Modeling of Gravitational Wave Emission

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Advances in Modeling High-Energy Astrophysical Sources:

Insights from recent multimessenger discoveries

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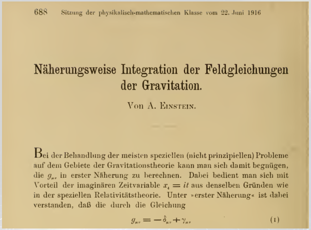
Outline

1. Introduction to GWs
   - General Relativity
   - Gravitational Waves
2. GW sources
   - Classes of GW sources
   - Transient GW sources
3. Modeling the GW emission
   - Compact binary coalescences
   - Example: GW190521
4. Conclusions

Gravitational Waves: generalities

- Gravitational waves (GWs) were predicted by Einstein's Theory of General Relativity (1916)
- Gravity as a consequence of the geometry of the spacetime



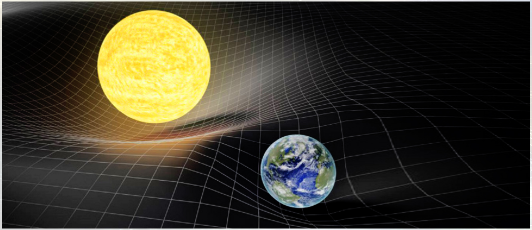
688 Sitzung der physikalisch-mathematischen Klasse vom 22. Juni 1916

Näherungsweise Integration der Feldgleichungen der Gravitation.

Von A. Einstein.

Bei der Behandlung der meisten speziellen (nicht prinzipiellen) Probleme auf dem Gebiete der Gravitationstheorie kann man sich damit begnügen, die $g_{\mu\nu}$ in erster Näherung zu berechnen. Dabei bedient man sich mit Vorteil der imaginären Zeitvariable $x_4 = it$ aus denselben Gründen wie in der speziellen Relativitätstheorie. Unter »erster Näherung« ist dabei verstanden, daß die durch die Gleichung

$$g_{\mu\nu} = -\delta_{\mu\nu} + \gamma_{\mu\nu} \quad (1)$$



Spacetime tells matter how to move; matter tells spacetime how to curve. (J. A. Wheeler)

Gravitational Waves: generalities

- **Einstein's Field Equations:**

$$\boxed{R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}}$$

Geometric part (Einstein tensor $G_{\mu\nu}$) $\rightarrow$ Geometry of spacetime

$R_{\mu\nu}$ is the Ricci Tensor; R is the scalar curvature; $g_{\mu\nu}$ is the metric

Stress-energy part (momentum-energy tensor $T_{\mu\nu}$) $\rightarrow$ Matter distribution

Gravitational Waves: generalities

- **Linearized Field Equations:**
  - In regions of spacetime far from the source (weak gravitational field) we can write the metrics as:

    $$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

    where $\eta_{\mu\nu}$ is the flat, Minkowski metric and $h_{\mu\nu} << 1$ is a small perturbation
  - By defining $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} h \eta_{\mu\nu}$, where h is the trace of $h_{\mu\nu}$ and using gauge invariance, the Einstein Equation becomes:

    $$\boxed{\Box \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu},}$$

    where $\Box = -(1/c^2)t + \nabla^2$

Gravitational Waves: generalities

- **Vacuum Solutions**

In vacuum (e.g., outside a source) $T_{\mu\nu} = 0 \Rightarrow \boxed{\Box \bar{h}_{\mu\nu} = 0}$

$\Rightarrow$ Solutions are plane waves propagating at the speed of light

$\rightarrow$ **Gravitational Waves!**

In the transverse traceless (TT) Gauge solutions read as

$$\bar{h}^{TT}_{\mu\nu}(t, z) = \begin{pmatrix} h_+ & h_x & 0 \\ h_x & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix} cos\left[\omega(t - z/c)\right]$$

with h_+ and h_x refer to the two different polarizations

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Gravitational Waves: generalities

