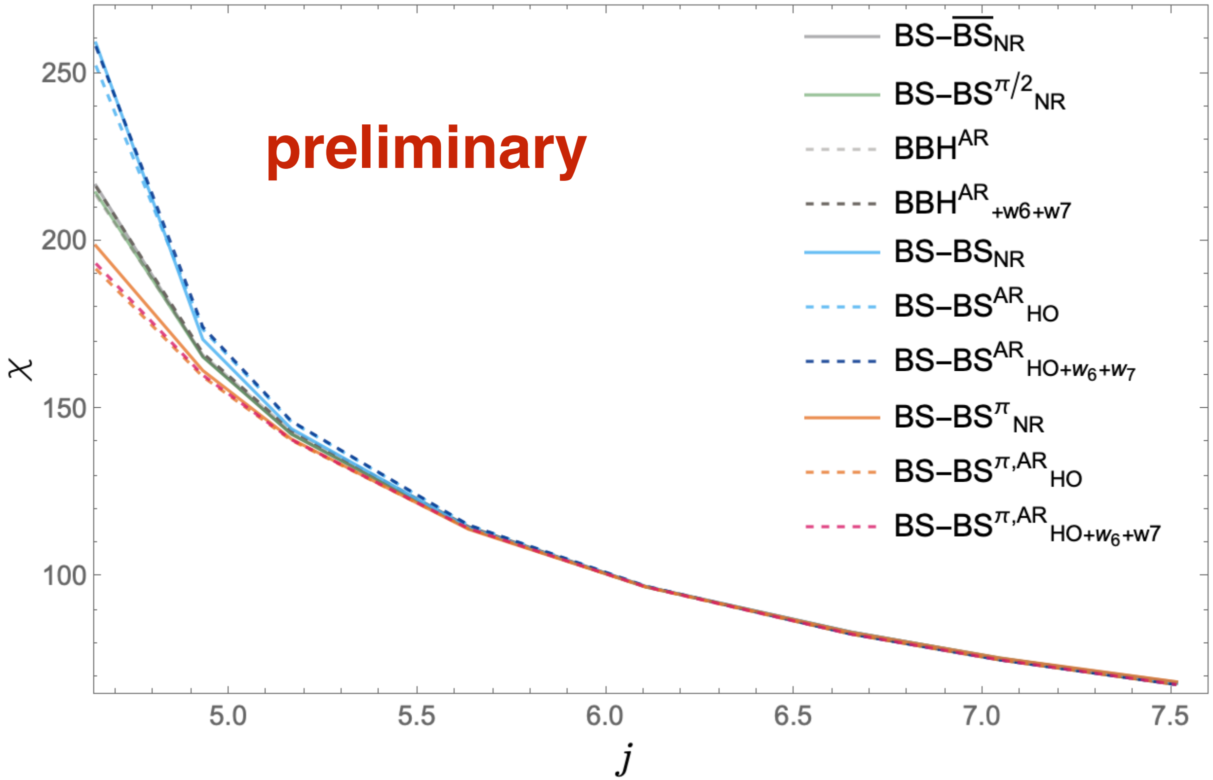
$$S_{\rm EFT} = \int \frac{\sqrt{-g}}{2}\left\{\frac{R}{8\pi G} - \left[g^{\mu\nu}\nabla_\mu\bar{\varphi}\nabla_\nu\varphi + \mu^2\bar{\varphi}\varphi\right]\right\}{\rm d}^4x - \sum_A \int \left\{m_A - 2\pi\left[\varphi(z_A)\bar{s}_A(\tau_A) + \bar{\varphi}(z_A)s_A(\tau_A)\right]\right\}{\rm d}\tau_A\,.$$

$$s_A(\tau_A) = c_A e^{i\omega_A\tau_A}$$



$$w(\gamma, \bar{r}) = w^{\rm BH}(\gamma, \bar{r}) + w^{\rm tidal}(\gamma, \bar{r}) + w^{\rm scalar}(\gamma, \bar{r})$$