- **Source Solutions**

In presence of matter $T_{\mu\nu} \neq 0$: the linearized Einstein equations can then be solved using the appropriate Green function; the solution is

$$\bar{h}_{ij}^{TT}(r,t) = -\frac{2G}{c^4}\,\frac{1}{r}\,\ddot{I}_{ij}(r,t)$$

where r is the distance from the source and $\ddot{I}_{ij}(r,t)$ is the second time derivative of the mass quadrupole moment (monopole and dipole disappear)

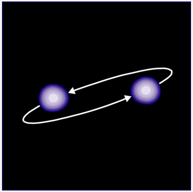
GW amplitude is proportional to:

- $\ddot{I}_{ij}(r,t) \Rightarrow$ asymmetrical acceleration of masses is needed!
- $\frac{1}{r} \Rightarrow$ the closer the source is, the higher is the amplitude of the GW signal
- $\frac{2G}{c^4} \sim 10^{-45}$ N^{-1} $\Rightarrow$ **Only astrophysical sources can produce detectable GWs!**

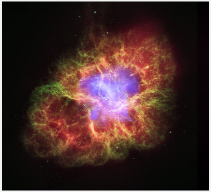
Which are the astrophysical sources of GWs?

Transients

Compact binary coalescences

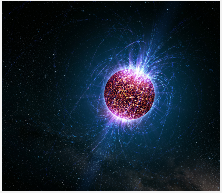


Burst

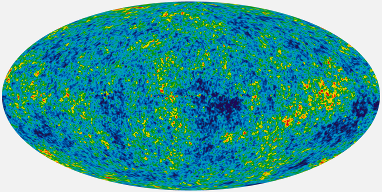


Non transients

Continuous

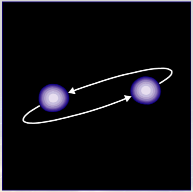


Stochastic



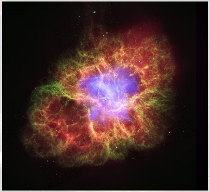
High frequency (10-1000 Hz) GW transient sources

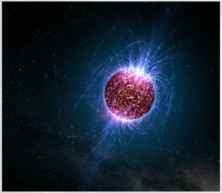
Coalescence of binary systems of Neutron Stars (NSs) and/or Black Holes (BHs)



- Accurate modeling of the GW signals → **Matched filter modeled searches**
- Energy emitted in GWs (NS-NS): $\sim 10^{-2}$ $M_{\odot}c^2$

Burst: Core collapse of massive stars and Isolated neutron stars





- The modeling of the GW signal is complicated → **Unmodeled searches**
- Energy emitted in GWs:
  - $\sim 10^{-11}$- 10^{-7} $M_{\odot}c^2$ for core collapse*
  - $\sim 10^{-16}$- 10^{-6} $M_{\odot}c^2$ for isolated NSs

* higher values are suggested by models exploring "extreme" GW emission scenarios

Modeling the GW emission: compact binary coalescences

GW emission from compact binary coalescences

Emission of GWs implies loss of orbital energy
$\Rightarrow$ the binary shrinks while emitting GWs untill it merges

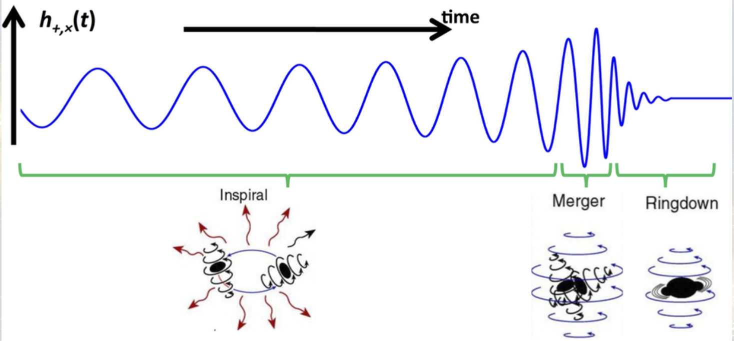


Figure from link

GW emission from compact binary coalescences

GWs carry information on the key source parameters

$$h_+(t) = A_{GW}(t)(1+\cos^2 i)\cos\phi_{GW}(t)$$

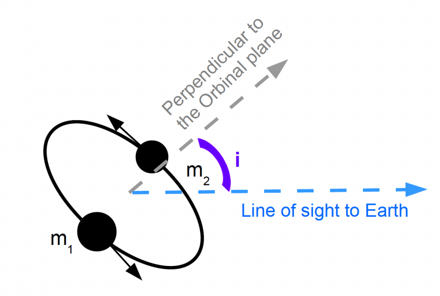
$$h_x(t) = -2A_{GW}(t)\cos i \sin\phi_{GW}(t)$$

- Extrinsic

  Sky position, luminosity distance, orbital orientation...

- Intrinsic

  masses, spins ...



The masses

- GW frequency (f_{GW}) is twice the orbital frequency (f_{orb})
- f_{GW} increases with time
- From frequency evolution we can infer the masses of the two compact objects:

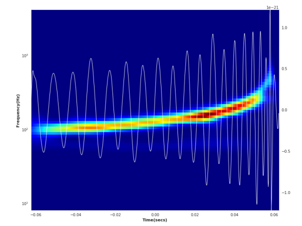


Figure from link

- Inspiral: the evolution of frequency with time is characterized by the chirp mass

$$M_c = \frac{(m_1 m_2)^{3/5}}{(m_1+m_2)^{1/5}} \propto \left(f_{\mathrm{GW}}^{-11/3} \dot{f}_{\mathrm{GW}}\right)^{3/5}$$

- Maximum amplitude of the GW signal: frequency of the innermost stable orbit

$$f_{\mathrm{ISCO}} \propto \frac{1}{M}$$

Important to understand the nature of the binary system!

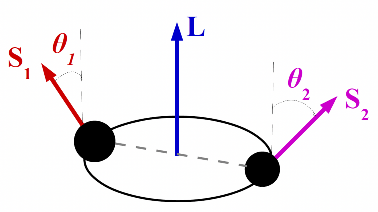
The effective spins

With GWs we can estimate two "effective spins": the **effective inspiral spin ($\chi_{\rm eff}$)** and the **effective precession spin ($\chi_{\rm p}$)**

$$\chi_{\rm eff} = \frac{c}{GM}\left(\frac{\mathbf{S_1}}{m_1} + \frac{\mathbf{S_2}}{m_2}\right)\hat{\mathbf{L}}$$

$$\chi_{\rm p} = \frac{c}{B_1 G m_1^2}\max\left(B_1 S_{1\perp}, B_2 S_{2\perp}\right);$$

$$B_1 = 2+3q/2,\, B_2 = 2+3/(2q),\, q = \frac{m_2}{m_1}$$



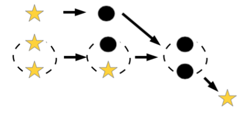
- $\chi_{\rm eff}$ quantifies the total spin parallel to the binary's orbital angular momentum
  ($\chi_{\rm eff} < 0$ implies at least one component spin tilted by $\theta > 90^\circ$ with respect to **L**)
- $\chi_{\rm p}$ depends on the component of the spins perpendicular to the binary's orbital angular momentum
  ($\chi_{\rm p} \neq 0$ implies spin-induced general relativistic precession of the orbital plane)

The spin

The spin orientations can help us to discriminate between different formation channels

Isolated binary in galactic fields

Dynamical interactions in clusters

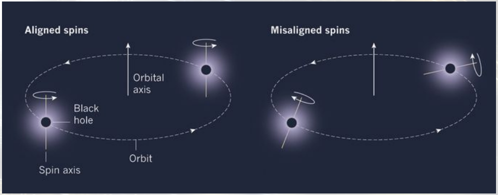


Isolated binary:

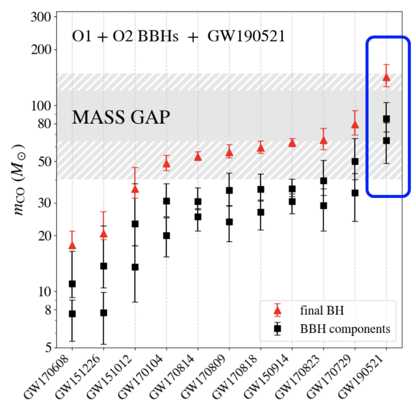
Spins preferentially aligned with the binary orbital angular momentum

Cluster binary:

Isotropic spin orientations



Example: GW190521



- GW event observed by the two LIGO detectors and Virgo
- m_1: 85^{+21}_{-14} $M_\odot$, m_2: 66^{+17}_{-18} $M_\odot$
- The primary falls in the mass gap by (pulsational) pair-instability SN

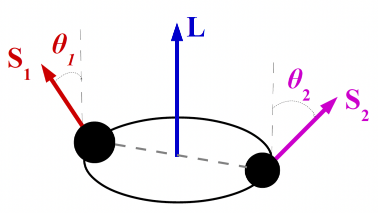
  $\Rightarrow$ **Challenge for stellar evolution**
- Isolated binary evolution is disfavoured

Dynamical scenario?

LVC 2020, PRL, 125, 101102

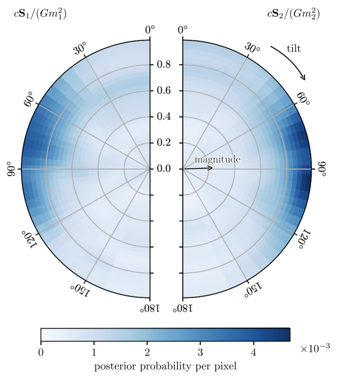
LVC 2020, ApJL, 900, 13

GW190521: the spin



$$\chi_i = \frac{c\mathbf{S_i}}{Gm_i^2} \quad \text{Dimensionless spin}$$

$$\theta_i : \quad \text{Tilt angle}$$



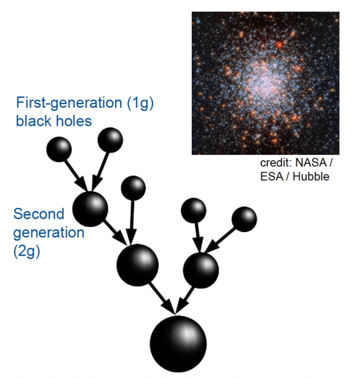
Mild evidence for large spins nearly in the orbital plane
... **dynamical origin of the system?**

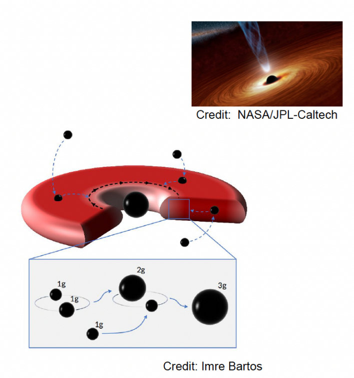
LVK Collaboration 2020, PRL, 125, 101102

LVK Collaboration 2020, ApJL, 900, 13

Dynamical scenarios for GW190521

Example: hierarchical mergers in AGN disks



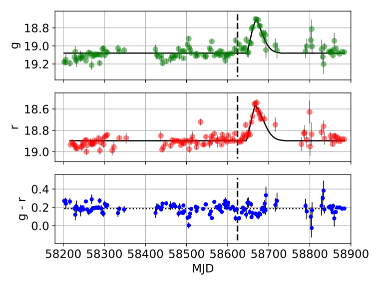


Several studies predict EM emission in association with a BBH merger when it take place in the disks of AGNs (e.g. Bartos et al. 2017, McKernan et al. 2019)

GW190521: an EM counterpart?