good agreement using w-EOB



Conclusions

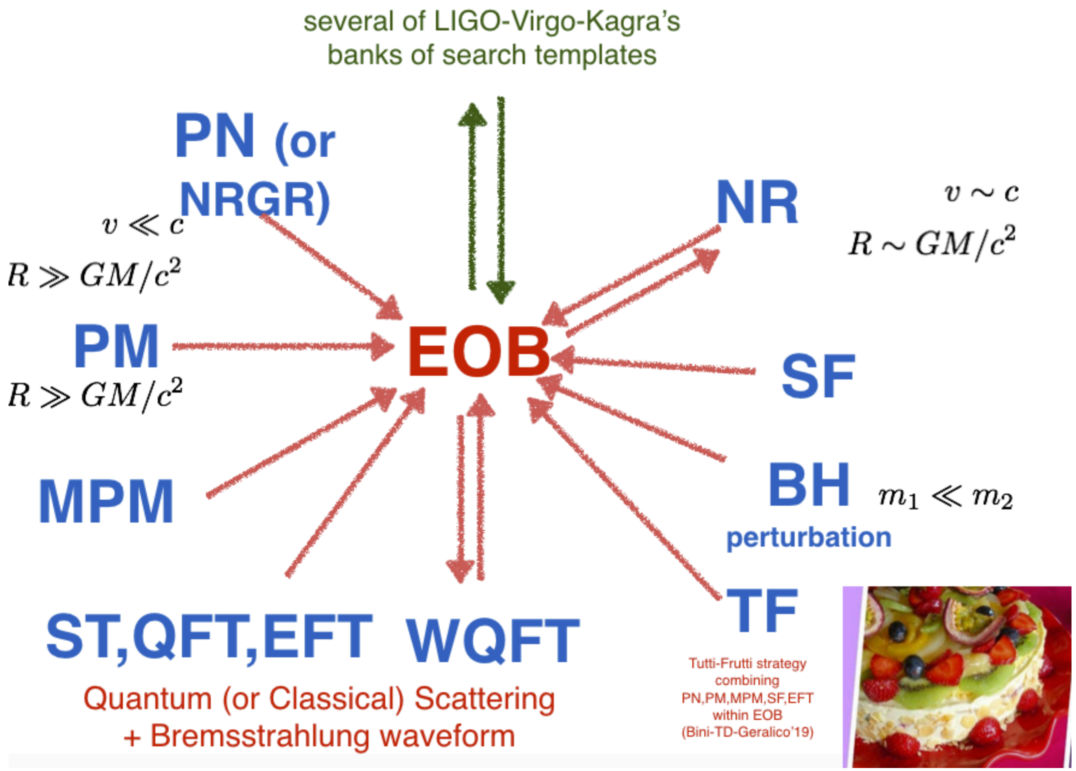
The recent synergy between various methods (time-honored and recent QFT-based ones) has led to many very interesting new vistas on the gravitational 2-body interaction.

Many impressive new results have been derived and more are in store, though one is close to reaching the limits of the new techniques

There remains puzzles to clarify

Though Numerical Relativity is and will remain very important and useful, analytical approaches will continue to play an important role.

Some improved avatar of the time-honored PN+MPM (+EFT) approach might remain most useful.



The flexible analytical nature of the EOB formalism makes it useful for incorporating new information in LIGO-Virgo-Kagra useful form.

Current Puzzles

high-energy limits?

G^3 energy loss too large

G^3 angular momentum loss too large (Manohar-Ridgway-Shen'22)

Rad-reacted G^4 scattering diverges (Porto..,Damgaard..)

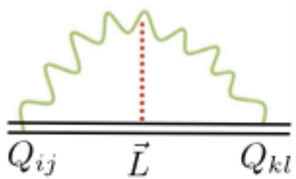
cf ACV motivation: BH formation in HE scattering

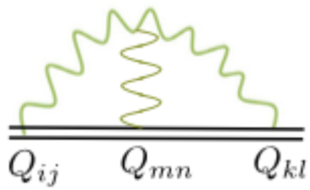
Subtleties in defining/computing angular momentum flux

(Ashtekar et al., Veneziano-Vilkovisky, Riva-Vernizzi,…)

low-energy discrepancy at 5PN between

Foffa-Sturani'19,21,22 Bluemlein et al'21 and Bini-TD-Geralico





$$
\begin{aligned}
S_{QQL} &= C_{QQL} G^2 \int dt I^{(4)}_{is} I^{(3)}_{js} \varepsilon_{ijk} L_k \\
S_{QQQ_1} &= C_{QQQ_1} G^2 \int dt I^{(4)}_{is} I^{(4)}_{js} I_{ij} \, , \\
S_{QQQ_2} &= C_{QQQ_2} G^2 \int dt I^{(3)}_{is} I^{(3)}_{js} I^{(2)}_{ij} \, .
\end{aligned}
$$

TF-constraint on 5PN O(nu^2)
EFT radiative terms

$$
0 = \frac{2973}{350} - \frac{69}{2} C_{QQL} + \frac{253}{18} C_{QQQ_1} + \frac{85}{9} C_{QQQ_2}
$$

solved (only) at G^4 by Porto-Riva-Yang'24