The Zwicky Transient Facility (ZTF) detected a candidate optical counterpart in AGN J124942.3+344929

- GW sky localization: 765 deg^2 (90% C.R.)
- ZTF observed 48% of the 90% C.R. of the GW skymap
- An EM flare observed $\sim$ 34 days after the GW event
- It is consistent with expectations for a **BBH merger in the accretion disk of an AGN** (see McKernan et al. 2019, ApJL, 884, 50)



Graham et al. 2020, PRL, 124, 251102

Common origin of the two transients seems to be preferred with respect to random coincidence (Morton et al. 2023; see, however, Ashton et al. 2021, Palmese et al. 2021)

Conclusions

Gravitational Waves:

- Predicted by Einstein in 1916 in the theory of General Relativity
- Many astrophysical sources can emit GWs
- Accurate modeling for coalescing compact binaries

  $\Rightarrow$ key tool to constrain key parameters of these sources
- Complicate modeling for other sources (supernovae ...)

  $\Rightarrow$ future detections could provide valuable insights into these events

GWs are revolutionizing our way to study the Universe...

stay tuned for new discoveries!

Backup

GW emission from core collapse of massive stars

- Massive stars: stars with masses $> 8\ M_\odot$
- Core-collapsing massive stars at their final evolutionary stage are expected to emit GWs if there is some asymmetry in the stellar envelope ejection phase
- Large uncertainties on the collapsing phase $\Rightarrow$ the amount of GW-released energy and the expected GW waveforms are highly uncertain

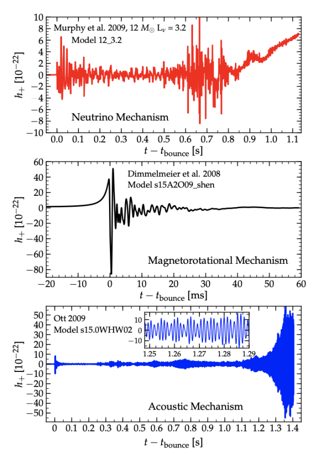
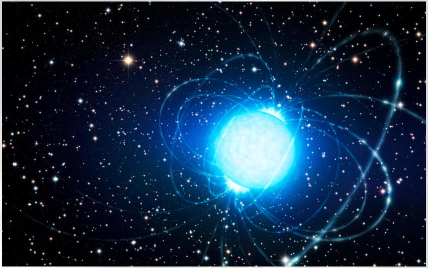


Figure from Logue et al. 2012, PRD, 86, 044023

GW emission from Isolated neutron stars

A class of isolated NSs that can emit transient GW signals is represented by magnetars

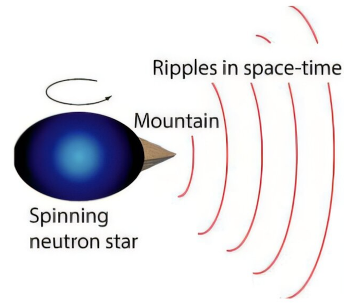
- Magnetars are slow spinning (P $\sim$ 2 -12 s), highly magnetized (B $\sim 10^{13}$ - 10^{15} G) isolated NSs
- Occasionally emit flares of soft gamma rays, with energies up to $\sim 10^{47}$ erg (giant flares)
- Thought to be associated with cracking of the crust ("starquake") or magnetic reconnection
- Some of the energy of the cracking event may excite non-radial oscillation modes in the star, which radiate GWs (e.g. Corsi & Owen 2011)



Credit: ESO/L. Calçada (CC BY 4.0)

Continuous GW emission from Isolated NSs

- NSs with non-axisymmetric deformation emit continuous GWs
- Typical amplitude:



$$h_0 = \frac{4\pi^2 G}{c^4}\frac{\epsilon I_{zz} f_{GW}^2}{d} \approx 1.06 \times 10^{-26}\left(\frac{\epsilon}{10^{-6}}\right)$$
$$\times \left(\frac{I_{zz}}{10^{38}\ \mathrm{kg\ m}^2}\right)\left(\frac{f_{GW}}{100\ \mathrm{Hz}}\right)^2\left(\frac{1\ \mathrm{kpc}}{d}\right),$$

where

- d is the distance from the detector to the source
- f_{GW} is the GW frequency $= 2\ \times f_{rot}$
- I_{zz} is the moment of inertia with respect to rotation axis
- ϵ is the ellipticity (or asymmetry of the star): $(I_{xx} - I_{yy})/I_{zz}$

Stochastic GW emission

- It could be generated by the superposition from various unresolved astrophysical and cosmological sources, such as, e.g.:
  - Core-collapse supernovae
  - Compact binary coalescences
  - Cosmic strings
  - GWs from inflation
- It cannot be characterized by a waveform, but only through statistical methods

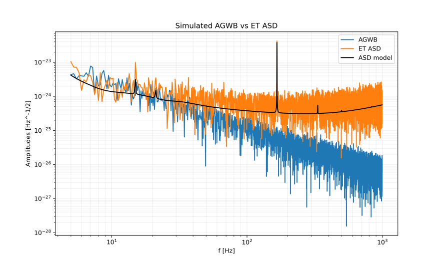


Image credit: A. Ricciardone

GW emission from compact binary coalescences

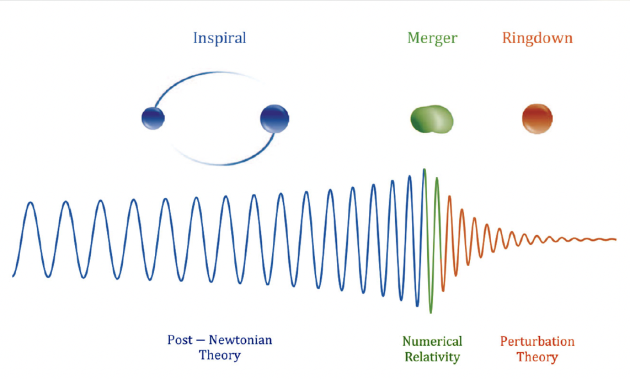
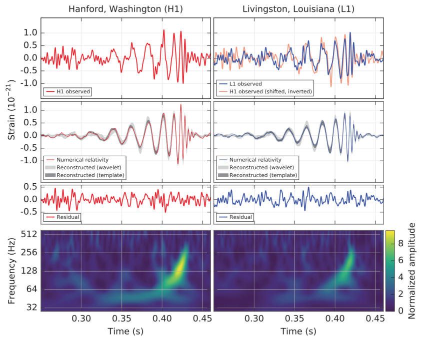


Figure from Antelis and Moreno 2017

Example - GW150914: how do we know this is a BBH?



LVC 2016, PRL, 116, 061102

- Estimating f_{GW} and $\dot{f}_{GW}$ from the data we have M_c ~30 M$_\odot$

  $\Rightarrow M = m_1 + m_2 \gtrsim$ 70 M$_\odot$

  $\Rightarrow$ **NS-NS systems cannot have $M \geq 70$ M$_\odot$**

- From the data we have $f_{\rm ISCO}$=75 Hz
- BH-NS systems can have the estimated value of M_c if the BH has at least

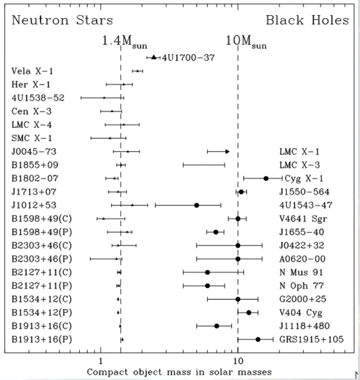
  m$_1$=1000 M$_\odot$ $\Rightarrow M > 1000$ M$_\odot$

  $\Rightarrow$ **BH-NS systems would merge at $f_{\rm ISCO}$ much lower than 75 Hz**

The BH population with EM observations

What did we know about stellar-mass BHs before GW observations?

- Our knowledge of stellar-mass BHs was limited to electromagnetic (EM) observations of X-ray binaries
- $\sim$ 20 stellar-origin BHs with dynamical mass measure
- Measured masses between 5 $M_\odot$ and 20 $M_\odot$
- No BBHs or BH-NSs known

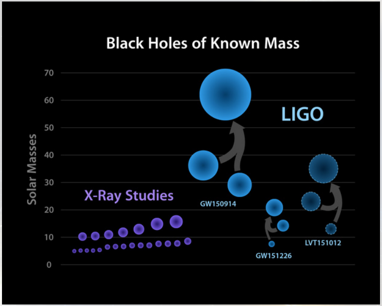


Clark, J.S. et al. 2002, A&A 392, 909

The BH population with GW observations: O1

- BBHs can form in nature and merge within a Hubble time
- First direct evidences for “heavy” stellar mass BHs (> 25 M$_\odot$)
- From the masses we can infer information on the environment:

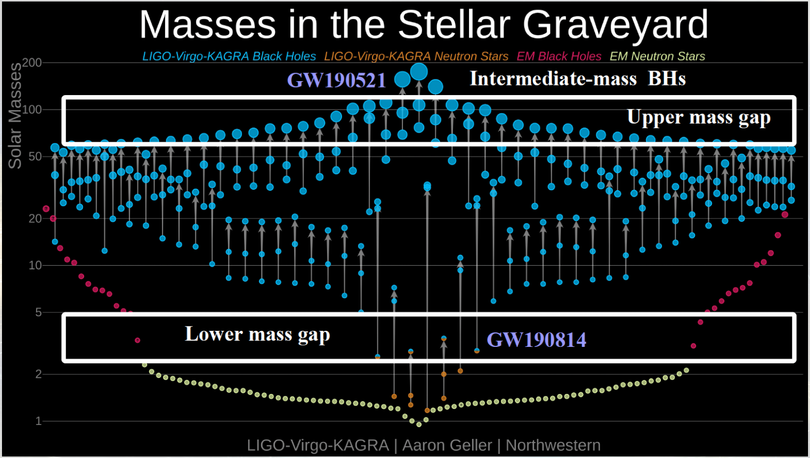
  $\rightarrow$ events like GW150914 most likely formed in low-metallicity environment (≤ 0.5 Z$_\odot$)
- BBH merger rate: 9 - 240 Gpc^{-3} yr^{-1}



LVT151012 was later re-labeled as GW151012

LVK Collaboration 2016, ApJL, 818, 22
LVK Collaboration 2016, PRX, 6, 041015
LVK Collaboration 2017, PRL 118, 221101

The BH population with GW observations: O1 + O2 + O3



LVK Collaboration 2021, arXiv:2111.03606

Population studies

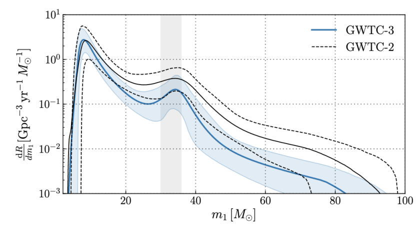
From single events to a population

- Individual GW events can reveal the properties of unique single sources, but ...
- ... a population of events is needed to shed light on how these systems form and evolve throughout the Universe

  $\rightarrow$ the statistical distribution of BH source properties such as their **mass**, **spin** and **redshift** can be used to probe the astrophysics of BBH formation and evolution

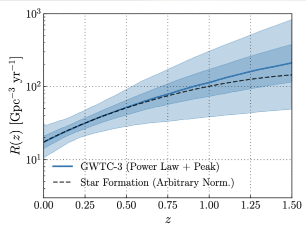
The population of BBH merging systems

Primary BH mass distribution



- BH mass distribution is non-uniform, with overdensities at BH masses of 10 $M_\odot$ and 35 $M_\odot$; tail up to 80 $M_\odot$

BBH merger rate



- BBH merger rate is observed to increase with redshift

LVK Collaboration 2023, PRX, 13, 011048; LVC 2021, ApJL, 913, L7